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Mathematical Justification of an Obstacle Problem in the Case of a Plate*

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Abstract In this paper the modeling of a thin plate in unilateral contact with a rigid plane is properly justified. Starting from the three-dimensional nonlinear Signorini problem, by an asymptotic approach the convergence of the displacement field as the thickness of the plate goes to zero is studied. It is shown that the transverse mechanical displacement field decouples from the in-plane components and solves an obstacle problem.

Keywords Signorini problem, Obstacle problem, Asymptotic analysis, Plate
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1 Introduction

In this paper, we consider the so-called Signorini problem, also called unilateral contact problem, of an elastic body in contact with a rigid support. One of the major interests of this modeling is to keep the full elastic tensor, namely, there is no assumption on the elastic isotropy (see [1–2]).

Bilateral models for plates and shells were studied by formal asymptotic methods or by variational analysis (see [3–4] and the references therein). The contact problem can be stated as the minimization of some energy functional under an inequality constraint. The modelling of unilateral contact problems of elastic bodies was established by Signorini in 1933. The first mathematical properties of the solution to such a problem can be found in [5–6]. Later Paumier gave, by an asymptotic approach, the model of an elastic Kirchhoff-Love plate in unilateral contact (see [7]). Léger and Miara generalized Paumier's work to elastic shallow shell. They obtained the limit model written in terms of a variational inequality in the framework of Cartesian (see [1]) and curvilinear coordinates (see [2]), respectively.

In this paper we properly justify the modeling of a thin plate in unilateral contact with a rigid plane. By an asymptotic approach, we study the convergence of the displacement field as the thickness of the plate goes to zero. We establish that the transverse mechanical displacement field decouples from the in-plane components and solves an obstacle problem. In Section 2, we study a Signorini problem arising in the case of three-dimensional plate. In Section 3, by using

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appropriate scalings, we give the new scaled variational inequality problem. In Section 4, we prove the convergence of the solution when the thickness of the plate tends to zero and establish the limit problem of a elastic plate in unilateral contact.

2 Thin Plate

In this paper, Latin indices take their values in the set {1,2,3}, Greek indices take their values in the set {1,2}; and the Einstein summation convention is used. Bold letters are used for vectors or vector spaces. We denote by $\boldsymbol{a}\cdot\boldsymbol{b}$ the vector product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. $|\cdot|_{0,\Omega}$, $\|\cdot\|_{1,\Omega}$ stand for the classical norms in $L^2(\Omega)$, $H^1(\Omega)$, respectively, for both scalar-valued and vector-valued functions. Moreover, for simplicity let c denote different positive constants.

In order to establish properly the bi-dimensional model of a thin plate in contact with a rigid plane, we take the reference configuration to be a cylinder with middle surface ω and thickness 2ε. More precisely, let $\varepsilon > 0$ be a small parameter and ω be an open bounded and connected subset of $\mathbb{R}^2$ with Lipschitz-continuous boundary γ. Then the reference configuration of the plate under consideration is denoted by $\overline{\Omega}^\varepsilon$, where $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$. We define a new partition of the boundary $\partial\Omega^\varepsilon = \Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_- \cup \Gamma^\varepsilon_0$ with the upper and lower faces $\Gamma^\varepsilon_+ = \omega \times \{\varepsilon\}$, $\Gamma^\varepsilon_- = \omega \times \{-\varepsilon\}$ and the lateral boundary $\Gamma^\varepsilon_0 = \gamma \times [-\varepsilon, \varepsilon]$.

2.1 Three dimensional problem

We consider a family of plates with reference configuration $\overline{\Omega}^\varepsilon$, made of elastic material with elastic characteristic tensors $\mathbb{C} = (C_{ijkl})$. There exists a positive number c such that, for every second order 3×3 symmetric tensor $\mathbb{M} = (M_{ij})$ we have

$$C_{ijkl} = C_{ijlk} = C_{klij}, \quad C_{ijkl}M_{kl}M_{ij} \geq c\sum_{i,j=1}^{3} M_{ij}^2, \tag{2.1}$$

We denote, respectively, $\boldsymbol{\sigma}^\varepsilon$ and $\boldsymbol{e}^\varepsilon$ the stress and strain tensors, $\boldsymbol{u}^\varepsilon$ the mechanical displacement field. The constitutive equation posed in Ω^ε is given by

$$\boldsymbol{\sigma}^\varepsilon(\boldsymbol{u}^\varepsilon) = \mathbb{C}\boldsymbol{e}^\varepsilon(\boldsymbol{u}^\varepsilon). \tag{2.2}$$

Let $x^\varepsilon = (x^\varepsilon_k)$ be a generic point on $\overline{\Omega}^\varepsilon$ with $x^\varepsilon_\alpha \in \omega$ and $x^\varepsilon_3 \in (-\varepsilon, \varepsilon)$, and let $\partial^\varepsilon_i = \dfrac{\partial}{\partial x^\varepsilon_i}$. Then the linear strain tensor $\boldsymbol{e}^\varepsilon(\boldsymbol{u}^\varepsilon)$ is defined by $\boldsymbol{e}^\varepsilon(\boldsymbol{u}^\varepsilon) = \frac{1}{2}(\nabla^\varepsilon\boldsymbol{u}^\varepsilon + (\nabla^\varepsilon\boldsymbol{u}^\varepsilon)^T)$ or component-wisely

$$\mathrm{e}^\varepsilon_{ij}(\boldsymbol{u}^\varepsilon) = \frac{1}{2}(\partial^\varepsilon_i u^\varepsilon_j + \partial^\varepsilon_j u^\varepsilon_i).$$

For a plate subjected by applied body forces with density $\boldsymbol{f}^\varepsilon$, the equilibrium problem posed in Ω^ε reads

$$\mathrm{Div}^\varepsilon\boldsymbol{\sigma}^\varepsilon(\boldsymbol{u}^\varepsilon) = -\boldsymbol{f}^\varepsilon. \tag{2.3}$$

We consider the situation that the body is clamped on the whole lateral surface Γ^ε_0, and is subjected to applied surface forces with density $\boldsymbol{g}^\varepsilon$ on the upper surface and is in mechanical contact with the lower face Γ^ε_-.

2.2 The boundary conditions for a body in contact with a plane

We focus now on the unilateral contact with a horizontal plane set at level $-\varepsilon$. Let $x_-^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, -\varepsilon)$ be a point on the lower face Γ_-^ε. The unilateral contact conditions first mean that the displacement on Γ_-^ε must satisfy a nonpenetrability condition $(\boldsymbol{x}_-^\varepsilon + \boldsymbol{u}^\varepsilon(x_-^\varepsilon)) \cdot \boldsymbol{e}_3 \geq -\varepsilon$, in other words,

$$\forall x_-^\varepsilon \in \Gamma_-^\varepsilon, \qquad -\varepsilon + u_3^\varepsilon(x_-^\varepsilon) \geq -\varepsilon \Longrightarrow u_3^\varepsilon(x_-^\varepsilon) \geq 0, \tag{2.4}$$

where $\boldsymbol{e}_3 = (0, 0, 1)$.

The so-called Signorini conditions which give the full description of the unilaterality are classically obtained by adding the following constraints to the nonpenetrability condition:

(1) No tensile forces but only compressive forces are exerted on the boundary by the obstacle;

(2) all points in contact are on Γ_-^ε so that conditions (2.4) is an equality.

These constraints read

$$\sigma_3^\varepsilon(x_-^\varepsilon) = -(\boldsymbol{\sigma}^\varepsilon(x_-^\varepsilon)\boldsymbol{\nu}) \cdot \boldsymbol{e}_3 \geq 0, \quad \forall x^\varepsilon \in \Gamma_-^\varepsilon, \tag{2.5}$$

$$\sigma_3^\varepsilon(x_-^\varepsilon) u_3^\varepsilon(x_-^\varepsilon) = 0, \qquad\qquad \forall x^\varepsilon \in \Gamma_-^\varepsilon, \tag{2.6}$$

where $\boldsymbol{\nu}$ is the unit normal vector to $\partial\Omega$, so the contact condition reads

$$u_3^\varepsilon(x_-^\varepsilon) \geq 0, \quad \sigma_3^\varepsilon(x_-^\varepsilon) \geq 0, \quad \sigma_3^\varepsilon(x_-^\varepsilon) u_3^\varepsilon(x_-^\varepsilon) = 0 \quad \text{on} \quad \Gamma_-^\varepsilon, \tag{2.7}$$

and $\sigma_3^\varepsilon(x_-^\varepsilon)$ is the Kuhn and Tucker multiplier associated to the contact condition. Eventually we gather all the conditions and get

$$\begin{cases} \boldsymbol{u}^\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ \boldsymbol{\sigma}^\varepsilon(\boldsymbol{u}^\varepsilon)\boldsymbol{\nu} = \boldsymbol{g}^\varepsilon & \text{on } \Gamma_+^\varepsilon, \\ u_3^\varepsilon \geq 0,\ \sigma_3^\varepsilon \geq 0,\ \sigma_3^\varepsilon u_3^\varepsilon = 0 & \text{on } \Gamma_-^\varepsilon. \end{cases} \tag{2.8}$$

The equilibrium problem which we are dealing with is finally written as

$$\begin{cases} \mathrm{div}\boldsymbol{\sigma}^\varepsilon = -\boldsymbol{f}^\varepsilon & \text{in } \Omega^\varepsilon, \\ \boldsymbol{u}^\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ \boldsymbol{\sigma}^\varepsilon(\boldsymbol{u}^\varepsilon)\boldsymbol{\nu} = \boldsymbol{g}^\varepsilon & \text{on } \Gamma_+^\varepsilon, \\ u_3^\varepsilon \geq 0,\ \sigma_3^\varepsilon \geq 0,\ \sigma_3^\varepsilon u_3^\varepsilon = 0 & \text{on } \Gamma_-^\varepsilon. \end{cases} \tag{2.9}$$

2.3 The variational inequality in $\boldsymbol{\Omega}^\varepsilon$

The natural functional framework for (2.9) is the vector space $\mathbb{K}^\varepsilon(\Omega^\varepsilon)$, where

$$\mathbb{K}^\varepsilon(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon \in \mathbb{H}^1(\Omega^\varepsilon),\ \ \boldsymbol{v}^\varepsilon = 0 \ \text{ on } \ \Gamma_0^\varepsilon,\ \ v_3^\varepsilon \geq 0 \ \text{ on } \ \Gamma_-^\varepsilon\} \tag{2.10}$$

is a convex set. Hence the weak solution $\boldsymbol{u}^\varepsilon$ to (2.9) is given by the following variational inequality

$$\begin{cases} \text{Find } \boldsymbol{u}^\varepsilon \in \mathbb{K}^\varepsilon(\Omega^\varepsilon), \text{ such that} \\ \displaystyle\int_{\Omega^\varepsilon} C^\varepsilon_{ijkl} e^\varepsilon_{kl}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{ij}(\boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon) \mathrm{d}x^\varepsilon \\ \displaystyle\geq \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon(\boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon)\mathrm{d}x^\varepsilon + \int_{\Gamma^\varepsilon_+} \boldsymbol{g}^\varepsilon(\boldsymbol{v}^\varepsilon - \boldsymbol{u}^\varepsilon)\mathrm{d}a^\varepsilon, \quad \forall \boldsymbol{v}^\varepsilon \in \mathbb{K}^\varepsilon(\Omega^\varepsilon), \end{cases} \tag{2.11}$$

where $\mathrm{d}a^\varepsilon$ is the area element of the boundary $\partial\Omega^\varepsilon$.

Based on classical arguments, (2.9) and (2.11) are equivalent. Moreover, (2.11) has a unique solution for any fixed $\varepsilon > 0$, and the weak solution associated to (2.9) is given by this unique solution (for proof, see [8]).

Let us now introduce the scaling procedure in order to establish the convergence theorem as $\varepsilon \to 0$.

3 Scaling and Equilibrium Equation on the Fixed Domain Ω

3.1 Scalings of the unknowns and test functions

We now change the domain Ω^ε having the middle surface ω and the thickness 2ε into a fixed domain Ω with the same middle surface and the thickness 2 independent of ε by means of the simple geometrical transformation defined as follows: Let $x^\varepsilon = (x^\varepsilon_k)$ be a generic point on $\overline{\Omega}^\varepsilon$. The corresponding point $x = (x_k)$ on $\overline{\Omega}$ with $x^\varepsilon_\alpha = x_\alpha \in \omega$ and $x^\varepsilon_3 = \varepsilon x_3 \in (-\varepsilon, \varepsilon)$. This induces $\partial^\varepsilon_\alpha = \frac{\partial}{\partial x^\varepsilon_\alpha} = \frac{\partial}{\partial x_\alpha}$ and $\partial^\varepsilon_3 = \frac{\partial}{\partial x^\varepsilon_3} = \frac{1}{\varepsilon}\frac{\partial}{\partial x_3}$. By analogy, the boundary of the domain Ω is divided into three parts: $\partial\Omega = \Gamma_- \cup \Gamma_+ \cup \Gamma_0$, $\Gamma_- = \omega \times \{-1\}$, $\Gamma_+ = \omega \times \{1\}$, $\Gamma_0 = \gamma \times [-1, 1]$.

We give the scaled displacement $\boldsymbol{u}(\varepsilon)$ and the scaled test functions $\boldsymbol{v}$ defined on $\overline{\Omega}$ as

$$\begin{cases} u^\varepsilon_\alpha = \varepsilon^2 u_\alpha(\varepsilon), & u^\varepsilon_3 = \varepsilon u_3(\varepsilon), \\ v^\varepsilon_\alpha = \varepsilon^2 v_\alpha, & v^\varepsilon_3 = \varepsilon v_3. \end{cases} \tag{3.1}$$

Along with the scaling procedure, we set $\boldsymbol{e} = (e_{ij})$ to denote the scaled linearized strain tensor, the components of which are

$$\begin{cases} e^\varepsilon_{\alpha\beta}(\boldsymbol{v}^\varepsilon) = \varepsilon^2 e_{\alpha\beta}(\boldsymbol{v}), \\ e^\varepsilon_{\alpha 3}(\boldsymbol{v}^\varepsilon) = \varepsilon e_{\alpha 3}(\boldsymbol{v}), \\ e^\varepsilon_{33}(\boldsymbol{v}^\varepsilon) = e_{33}(\boldsymbol{v}), \end{cases} \tag{3.2}$$

where $e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$.

3.2 Assumptions on the data

In order to obtain a nontrivial limit problem by asymptotic analysis, it is essential to scale the data in accordance with the scalings of the unknowns. More precisely, we assume that there exist functions $\boldsymbol{f} \in \mathbb{L}^2(\Omega)$ and $\boldsymbol{g} \in \mathbb{L}^2(\Gamma_+)$ independent of ε, such that

$$\begin{cases} f_\alpha^\varepsilon = \varepsilon^2 f_\alpha, \quad f_3^\varepsilon = \varepsilon^3 f_3, \quad \forall x \in \Omega, \\ g_\alpha^\varepsilon = \varepsilon^3 g_\alpha, \quad g_3^\varepsilon = \varepsilon^4 g_3, \quad \forall x \in \Gamma_+. \end{cases} \tag{3.3}$$

3.3 Contact condition on the fixed domain

After the scaling process, the non-penetrability condition holds now on Γ_- and reads

$$v_3(x_1, x_2, -1) \geq 0, \quad \forall (x_1, x_2) \in \omega, \tag{3.4}$$

and the corresponding functional space is

$$\mathbb{K}(\Omega) = \{\boldsymbol{v} \in \mathbb{H}^1(\Omega),\ \ \boldsymbol{v} = 0 \text{ on } \Gamma_0,\ \ v_3(x_1, x_2, -1) \geq 0 \text{ on } \Gamma_-\}.$$

3.4 Equilibrium problem on the fixed domain $\boldsymbol{\Omega = \omega \times (-1, 1)}$

Replacing $\boldsymbol{u}^\varepsilon$ and $\boldsymbol{v}^\varepsilon$ by their scaled values $\boldsymbol{u}(\varepsilon)$ and $\boldsymbol{v}$ given by (3.1) in the problem (2.11), respectively, we get the following problem posed over the fixed domain Ω

$$\begin{cases} \text{Find } \boldsymbol{u}(\varepsilon) \in \mathbb{K}(\Omega), \text{ such that} \\ \quad \varepsilon^5 \displaystyle\int_\Omega C_{\alpha\beta\sigma\tau} e_{\sigma\tau}(\boldsymbol{u}(\varepsilon)) e_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \mathrm{d}x \\ \quad +2\varepsilon^4 \displaystyle\int_\Omega C_{\alpha\beta\sigma3}(e_{\sigma3}(\boldsymbol{u}(\varepsilon)) e_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) + e_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) e_{\sigma3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))) \mathrm{d}x \\ \quad +4\varepsilon^3 \displaystyle\int_\Omega C_{\alpha3\sigma3} e_{\sigma3}(\boldsymbol{u}(\varepsilon)) e_{\alpha3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \mathrm{d}x \\ \quad +\varepsilon^3 \displaystyle\int_\Omega C_{\alpha\beta33}(e_{33}(\boldsymbol{u}(\varepsilon)) e_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) + e_{\alpha\beta}(\boldsymbol{u}(\varepsilon)) e_{33}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))) \mathrm{d}x \\ \quad +2\varepsilon^2 \displaystyle\int_\Omega C_{\alpha333}(e_{33}(\boldsymbol{u}(\varepsilon)) e_{\alpha3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) + e_{\alpha3}(\boldsymbol{u}(\varepsilon)) e_{33}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))) \mathrm{d}x \\ \quad +\varepsilon \displaystyle\int_\Omega C_{3333} e_{33}(\boldsymbol{u}(\varepsilon)) e_{33}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \mathrm{d}x \\ \geq \varepsilon^5 \displaystyle\int_\Omega \boldsymbol{f}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \mathrm{d}x + \varepsilon^5 \int_{\Gamma_+} \boldsymbol{g}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)) \mathrm{d}a, \quad \forall \boldsymbol{v} \in \mathbb{K}(\Omega), \end{cases} \tag{3.5}$$

where $\mathrm{d}a$ is the area element of the boundary $\partial\Omega$.

Classical arguments (see [8]) can be applied to prove the existence and uniqueness of the weak solution to the variational inequality problem (3.5). We have the following theorem.

Theorem 3.1 *For any fixed $\varepsilon > 0$, the problem* (3.5) *has a unique weak solution.*

4 Convergence

The aim of this section is to show that when ε tends to zero, the sequence $\{\boldsymbol{u}(\varepsilon)\}$ converges to a limit $\boldsymbol{u}$ which solves a two-dimensional obstacle problem. An important preliminary point here is the following lemma, which is a version of Korn's inequality.

Lemma 4.1 *For all $\boldsymbol{v} \in \mathbb{H}^1(\Omega)$, the mapping $\boldsymbol{v} \to \big\{ \sum_{ij} |e_{ij}(\boldsymbol{v})|^2_{0,\Omega} \big\}^{\frac{1}{2}}$ is a norm over the set $\mathbb{K}(\Omega)$, which is equivalent to the norm induced by $\|\cdot\|_{1,\Omega}$*

Proof The proof follows from the fact that the set $\mathbb{K}(\Omega)$ is a closed subset of the vector space $\{\boldsymbol{v} \in \mathbb{H}^1(\Omega),\ \ \boldsymbol{v} = 0 \ \text{ on } \ \Gamma_0\}$.

Theorem 4.1 *Assume that $\boldsymbol{f} \in \mathbb{L}^2(\Omega)$ and $\boldsymbol{g} \in \mathbb{L}^2(\Gamma_+)$. Then*

(i) *As ε tends to 0, the family $\{\boldsymbol{u}(\varepsilon)\}$ converges strongly in the set $\mathbb{K}(\Omega)$ to a limit $\boldsymbol{u}$.*

(ii) *The limit $\boldsymbol{u}$ is a Kirchhoff-Love displacement field, namely, there exists $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_H, \zeta_3) \in \mathbb{V}_H(\omega) \times K_3(\omega)$, such that*

$$u_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3, \ \ u_3 = \zeta_3,$$

where the bi-dimensional functional spaces $\mathbb{V}_H(\omega)$ and $K_3(\omega)$ are

$$\begin{cases} \mathbb{V}_H(\omega) = \{\boldsymbol{\eta}_H = (\eta_\alpha) \in \mathbb{H}^1(\omega), \ \ \boldsymbol{\eta}_H = \boldsymbol{0} \ \ on \ \ \gamma\}, \\ K_3(\omega) = \{\eta_3 \in H^2(\omega), \ \ \eta_3 = \partial_\nu\eta_3 = 0 \ \ on \ \ \gamma, \ \ \eta_3 \geq 0 \ \ in \ \ \omega\}. \end{cases}$$

(iii) *The function $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_H, \zeta_3)$ solves the following problem: Find $(\boldsymbol{\zeta}_H, \zeta_3) \in \mathbb{V}_H(\omega) \times K_3(\omega)$, such that*

$$\begin{cases} \quad \dfrac{2}{3}\displaystyle\int_\omega \widetilde{C}_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3\partial_{\alpha\beta}(\eta_3 - \zeta_3)\mathrm{d}\omega \\ \geq \displaystyle\int_\omega \Big(p_3(\eta_3 - \zeta_3) - s_\alpha\partial_\alpha(\eta_3 - \zeta_3)\Big)\mathrm{d}\omega, \quad \forall\eta_3 \in K_3(\omega), \\ \quad 2\displaystyle\int_\omega \widetilde{C}_{\alpha\beta\sigma\tau}e_{\sigma\tau}(\boldsymbol{\zeta}_H)e_{\alpha\beta}(\boldsymbol{\eta}_H)\mathrm{d}\omega = \int_\omega p_\alpha\eta_\alpha\mathrm{d}\omega, \quad \forall\boldsymbol{\eta}_H \in \mathbb{V}_H(\omega), \end{cases} \tag{4.1}$$

where the mechanical forces are given by

$$p_i(x_1, x_2) := \int_{-1}^{1} f_i\mathrm{d}x_3 + g_i, \qquad s_\alpha(x_1, x_2) := \int_{-1}^{1} x_3 f_\alpha\mathrm{d}x_3 + g_\alpha, \tag{4.2}$$

and the new bi-dimensional elastic tensor $\widetilde{\mathbb{C}} = (\widetilde{C}_{\alpha\beta\sigma\tau})$ is given by

$$\widetilde{C}_{\alpha\beta\sigma\tau} = C_{\alpha\beta\sigma\tau} - \frac{1}{\triangle}C_{\alpha\beta k3}\triangle^k_{\sigma\tau}, \tag{4.3}$$

where

$$\begin{cases} \triangle = \epsilon_{pqr}C_{13p3}C_{23q3}C_{33r3}, \\ \triangle^1_{\zeta\eta} = \epsilon_{pqr}C_{3p\zeta\eta}C_{23q3}C_{33r3}, \\ \triangle^2_{\zeta\eta} = \epsilon_{pqr}C_{13p3}C_{3q\zeta\eta}C_{33r3}, \\ \triangle^3_{\zeta\eta} = \epsilon_{pqr}C_{13p3}C_{23q3}C_{3r\zeta\eta}, \end{cases} \tag{4.4}$$

in which ϵ_{ijk} denote the Levi-Civitta symbol

$$\epsilon_{ijk} = \begin{cases} 1, & (i,j,k) \ \ is \ \ even \ \ permutation \ \ of \ (1,2,3), \\ -1, & (i,j,k) \ \ is \ \ odd \ \ permutation \ \ of \ (1,2,3). \end{cases} \tag{4.5}$$

Proof The proof is divided into four steps. In the first step, we introduce a new scaled strain tensor $\mathbb{R}(\varepsilon)$. By means of some boundness results we get that the sequence $\{\boldsymbol{u}(\varepsilon)\}$ converges weakly to a limit $\boldsymbol{u}$ which is a Kirchhoff-Love field. The second step deals with certain technical results about the components of this strain tensor. In the third step we show that the convergence of the family $\{\boldsymbol{u}(\varepsilon)\}$ towards the Kirchhoff-Love field $\boldsymbol{u}$ is actually strong. The fourth step completes the proof by deducing the variational problem.

Step I. Let us introduce the following symmetric tensor $\mathbb{R}(\varepsilon) = (R_{ij}(\varepsilon)) \in \mathbb{L}^2(\Omega)$ by

$$\begin{cases} R_{\alpha\beta}(\varepsilon)(\boldsymbol{v}) = e_{\alpha\beta}(\boldsymbol{v}), \\ R_{\alpha 3}(\varepsilon)(\boldsymbol{v}) = \dfrac{1}{\varepsilon} e_{\alpha 3}(\boldsymbol{v}), \\ R_{33}(\varepsilon)(\boldsymbol{v}) = \dfrac{1}{\varepsilon^2} e_{33}(\boldsymbol{v}). \end{cases}$$

By introducing $\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))$ in the variational inequality (3.5), we get

$$\begin{cases} \displaystyle\int_\Omega C_{\alpha\beta\sigma\tau} R_{\sigma\tau}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \\ \displaystyle +2\int_\Omega C_{\alpha\beta\sigma 3} R_{\sigma 3}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \\ \displaystyle +\int_\Omega C_{\alpha\beta 33} R_{33}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \\ \displaystyle +\frac{2}{\varepsilon}\int_\Omega C_{\alpha 3\sigma\tau} R_{\sigma\tau}(\boldsymbol{u}(\varepsilon))(\varepsilon) e_{\alpha 3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \\ \displaystyle +\frac{4}{\varepsilon}\int_\Omega C_{\alpha 3\sigma 3} R_{\sigma 3}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{\alpha 3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \\ \displaystyle +\frac{2}{\varepsilon}\int_\Omega C_{\alpha 333} R_{33}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{\alpha 3}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \\ \displaystyle +\frac{1}{\varepsilon^2}\int_\Omega C_{33\sigma\tau} R_{\sigma\tau}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{33}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \\ \displaystyle +\frac{2}{\varepsilon^2}\int_\Omega C_{33\sigma 3} R_{\sigma 3}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{33}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \\ \displaystyle +\frac{1}{\varepsilon^2}\int_\Omega C_{3333} R_{33}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{33}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \\ \displaystyle \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x + \int_{\Gamma_+} \boldsymbol{g}\cdot(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}a. \end{cases} \tag{4.6}$$

By introducing the new tensors $\mathbb{R}(\varepsilon)(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))$ into the inequality (4.6), we obtain

$$\int_\Omega C_{ijkl} R_{kl}(\varepsilon)(\boldsymbol{u}(\varepsilon)) R_{ij}(\varepsilon)(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \geq \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x + \int_{\Gamma_+} \boldsymbol{g}\cdot(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}a. \tag{4.7}$$

Taking $\boldsymbol{v} = \boldsymbol{0}$ in (4.7), we get

$$-\int_\Omega C_{ijkl} R_{kl}(\varepsilon)(\boldsymbol{u}(\varepsilon)) R_{ij}(\varepsilon)(\boldsymbol{u}(\varepsilon))\mathrm{d}x \geq -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}(\varepsilon)\mathrm{d}x - \int_{\Gamma_+} \boldsymbol{g}\cdot\boldsymbol{u}(\varepsilon)\mathrm{d}a,$$

which yields

$$\int_\Omega C_{ijkl} R_{kl}(\varepsilon)(\boldsymbol{u}(\varepsilon)) R_{ij}(\varepsilon)(\boldsymbol{u}(\varepsilon))\mathrm{d}x \leq \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}(\varepsilon)\mathrm{d}x + \int_{\Gamma_+} \boldsymbol{g}\cdot\boldsymbol{u}(\varepsilon)\mathrm{d}a.$$

Using the coerciveness properties of tensors $\mathbb{C}$, it follows from this inequality that

$$|\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))|_{0,\Omega}^2 \le c|\boldsymbol{u}(\varepsilon)|_{0,\Omega} \le c\|\boldsymbol{u}(\varepsilon)\|_{1,\Omega}.$$

Recalling Korn's inequality: there exists $c > 0$ such that

$$\|\boldsymbol{u}(\varepsilon)\|_{1,\Omega}^2 \le c|\boldsymbol{e}(\boldsymbol{u}(\varepsilon))|_{0,\Omega}^2,$$

we get that for $0 < \varepsilon \le 1$, there exist $c > 0$ such that

$$\|\boldsymbol{u}(\varepsilon)\|_{1,\Omega}^2 \le c|\boldsymbol{e}(\boldsymbol{u}(\varepsilon))|_{0,\Omega}^2 \le c|\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))|_{0,\Omega}^2 \le c\|\boldsymbol{u}(\varepsilon)\|_{1,\Omega}. \tag{4.8}$$

These inequalities imply that the norms $\|\boldsymbol{u}(\varepsilon)\|_{1,\Omega}$ and $|\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))|_{0,\Omega}$ are uniformly bounded with respect to ε. Then there exists a subsequence, still denoted by $\boldsymbol{u}(\varepsilon)$, and functions $\boldsymbol{u} \in \mathbb{H}^1(\Omega)$ and $\mathbb{R} \in \mathbb{L}^2(\Omega)$ such that as $\varepsilon \to 0$, we have

$$\begin{cases} \boldsymbol{u}(\varepsilon) \rightharpoonup \boldsymbol{u} & \text{in } \mathbb{H}^1(\Omega), \\ \mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon)) \rightharpoonup \mathbb{R}(\boldsymbol{u}) & \text{in } \mathbb{L}^2(\Omega). \end{cases}$$

Moreover, from the definition of $\mathbb{R}(\varepsilon)$, we have

$$|e_{\alpha 3}(\boldsymbol{u}(\varepsilon))|_{0,\Omega} \le c\varepsilon, \quad |e_{33}(\boldsymbol{u}(\varepsilon))|_{0,\Omega} \le c\varepsilon^2.$$

Hence $e_{i3}(\boldsymbol{u}(\varepsilon)) \rightharpoonup 0$ in $L^2(\Omega)$ as $\varepsilon \to 0$, thus

$$|e_{i3}(\boldsymbol{u})|_{0,\Omega} \le \liminf_{\varepsilon \to 0} |e_{i3}(\boldsymbol{u}(\varepsilon))|_{0,\Omega} = 0.$$

Since $e_{i3}(\boldsymbol{u}) = 0$, we deduce that there exists a bi-dimensional field $\boldsymbol{\zeta} = (\zeta_i)$ such that $\zeta_\alpha \in H^1(\omega)$ and $\zeta_3 \in H^2(\omega)$, and $\boldsymbol{u}$ is a Kirchhoff-Love displacement field

$$u_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3, \quad u_3 = \zeta_3.$$

The following lemma will be frequently used in the next step.

Lemma 4.2 *Let the operator $\mathbb{A}(\varepsilon) : \boldsymbol{u}(\varepsilon) \longrightarrow \mathbb{H}^1(\Omega)$ which satisfies the weak convergence*

$$\mathbb{A}(\varepsilon)(\boldsymbol{u}(\varepsilon)) \rightharpoonup \mathbb{A}(\boldsymbol{u}) \in \mathbb{L}^2(\Omega) \quad \text{as } \varepsilon \to 0.$$

If $\boldsymbol{u}(\varepsilon) \in \mathbb{K}(\Omega)$ solves the variational inequality

$$\int_\Omega \mathbb{A}(\varepsilon)(\boldsymbol{u}(\varepsilon)) \cdot \partial_3\big(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)\big)\mathrm{d}x \ge \varepsilon \int_\Omega \boldsymbol{f} \cdot \big(\boldsymbol{v} - \boldsymbol{u}(\varepsilon)\big)\mathrm{d}x, \quad \forall \boldsymbol{v} \in \mathbb{K}(\Omega), \tag{4.9}$$

then $\mathbb{A}(\boldsymbol{u}) = \boldsymbol{0}$.

Proof First taking $\boldsymbol{v} = \boldsymbol{0}$ and then $\boldsymbol{v} = 2\boldsymbol{u}(\varepsilon)$ in (4.9), we get

$$\int_\Omega \mathbb{A}(\varepsilon)(\boldsymbol{u}(\varepsilon)) \cdot \partial_3\boldsymbol{u}(\varepsilon)\mathrm{d}x = \varepsilon \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}(\varepsilon)\mathrm{d}x.$$

Moreover, (4.9) now reads

$$\int_\Omega \mathbb{A}(\varepsilon)(\boldsymbol{u}(\varepsilon)) \cdot \partial_3 \boldsymbol{v} \mathrm{d}x \geq \varepsilon \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \mathrm{d}x, \quad \forall \boldsymbol{v} \in \mathbb{K}(\Omega),$$

which implies

$$\int_\Omega \mathbb{A}(\boldsymbol{u}) \cdot \partial_3 \boldsymbol{v} \mathrm{d}x \geq \boldsymbol{0}, \quad \forall \boldsymbol{v} \in \mathbb{K}(\Omega).$$

We consider the following 2 cases for the in-plane components and the vertical component of the displacement fields, respectively.

(1) Since v_α belongs to a vector space, the previous inequality becomes an equality

$$\int_\Omega A_\alpha(\boldsymbol{u})\ \partial_3 v_\alpha \mathrm{d}x = 0, \quad \forall v_\alpha \in H^1(\Omega),$$

then, following [1] we get $A_\alpha(\boldsymbol{u}) = 0$.

(2) For the transverse component we consider $z, t \in D(\Omega),\ z > 0$ and choose $v_3 \geq 0$ under the form

$$v_3(x) = z(x_1, x_2) + \max_{-1<s<1} |t(x_1, x_2, s)| + \int_{-1}^{x_3} t(x_1, x_2, s) \mathrm{d}s.$$

By a direct computation we can show that $\int_\Omega A_3(\boldsymbol{u})\ \partial_3 v_3 \mathrm{d}x = \int_\Omega A_3(\boldsymbol{u})\ t(x_1, x_2, x_3) \mathrm{d}x = 0$ for all $t \in D(\Omega)$ implies $A_3(\boldsymbol{u}) = 0$.

Step II. Before establishing the strong convergence, we compute $R_{i3}(\boldsymbol{u})$.

The weak convergence established in the Step I implies

$$R_{\alpha\beta}(\varepsilon)(\boldsymbol{u}(\varepsilon)) \rightharpoonup R_{\alpha\beta}(\boldsymbol{u}) = e_{\alpha\beta}(\boldsymbol{u}) \text{ in } L^2(\Omega) \text{ as } \varepsilon \to 0.$$

Taking $v_3 = u_3(\varepsilon)$ and multiplying by ε in (4.6), we get

$$\left\{\begin{aligned} &\int_\Omega (2C_{\alpha 3\sigma\tau} R_{\sigma\tau}(\varepsilon)(\boldsymbol{u}(\varepsilon)) + 4C_{\alpha 3\sigma 3} R_{\sigma 3}(\varepsilon)(\boldsymbol{u}(\varepsilon)) + 2C_{\alpha 333} R_{33}(\varepsilon)(\boldsymbol{u}(\varepsilon)))\partial_3(v_\alpha - u_\alpha(\varepsilon))\mathrm{d}x \\ &\geq -\varepsilon \int_\Omega C_{\alpha\beta\sigma\tau} R_{\sigma\tau}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x - 2\varepsilon \int_\Omega C_{\alpha\beta\sigma 3} R_{\sigma 3}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x \\ &\quad -\varepsilon \int_\Omega C_{\alpha\beta 33} R_{33}(\varepsilon)(\boldsymbol{u}(\varepsilon)) e_{\alpha\beta}(\boldsymbol{v} - \boldsymbol{u}(\varepsilon))\mathrm{d}x + \varepsilon \int_\Omega f_\alpha(v_\alpha - u_\alpha(\varepsilon))\mathrm{d}x \\ &\quad +\varepsilon \int_{\Gamma_+} g_\alpha(v_\alpha - u_\alpha(\varepsilon))\mathrm{d}a, \quad \forall v_\alpha \in H^1(\Omega),\ \ v_\alpha = 0 \text{ on } \Gamma_0, \end{aligned}\right.$$

then, we obtain

$$2C_{\alpha 3\alpha 3} R_{\alpha 3}(\boldsymbol{u}) + C_{\alpha 333} R_{33}(\boldsymbol{u}) = -C_{\alpha 3\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}).$$

Finally, taking $v_\alpha = u_\alpha(\varepsilon)$ and multiplying by ε^2 in (4.6), we obtain

$$\left\{\begin{aligned} &\int_\Omega (C_{33\sigma\tau} R_{\sigma\tau}(\varepsilon)(\boldsymbol{u}(\varepsilon)) + 2C_{33\sigma 3} R_{\sigma 3}(\varepsilon)(\boldsymbol{u}(\varepsilon)) + C_{3333} R_{33}(\varepsilon)(\boldsymbol{u}(\varepsilon)))\partial_3(v_3 - u_3(\varepsilon))\mathrm{d}x \\ &\geq -\varepsilon \int_\Omega C_{\alpha 3\sigma\tau} R_{\sigma\tau}(\boldsymbol{u}(\varepsilon))(\varepsilon)\partial_\alpha(v_3 - u_3(\varepsilon))\mathrm{d}x - 2\varepsilon \int_\Omega C_{\alpha 3\sigma 3} R_{\sigma 3}(\varepsilon)(\boldsymbol{u}(\varepsilon))\partial_\alpha(v_3 - u_3(\varepsilon))\mathrm{d}x \\ &\quad -\varepsilon \int_\Omega C_{\alpha 333} R_{33}(\varepsilon)(\boldsymbol{u}(\varepsilon))\partial_\alpha(v_3 - u_3(\varepsilon))\mathrm{d}x + \varepsilon^2 \int_\Omega f_3(v_3 - u_3(\varepsilon))\mathrm{d}x \\ &\quad +\varepsilon^2 \int_{\Gamma_+} g_3(v_3 - u_3(\varepsilon))\mathrm{d}a, \quad \forall v_3 \in H^1(\Omega),\ \ v_3 = 0 \text{ on } \Gamma_0,\ \ v_3 \geq 0 \text{ on } \Gamma_- \end{aligned}\right.$$

and

$$2C_{33\alpha 3}R_{\alpha 3}(\boldsymbol{u})+C_{3333}R_{33}(\boldsymbol{u})=-C_{33\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}).$$

Thus, $R_{i3}(\boldsymbol{u})$ satisfies the following linear system

$$2C_{i3\alpha 3}R_{\alpha 3}(\boldsymbol{u})+C_{i333}R_{33}(\boldsymbol{u})=-C_{i3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}). \tag{4.10}$$

To show that this system has a unique solution, first by the symmetry and positivity of tensor $\mathbf{C}$ in (2.1), for every second order 3×3 symmetric tensor $\mathbb{A}=(A_{ij})$, we have

$$C_{ijkl}A_{kl}A_{ij}\geq cA_{kl}A_{ij}.$$

Next we note that the system (4.10) can be written as $2C_{i3\alpha 3}x_\alpha+C_{i333}x_3=f_i$, and the determinant of this linear system is

$$\begin{vmatrix} 2C_{1313} & 2C_{1323} & C_{1333}\\ 2C_{2313} & 2C_{2323} & C_{2333}\\ 2C_{3313} & 2C_{3323} & C_{3333}\end{vmatrix}=4\begin{vmatrix} C_{1313} & C_{1323} & C_{1333}\\ C_{2313} & C_{2323} & C_{2333}\\ C_{3313} & C_{3323} & C_{3333}\end{vmatrix}.$$

With $A_{k\alpha}=0$ we get $C_{ijkl}A_{kl}A_{ij}=C_{i3j3}A_{i3}A_{j3}>0$, therefore the system (4.10) has a unique solution

$$\begin{cases} R_{\alpha 3}(\boldsymbol{u})=-\dfrac{1}{2\triangle}\triangle^{\alpha}_{\zeta\eta}e_{\zeta\eta}(\boldsymbol{u}),\\ R_{33}(\boldsymbol{u})=-\dfrac{1}{\triangle}\triangle^{3}_{\zeta\eta}e_{\zeta\eta}(\boldsymbol{u}),\end{cases} \tag{4.11}$$

where $\triangle$, $\triangle^{\alpha}_{\zeta\eta}$ $(\alpha=1,2)$ and $\triangle^{3}_{\zeta\eta}$ are given by (4.4).

Step III. The whole family $\{\boldsymbol{u}(\varepsilon)\}$ converges strongly.

Introduce the notation $\int_\Omega \mathbb{CA}:\mathbb{A}\mathrm{d}x=\int_\Omega C_{ijkl}A_{kl}A_{ij}\mathrm{d}x$ for all second order symmetric tensor $\mathbb{A}$. We have

$$\begin{aligned} & c|\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))-\mathbb{R}(\boldsymbol{u})|^2_{0,\Omega}\\ \leq & \int_\Omega \mathbb{C}\big(\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))-\mathbb{R}(\boldsymbol{u})\big):\big(\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))-\mathbb{R}(\boldsymbol{u})\big)\mathrm{d}x\\ \leq & \int_\Omega \mathbb{C}\mathbb{R}(\boldsymbol{u}):(\mathbb{R}(\boldsymbol{u})-2\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon)))\mathrm{d}x+\int_\Omega \mathbb{C}\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon)):\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))\mathrm{d}x. \end{aligned}$$

Since we have already established the weak convergence $\mathbb{R}(\varepsilon)\rightharpoonup\mathbb{R}$ in $\mathbb{L}^2(\Omega)$ as $\varepsilon\to 0$,

$$\lim_{\varepsilon\to 0}c|\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))-\mathbb{R}(\boldsymbol{u})|^2_{0,\Omega}\leq-\int_\Omega \mathbb{C}\mathbb{R}(\boldsymbol{u}):\mathbb{R}(\boldsymbol{u})\mathrm{d}x+\lim_{\varepsilon\to 0}\int_\Omega C\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon)):\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))\mathrm{d}x.$$

Taking $\boldsymbol{v}=\boldsymbol{u}$, and passing to the limit as $\varepsilon\to 0$ in (4.7), we get

$$\int_\Omega \mathbb{C}\mathbb{R}(\boldsymbol{u}):\mathbb{R}(\boldsymbol{u})\mathrm{d}x-\lim_{\varepsilon\to 0}\int_\Omega C\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon)):\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))\mathrm{d}x\geq 0,$$

so we have $\lim\limits_{\varepsilon\to 0}|\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))-\mathbb{R}(\boldsymbol{u})|_{0,\Omega}^2\le 0$, $\{\mathbb{R}(\varepsilon)(\boldsymbol{u}(\varepsilon))\}$ converges strongly to $\mathbb{R}(\boldsymbol{u})$ in $\mathbb{L}^2(\Omega)$. By the definition of $\mathbb{R}(\varepsilon)$, we have

$$\begin{aligned}|\boldsymbol{e}(\boldsymbol{u}(\varepsilon))-\boldsymbol{e}(\boldsymbol{u})|_{0,\Omega}^2 \le \sum_{\alpha,\beta}|R_{\alpha\beta}(\varepsilon)(\boldsymbol{u}(\varepsilon))-R_{\alpha\beta}(\boldsymbol{u})|_{0,\Omega}^2+2\varepsilon^2\sum_{\alpha}|R_{\alpha3}(\varepsilon)(\boldsymbol{u}(\varepsilon))-R_{\alpha3}(\boldsymbol{u})|_{0,\Omega}^2\\ +\varepsilon^4|R_{33}(\varepsilon)(\boldsymbol{u}(\varepsilon))-R_{33}(\boldsymbol{u})|_{0,\Omega}^2,\end{aligned}$$

which implies that the sequence $\{\boldsymbol{e}(\boldsymbol{u}(\varepsilon))\}$ converges strongly to $\boldsymbol{e}(\boldsymbol{u})$ in $\mathbb{L}^2(\Omega)$. Then by Korn's inequality the sequence $\boldsymbol{u}(\varepsilon)$ converges strongly to $\boldsymbol{u}$ in $\mathbb{H}^1(\Omega)$.

Step IV. For any Kirchhoff-Love vector field $\boldsymbol{v}=(\boldsymbol{\zeta}_H,\zeta_3)\in\mathbb{V}_H(\omega)\times K_3(\omega)$, we have

$$e_{\alpha\beta}(\boldsymbol{v})=e_{\alpha\beta}(\boldsymbol{\eta}_H)-x_3\partial_{\alpha\beta}\eta_3,\quad e_{i3}(\boldsymbol{v})=0.$$

Passing to the limit as $\varepsilon\to 0$ in (4.7), for all vector field $\boldsymbol{v}=(\boldsymbol{\zeta}_H,\zeta_3)\in\mathbb{V}_H(\omega)\times K_3(\omega)$ we get the following variational inequality:

$$\begin{cases}\displaystyle\int_\Omega C_{\alpha\beta\sigma\tau}(e_{\sigma\tau}(\boldsymbol{\zeta}_H)-x_3\partial_{\sigma\tau}\zeta_3)(e_{\alpha\beta}(\boldsymbol{\eta}_H-\boldsymbol{\zeta}_H)-x_3\partial_{\alpha\beta}(\eta_3-\zeta_3))\mathrm{d}x\\ \displaystyle +2\int_\Omega C_{\alpha\beta\sigma3}\Big[-\frac{1}{2\triangle}(\triangle^{\sigma}_{\sigma\tau}(e_{\sigma\tau}(\boldsymbol{\zeta}_H)-x_3\partial_{\sigma\tau}\zeta_3))\Big](e_{\alpha\beta}(\boldsymbol{\eta}_H-\boldsymbol{\zeta}_H)-x_3\partial_{\alpha\beta}(\eta_3-\zeta_3))\mathrm{d}x\\ \displaystyle +\int_\Omega C_{\alpha\beta33}\Big[-\frac{1}{\triangle}(\triangle^{3}_{\sigma\tau}(e_{\sigma\tau}(\boldsymbol{\zeta}_H)-x_3\partial_{\sigma\tau}\zeta_3))\Big](e_{\alpha\beta}(\boldsymbol{\eta}_H-\boldsymbol{\zeta}_H)-x_3\partial_{\alpha\beta}(\eta_3-\zeta_3))\mathrm{d}x\\ \displaystyle \ge\int_\Omega\Big[f_\alpha(\eta_\alpha-\zeta_\alpha-x_3\partial_\alpha(\eta_3-\zeta_3))+f_3(\eta_3-\zeta_3)\Big]\mathrm{d}x\\ \displaystyle +\int_{\Gamma_+}\Big[g_\alpha(\eta_\alpha-\zeta_\alpha-x_3\partial_\alpha(\eta_3-\zeta_3))+g_3(\eta_3-\zeta_3)\Big]\mathrm{d}a.\end{cases}$$

Then we get

$$\begin{cases}\displaystyle 2\int_\omega\widetilde{C}_{\alpha\beta\sigma\tau}e_{\sigma\tau}(\boldsymbol{\zeta}_H)e_{\alpha\beta}(\boldsymbol{\eta}_H-\boldsymbol{\zeta}_H)\mathrm{d}\omega+\frac{2}{3}\int_\omega\widetilde{C}_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3\partial_{\alpha\beta}(\eta_3-\zeta_3)\mathrm{d}\omega\\ \displaystyle \ge\int_\omega\boldsymbol{p}_H\cdot(\boldsymbol{\eta}_H-\boldsymbol{\zeta}_H)\mathrm{d}\omega+\int_\omega p_3(\eta_3-\zeta_3)\mathrm{d}\omega-\int_\omega s_\alpha\partial_\alpha(\eta_3-\zeta_3)\mathrm{d}\omega.\end{cases}$$

This inequality can be decoupled as

$$\begin{cases}\displaystyle \frac{2}{3}\int_\omega\widetilde{C}_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3\partial_{\alpha\beta}(\eta_3-\zeta_3)\mathrm{d}\omega\\ \displaystyle \ge\int_\omega p_3(\eta_3-\zeta_3)\mathrm{d}\omega-\int_\omega s_\alpha\partial_\alpha(\eta_3-\zeta_3)\mathrm{d}\omega,\quad\forall\eta_3\in K_3(\omega),\\ \displaystyle 2\int_\omega\widetilde{C}_{\alpha\beta\sigma\tau}e_{\sigma\tau}(\boldsymbol{\zeta}_H)e_{\alpha\beta}(\boldsymbol{\eta}_H)\mathrm{d}\omega=\int_\omega\boldsymbol{p}_H\cdot\boldsymbol{\eta}_H\mathrm{d}\omega,\quad\forall\boldsymbol{\eta}_H\in\mathbb{V}_H(\omega),\end{cases}\tag{4.12}$$

where the mechanical forces and electric charges are given by

$$p_i:=\int_{-1}^{1}f_i\mathrm{d}x_3+g_i,\quad s_\alpha:=\int_{-1}^{1}x_3f_\alpha\mathrm{d}x_3+g_\alpha,$$

respectively, and the new characteristic tensor $\widetilde{C}=(\widetilde{C}_{\alpha\beta\sigma\tau})$ of the elastic plate is given by

$$\widetilde{C}_{\alpha\beta\sigma\tau}=C_{\alpha\beta\sigma\tau}-\frac{1}{\triangle}C_{\alpha\beta k3}\triangle^{k}_{\sigma\tau}.\tag{4.13}$$

Remark 4.1 It is interesting that (4.1) consists of an equality in a vector space for the ζ_α components, namely, for the membrane part of the solution, and an inequality in a cone for the ζ_3 component, namely, for the bending part of the solution. Thus, the obstacle condition deals only with the bending part.

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