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Global Existence and Blow-up for Semilinear Wave Equations with Variable Coefficients*

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Abstract The authors consider the critical exponent problem for the variable coefficients wave equation with a space dependent potential and source term. For sufficiently small data with compact support, if the power of nonlinearity is larger than the expected exponent, it is proved that there exists a global solution. Furthermore, the precise decay estimates for the energy, L^2 and L^{p+1} norms of solutions are also established. In addition, the blow-up of the solutions is proved for arbitrary initial data with compact support when the power of nonlinearity is less than some constant.

Keywords Semilinear wave equations, Global existence, Energy decay, L^2 and L^{p+1} estimates, Blow up
2000 MR Subject Classification 35L05, 35L70

1 Introduction

We consider the following Cauchy problem for the semilinear wave equation with variable coefficients:

$$u_{tt} - \operatorname{div}(b(x)\nabla u) + a(x)u_t = |u|^{p-1}u, \quad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$u(0,x) = \varepsilon u_0(x), \quad u_t(0,x) = \varepsilon u_1(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $\varepsilon > 0$, the coefficients $a(x) \in C^0(\mathbb{R}^n), b(x) \in C^1(\mathbb{R}^n)$ are positive functions which will be specified later and the initial data $u_0 \in H^1(\mathbb{R}^n)$, $u_1 \in L^2(\mathbb{R}^n)$ have compact support

$$u_0(x) = u_1(x) = 0 \quad \text{for } |x| > R,$$

where the exponent p of nonlinearity satisfies

$$1 < p < \frac{n+2}{n-2} \quad \text{for } n \geq 3, \quad 1 < p < \infty \quad \text{for } n = 1, 2.$$

Such a system is generally accepted as models for travelling waves in a nonhomogeneous gas with damping changing with the position. The unknown u denotes the displacement, the

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coefficient b called the bulk modulus, accounts for changes of the temperature depending on the location, while a is referred as the friction coefficient or potential (see [5]). This problem has been studied intensively for the homogenous medium, but the result are scarce for the variable coefficient case. In [11] there is the authors find decay estimates for wave equations with variable coefficient, however, no nonlinearity is present. In addition, [1], looked at an equation with nonlinear internal damping but for bounded domains. To our knowledge, the results of this paper are the first to be obtained for semilinear wave equations which exhibit space dependent hyperbolic operators and space dependent potential on the entire space $\mathbb{R}^n$.

Our aim is to determine the critical exponent p_c, which is a number defined by the following property:

If $p_c < p$, all the small data solutions of (1.1)–(1.2) are global; if $1 < p \le p_c$, the time-local solution cannot be extended time-globally for some data. What is more, it is expected that the critical exponent agrees with that of only the hyperbolic operators and constant coefficient case.

It is of interest to compare the semilinear wave equations (1.1)–(1.2) for different coefficients. When $a(x) = 0$ namely the damping term is missing and $b(x) = 1$, that is

$$u_{tt} - \Delta u = |u|^{p-1}u, \quad x \in \mathbb{R}^n, \ t > 0, \tag{1.3}$$

$$u(0,x) = \varepsilon u_0(x), \quad u_t(0,x) = \varepsilon u_1(x), \quad x \in \mathbb{R}^n, \tag{1.4}$$

for small data with compact support, there exists a critical exponent $p_w(n)$ such that the solutions of (1.3)–(1.4) are global if $p > p_w(n)$, and the solutions of (1.3)–(1.4) blow up if $1 < p \le p_w(n)$. Actually, the critical exponent $p_w(n)$ is the positive root of the quadratic equation $(n-1)p^2 - (n+1)p - 2 = 0$ for $n \ge 2$ ($p_w(1) = \infty$). For the details, one can see Takamura and Wakasa [15]. This is the famous Strauss conjecture and the proof was completed by the effort of many mathematicians (see [3, 4, 8–9, 12–14, 16, 22]).

There are many results for the semilinear damped wave equation. Todorova and Yordanov [17–18] studied the constant coefficients case of (1.1)–(1.2), that is

$$u_{tt} - \Delta u + u_t = |u|^{p-1}u, \quad x \in \mathbb{R}^n, \ t > 0, \tag{1.5}$$

$$u(0,x) = \varepsilon u_0(x), \quad u_t(0,x) = \varepsilon u_1(x), \quad x \in \mathbb{R}^n. \tag{1.6}$$

They developed a weighted energy method and determined that the critical exponent is $p_c(n) = 1 + \frac{2}{n}$; more precisely, they proved small data global existence in the case $p > p_c(n)$ and blow-up for all solutions of (1.5)–(1.6) with positive on average data in the case $1 < p < p_c(n)$. Later on Zhang [21] showed that the critical case $p = p_c(n)$ belongs to the blow-up region. We mention that Todorova and Yordanov [17–18] assumed that data have compact support and essentially used this property. However, Ikehata and Tanizawa [6] extended the global existence in [17–18] to certain non-compactly supported initial data. We remarked that the critical exponent $p_c(n)$ of (1.5)–(1.6) is the same as the famous Fujita's critical exponent (see [2]) for the heat equation $u_t - \triangle u = u^p$.

On the other hand, Ikehata, Todorova and Yordanov [7] solved the critical exponent problem for the wave equation (1.1)–(1.2) when $b(x) = 1$,

$$u_{tt} - \Delta u + a(x)u_t = |u|^{p-1}u, \quad x \in \mathbb{R}^n,\ t > 0,$$
$$u(0,x) = \varepsilon u_0(x), \quad u_t(0,x) = \varepsilon u_1(x), \quad x \in \mathbb{R}^n.$$

They determined that the critical exponent is $p_c(n,\alpha) = 1 + \frac{2}{n-\alpha}$ by using a refined multiplier method, where $a(x) \in C^1(\mathbb{R}^n)$ is a radially symmetric function satisfying

$$a(x) \sim a_0(1+|x|)^{-\alpha}, \quad |x| \to \infty$$

with $a_0 > 0$ and $\alpha \in [0,1)$. They derived the global existence of the sufficiently small data for $p > p_c(n,\alpha)$, also obtained precise decay estimates for the energy, L^2 and L^{p+1} norms of solutions. Moreover, they proved that the solutions blow up for $1 < p \le p_c(n,\alpha)$ by applying the method of Zhang [21]. Recently, Nishihara [10] considered the semilinear wave equation with time-dependent damping

$$u_{tt} - \Delta u + a(t)u_t = |u|^{p-1}u,$$

where $a(t) = a_0(1+t)^{-\beta},\ \beta \in (-1,1)$. He proved that the critical exponent is $p_c(n) = 1 + \frac{2}{n}$. Wakasugi [20] considered the Cauchy problem for the semilinear wave equation with space-time dependent damping

$$u_{tt} - \Delta u + a(x)b(t)u_t = |u|^{p-1}u, \quad x \in \mathbb{R}^n,\ t > 0,$$

where $a(x) = a_0(1+|x|^2)^{-\frac{\alpha}{2}},\ b(t) = (1+t)^{-\beta}$, with $a_0 > 0,\ \alpha,\beta \ge 0,\ \alpha+\beta < 1$, and proved that the expected exponent is given by

$$p_c = 1 + \frac{2}{n-\alpha},$$

which is the critical exponent for the semilinear wave equations with space dependent potential. This shows that, roughly speaking, time-dependent coefficient of damping term does not influence the critical exponent. This is also why we consider wave equation with only space dependent potential.

The main innovation in this paper is that we find the exponent $p_{cr}(n,\alpha,\beta)$ such that for sufficiently small data and $p_{cr}(n,\alpha,\beta) < p$, the solutions of (1.1)–(1.2) are global. In addition, we also determine the exponent $p_2(n,\alpha,\beta)$ such that $1 < p \le p_2(n,\alpha,\beta)$, where $p_2(n,\alpha,\beta) \le p_{cr}(n,\alpha,\beta)$, the solution of problem (1.1)–(1.2) does not exist globally for some data. However, when $p_2(n,\alpha,\beta) < p \le p_{cr}(n,\alpha,\beta)$, the solution of (1.1)–(1.2) exists globally or not, we have no result. In the future paper, we aim to study it.

For the potential $a(x)$ and the bulk modulus $b(x)$, we assume that

$$\frac{a_0}{(1+|x|)^\alpha} \le a(x) \le \frac{a_1}{(1+|x|)^\alpha} \tag{1.7}$$

and

$$b_0(1+|x|)^\beta \le b(x) \le b_1(1+|x|)^\beta, \tag{1.8}$$

where a_0, a_1, b_0, $b_1 > 0$ are constants, and α and β belong to the following range of exponents

$$0 \le \alpha < 1, \quad 0 \le \beta < 2, \quad 2\alpha + \beta \le 2, \tag{1.9}$$

the exponent of focusing nonlinearity is given explicitly by

$$p_{cr}(n, \alpha, \beta) := \frac{4(2-\beta)}{4n - 4\alpha - \beta n} + 1, \quad p_2(n, \alpha, \beta) := \frac{2-\beta}{n-\alpha} + 1. \tag{1.10}$$

In the case of variable coefficient wave equation we observe some new phenomena. The decay rates pinpoint the interaction between the coefficients a and b. It is worthwhile to mention that the energy decay rate goes to infinity (see Corollary 1.3) when $\beta \to 2^-$ and $\alpha \to 0^+$. This shows that the range of the exponent β in (1.9) is natural. On the other hand, in the case of $\beta \to 0^+$, $\alpha \to 1^-$, the energy of global solutions decays polynomially like t^{-n} as $t \to \infty$ (see Corollary 1.3).

Before stating the main results, we need some preparations concerning the space dependent on factors $a(x)$ and $b(x)$.

Hypothesis A (see [11]) Under the above assumptions (1.7) and (1.8), there exists a subsolution $A(x)$ which satisfies

$$\operatorname{div}(b(x)\nabla A(x)) = a(x), \quad x \in \mathbb{R}^n, \tag{1.11}$$

and has the following properties:

$$(a1) \quad A(x) \ge 0 \quad \text{for all } x, \tag{1.12}$$

$$(a2) \quad A(x) = O(|x|^{2-\alpha-\beta}) \quad \text{for large } |x|, \tag{1.13}$$

$$(a3) \quad \mu := \liminf_{x\to\infty} \frac{a(x)A(x)}{b(x)|\nabla A(x)|^2} > 0. \tag{1.14}$$

Here we announce our main results for the existence of the global solutions for sufficiently small data.

Let us denote $X_1(0,T) := C([0,T]; H^1(\mathbb{R}^n)) \cap C^1([0,T]; L^2(\mathbb{R}^n))$.

Theorem 1.1 *Let $p_{cr}(n, \alpha, \beta)$ be defined in (1.10), $a(x)$, $b(x)$ satisfy (1.7), (1.8) respectively and the exponents α, β belong to (1.9). If $p_{cr}(n, \alpha, \beta) < p < \frac{n+2}{n-2}$ for $n \ge 3$ and $p_{cr}(n, \alpha, \beta) < p < \infty$ for $n = 1, 2$. Then there exists a number $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, the problems (1.1)–(1.2) has a solution $u \in X_1(0, \infty)$ satisfying*

$$\int_{\mathbb{R}^n} \mathrm{e}^{(\mu-\delta)\frac{A(x)}{t}} u^2 \mathrm{d}x \le C t^{\frac{\alpha}{2-\alpha-\beta}+\delta-\mu},$$

$$\int_{\mathbb{R}^n} \mathrm{e}^{(\mu-\delta)\frac{A(x)}{t}} (u_t^2 + b|\nabla u|^2) \mathrm{d}x \le C t^{\delta-\mu-1},$$

$$\int_{\mathbb{R}^n} \mathrm{e}^{(\mu-\delta)\frac{A(x)}{t}} |u|^{p+1} \mathrm{d}x \le C t^{-\rho+\delta-\mu-1}$$

for large $t \gg 1$, where

$$\rho := \frac{(p-1)(4n - 4\alpha - \beta n) - 4(2-\beta)}{4(2-\alpha-\beta)} - (p-1)\delta,$$

and $\delta > 0$ is an arbitrarily small number.

Remark 1.1 In the case of constant coefficients $a(x) = 1$, $b(x) = 1$, we have $\alpha = 0$, $\beta = 0$. Thus $p_{cr}(n, \alpha, \beta)$ becomes the Fujita's critical exponent $1 + \frac{2}{n}$. Furthermore, in the case of constant bulk modulus $b(x) = 1$, namely, $\beta = 0$, then the exponent agrees with that of only the space dependent coefficient case in the literature [7].

Proposition 1.1 (see [11]) *Let $a(x)$, $b(x)$ satisfy* (1.7), (1.8) *respectively.*

(i) (1.11) *admits a solution $A(x)$ such that*

$$\begin{aligned} &(A1) \quad A_0(1 + |x|)^{2-\alpha-\beta} \le A(x) \le A_1(1 + |x|)^{2-\alpha-\beta}, \\ &(A2) \quad \mu > 0, \end{aligned}$$

where A_0 and A_1 are positive constants.

(ii) *In the special case,*

$$a(x) \sim a_2|x|^{-\alpha}, \quad b(x) \sim b_2|x|^{\beta} \quad \text{for large } x \tag{1.15}$$

with $a_2 > 0$, $b_2 > 0$, (1.11) *has a solution with the following properties:*

$$\begin{aligned} &(A3) \quad A(x) \sim \frac{a_2}{b_2(n-\alpha)(2-\alpha-\beta)}|x|^{2-\alpha-\beta}, \\ &(A4) \quad \mu = \frac{n-\alpha}{2-\alpha-\beta}. \end{aligned}$$

Combining the above proposition with Theorem 1.1, we can give more explicit weighted estimates.

Corollary 1.1 *Under the assumptions in Theorem* 1.1, *the following estimates hold:*

$$\begin{aligned} &\int_{\mathbb{R}^n} \mathrm{e}^{A_0(\mu-\delta)\frac{|x|^{2-\alpha-\beta}}{t}} u^2 \mathrm{d}x \le Ct^{\frac{\alpha}{2-\alpha-\beta}+\delta-\mu}, \\ &\int_{\mathbb{R}^n} \mathrm{e}^{A_0(\mu-\delta)\frac{|x|^{2-\alpha-\beta}}{t}} (u_t^2 + b|\nabla u|^2) \mathrm{d}x \le Ct^{\delta-\mu-1}, \\ &\int_{\mathbb{R}^n} \mathrm{e}^{A_0(\mu-\delta)\frac{|x|^{2-\alpha-\beta}}{t}} |u|^{p+1} \mathrm{d}x \le Ct^{-\rho+\delta-\mu-1} \end{aligned}$$

for large $t \gg 1$, where ρ is defined as Theorem 1.1.

Corollary 1.2 *Assume that $a(x)$, $b(x)$ satisfy the condition* (1.15). *Then for every $\delta > 0$, the solution of* (1.1)–(1.2) *satisfies*

$$\begin{aligned} &\int_{\mathbb{R}^n} \mathrm{e}^{\frac{a_2}{b_2}(2-\alpha-\beta+\delta)^{-2}\frac{|x|^{2-\alpha-\beta}}{t}} u^2 \mathrm{d}x \le Ct^{\delta-\frac{n-2\alpha}{2-\alpha-\beta}}, \\ &\int_{\mathbb{R}^n} \mathrm{e}^{\frac{a_2}{b_2}(2-\alpha-\beta+\delta)^{-2}\frac{|x|^{2-\alpha-\beta}}{t}} (u_t^2 + b|\nabla u|^2) \mathrm{d}x \le Ct^{\delta-\frac{n-\alpha}{2-\alpha-\beta}-1}, \\ &\int_{\mathbb{R}^n} \mathrm{e}^{\frac{a_2}{b_2}(2-\alpha-\beta+\delta)^{-2}\frac{|x|^{2-\alpha-\beta}}{t}} |u|^{p+1} \mathrm{d}x \le Ct^{-\rho+\delta-\frac{n-\alpha}{2-\alpha-\beta}-1} \end{aligned}$$

for large $t \gg 1$, where ρ is defined as Theorem 1.1.

Corollary 1.3 *Assume that* $a(x)$, $b(x)$ *satisfy the condition* (1.15). *Then for every* $\delta > 0$, *the solution of* (1.1)–(1.2) *satisfies*

$$\begin{aligned}
&\int_{\mathbb{R}^n} u^2 \mathrm{d}x \leq C t^{\delta - \frac{n-2\alpha}{2-\alpha-\beta}}, \\
&\int_{\mathbb{R}^n} (u_t^2 + b|\nabla u|^2) \mathrm{d}x \leq C t^{\delta - \frac{n-\alpha}{2-\alpha-\beta} - 1}, \\
&\int_{\mathbb{R}^n} |u|^{p+1} \mathrm{d}x \leq C t^{-\rho + \delta - \frac{n-\alpha}{2-\alpha-\beta} - 1}
\end{aligned}$$

for large $t \gg 1$, *where* ρ *is defined as Theorem* 1.1.

Another important consequence of main conclusions is that the energy estimate under consideration, restricted to $\{x : A(x) \geq t^{1+\kappa}\}$ with $\kappa > 0$, decays exponentially.

Corollary 1.4 *Under the assumptions in Theorem* 1.1, *for arbitrary fixed* $\kappa > 0$ *and every* $\delta > 0$, *the solution of* (1.1)–(1.2) *satisfies*

$$\int_{A(x) \geq t^{1+\kappa}} (u_t^2 + b|\nabla u|^2) \mathrm{d}x \leq C \mathrm{e}^{-(\mu - \delta) t^{\kappa}}$$

for large $t \gg 1$, *where* $A(x)$ *and* μ *are given by Hypothesis A.*

Thus, the local energy in $\{x : A(x) \geq t^{1+\kappa}\}$ decays exponentially fast as $t \to \infty$. This observation confirms that for small data the global solutions of (1.1)–(1.2) have parabolic asymptotic profiles.

The blow up result in the case when $1 < p \leq p_2(n, \alpha, \beta) := \frac{2-\beta}{n-\alpha} + 1$ is as follows.

Theorem 1.2 *Let* $a(x)$ *and* $b(x)$ *satisfy* (1.7) *and* (1.8) *respectively, and let the exponents* α, β *belong to*

$$0 \leq \alpha < 1, \quad 0 \leq \beta < 2, \quad 2\alpha + \beta \leq 2, \quad \alpha + \beta > 1.$$

When $1 < p \leq p_2(n, \alpha, \beta)$, *if the initial data* (u_0, u_1) *satisfy*

$$\int_{\mathbb{R}^n} (a(x) u_0 + u_1) \mathrm{d}x > 0,$$

then the solution of problem (1.1)–(1.2) *does not exist globally for any* $\varepsilon > 0$.

Remark 1.2 In the case of constant $a(x) = 1,\ b(x) = 1$, we obtain $\alpha = 0,\ \beta = 0$. Then the exponent $p_2(n, 0, 0) = 1 + \frac{2}{n}$ becomes the Fujita's critical exponent. In addition, when $b(x) = 1$, namely, $\beta = 0$, the exponent $p_2(n, \alpha, 0) = 1 + \frac{2}{n-\alpha}$ coincides with the blow up result in the literature [7].

Remark 1.3 When $p_2(n, \alpha, \beta) < p \leq p_{cr}(n, \alpha, \beta)$, it is expected that the solution of (1.1)–(1.2) either exists globally or blows up in the finite time. However, we have no result. In the future paper, we aim to study it.

2 Small Data Global Solutions

We first state a proposition about the support of the solutions for the wave equation with variable coefficients. Fortunately, the argument has presented in [11].

Proposition 2.1 (Finite Speed of Propagation) *Assume that $b(x)$ satisfies* (1.8). *If u_0, u_1 are supported inside the ball $|x| < R$, then $u(x,t) = 0$ when*

$$|x| > R_t := [(1+R)^{\frac{2-\beta}{2}} + t\sqrt{b_1}]^{\frac{2}{2-\beta}}.$$

Moreover, one has that the radius R_t for a general $b(x)$ satisfies the following estimates:

$$R_t \sim R + Ct^{\frac{2-\beta}{2}}.$$

Proposition 2.2 (see [19]) *Define $\gamma := \frac{2\alpha}{2-\beta}$, then $\gamma \in [0,1]$ and*

$$g(t) := \inf\{a(x) : x \in \text{supp } u(\cdot,t)\}, \tag{2.1}$$

$$G(t) := \sup\{A(x) : x \in \text{supp } u(\cdot,t)\}. \tag{2.2}$$

Then

$$g(t) \geq g_0 t^{-\gamma}, \qquad t \geq T, \tag{2.3}$$

$$G(t) \leq G_0 t^{2-\gamma}, \quad t \geq T, \tag{2.4}$$

where g_0 and G_0 are positive constants.

To show the global existence of solutions of problem (1.1)–(1.2) for sufficiently small data, we rely on a modification of technique developed by Todorova and Yordanov [17–18]. Indeed, for the solution $u(x,t)$ of problem (1.1)–(1.2) we set $v = uw^{-1}$, where w is an approximate solution of linear part of (1.1)–(1.2) and can be defined by

$$w(t,x) := t^{-m} \mathrm{e}^{-m_1 \frac{A(x)}{t}},$$

where the parameters $m := \mu - 2\delta$, $m_1 := \mu - \delta$ and $A(x)$ is determined in Hypothesis A, where $\delta \in (0, \frac{1}{2}\mu)$ is a small number.

We also set

$$w_1(t,x) := \frac{3}{4}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1} w(t,x),$$

where

$$\sigma(x) := (\mu - \delta)A(x).$$

We consider the semilinear wave equation of the form

$$u_{tt} - \operatorname{div}(b(x)\nabla u) + a(x)u_t = |u|^{p-1}u, \tag{2.5}$$

where the coefficients $a(x) \in C^0(\mathbb{R}^n),\ b(x) \in C^1(\mathbb{R}^n)$.

Our goal is to derive a weighted energy identity for u. Let $v = w^{-1}u$ and substitute $u = wv$ into (2.5), we have

$$v_{tt} - b\nabla v - (\nabla b + 2bw^{-1}\nabla w)\nabla v + (2w^{-1}w_t + a)v_t + Q(t,x)v = w^{p-1}|v|^p, \tag{2.6}$$

where

$$Q(t,x) = w^{-1}(w_{tt} - \operatorname{div}(b\nabla w) + aw_t).$$

Multiplying both sides of (2.6) by $wv + w_1 v_t$ and integrating over $\mathbb{R}^n$, we have the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}E(v_t,\nabla v, v) + F(v_t,\nabla v) + G(v) = H(t) + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{p+1}\int_{\mathbb{R}^n} w_1 w^{p-1}|v|^{p+1}\mathrm{d}x\Big), \tag{2.7}$$

where the weighted energy

$$E(v_t,\nabla v, v) = \frac{1}{2}\int_{\mathbb{R}^n}[w_1(v_t^2 + b|\nabla v|^2) + 2wv_t v + (Qw_1 + w_t + aw)v^2]\mathrm{d}x \tag{2.8}$$

and

$$\begin{aligned} F(v_t,\nabla v) &= \frac{1}{2}\int_{\mathbb{R}^n}(-\partial_t w_1 + 2(a + 2w^{-1}w_t)w_1 - 2w)v_t^2\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^n} b(\nabla w_1 - 2w_1 w^{-1}\nabla w)v_t\nabla v\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} b(-\partial_t w_1 + 2w)|\nabla v|^2\mathrm{d}x, \qquad (2.9) \\ G(v) &= \frac{1}{2}\int_{\mathbb{R}^n}[Qw - (Qw_1)_t]v^2\mathrm{d}x, \qquad (2.10) \\ H(t) &= \int_{\mathbb{R}^n}\underbrace{\Big[w^p - \frac{1}{p+1}(w_1 w^{p-1})_t\Big]}_{K(t,x)}|v|^{p+1}\mathrm{d}x. \qquad (2.11) \end{aligned}$$

Different conditions are needed for the damping weights w_1, w to ensure that $F(v_t,\nabla v) + G(v) > 0$, and hence the weighted energy $E(v_t,\nabla v, v)$ is bounded.

Lemma 2.1 *Let $a(x)$ and $b(x)$ satisfy conditions* (1.7) *and* (1.8)*. There exists a large number $t_0 > 0$ such that for $t \geq t_0$ the following conditions hold:*

(i) $Q \geq 0$, $Q_t \leq 0$,

(ii) $-\partial_t w_1 + w \geq 0$,

(iii) $(-\partial_t w_1 + 2(a + 2w^{-1}w_t)w_1 - 2w)(-\partial_t w_1 + 2w) \geq b(\nabla w_1 - 2w_1 w^{-1}\nabla w)^2$.

If u is a solution of (1.1)–(1.2)*, where $t \in (t_0, T_m)$, we have*

$$E(v_t,\nabla v, v)(t) \leq E(v_t,\nabla v, v)(t_0) + \underbrace{\frac{1}{p+1}\int_{\mathbb{R}^n} w_1 w^{p-1}|v|^{p+1}\mathrm{d}x}_{\text{(I)}} + \underbrace{\int_{t_0}^{t} H(s)\mathrm{d}s}_{\text{(II)}}. \tag{2.12}$$

Proof The proof of conditions (i)–(iii) is similar to [19], so we omit it. Notice that conditions (i) and (ii) imply

$$Qw - \partial_t(Qw_1) = Q(w - \partial_t w_1) - Q_t w_1 \geq 0,$$

hence $G(v) \geq 0$. Condition (iii) and $-\partial_t w_1 + 2w \geq 0$, which follows from (ii), guarantee that the quadratic form $F(v_t, \nabla v) \geq 0$. Therefore, after integrating (2.7) over $[t_0, t]$, we can obtain the final inequality, where $t_0 < t < T_m$.

We need estimates (I) and (II) in the right side of (2.12). Now we introduce a new function:

$$W(t) := \int_{\mathbb{R}^n} w_1(t,x)(v_t^2 + b|\nabla v|^2)\mathrm{d}x + \int_{\mathbb{R}^n} a(x)w(t,x)v^2(t,x)\mathrm{d}x.$$

For (I) in the right side of (2.12), we have the following crucial estimate.

Lemma 2.2 *Let $a(x)$ and $b(x)$ satisfy conditions* (1.7)–(1.8). *If $p > p_{cr}(n,\alpha,\beta)$, then there exists a number $\rho > 0$, which depends on p, n, α, β and δ such that*

$$\int_{\mathbb{R}^n} \Big(1 + \frac{\sigma(x)}{t}\Big) w_1 w^{p-1}|v|^{p+1}\mathrm{d}x \leq Ct^{-\rho} W(t)^{\frac{p+1}{2}}, \quad t \in [t_0, T_m), \tag{2.13}$$

where

$$\rho := \frac{(p-1)(4n - 4\alpha - \beta n) - 4(2-\beta)}{4(2-\alpha-\beta)} - (p-1)\delta.$$

Proof Using definitions of $w(t,x)$ and $w_1(t,x)$, we have

$$w_1 w^{p-1}|v|^{p+1} = \frac{3}{4}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1} w^p|v|^{p+1} \leq Ctw^p|v|^{p+1}$$

and

$$\frac{\sigma(x)}{t} w_1 w^{p-1}|v|^{p+1} = \frac{3}{4} t \frac{\sigma(x)}{t}\Big(6 + \frac{\sigma(x)}{t}\Big)^{-1} w^p|v|^{p+1} \leq Ctw^p|v|^{p+1}.$$

We add the above two estimates and integrate it over $\mathbb{R}^n$, then get

$$\int_{\mathbb{R}^n} \Big(1 + \frac{\sigma(x)}{t}\Big) w_1 w^{p-1}|v|^{p+1}\mathrm{d}x \leq Ct^{-(pm-1)} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{p\sigma(x)}{t}}|v|^{p+1}\mathrm{d}x. \tag{2.14}$$

By setting

$$\psi(t,x) = \frac{\sigma(x)}{2t}, \quad \eta = \frac{2p}{p+1},$$

we can rewrite (2.14) in the form

$$\int_{\mathbb{R}^n} \Big(1 + \frac{\sigma(x)}{t}\Big) w_1 w^{p-1}|v|^{p+1}\mathrm{d}x \leq Ct^{-(pm-1)} \|\mathrm{e}^{-\eta\psi(t,\cdot)}v\|_{p+1}^{p+1}. \tag{2.15}$$

To estimate the weighted norm $\|\mathrm{e}^{-\eta\psi(t,\cdot)}v\|_{p+1}$, we use the Gagliardo-Nirenberg inequality

$$\|\mathrm{e}^{-\eta\psi(t,\cdot)}v\|_{p+1} \leq \underbrace{\|\mathrm{e}^{-\eta\psi(t,\cdot)}v\|_2^{\theta}}_{\text{(A)}} \underbrace{\|\nabla(\mathrm{e}^{-\eta\psi(t,\cdot)}v)\|_2^{1-\theta}}_{\text{(B)}}, \tag{2.16}$$

where

$$\theta = 1 - n\Big(\frac{1}{2} - \frac{1}{p+1}\Big).$$

We estimate the first term (A) in the right side of inequality (2.16), beginning with following decomposition

$$\begin{aligned}\mathrm{e}^{-2\eta\psi}v^2(t,x) &= \mathrm{e}^{-\frac{\sigma(x)}{t}}t^{-m}t^m\mathrm{e}^{-\frac{(p-1)\sigma(x)}{(p+1)t}}t^{-\frac{\alpha}{2-\alpha-\beta}}t^{\frac{\alpha}{2-\alpha-\beta}}v^2\\ &= t^m w(t,x)v^2(t,x)(\mathrm{e}^{-\frac{(p-1)\sigma(x)}{(p+1)t}}t^{-\frac{\alpha}{2-\alpha-\beta}})t^{\frac{\alpha}{2-\alpha-\beta}}. \end{aligned}\tag{2.17}$$

Furthermore, there exists a constant $C>0$, such that for any $x>0$, it is true that

$$x^{\frac{\alpha}{2-\alpha-\beta}} \le C\mathrm{e}^{\frac{x(p-1)}{p+1}}, \quad 0\le\alpha<1, \quad \frac{\alpha}{2-\alpha-\beta}\le 1.$$

Thus,

$$\begin{aligned}\mathrm{e}^{-\frac{(p-1)\sigma(x)}{(p+1)t}}t^{-\frac{\alpha}{2-\alpha-\beta}} &\le t^{-\frac{\alpha}{2-\alpha-\beta}}\Big(\frac{\sigma(x)}{t}\Big)^{-\frac{\alpha}{2-\alpha-\beta}}\\ &= \sigma(x)^{-\frac{\alpha}{2-\alpha-\beta}} = (\mu-\delta)^{-\frac{\alpha}{2-\alpha-\beta}}A(x)^{-\frac{\alpha}{2-\alpha-\beta}}\\ &\le Ca(x),\end{aligned}$$

this inequality combined with (2.17) implies

$$\|\mathrm{e}^{-\eta\psi}v\|_2^2 \le Ct^{m+\frac{\alpha}{2-\alpha-\beta}}\int_{\mathbb{R}^n}a(x)w(t,x)v^2(t,x)\mathrm{d}x. \tag{2.18}$$

To estimate the second term (B) in the right of inequality (2.16), notice that

$$|\nabla(\mathrm{e}^{-\eta\psi}v)|^2 = \underbrace{\eta^2\mathrm{e}^{-2\eta\psi}|\nabla\psi|^2v^2}_{\text{(B1)}} - \underbrace{2\eta\mathrm{e}^{-2\eta\psi}v\nabla\psi\nabla v}_{\text{(B2)}} + \underbrace{\mathrm{e}^{-2\eta\psi}|\nabla v|^2}_{\text{(B3)}}.$$

Integrating (B2) by parts

$$\begin{aligned}2\eta\int_{\mathbb{R}^n}\mathrm{e}^{-2\eta\psi}v\nabla v\nabla\psi\mathrm{d}x &= \eta\int_{\mathbb{R}^n}(\mathrm{e}^{-2\eta\psi}\nabla\psi)\nabla v^2\mathrm{d}x\\ &= -\eta\int_{\mathbb{R}^n}v^2(\mathrm{e}^{-2\eta\psi}\Delta\psi - 2\eta\mathrm{e}^{-2\eta\psi}|\nabla\psi|^2)\mathrm{d}x\\ &= 2\eta^2\int_{\mathbb{R}^n}\mathrm{e}^{-2\eta\psi}v^2|\nabla\psi|^2\mathrm{d}x - \eta\int_{\mathbb{R}^n}v^2\mathrm{e}^{-2\eta\psi}\Delta\psi\mathrm{d}x,\end{aligned}$$

combining (B1)–(B3) and integrating over $\mathbb{R}^n$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}|\nabla(\mathrm{e}^{-\eta\psi}v)|^2\mathrm{d}x &= \int_{\mathbb{R}^n}\mathrm{e}^{-2\eta\psi}|\nabla v|^2\mathrm{d}x + \int_{\mathbb{R}^n}\mathrm{e}^{-2\eta\psi}(\eta\Delta\psi - \eta^2|\nabla\psi|^2)v^2\mathrm{d}x\\ &\le \underbrace{\eta\int_{\mathbb{R}^n}\mathrm{e}^{-2\eta\psi}(\Delta\psi)v^2\mathrm{d}x}_{\text{(B4)}} + \underbrace{\int_{\mathbb{R}^n}\mathrm{e}^{-2\eta\psi}|\nabla v|^2\mathrm{d}x}_{\text{(B5)}}.\end{aligned}\tag{2.19}$$

To estimate the first term (B4) in the right side of inequality (2.19), we use

$$\Delta\psi(t,x) = \frac{1}{2t}\Delta\sigma(x) = \frac{\mu-\delta}{2t}\Delta A(x) \le \frac{\mu-\delta}{2a_0t}a(x).$$

So, (B4) of inequality (2.19) will be

$$\int_{\mathbb{R}^n}\mathrm{e}^{-2\eta\psi}(\Delta\psi)v^2\mathrm{d}x \le Ct^{-1}\int_{\mathbb{R}^n}a(x)\mathrm{e}^{-2\eta\psi}v^2\mathrm{d}x.$$

Since the exponential term satisfies

$$e^{-2\eta\psi(t,x)} = e^{-\frac{2p\sigma(x)}{(p+1)t}} = e^{-\frac{\sigma(t)}{t}} t^{-m} e^{-\frac{(p-1)\sigma(x)}{(p+1)t}} t^m \leq t^m w(t,x), \tag{2.20}$$

(B4) is estimated as follows

$$\eta \int_{\mathbb{R}^n} e^{-2\eta\psi}(\Delta\psi)v^2 dx \leq Ct^{m-1} \int_{\mathbb{R}^n} a(x)w(t,x)v^2(t,x)dx. \tag{2.21}$$

To proceed, we estimate the second term (B5) in the right side of inequality (2.19). From the definition of w_1 and (2.20), we see that

$$\begin{aligned} e^{-2\eta\psi(t,x)} &= t^m w(t,x) e^{-\frac{(p-1)\sigma(x)}{(p+1)t}} = t^m \frac{4}{3}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big) w_1(t,x) e^{-\frac{(p-1)\sigma(x)}{(p+1)t}} \\ &= \frac{4}{3} t^{m-1} w_1(t,x) \Big[\Big(6 + \frac{\sigma(x)}{t}\Big) e^{-\frac{(p-1)\sigma(x)}{(p+1)t}}\Big] \\ &\leq \frac{C}{b_0} t^{m-1} b_0 (1+|x|)^\beta w_1(t,x) \\ &\leq Ct^{m-1} b(x) w_1(t,x), \end{aligned}$$

where $0 \leq \beta < 2$, $C > 0$, $x \mapsto (6+x)e^{-kx} (k > 0)$ is bounded above. Then the final estimate of (B5) is

$$\int_{\mathbb{R}^n} e^{-2\eta\psi} |\nabla v|^2 dx \leq Ct^{m-1} \int_{\mathbb{R}^n} w_1(t,x) b(x) |\nabla v|^2 dx. \tag{2.22}$$

Therefore, by using (2.18) and (2.21)–(2.22), we rewrite inequality (2.16) as

$$\begin{aligned} \|e^{-\eta\psi(t,\cdot)} v\|_{p+1}^{p+1} &\leq C\Big(t^{m+\frac{\alpha}{2-\alpha-\beta}} \int_{\mathbb{R}^n} a(x)w(t,x)v^2 dx\Big)^{\frac{\theta(p+1)}{2}} \\ &\quad \times \Big(t^{m-1} \int_{\mathbb{R}^n} a(x)w(t,x)v^2 dx + t^{m-1} \int_{\mathbb{R}^n} w_1(t,x)b(x)|\nabla v|^2 dx\Big)^{\frac{(1-\theta)(p+1)}{2}} \\ &\leq C(t^{m+\frac{\alpha}{2-\alpha-\beta}} W(t))^{\frac{\theta(p+1)}{2}} (t^{m-1} W(t))^{\frac{(1-\theta)(p+1)}{2}} \\ &= Ct^{[(m+\frac{\alpha}{2-\alpha-\beta})\frac{\theta}{2} + (m-1)\frac{1-\theta}{2}](p+1)} W(t)^{\frac{p+1}{2}}. \end{aligned}$$

This inequality together with (2.15) gives

$$\int_{\mathbb{R}^n} \Big(1 + \frac{\sigma(x)}{t}\Big) w_1 w^{p-1} |v|^{p+1} dx \leq Ct^{-\rho} W(t)^{\frac{p+1}{2}},$$

where

$$\begin{aligned} \rho : &= \frac{1}{2}(p-1)(m+1) - \frac{2-\beta}{2-\alpha-\beta} \frac{p+1}{2} \theta \\ &= \frac{(p-1)(4n-4\alpha-\beta n) - 4(2-\beta)}{4(2-\alpha-\beta)} - (p-1)\delta. \end{aligned}$$

Finally, from the assumption $p_{cr}(n,\alpha,\beta) < p$ we find that $\rho > 0$, which implies the desired estimate.

Now using the result of Lemma 2.2, we are able to estimate the second term (II) to the right side of (2.12).

Lemma 2.3 *Under the assumptions in Lemma 2.2, we have*

$$\int_{t_0}^t H(s)\mathrm{d}s \le C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s, \quad t\in[t_0,T_m) \tag{2.23}$$

with some constant $C>0$, where $\rho>0$ is the constant determined in Lemma 2.2.

Proof It follows from the definitions of w_1 and w that

$$K(t,x) := w^p - \frac{1}{p+1}(w_1w^{p-1})_t = \frac{1}{p+1}w_1w^{p-1}\Big[(p+1)\frac{w}{w_1} - \frac{\partial_t w_1}{w_1} - (p-1)\frac{w_t}{w}\Big].$$

It is easy to see that

$$\frac{w_t}{w} = -\frac{m}{t} + \frac{\sigma(x)}{t^2}, \quad \frac{w}{w_1} = \frac{4}{3}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big),$$

moreover,

$$\frac{\partial_t w_1}{w_1} = \Big(\frac{6}{t} + \frac{2\sigma(x)}{t^2}\Big)\Big(6 + \frac{\sigma(x)}{t}\Big)^{-1} + \frac{w_t}{w}.$$

Hence, we obtain

$$\begin{aligned} K(t,x) &= w_1w^{p-1}\Big[\frac{4}{3}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big) - \frac{p}{p+1}\Big(\frac{\sigma(x)}{t^2} - \frac{m}{t}\Big) \\ &\quad - \frac{1}{p+1}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big)^{-1}\Big(\frac{6}{t^2} + \frac{2\sigma(x)}{t^3}\Big)\Big] \\ &\le w_1w^{p-1}\Big[\frac{4}{3}\Big(\frac{6}{t} + \frac{\sigma(x)}{t^2}\Big) + \frac{pm}{(p+1)t}\Big] \\ &= t^{-1}w_1w^{p-1}\Big[\Big(8 + \frac{pm}{p+1}\Big) + \frac{4\sigma(x)}{3t}\Big] \\ &\le Ct^{-1}w_1w^{p-1}\Big(1 + \frac{\sigma(x)}{t}\Big), \end{aligned}$$

which implies

$$K(t,x) = w^p - \frac{1}{p+1}(w_1w^{p-1})_t \le Ct^{-1}w_1w^{p-1}\Big(1 + \frac{\sigma(x)}{t}\Big).$$

Integrating over $[t_0,t]$, one has

$$\begin{aligned} \int_{t_0}^t H(s)\mathrm{d}s &= \int_{t_0}^t\int_{\mathbb{R}^n}\Big[w^p - \frac{1}{p+1}(w_1w^{p-1})_t\Big]|v|^{p+1}\mathrm{d}x\mathrm{d}s \\ &\le C\int_{t_0}^t s^{-1}\int_{\mathbb{R}^n}\Big(1 + \frac{\sigma(x)}{s}\Big)w_1(s,x)w^{p-1}(s,x)|v(s,x)|^{p+1}\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{2.24}$$

Thus, by using Lemma 2.2 and (2.24), we derive the estimate in Lemma 2.3.

From Lemmas 2.2–2.3 and the weighted energy inequality (2.12), we get the following estimate

$$\begin{aligned} E(v_t,\nabla v,v)(t) &= \frac{1}{2}\int_{\mathbb{R}^n} w_1(v_t^2 + b|\nabla v|^2)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n}[2wv_tv + (Qw_1 + w_t + aw)v^2]\mathrm{d}x \\ &\le E(v_t,\nabla v,v)(t_0) + CW(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s. \end{aligned} \tag{2.25}$$

Using $w_1 > 0$ and $Q \geq 0$, which follows from Lemma 2.1(iii), we obtain $Qw_1 \geq 0$. Since $\frac{\mathrm{d}}{\mathrm{d}t}(wv^2) = w_t v^2 + 2wvv_t$, we can rewrite (2.25) as follows

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^n} w_1(v_t^2 + b|\nabla v|^2)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} awv^2\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} wv^2\mathrm{d}x\\ &\leq E(v_t, \nabla v, v)(t_0) + CW(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s. \qquad (2.26)\end{aligned}$$

We need one more preparation.

Lemma 2.4 *Let $\gamma \in [0, 1]$, $c_0 > 0$, $E_0 > 0$ be given real numbers, and let $f \in C([t_0, T_m))$ be a monotone increasing function. If a function $h \in C^1([t_0, T_m))$ satisfies the inequality*

$$h'(t) + c_0 t^{-\gamma} h(t) \leq E_0 + f(t),$$

then the following estimate holds

$$h(t) \leq h(t_0) + C(E_0 + f(t))t^{\gamma}$$

with some constant $C > 0$.

Proof The proof is similar to [7], so we omit it.

Let

$$M(t) := \max_{0\leq s\leq t} W(s). \qquad (2.27)$$

Note that the function $t \mapsto M(t)$ is monotone increasing. Under these preparations one can prove the following lemma.

Lemma 2.5 *Let $\gamma \in [0, 1]$, then the following bound holds*

$$\int_{\mathbb{R}^n} wv^2\mathrm{d}x \leq \int_{\mathbb{R}^n} wv^2\mathrm{d}x|_{t=t_0} + Ct^{\gamma}\Big[E(v_t, \nabla v, v)(t_0) + M(t)^{\frac{p+1}{2}} + \int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s\Big]$$

for $t \in [t_0, T_m)$ with large $t_0 > 0$.

Proof Since $a(x) \geq g(t) \geq g_0 t^{-\gamma}$ as it was presented in Proposition 2.2, it follows from (2.26) that

$$g_0 t^{-\gamma}\int_{\mathbb{R}^n} wv^2\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} wv^2\mathrm{d}x \leq 2E(v_t, \nabla v, v)(t_0) + CM(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s.$$

Note that the function

$$t \mapsto CM(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s$$

is monotone increasing. We can apply Lemma 2.4 with

$$h(t) = \int_{\mathbb{R}^n} wv^2\mathrm{d}x, \quad E_0 = 2E(v_t, \nabla v, v)(t_0), \quad c_0 = g_0,$$

$$f(t) = CM(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s$$

and obtain the desired estimate.

Denote by

$$E_u(t) := \frac{1}{2}\int_{\mathbb{R}^n} |u_t|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} b(x)|\nabla u|^2 \mathrm{d}x,$$

where $u \in X_1[0, T_m)$ is the weak solutions of (1.1)–(1.2). We are in need of the following lemmas.

Lemma 2.6 *For each $t \in [0, T_m)$, it is true that*

$$\int_{\mathbb{R}^n} w(t,x)v^2(t,x)\mathrm{d}x \le C_R(t)\|\nabla u\|^2,$$
$$E(v_t, \nabla v, v)(t) \le C_R(t)E_u(t),$$

with some t-dependent constant $C_R(t)$ satisfying $\lim_{t\to+\infty} C_R(t) = +\infty$.

Proof The proof is omitted since it elementary follows from the fact that $v = w^{-1}u$, $w = t^{-m}\mathrm{e}^{-m_1\frac{A(x)}{t}}$, the compact support of the data and the Poincaré inequality.

The standard energy inequality associated with the problem (1.1)–(1.2) is

$$E_u(t) \le E_u(0) + \frac{1}{p+1}\|u(t,\cdot)\|_{p+1}^{p+1}. \tag{2.28}$$

Lemma 2.7 *Let $t_0 > 0$ be the time defined in Lemmas 2.1–2.6, then there exists $T \in (t_0, T_m)$, which depends on $\varepsilon > 0$, such that for all $t \in [0, T]$*

$$E_u(t) \le 2E_u(0) \le (\|u_1\|^2 + b_1(1+R)^2\|\nabla u_0\|^2)\varepsilon^2,$$
$$\lim_{\varepsilon\to 0^+} T = +\infty.$$

Proof With a simple modification for [7], the lemma can be easily proved, so we omit it here.

3 Global Existence

In this section, we are going to prove the main theorem and corollaries. However, we need to state several lemmas in order to find the decay estimates for the energy, L^2 and L^{p+1} norms.

Lemma 3.1 *Assume that the $a(x)$ and $b(x)$ satisfy (1.7)–(1.8), and $\gamma = \frac{2\alpha}{2-\beta} \in [0, 1]$, if the weights w and w_1 satisfy conditions (i)–(iii) and*

$$\text{(iv)}\quad w(t,x) \le C_1 t^{-\gamma} w_1(t,x),$$
$$\text{(v)}\quad |w_t(t,x)| \le C_1 t^{-\gamma} w(t,x),$$

then

$$\int_{\mathbb{R}^n} w_1(v_t^2 + b|\nabla v|^2)\mathrm{d}x \le C\varepsilon^2,$$
$$\int_{\mathbb{R}^n} awv^2 \mathrm{d}x \le C\varepsilon^2$$

for all $t \geq t_0$.

Proof For the proof of conditions (iv)–(v) (see [19]), Lemma 2.7 shows that we can consider the case $t_0 < T < T_m$. Using the inequality (2.25), since $Qw_1 \geq 0$, we have

$$\int_{\mathbb{R}^n} w_1(v_t^2 + b|\nabla v|^2)\mathrm{d}x + \int_{\mathbb{R}^n} (2wv_tv + w_tv^2 + awv^2)\mathrm{d}x \\ \leq 2E(v_t, \nabla v, v)(t_0) + CW(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s. \tag{3.1}$$

For any $\epsilon \in (0,1)$, since

$$|2wv_tv| \leq \epsilon t^\gamma wv_t^2 + \epsilon^{-1}t^{-\gamma}wv^2,$$

then

$$2wv_tv \geq -\epsilon t^\gamma wv_t^2 - \epsilon^{-1}t^{-\gamma}wv^2.$$

Hence, (3.1) becomes

$$\int_{\mathbb{R}^n} w_1(v_t^2 + b|\nabla v|^2)\mathrm{d}x - \epsilon t^\gamma \int_{\mathbb{R}^n} wv_t^2\mathrm{d}x - \epsilon^{-1}t^{-\gamma}\int_{\mathbb{R}^n} wv^2\mathrm{d}x + \int_{\mathbb{R}^n} (w_t + aw)v^2\mathrm{d}x \\ \leq 2E(v_t, \nabla v, v)(t_0) + CW(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s.$$

That is,

$$\int_{\mathbb{R}^n} (w_1 - \epsilon t^\gamma w)(v_t^2 + b|\nabla v|^2)\mathrm{d}x + \int_{\mathbb{R}^n} (w_t + aw - \epsilon^{-1}t^{-\gamma}w)v^2\mathrm{d}x \\ \leq 2E(v_t, \nabla v, v)(t_0) + CW(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s,$$

conditions (iv) and (v) yield

$$(1 - \epsilon C_1)\int_{\mathbb{R}^n} w_1(v_t^2 + b|\nabla v|^2)\mathrm{d}x + \int_{\mathbb{R}^n} awv^2\mathrm{d}x - (C_1 + \epsilon^{-1})t^{-\gamma}\int_{\mathbb{R}^n} wv^2\mathrm{d}x \\ \leq 2E(v_t, \nabla v, v)(t_0) + CW(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s.$$

Having this together with Lemma 2.5, we obtain

$$(1 - \epsilon C_1)\int_{\mathbb{R}^n} w_1(v_t^2 + b|\nabla v|^2)\mathrm{d}x + \int_{\mathbb{R}^n} awv^2\mathrm{d}x \\ \leq CE(v_t, \nabla v, v)(t_0) + Ct^{-\gamma}\int_{\mathbb{R}^n} wv^2\mathrm{d}x|_{t=t_0} + CM(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s,$$

where $C > 0$ is a constant independent of ϵ. By taking $\epsilon > 0$ sufficiently small, one has

$$\int_{\mathbb{R}^n} w_1(v_t^2 + b|\nabla v|^2)\mathrm{d}x + \int_{\mathbb{R}^n} awv^2\mathrm{d}x \\ \leq CE(v_t, \nabla v, v)(t_0) + Ct^{-\gamma}\int_{\mathbb{R}^n} wv^2\mathrm{d}x|_{t=t_0} + CM(t)^{\frac{p+1}{2}} + C\int_{t_0}^t s^{-1-\rho}W(s)^{\frac{p+1}{2}}\mathrm{d}s$$

for all $t \in [t_0, T_m)$. From this inequality and Lemma 2.6, we obtain the final estimate of $W(t)$

$$W(t) \le C_{t_0} E_u(t_0) + CM(t)^{\frac{p+1}{2}} + \frac{t_0^{-\rho}}{\rho}\Big(\max_{0\le s\le t} W(s)\Big)^{\frac{p+1}{2}}, \quad t \in [t_0, T_m).$$

Applying Lemma 2.7, we have

$$M(t) \le C_{t_0}\varepsilon^2 + CM(t)^{\frac{p+1}{2}}, \quad t \in [t_0, T_m)$$

for sufficiently small ε, then continuous non-decreasing function $M(t)$ must remain bounded.

Indeed, if $C2^{\frac{p+1}{2}}(C_{t_0}\varepsilon^2)^{\frac{p-1}{2}} < 1$, then $M(t)$ can never equal $2C_{t_0}\varepsilon^2$. If it does, we would have

$$2C_{t_0}\varepsilon^2 \le C_{t_0}\varepsilon^2 + C(2C_{t_0}\varepsilon^2)^{\frac{p+1}{2}},$$

that is

$$1 \le C2^{\frac{p+1}{2}}(C_{t_0}\varepsilon^2)^{\frac{p-1}{2}}$$

which is false. Therefore

$$M(t) < 2C_{t_0}\varepsilon^2, \quad t \in [t_0, T_m). \tag{3.2}$$

This implies that $T_m = \infty$, in other words, we have global solutions.

Lemma 3.2 *Let $a(x)$ and $b(x)$ satisfy* (1.7)–(1.8), *assume that the weights w and w_1 satisfy conditions* (i)–(v) *together with*

$$\text{(vi)} \quad w_1 w^{-3}(w_t^2 + b|\nabla w|^2) \le Ca(x),$$

where C is a constant. Then, the solution u of (1.1)–(1.2) *satisfies*

$$\int_{\mathbb{R}^n} a(x)w^{-1}u^2 \mathrm{d}x \le C\varepsilon^2,$$
$$\int_{\mathbb{R}^n} w_1 w^{-2}(u_t^2 + b|\nabla u|^2)\mathrm{d}x \le C\varepsilon^2,$$

for all $t \ge t_0$.

Proof For the proof of condition (vi), see [19]. To prove the first estimate, we use the second estimate of Lemma 3.1 with $v = w^{-1}u$. It is left to prove.

We have the second estimate

$$v_t^2 = (-w^{-2}w_t u + w^{-1}u_t)^2 \ge \frac{1}{2}w^{-2}u_t^2 - 3w^{-4}w_t^2u^2$$

and

$$|\nabla v|^2 = |-w^{-2}u\nabla w + w^{-1}\nabla u|^2 \ge \frac{1}{2}w^{-2}|\nabla u|^2 - 3w^{-4}|\nabla w|^2u^2.$$

These equalities imply

$$\frac{1}{2}w_1 w^{-2}(u_t^2 + b|\nabla u|^2) \le w_1(v_t^2 + b|\nabla v|^2) + 3w_1 w^{-4}(w_t^2 + b|\nabla w|^2)u^2.$$

Integrating this inequality over $\mathbb{R}^n$ and using (vi), Lemma 3.1, we obtain

$$\frac{1}{2}\int_{\mathbb{R}^n} w_1 w^{-2}(u_t^2 + b|\nabla u|^2)\mathrm{d}x \le \int_{\mathbb{R}^n} w_1(v_t^2 + b|\nabla v|^2)\mathrm{d}x + C\int_{\mathbb{R}^n} w^{-1}a(x)u^2\mathrm{d}x \le C\varepsilon^2.$$

Here we are in a position to proof the main theorem and corollaries.

Proof of Theorem 1.1 Using the first estimate of Lemma 3.2, we have the following weighted estimates

$$\int_{\mathbb{R}^n} a(x)\mathrm{e}^{(\mu-\delta)\frac{A(x)}{t}}u^2\mathrm{d}x \le C\varepsilon^2 t^{2\delta-\mu}. \tag{3.3}$$

Further by using the bounds for $A(x)$, namely,

$$A_1(1+|x|)^{2-\alpha-\beta} \le A(x) \le A_2(1+|x|)^{2-\alpha-\beta}, \quad \text{for } x \in \mathbb{R}^n,$$

together with (1.7) we get the estimate

$$\begin{aligned} a(x) &\ge CA(x)^{-\frac{\alpha}{2-\alpha-\beta}} = Ct^{-\frac{\alpha}{2-\alpha-\beta}}\Big(\frac{A(x)}{t}\Big)^{-\frac{\alpha}{2-\alpha-\beta}} \\ &\ge Ct^{-\frac{\alpha}{2-\alpha-\beta}}\mathrm{e}^{-\delta\frac{A(x)}{t}}, \end{aligned}$$

where $C > 0$ and $t \ge t_0$ is sufficiently large. Substituting this lower bound for $a(x)$ into inequality (3.3), we have the decay estimate for the L^2-norm of solution

$$\int_{\mathbb{R}^n} \mathrm{e}^{(\mu-2\delta)\frac{A(x)}{t}}u^2\mathrm{d}x \le C\varepsilon^2 t^{\frac{\alpha}{2-\alpha-\beta}+2\delta-\mu}.$$

To prove the decay estimate for the energy of solution u, we use the second estimate of Lemma 3.2,

$$\int_{\mathbb{R}^n} w_1 w^{-2}(u_t^2 + b|\nabla u|^2)\mathrm{d}x \le C\varepsilon^2,$$

which is equivalent to

$$\int_{\mathbb{R}^n} \mathrm{e}^{(\mu-\delta)\frac{A(x)}{t}}\Big(\frac{1}{t} + \frac{A(x)}{t^2}\Big)^{-1}(u_t^2 + b|\nabla u|^2)\mathrm{d}x \le C\varepsilon^2 t^{2\delta-\mu}.$$

To simplify the estimate, we notice that

$$\Big(\frac{1}{t} + \frac{A(x)}{t^2}\Big)^{-1} = t\Big(1 + \frac{A(x)}{t}\Big)^{-1} \ge C_0 t\mathrm{e}^{-\delta\frac{A(x)}{t}} \tag{3.4}$$

with some $C_0 > 0$ depending on δ. Hence,

$$\int_{\mathbb{R}^n} \mathrm{e}^{(\mu-2\delta)\frac{A(x)}{t}}(u_t^2 + b|\nabla u|^2)\mathrm{d}x \le C\varepsilon^2 t^{2\delta-\mu-1}.$$

Finally, we can show the decay estimate for the L^{p+1} norm of the solution u. From Lemma 2.2 and (3.2), we find that

$$\int_{\mathbb{R}^n} w_1 w^{p-1}|v|^{p+1}\mathrm{d}x \le C\varepsilon^2 t^{-\rho}.$$

Applying $v=w^{-1}u$ and the weights w, w_1, one obtains

$$\int_{\mathbb{R}^n}\Big(\frac{1}{t}+\frac{A(x)}{t^2}\Big)^{-1}t^{\mu-2\delta}\mathrm{e}^{(\mu-\delta)\frac{A(x)}{t}}|u|^{p+1}\mathrm{d}x\le C\varepsilon^2t^{-\rho}.$$

Using (3.4) again, we have

$$\int_{\mathbb{R}^n}\mathrm{e}^{(\mu-2\delta)\frac{A(x)}{t}}|u|^{p+1}\mathrm{d}x\le C\varepsilon^2t^{-\rho+2\delta-\mu-1}.$$

These yield the estimates in Theorem 1.1 with a loss of decay δ. To obtain the final form we only replace 2δ by δ.

Proof of Corollary 1.1 To obtain the weighted energy estimate, we combine Theorem 1.1 with the lower bound of $A(x)$ from Proposition 1.1, namely,

$$A_0(1+|x|)^{2-\alpha-\beta}\le A(x)\le A_1(1+|x|)^{2-\alpha-\beta}. \tag{3.5}$$

We complete the proof by substituting this lower bound of $A(x)$ into the result of Theorem 1.1.

Proof of Corollary 1.2 The result is similar to but more precise than the first corollary. Using properties (A3) and (A4) in Proposition 1.1, we write the main decay estimates as

$$\int_{\mathbb{R}^n}\mathrm{e}^{(\frac{n-\alpha}{2-\alpha-\beta}-\delta)\frac{A(x)}{t}}u^2\mathrm{d}x\le Ct^{\delta-\frac{n-\alpha}{2-\alpha-\beta}},$$
$$\int_{\mathbb{R}^n}\mathrm{e}^{(\frac{n-\alpha}{2-\alpha-\beta}-\delta)\frac{A(x)}{t}}(u_t^2+b|\nabla u|^2)\mathrm{d}x\le Ct^{\delta-\frac{n-\alpha}{2-\alpha-\beta}-1},$$
$$\int_{\mathbb{R}^n}\mathrm{e}^{(\frac{n-\alpha}{2-\alpha-\beta}-\delta)\frac{A(x)}{t}}|u|^{p+1}\mathrm{d}x\le Ct^{-\rho+\delta-\frac{n-\alpha}{2-\alpha-\beta}-1},$$

where

$$A(x)\sim\frac{a_2}{b_2(n-\alpha)(2-\alpha-\beta)}|x|^{2-\alpha-\beta},\quad |x|\to\infty.$$

Hence there exists a $C>0$, such that

$$A(x)+C\ge\frac{a_2}{b_2(n-\alpha)(2-\alpha-\beta+\delta)}|x|^{2-\alpha-\beta},\quad |x|\in\mathbb{R}^n.$$

The remain part is a lower bound of $A(x)$ which is similar to (3.5).

Proof of Corollary 1.3 It can be concluded from Corollary 1.2 easily, and so we omit it.

Proof of Corollary 1.4 For the energy estimate in Theorem 1.1, we restrict the integration to $\{x:A(x)\ge t^{1+\kappa}\}$ and complete the proof.

4 Blow-up

In this section, we prove the blow-up part of Theorem 1.2. We adopt on the method of test functions developed by Zhang [21].

Proof of Theorem 1.2 First we find a non-negative $\phi \in C_o^\infty(\mathbb{R}\times\mathbb{R}^n)$ such that

$$\phi(t,x) = \begin{cases} 1, & (t,x)\in[-1,1]\times B(R), \\ 0, & (t,x)\in \dfrac{\mathbb{R}\times\mathbb{R}^n}{[-2,2]\times B(R)}. \end{cases}$$

The function ϕ also satisfies the addition condition

$$|D^2\phi(t,x)|^4 + |D\phi(t,x)|^2 \le C\phi(t,x), \quad (t,x)\in(\mathbb{R}\times\mathbb{R}^n),$$

where $D=(\partial_t,\nabla)$ and $C>0$ is some constant, see [21] for the existence of such function. Then the test function ϕ_T is defined by

$$\phi_T(t,x) = \phi\Big(\frac{t}{T^{2-\alpha-\beta}}, \frac{x}{T}\Big), \quad (t,x)\in(\mathbb{R}\times\mathbb{R}^n),$$

where T is some large parameter. Let P_T be the subset of $\mathbb{R}\times\mathbb{R}^n$, where $\phi_T=1$ and

$$Q_T = (\mathrm{supp}(D^2\phi_T)\cup\mathrm{supp}(D\phi_T))\cap([0,+\infty)\times\mathbb{R}^n),$$

that is, Q_T is the support of derivatives restricted to $t\ge 0$. It is easy to see that

$$\begin{aligned} &P_T \supset \{(t,x): t\le T^{2-\alpha-\beta} \text{ and } |x|\le RT\}, \\ &Q_T \subset \{(t,x): t> T^{2-\alpha-\beta} \text{ or } |x|> RT\}. \end{aligned} \tag{4.1}$$

Assume that a global solution u exists. To obtain a contradiction, we multiply the equation (1.1) by ϕ_T^q, with $q=\frac{2p}{p-1}$. First using integration by part over $[0,+\infty]\times\mathbb{R}^n$, it is easy to see that

$$\begin{aligned} &\int_0^\infty\int_{\mathbb{R}^n} u(\partial_t^2\phi_T^q - b(x)\Delta\phi_T^q - \nabla b(x)\cdot\nabla\phi_T^q - a(x)\partial_t\phi_T^q)\mathrm{d}x\mathrm{d}t \\ &= \int_0^\infty\int_{\mathbb{R}^n}|u|^{p-1}u\phi_T^q\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n}(a(x)u_0+u_1)\mathrm{d}x. \end{aligned} \tag{4.2}$$

Here we use $\phi_T(0,x)=1,\ \partial\phi_T(0,x)=1$ and the initial conditions on u to evaluate boundary integral at $t=0$.

Next, we estimate the integral on the left side of (4.2) and compare it with the integrals on the right side of (4.2). A straightforward calulation yields

$$\begin{aligned} &|\partial_t^2\phi_T^q - b(x)\Delta\phi_T^q - \nabla b(x)\cdot\nabla\phi_T^q - a(x)\partial_t\phi_T^q| \\ &\le C[\phi_T^{q-2}(\partial_t\phi_T)^2 + \phi_T^{q-1}(\partial_t^2\phi_T) + b(x)\phi_T^{q-2}(\nabla\phi_T)^2 + b(x)\phi_T^{q-1}\triangle\phi_T \\ &\quad + \nabla b(x)\cdot\nabla\phi_T\phi_T^{q-1} + a(x)\phi_T^{q-1}\partial_t\phi_T] \\ &\le C[T^{-2(2-\alpha-\beta)}\phi_T^{q-\frac{3}{2}} + T^{-2}b(x)\phi_T^{q-1} + T^{-1}\nabla b(x)\cdot\phi_T^{q-\frac{1}{2}} + T^{-(2-\alpha-\beta)}a(x)\phi_T^{q-\frac{1}{2}}]. \end{aligned}$$

By Holder's inequality, we have

$$\begin{aligned} &\Big|\int_0^\infty\int_{\mathbb{R}^n} u(\partial_t^2\phi_T^q - b(x)\Delta\phi_T^q - \nabla b(x)\cdot\nabla\phi_T^q - a(x)\partial_t\phi_T^q)\mathrm{d}x\mathrm{d}t\Big| \\ &\le C\int_0^\infty\int_{Q_T}|u|(T^{-2(2-\alpha-\beta)}\phi_T^{q-\frac{3}{2}} + T^{-2}b(x)\phi_T^{q-1} + T^{-1}\nabla b(x)\cdot\phi_T^{q-\frac{1}{2}} \\ &\quad + T^{-(2-\alpha-\beta)}a(x)\phi_T^{q-\frac{1}{2}})\mathrm{d}x\mathrm{d}t \\ &\le C\Big(\int_0^\infty\int_{Q_T}|u|^p\phi_T^q\mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p}} I^{\frac{p-1}{p}}(T), \end{aligned} \tag{4.3}$$

where $\mathrm{I}(T)=\mathrm{I}_1(T)+\mathrm{I}_2(T)+\mathrm{I}_3(T)+\mathrm{I}_4(T)$, $\mathrm{I}_i(T)$ (i=1,2,3,4) will be given as follows, and we will estimate $\mathrm{I}_i(T)$ (i=1,2,3,4) separately.

$$\begin{aligned}\mathrm{I}_1(T)&\le\int_0^\infty\int_{\mathbb{R}^n}|T^{-2(2-\alpha-\beta)}\phi_T^{\frac{1}{2}}|^{\frac{p}{p-1}}\mathrm{d}x\mathrm{d}t\le CT^{-2(2-\alpha-\beta)\frac{p}{p-1}}\int_0^{2T^{2-\alpha-\beta}}\int_{|x|\le 2RT}1\mathrm{d}x\mathrm{d}t\\&\le CT^{-2(2-\alpha-\beta)\frac{p}{p-1}+(2-\alpha-\beta)+n}\le CT^{-(2-\beta)\frac{p}{p-1}+(2-\alpha-\beta)+n},\end{aligned}\tag{4.4}$$

where we use $2(2-\alpha-\beta)\ge 2-\beta$ from the assumption $2\alpha+\beta\le 2$.

$$\mathrm{I}_2(T)\le CT^{-2\frac{p}{p-1}}\int_0^{2T^{2-\alpha-\beta}}\int_{|x|\le 2RT}(1+|x|)^{\frac{\beta p}{p-1}}\mathrm{d}x\mathrm{d}t\le CT^{-(2-\beta)\frac{p}{p-1}+(2-\alpha-\beta)+n}.\tag{4.5}$$

To proceed, we estimate the $\mathrm{I}_3(T)$ and $\mathrm{I}_4(T)$.

$$\begin{aligned}\mathrm{I}_3(T)&\le CT^{-\frac{p}{p-1}}\int_0^{2T^{2-\alpha-\beta}}\int_{|x|\le 2RT}(1+|x|)^{\frac{(\beta-1)p}{p-1}}\mathrm{d}x\mathrm{d}t\\&\le C\begin{cases}T^{-\frac{p}{p-1}+(2-\alpha-\beta)}, & \beta<1, \dfrac{(1-\beta)p}{p-1}>n,\\ T^{-\frac{p}{p-1}+(2-\alpha-\beta)}\ln T, & \beta<1, \dfrac{(1-\beta)p}{p-1}=n,\\ T^{-(2-\beta)\frac{p}{p-1}+(2-\alpha-\beta)+n}, & \beta<1, \dfrac{(1-\beta)p}{p-1}<n \text{ or } \beta\ge 1.\end{cases}\end{aligned}$$

Using $2-\alpha-\beta<1$, which follows from the assumption $\alpha+\beta>1$, we obtain

$$-\frac{p}{p-1}+(2-\alpha-\beta)<-\frac{p}{p-1}+1=-\frac{1}{p-1}<0,$$

thus an upper bound is

$$\mathrm{I}_3(T)\le CT^{-(2-\beta)\frac{p}{p-1}+(2-\alpha-\beta)+n}.\tag{4.6}$$

With the same reason of estimating $\mathrm{I}_3(T)$, we have

$$\begin{aligned}\mathrm{I}_4(T)&\le CT^{-(2-\alpha-\beta)\frac{p}{p-1}}\int_0^{2T^{2-\alpha-\beta}}\int_{|x|\le 2RT}(1+|x|)^{-\frac{\alpha p}{p-1}}\mathrm{d}x\mathrm{d}t\\&\le C\begin{cases}T^{-(2-\alpha-\beta)\dfrac{p}{p-1}+(2-\alpha-\beta)}, & \dfrac{\alpha p}{p-1}>n,\\ T^{-(2-\alpha-\beta)\dfrac{p}{p-1}+(2-\alpha-\beta)}\ln T, & \dfrac{\alpha p}{p-1}=n,\\ T^{-(2-\beta)\dfrac{p}{p-1}+(2-\alpha-\beta)+n}, & \dfrac{\alpha p}{p-1}<n\end{cases}\\&\le CT^{-(2-\beta)\frac{p}{p-1}+(2-\alpha-\beta)+n}.\end{aligned}\tag{4.7}$$

Therefore, by using (4.4)–(4.7), we derive the final estimate of $\mathrm{I}(T)$,

$$\mathrm{I}(T)\le CT^{-(2-\beta)\frac{p}{p-1}+(2-\alpha-\beta)+n}.$$

Combing with (4.2)–(4.3) and the assumption of initial data, we have

$$\int_0^\infty \int_{\mathbb{R}^n} |u|^p \phi_T^q \mathrm{d}x\mathrm{d}t \le C\Big(\int_0^\infty \int_{Q_T} |u|^p \phi_T^q \mathrm{d}x\mathrm{d}t\Big)^{\frac{1}{p}} T^{-(2-\beta)+(2-\alpha-\beta+n)\frac{p-1}{p}}, \tag{4.8}$$

where C is indepent of T. Finally, we show that the above inequality cannot hold as $T \to \infty$. If $p \le p_2(n, \alpha, \beta)$, then

$$-(2-\beta) + (2-\alpha-\beta+n)\frac{p-1}{p} \le 0,$$

and (4.8) shows that

$$\Big(\int_0^\infty \int_{P_T} |u|^p \mathrm{d}x\mathrm{d}t\Big)^{\frac{p-1}{p}} \le C.$$

Letting $T \to \infty$ and using (4.1), we conclude that $u \in L^p([0, +\infty)\times\mathbb{R}^n)$. Hence (4.1) also implies that $\|u\|_{L^p(Q_T)} \to 0$ as $T \to \infty$. Passing to the limit in (4.8), we obtain $\|u\|_{L^p([0,+\infty)\times\mathbb{R}^n)} \le 0$ for any $1 < p \le p_2(n, \alpha, \beta)$. Since u is a non-trivial solution, that is impossible.

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