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Nonlinear Maps Preserving the Jordan Triple ∗-Product on Factor von Neumann Algebras*

Changjing LI[1] Quanyuan CHEN[2] Ting WANG[3]

Abstract Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. For $A, B \in \mathcal{A}$, define by $[A, B]_* = AB - BA^*$ the skew Lie product of A and B. In this article, it is proved that a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([[A, B]_*, C]_*) = [[\Phi(A), \Phi(B)]_*, \Phi(C)]_*$ for all $A, B, C \in \mathcal{A}$ if and only if Φ is a linear ∗-isomorphism, or a conjugate linear ∗-isomorphism, or the negative of a linear ∗-isomorphism, or the negative of a conjugate linear ∗-isomorphism.

Keywords Jordan triple ∗-product, Isomorphism, von Neumann algebras
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1 Introduction

Let $\mathcal{A}$ be a ∗-algebra and η be a non-zero scalar. For $A, B \in \mathcal{A}$, define the Jordan η-∗-product of A and B by $A \diamondsuit_\eta B = AB + \eta BA^*$. The Jordan (-1)-∗-product, which is customarily called the skew Lie product, was extensively studied because it naturally arises in the problem of representing quadratic functionals with sesquilinear functionals (see [9–11]) and in the problem of characterizing ideals (see [2, 8]). We often write the Jordan (-1)-∗-product by $[A, B]_*$, that is $[A, B]_* = AB - BA^*$. A not necessarily linear map Φ between ∗-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to preserve the Jordan η-∗-product if $\Phi(A \diamondsuit_\eta B) = \Phi(A) \diamondsuit_\eta \Phi(B)$ for all $A, B \in \mathcal{A}$. Recently, many authors have started to pay more attention to the maps preserving the Jordan η-∗-product between ∗-algebras (see [1, 3, 6–7]). In [3], Dai and Lu proved that if Φ is a bijective map preserving the Jordan η-∗-product between two von Neumann algebras, one of which has no central abelian projections, then Φ is a linear ∗-isomorphism if η is not real and Φ is a sum of a linear ∗-isomorphism and a conjugate linear ∗-isomorphism if η is real.

Recently, Huo et al. [5] studied a more general problem. They considered the Jordan triple η-∗-product of three elements A, B and C in a ∗-algebra $\mathcal{A}$ defined by $A \diamondsuit_\eta B \diamondsuit_\eta C = (A \diamondsuit_\eta B) \diamondsuit_\eta C$ (we should be aware that $\diamondsuit_\eta$ is not necessarily associative). A map Φ between ∗-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to preserve the Jordan triple η-∗-product if $\Phi(A \diamondsuit_\eta B \diamondsuit_\eta C) = \Phi(A) \diamondsuit_\eta \Phi(B) \diamondsuit_\eta \Phi(C)$ for all $A, B, C \in \mathcal{A}$. Clearly a map between ∗-algebras preserving the Jordan η-∗-product also

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[1]Corresponding author. School of Mathematical Sciences, Shandong Normal University, Jinan 250014, China. E-mail: lcjbxh@163.com
[2]College of Information, Jingdezhen Ceramic Institute, Jingdezhen 333403, Jiangxi, China. E-mail: cqy0798@163.com
[3]Department of Mathematics and Statistics, Nanyang Normal University, Nanyang 473061, Henan, China. E-mail: tingwang526@126.com
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preserves the Jordan triple η-$*$-product. However, the map $\Phi : \mathbb{C} \to \mathbb{C}$, $\Phi(\alpha+\beta\mathrm{i}) = -4(\alpha^3+\beta^3\mathrm{i})$ is a bijection which preserves the Jordan triple (-1)-$*$-product and Jordan triple 1-$*$-product, but it does not preserve the Jordan (-1)-$*$-product or Jordan 1-$*$-product. So, the class of those maps preserving the Jordan triple η-$*$-product is, in principle wider than the class of maps preserving the Jordan η-$*$-product.

In [5], let $\eta \neq -1$ be a non-zero complex number, and let Φ be a bijection between two von Neumann algebras, one of which has no central abelian projections, satisfying $\Phi(I) = I$ and preserving the Jordan triple η-$*$-product. Huo et al. showed that Φ is a linear $*$-isomorphism if η is not real and Φ is the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism if η is real. On the one hand, Huo et al. did not consider the case $\eta = -1$. However, the Jordan (triple) (-1)-$*$-product is the most meaningful and important in Jordan (triple) η-$*$-products. On the other hand, it is easy to see that a map Φ preserving the Jordan triple η-$*$-product does not need to satisfy $\Phi(I) = I$. Indeed, let $\Phi(A) = -A$ for all $A \in \mathcal{A}$. Then Φ preserves the Jordan triple η-$*$-product but $\Phi(I) = -I$. Because of the above two reasons, in this paper, we will discuss maps preserving the Jordan triple (-1)-$*$-product without the assumption $\Phi(I) = I$. We mainly prove that a bijective map Φ between two factor von Neumann algebras preserves the Jordan triple (-1)-$*$-product if and only if Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.

As usual, $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real field and complex field. Throughout, algebras and spaces are over $\mathbb{C}$. A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity operator I. $\mathcal{A}$ is a factor von Neumann algebra means that its center contains only the scalar operators. It is clear that if $\mathcal{A}$ is a factor von Neumann algebra, then $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$ if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$.

2 The Main Result and Its Proof

To complete the proof of main theorem, we need two lemmas.

Lemma 2.1 *Let $\mathcal{A}$ be an arbitrary factor von Neumann algebra with the identity operator I and $A \in \mathcal{A}$. If $AB = BA^*$ for all $B \in \mathcal{A}$, then $A \in \mathbb{R}I$.*

Proof In fact, take $B = I$, then $A = A^*$. So $AB = BA$ for all $B \in \mathcal{A}$, which implies A belongs to the center of $\mathcal{A}$. Note that $\mathcal{A}$ is a factor, it follows that $A \in \mathbb{R}I$.

Lemma 2.2 (see [4, Problem 230]) *Let $\mathcal{A}$ be a Banach algebra with the identity I. If $A, B \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ are such that $[A, B] = \lambda I$, where $[A, B] = AB - BA$, then $\lambda = 0$.*

The main result in this paper is as follows.

Theorem 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. Then a bijective map $\Phi : \mathcal{A} \to \mathcal{B}$ satisfies $\Phi([[A, B]_*, C]_*) = [[\Phi(A), \Phi(B)]_*, \Phi(C)]_*$ for all $A, B, C \in \mathcal{A}$ if and only if Φ is a linear $*$-isomorphism, or a conjugate linear $*$-isomorphism, or the negative of a linear $*$-isomorphism, or the negative of a conjugate linear $*$-isomorphism.*

Proof Clearly, we only need to prove the necessity. First we give a key technique. Suppose that $A_1, A_2, \cdots, A_n$ and T are in $\mathcal{A}$ such that $\Phi(T) = \sum\limits_{i=1}^{n} \Phi(A_i)$. Then for all $S_1, S_2 \in \mathcal{A}$, we have

$$\Phi([[S_1, S_2]_*, T]_*) = [[\Phi(S_1), \Phi(S_2)]_*, \Phi(T)]_* = \sum_{i=1}^{n} \Phi([[S_1, S_2]_*, A_i]_*), \tag{2.1}$$

$$\Phi([[S_1, T]_*, S_2]_*) = [[\Phi(S_1), \Phi(T)]_*, \Phi(S_2)]_* = \sum_{i=1}^{n} \Phi([[S_1, A_i]_*, S_2]_*) \tag{2.2}$$

and

$$\Phi([[T, S_1]_*, S_2]_*) = [[\Phi(T), \Phi(S_1)]_*, \Phi(S_2)]_* = \sum_{i=1}^{n} \Phi([[A_i, S_1]_*, S_2]_*). \tag{2.3}$$

In the following, we will complete the proof of Theorem 2.1 by proving several claims.

Claim 1 $\Phi(0) = 0$.

Since Φ is surjective, there exists $A \in \mathcal{A}$ such that $\Phi(A) = 0$. Then we obtain $\Phi(0) = \Phi([[0, A]_*, A]_*) = [[\Phi(0), \Phi(A)]_*, \Phi(A)]_* = 0$.

Claim 2 $\Phi(\mathbb{R}I) = \mathbb{R}I$, $\Phi(\mathbb{C}I) = \mathbb{C}I$ and Φ preserves self-adjoint elements in both directions.

Let $\lambda \in \mathbb{R}$ be arbitrary. Since Φ is surjective, there exists $B \in \mathcal{A}$ such that $\Phi(B) = I$. By Claim 1, we have that

$$\begin{aligned} 0 &= \Phi([[\lambda I, A]_*, B]_*) \\ &= [[\Phi(\lambda I), \Phi(A)]_*, I]_* \\ &= \Phi(\lambda I)\Phi(A) - \Phi(A)\Phi(\lambda I)^* - \Phi(A)^*\Phi(\lambda I)^* + \Phi(\lambda I)\Phi(A)^* \end{aligned}$$

holds true for all $A \in \mathcal{A}$. That is,

$$\Phi(\lambda I)(\Phi(A) + \Phi(A)^*) = (\Phi(A) + \Phi(A)^*)\Phi(\lambda I)^*$$

holds true for all $A \in \mathcal{A}$. So

$$\Phi(\lambda I)B = B\Phi(\lambda I)^*$$

holds true for all $B = B^* \in \mathcal{B}$. Since for every $B \in \mathcal{B}$, $B = B_1 + \mathrm{i}B_2$ with $B_1 = \frac{B+B^*}{2}$ and $B_2 = \frac{B-B^*}{2\mathrm{i}}$, it follows that

$$\Phi(\lambda I)B = B\Phi(\lambda I)^*$$

holds true for all $B \in \mathcal{B}$. It follows from Lemma 2.1 that $\Phi(\lambda I) \in \mathbb{R}I$. Note that Φ^{-1} has the same properties as Φ. Similarly, if $\Phi(A) \in \mathbb{R}I$, then $A \in \mathbb{R}I$. Therefore, $\Phi(\mathbb{R}I) = \mathbb{R}I$.

Let $A = A^* \in \mathcal{A}$. Since $\Phi(\mathbb{R}I) = \mathbb{R}I$, there exists $\lambda \in \mathbb{R}$ such that $\Phi(\lambda I) = I$. Then

$$\begin{aligned} 0 &= \Phi([[A, \lambda I]_*, \lambda I]_*) = [[\Phi(A), I]_*, I]_* \\ &= 2\Phi(A) - 2\Phi(A)^*. \end{aligned}$$

Hence $\Phi(A) = \Phi(A)^*$. Similarly, if $\Phi(A) = \Phi(A)^*$, then $A = A^* \in \mathcal{A}$. Therefore Φ preserves self-adjoint elements in both directions.

Let $\lambda \in \mathbb{C}$ be arbitrary. For every $A = A^* \in \mathcal{A}$, we obtain that

$$\begin{aligned} 0 &= \Phi([[A, \lambda I]_*, C]_*) \\ &= [[\Phi(A), \Phi(\lambda I)]_*, \Phi(C)]_* \end{aligned}$$

holds true for all $C \in \mathcal{A}$. It follows from Lemma 2.1 that

$$[\Phi(A), \Phi(\lambda I)]_* \in \mathbb{R}I.$$

Since $A = A^*$, we have $\Phi(A) = \Phi(A)^*$. Hence

$$[\Phi(A), \Phi(\lambda I)] \in \mathbb{R}I.$$

It follows from Lemma 2.2 that

$$[\Phi(A), \Phi(\lambda I)] = 0,$$

and then

$$B\Phi(\lambda I) = \Phi(\lambda I)B$$

for all $B = B^* \in \mathcal{B}$. Thus for every $B \in \mathcal{B}$, since $B = B_1 + \mathrm{i}B_2$ with $B_1 = \frac{B+B^*}{2}$ and $B_2 = \frac{B-B^*}{2\mathrm{i}}$, we get

$$\Phi(\lambda I)B = B\Phi(\lambda I)$$

holds true for all $B \in \mathcal{B}$. Hence $\Phi(\lambda I) \in \mathbb{C}I$. Similarly, if $\Phi(A) \in \mathbb{C}I$, then $A \in \mathbb{C}I$. Therefore, $\Phi(\mathbb{C}I) = \mathbb{C}I$.

Claim 3 $\Phi\big(\frac{1}{2}I\big) = \pm\frac{1}{2}I$, $\Phi\big(\frac{1}{2}\mathrm{i}I\big) = \pm\frac{1}{2}\mathrm{i}I$, $\Phi(iA) = \mathrm{i}\Phi(A)$ ($\forall A \in \mathcal{A}$) or $\Phi(\mathrm{i}A) = -\mathrm{i}\Phi(A)$ ($\forall A \in \mathcal{A}$), where i is the imaginary unit.

By Claim 2, we have

$$\Phi\Big(-\frac{1}{2}I\Big) = \alpha I, \quad \Phi\Big(\frac{1}{2}I\Big) = \beta I$$

and

$$\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big) = (\gamma_1 + \gamma\mathrm{i})I, \quad \Phi\Big(\frac{1}{2}\mathrm{i}I\Big) = (\omega_1 + \omega\mathrm{i})I,$$

where $\alpha, \beta, \gamma, \omega, \gamma_1, \omega_1 \in \mathbb{R}$ and $\alpha\beta\gamma\omega \neq 0$. It follows from $0 = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}\mathrm{i}I\big]_*, -\frac{1}{2}I\big]_*$ that

$$0 = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}\mathrm{i}I\Big)\Big]_*, \Phi\Big(-\frac{1}{2}I\Big)\Big]_* = [[(\gamma_1 + \gamma\mathrm{i})I, (\gamma_1 + \gamma\mathrm{i})I]_*, \alpha I]_* = 4\alpha\gamma\gamma_1\mathrm{i}I.$$

So $\gamma_1 = 0$. Similarly, by the equality $0 = \big[\big[\frac{1}{2}\mathrm{i}I, \frac{1}{2}\mathrm{i}I\big]_*, -\frac{1}{2}I\big]_*$, we get that $\omega_1 = 0$.

Now we get

$$\Phi\Big(-\frac{1}{2}I\Big) = \alpha I, \quad \Phi\Big(\frac{1}{2}I\Big) = \beta I, \quad \Phi\Big(-\frac{1}{2}\mathrm{i}I\Big) = \gamma\mathrm{i}I, \quad \Phi\Big(\frac{1}{2}\mathrm{i}I\Big) = \omega\mathrm{i}I.$$

It follows from $-\frac{1}{2}\mathrm{i}I = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}I\big]_*, -\frac{1}{2}I\big]_*$ that

$$\gamma\mathrm{i}I = \Phi\Big(-\frac{1}{2}\mathrm{i}I\Big) = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}I\Big)\Big]_*, \Phi\Big(-\frac{1}{2}I\Big)\Big]_* = 4\alpha^2\gamma\mathrm{i}I. \tag{2.4}$$

Also the equality $\frac{1}{2}\mathrm{i}I = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}I\big]_*, \frac{1}{2}I\big]_*$ implies

$$\omega\mathrm{i}I = \Phi\Big(\frac{1}{2}\mathrm{i}I\Big) = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}I\Big)\Big]_*, \Phi\Big(\frac{1}{2}I\Big)\Big]_* = 4\alpha\beta\gamma\mathrm{i}I, \tag{2.5}$$

the equality $\frac{1}{2}I = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}I\big]_*, -\frac{1}{2}\mathrm{i}I\big]_*$ implies

$$\beta I = \Phi\Big(\frac{1}{2}I\Big) = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}I\Big)\Big]_*, \Phi\Big(-\frac{1}{2}\mathrm{i}I\Big)\Big]_* = -4\alpha\gamma^2 I \tag{2.6}$$

and $-\frac{1}{2}I_{\mathcal{A}} = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}I\big]_*, \frac{1}{2}\mathrm{i}I\big]_*$ ensures that

$$\alpha I = \Phi\Big(-\frac{1}{2}I\Big) = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}I\Big)\Big]_*, \Phi\Big(\frac{1}{2}\mathrm{i}I\Big)\Big]_* = -4\alpha\gamma\omega I. \tag{2.7}$$

Now (2.4)–(2.7) ensures that

$$\alpha^2 = \gamma^2 = \frac{1}{4}, \quad \alpha = -\beta, \quad \gamma = -\omega. \tag{2.8}$$

For every $A \in \mathcal{A}$, it follows from $\mathrm{i}A = \big[\big[-\frac{1}{2}\mathrm{i}I, -\frac{1}{2}I\big]_*, A\big]_*$ that

$$\Phi(\mathrm{i}A) = \Big[\Big[\Phi\Big(-\frac{1}{2}\mathrm{i}I\Big), \Phi\Big(-\frac{1}{2}I\Big)\Big]_*, \Phi(A)\Big]_* = 4\alpha\gamma\mathrm{i}\Phi(A), \tag{2.9}$$

which together with (2.8) implies that $\Phi(\mathrm{i}A) = \mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$ or $\Phi(\mathrm{i}A) = -\mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$.

Choose an arbitrary nontrivial projection $P_1 \in \mathcal{A}$, and write $P_2 = I - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j,\ i,j = 1,2$, then $\mathcal{A} = \sum\limits_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$, we may write $A = \sum\limits_{i,j=1}^{2} A_{ij}$. In all that follows, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. The following Claims 4–9 are devoted to the additivity of Φ.

Claim 4 For every $A_{11} \in \mathcal{A}_{11}$ and $B_{22} \in \mathcal{A}_{22}$, we have

$$\Phi(A_{11} + B_{22}) = \Phi(A_{11}) + \Phi(B_{22}).$$

Since Φ is surjective, we may find an element $T = \sum\limits_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(B_{22}).$$

Since $[[\mathrm{i}P_1, I]_*, A_{22}]_* = 0$, it follows from (2.1) and Claim 1 that

$$\Phi([[\mathrm{i}P_1, I]_*, T]_*) = \Phi([[\mathrm{i}P_1, I]_*, A_{11}]_*).$$

By the injectivity of Φ, we obtain that

$$2\mathrm{i}(P_1T + TP_1) = [[\mathrm{i}P_1, I]_*, T]_* = [\mathrm{i}P_1, I]_*, A_{11}]_* = 4\mathrm{i}A_{11},$$

and then we get $T_{11} = A_{11},\ T_{12} = T_{21} = 0$. Similarly, $T_{22} = B_{22}$, proving the claim.

Claim 5 For every $A_{12} \in \mathcal{A}_{12},\ B_{21} \in \mathcal{A}_{21}$, we have

$$\Phi(A_{12} + B_{21}) = \Phi(A_{12}) + \Phi(B_{21}).$$

Choose $T = \sum\limits_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{12}) + \Phi(B_{21}).$$

Since

$$[[\mathrm{i}(P_2 - P_1), I]_*, A_{12}]_* = [[\mathrm{i}(P_2 - P_1), I]_*, B_{21}]_* = 0,$$

it follows from (2.1) that

$$\Phi([[\mathrm{i}(P_2 - P_1), I]_*, T]_*) = 0.$$

From this, we get $[[\mathrm{i}(P_2-P_1),I]_*,T]_*=0$. So $T_{11}=T_{22}=0$.

Since $[[A_{12},P_1]_*,I]_*=0$, it follows from (2.3) that

$$\Phi([[T,P_1]_*,I]_*)=\Phi([[B_{21},P_1]_*,I]_*).$$

By the injectivity of Φ, we obtain that

$$2(TP_1-P_1T^*)=[[T,P_1]_*,I]_*=[[B_{21},P_1]_*,I]_*=2(B_{21}-B_{21}^*).$$

Hence $T_{21}=B_{21}$. Similarly, $T_{12}=A_{12}$, proving the claim.

Claim 6 For every $A_{11}\in\mathcal{A}_{11},\ B_{12}\in\mathcal{A}_{12},\ C_{21}\in\mathcal{A}_{21},\ D_{22}\in\mathcal{A}_{22}$, we have

$$\Phi(A_{11}+B_{12}+C_{21})=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})$$

and

$$\Phi(B_{12}+C_{21}+D_{22})=\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22}).$$

Let $T=\sum\limits_{i,j=1}^{2}T_{ij}\in\mathcal{A}$ be such that

$$\Phi(T)=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21}).$$

It follows from (2.1) and Claim 5 that

$$\begin{aligned}
&\Phi(2\mathrm{i}(P_2T+TP_2))\\
&=\Phi([[\mathrm{i}P_2,I]_*,T]_*)\\
&=\Phi([[\mathrm{i}P_2,I]_*,A_{11}]_*)+\Phi([[\mathrm{i}P_2,I]_*,B_{12}]_*)+\Phi([[\mathrm{i}P_2,I]_*,C_{21}]_*)\\
&=\Phi(2\mathrm{i}B_{12})+\Phi(2\mathrm{i}C_{21})\\
&=\Phi(2\mathrm{i}(B_{12}+C_{21})).
\end{aligned}$$

Thus $P_2T+TP_2=B_{12}+C_{21}$, which implies $T_{22}=0,\ T_{12}=B_{12},\ T_{21}=C_{21}$. Now we get $T=T_{11}+B_{12}+C_{21}$.

Since

$$[[\mathrm{i}(P_2-P_1),I]_*,B_{12}]_*=[[\mathrm{i}(P_2-P_1),I]_*,C_{21}]_*=0,$$

it follows from (2.1) that

$$\Phi([[\mathrm{i}(P_2-P_1),I]_*,T]_*)=\Phi([[\mathrm{i}(P_2-P_1),I]_*,A_{11}]_*),$$

from which we get $T_{11}=A_{11}$. Consequently, $\Phi(A_{11}+B_{12}+C_{21})=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})$.

Similarly, we can get that $\Phi(B_{12}+C_{21}+D_{22})=\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22})$.

Claim 7 For every $A_{11}\in\mathcal{A}_{11},\ B_{12}\in\mathcal{A}_{12},\ C_{21}\in\mathcal{A}_{21},\ D_{22}\in\mathcal{A}_{22}$, we have

$$\Phi(A_{11}+B_{12}+C_{21}+D_{22})=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22}).$$

Let $T=\sum\limits_{i,j=1}^{2}T_{ij}\in\mathcal{A}$ be such that

$$\Phi(T)=\Phi(A_{11})+\Phi(B_{12})+\Phi(C_{21})+\Phi(D_{22}).$$

It follows from (2.1) and Claim 6 that

$$\begin{aligned}\Phi(2\mathrm{i}P_1T + 2\mathrm{i}TP_1) &= \Phi([[\mathrm{i}P_1, I]_*, T]_*)\\ &= \Phi([[\mathrm{i}P_1, I]_*, A_{11}]_*) + \Phi([[\mathrm{i}P_1, I]_*, B_{12}]_*)\\ &\quad + \Phi([[\mathrm{i}P_1, I]_*, C_{21}]_*) + \Phi([[\mathrm{i}P_1, I]_*, D_{22}]_*)\\ &= \Phi(4\mathrm{i}A_{11}) + \Phi(2\mathrm{i}B_{12}) + \Phi(2\mathrm{i}C_{21})\\ &= \Phi(4\mathrm{i}A_{11} + 2\mathrm{i}B_{12} + 2\mathrm{i}C_{21}).\end{aligned}$$

Thus

$$P_1T + TP_1 = 2A_{11} + B_{12} + C_{21},$$

it follows that $T_{11} = A_{11},\ T_{12} = B_{12},\ T_{21} = C_{21}$. Similarly, we can get

$$\Phi(2\mathrm{iP_2T} + 2\mathrm{iTP_2}) = \Phi(4\mathrm{iD_{22}} + 2\mathrm{iB_{12}} + 2\mathrm{iC_{21}}).$$

From this, we get $T_{22} = D_{22}$, proving the claim.

Claim 8 For every $A_{jk}, B_{jk} \in \mathcal{A}_{jk},\ 1 \le j \ne k \le 2$, we have

$$\Phi(A_{jk} + B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk}).$$

Since

$$\Big[\Big[\frac{\mathrm{i}}{2}I, P_j + A_{jk}\Big]_*, P_k + B_{jk}\Big]_* = \mathrm{i}(A_{jk} + B_{jk}) + \mathrm{i}(A_{jk}^*) + \mathrm{i}(B_{jk}A_{jk}^*),$$

we get from Claim 7 that

$$\begin{aligned}&\Phi(\mathrm{i}(A_{jk} + B_{jk})) + \Phi(\mathrm{i}A_{jk}^*) + \Phi(\mathrm{i}(B_{jk}A_{jk}^*))\\ &= \Phi\Big(\Big[\Big[\frac{\mathrm{i}}{2}I, P_j + A_{jk}\Big]_*, P_k + B_{jk}\Big]_*\Big)\\ &= \Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big), \Phi(P_j + A_{jk})\Big]_*, \Phi(P_k + B_{jk})\Big]_*\\ &= \Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big), \Phi(P_j) + \Phi(A_{jk})\Big]_*, \Phi(P_k) + \Phi(B_{jk})\Big]_*\\ &= \Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big), \Phi(P_j)\Big]_*, \Phi(P_k)\Big]_* + \Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big), \Phi(P_j)\Big]_*, \Phi(B_{jk})\Big]_*\\ &\quad + \Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big), \Phi(A_{jk})\Big]_*, \Phi(P_k)\Big]_* + \Big[\Big[\Phi\Big(\frac{\mathrm{i}}{2}I\Big), \Phi(A_{jk})\Big]_*, \Phi(B_{jk})\Big]_*\\ &= \Phi(\mathrm{i}B_{jk}) + \Phi(\mathrm{i}(A_{jk} + A_{jk}^*)) + \Phi(\mathrm{i}(B_{jk}A_{jk}^*))\\ &= \Phi(\mathrm{i}B_{jk}) + \Phi(\mathrm{i}A_{jk}) + \Phi(\mathrm{i}A_{jk}^*) + \Phi(\mathrm{i}(B_{jk}A_{jk}^*)),\end{aligned}$$

which implies $\Phi(\mathrm{i}(A_{jk}+B_{jk})) = \Phi(\mathrm{i}B_{jk})+\Phi(\mathrm{i}A_{jk})$. By Claim 3, we obtain that $\Phi(A_{jk}+B_{jk}) = \Phi(A_{jk}) + \Phi(B_{jk})$.

Claim 9 For every $A_{jj} \in \mathcal{A}_{jj}$ and $B_{jj} \in \mathcal{A}_{jj},\ 1 \le j \le 2$, we have

$$\Phi(A_{jj} + B_{jj}) = \Phi(A_{jj}) + \Phi(B_{jj}).$$

Let $T = \sum\limits_{i,j=1}^{2} T_{ij} \in \mathcal{A}$ be such that

$$\Phi(T) = \Phi(A_{jj}) + \Phi(B_{jj}).$$

For $1 \leq j \neq k \leq 2$, it follows from (2.1) that

$$\Phi([[\mathrm{i}P_k, I]_*, T]_*) = \Phi([[\mathrm{i}P_k, I]_*, A_{jj}]_*) + \Phi([[\mathrm{i}P_k, I_{\mathcal{A}}]_*, B_{jj}]_*) = 0.$$

Hence $P_kT + TP_k = 0$, which implies $T_{jk} = T_{kj} = T_{kk} = 0$. Now we get $T = T_{jj}$.

For every $C_{jk} \in \mathcal{A}_{jk},\ j \neq k$, it follows from (2.2) and Claim 8 that

$$\begin{aligned}\Phi(2\mathrm{i}T_{jj}C_{jk}) &= \Phi([[\mathrm{i}P_j, T_{jj}]_*, C_{jk}]_*)\\ &= \Phi([[\mathrm{i}P_j, A_{jj}]_*, C_{jk}]_*) + \Phi([[\mathrm{i}P_j, B_{jj}]_*, C_{jk}]_*)\\ &= \Phi(2\mathrm{i}A_{jj}C_{jk}) + \Phi(2\mathrm{i}B_{jj}C_{jk})\\ &= \Phi(2\mathrm{i}(A_{jj}C_{jk} + B_{jj}C_{jk})).\end{aligned}$$

Hence

$$(T_{jj} - A_{jj} - B_{jj})C_{jk} = 0$$

for all $C_{jk} \in \mathcal{A}_{jk}$. By the primeness of $\mathcal{A}$, we get that $T_{jj} = A_{jj} + B_{jj}$, proving the claim.

Claim 10 Φ is $*$-additive.

The additivity of Φ is an immediate consequence of Claims 7–9. For every $A \in \mathcal{A},\ A = A_1 + \mathrm{i}A_2$, where $A_1 = \frac{A+A^*}{2}$ and $A_2 = \frac{A-A^*}{2\mathrm{i}}$ are self-adjoint elements. By Claims 2–3 , if for every $A \in \mathcal{A}$, $\Phi(\mathrm{i}A) = \mathrm{i}\Phi(A)$, then

$$\begin{aligned}\Phi(A^*) &= \Phi(A_1 - \mathrm{i}A_2) = \Phi(A_1) - \Phi(\mathrm{i}A_2)\\ &= \Phi(A_1) - \mathrm{i}\Phi(A_2) = \Phi(A_1)^* - \mathrm{i}\Phi(A_2)^*\\ &= \Phi(A_1)^* + (\mathrm{i}\Phi(A_2))^* = \Phi(A_1 + \mathrm{i}A_2)^*\\ &= \Phi(A)^*.\end{aligned}$$

Similarly, if $\Phi(\mathrm{i}A) = -\mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$, we also have $\Phi(A^*) = \Phi(A)^*$.

By Claims 3 and 10, we get that $\Phi(I) = I$ or $\Phi(I) = -I$. In the rest of this section, we deal with these two cases respectively.

Case 1 If $\Phi(I) = I$, then Φ is either a linear $*$-isomorphism or a conjugate linear $*$-isomorphism.

If $\Phi(I) = I$, by (2.8)–(2.9) and Claim 10, then $\alpha = -\frac{1}{2},\ \beta = \frac{1}{2},\ \gamma = -\omega,\ \Phi(\mathrm{i}I) = 2\omega\mathrm{i}I$ and $\Phi(\mathrm{i}A) = -2\gamma\mathrm{i}\Phi(A)$ for all $A \in \mathcal{A}$. For all $A, B \in \mathcal{A}$, we can obtain that

$$\begin{aligned}-4\gamma\mathrm{i}\Phi(AB + BA^*) &= 2\Phi(\mathrm{i}(AB + BA^*)) = \Phi(2\mathrm{i}(AB + BA^*))\\ &= \Phi([[\mathrm{i}I, A]_*, B]_*) = [[\Phi(\mathrm{i}I), \Phi(A)]_*, \Phi(B)]_*\\ &= 4\omega\mathrm{i}(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)^*).\end{aligned}$$

From this, we get

$$\Phi(AB + BA^*) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)^*. \tag{2.10}$$

For all $A, B \in \mathcal{A}$, it follows from Claim 3 that

$$\begin{aligned}\Phi(AB - BA^*) &= \Phi((\mathrm{i}A)(-\mathrm{i}B) + (-\mathrm{i}B)(\mathrm{i}A)^*)\\ &= \Phi(\mathrm{i}A)\Phi(-\mathrm{i}B) + \Phi(-\mathrm{i}B)\Phi(\mathrm{i}A)^*\\ &= -\Phi(\mathrm{i}A)\Phi(\mathrm{i}B) - \Phi(\mathrm{i}B)\Phi(\mathrm{i}A)^*\\ &= \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^*.\end{aligned} \tag{2.11}$$

Summing (2.10) with (2.11), we get that $\Phi(AB) = \Phi(A)\Phi(B)$.

For every rational number q, we have $\Phi(qI) = qI$. Indeed, since q is a rational number, there exist two integers r and s such that $q = \frac{r}{s}$. Since $\Phi(I) = I$ and Φ is additive, we get that

$$\Phi(qI) = \Phi\Big(\frac{r}{s}I\Big) = r\Phi\Big(\frac{1}{s}I\Big) = \frac{r}{s}\Phi(I) = qI.$$

Let A be a positive element in $\mathcal{A}$. Then $A = B^2$ for some self-adjoint element $B \in \mathcal{A}$. It follows from Claim 11 that $\Phi(A) = \Phi(B)^2$. By Claim 2, we get that $\Phi(B)$ is self-adjoint. So $\Phi(A)$ is positive. This shows that Φ preserves positive elements.

Let $\lambda \in \mathbb{R}$. Choose sequence $\{a_n\}$ and $\{b_n\}$ of rational numbers such that $a_n \le \lambda \le b_n$ for all n and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \lambda$. It follows from

$$a_n I \le \lambda I \le b_n I$$

that

$$a_n I_{\mathcal{B}} \le \Phi(\lambda I) \le b_n I.$$

Taking the limit, we get that $\Phi(\lambda I_{\mathcal{A}}) = \lambda I_{\mathcal{B}}$. Hence for all $A \in \mathcal{A}$,

$$\Phi(\lambda A) = \Phi((\lambda I)A) = \Phi(\lambda I)\Phi(A) = \lambda\Phi(A).$$

Hence Φ is real linear. Therefore, if $\Phi(\mathrm{i}A) = \mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$, then Φ is a linear $*$-isomorphism. If $\Phi(\mathrm{i}A) = -\mathrm{i}\Phi(A)$ $(\forall A \in \mathcal{A})$, then Φ is a conjugate linear $*$-isomorphism.

Case 2 If $\Phi(I) = -I$, then Φ is either the negative of a linear $*$-isomorphism or the negative of a conjugate linear $*$-isomorphism.

Consider that the map $\Psi : \mathcal{A} \to \mathcal{B}$ defined by $\Psi(A) = -\Phi(A)$ for all $A \in \mathcal{A}$. It is easy to see that Ψ satisfies $\Psi([[A,B]_*,C]_*) = [[\Psi(A),\Psi(B)]_*,\Psi(C)]_*$ for all $A,B,C \in \mathcal{A}$ and $\Psi(I) = I$. Then the arguments for Case 1 ensure that Ψ is either a linear $*$-isomorphism or a conjugate linear $*$-isomorphism. So Φ is either the negative of a linear $*$-isomorphism or the negative of a conjugate linear $*$-isomorphism.

Combining Cases 1–2, the proof of Theorem 2.1 is finished.

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