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On a Class of Non-local Operators in Conformal Geometry*

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(*Dedicated to Professor Haim Brezis on his* 70*th birthday*)

Abstract In this expository article, the authors discuss the connection between the study of non-local operators on Euclidean space to the study of fractional GJMS operators in conformal geometry. The emphasis is on the study of a class of fourth order operators and their third order boundary operators. These third order operators are generalizations of the Dirichlet-to-Neumann operator.

Keywords High order fractional GJMS operator, Generalized boundary Yamabe problem, Sobolev trace extension
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1 Introduction

In recent literature, there have been some parallel developments. On the one hand, there is a vast literature in the study of "non-local operators" on Euclidean domains, led by Caffarelli and many others. On the other hand, there is the study of classes of pseudo-differential operators defined on the boundary of manifolds, which are generalizations of the Dirichlet-to-Neumann operator; they are also non-local in nature. The study of the latter led to the recent study of the "fractional Yamabe problem" in conformal geometry. Obviously, there are interesting connections between the two. In this article, we describe one of them.

This is an expository paper which summarizes some of the results in the papers [1, 6–7, 26]. One key element which has played a major role in all above papers is the extension theorem of Caffarelli-Silvestre (see [8]) and a higher order generalization of the theorem by the second author (see [26]). For expository purposes, here we derive the extension theorem part of the paper [26], and the later generalization to the manifolds setting in [6].

This article is organized as follows. In Section 2, we cite the extension theorem of [8] for the fractional order Laplace operator of order 2γ, where $0 < \gamma < 1$, defined on the Euclidean space. In Section 3, we derive a generalization of the extension theorem to fractional order Laplace operators of order $\gamma > 1$, again on Euclidean spaces. In Section 4, we briefly quote the work of Graham and Zworski (see [16]) for the existence of a class of fractional order conformal

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covariant operators defined on conformal compact Einstein manifolds, and explain how in the special case when the manifold is the hyperbolic ball, the operators defined on the boundary of the ball correspond to the fractional order Laplace operators defined on Euclidean space (see [7]). In Section 5, we discuss the generalization of the extension theorems in [8, 26] to the manifold setting (we call this the "curved" setting) when $1 < \gamma < 2$ (see [6]). In Section 6, we explain how to apply the extension theorem in the special case $\gamma = \frac{3}{2}$ to derive some sharp Sobolev trace inequalities for the bi-Laplace operators on the Euclidean ball (see [1]).

To make the paper easier to understand and the picture more transparent, in most part of the paper we only derive results for the boundary operator of order 3, that is, when $\gamma = \frac{3}{2}$.

In the settings of Sections 3–4, the solution of the Poisson equation of order 2 in both the settings of Euclidean space and in the settings of the conformal compact Einstein manifolds is equivalent to the solution of a fourth order PDE, which is a product of the second order Poisson equation with another second order equation. This crucial equivalence is summarized in Remarks 3.1–3.2, 5.3 and 5.5.

2 Extension Theorem of Caffarelli-Silvestre

In [8], Caffarelli and Silvestre showed that, for $0 < \gamma < 1$, the fractional Laplacian $(-\Delta)^\gamma$ of a function f living on $\mathbb{R}^n$ can be understood as the Dirichlet-to-Neumann map for a function U living on the upper half-space $\mathbb{R}^{n+1}_+$, where U coincided with f on $\mathbb{R}^n$, and U satisfied a particular 2nd-order elliptic equation. The more precise statement of the theorem is the following.

First we recall a well-known result: f smooth on $\mathbb{R}^n$,

$$\Delta_{x,y} U(x,y) = 0 \quad \text{on } \mathbb{R}^{n+1}_+ \text{ with } U\big|_{\mathbb{R}^n}(x) \,=\, f(x),$$

then $(-\Delta_x)^{\frac{1}{2}} f(x) = -U_y(x,0)$.

Theorem 2.1 (see [8]) *For $0 < \gamma < 1$, $a = 1 - 2\gamma$,*

$$(*) \quad \begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{on } \mathbb{R}^{n+1}_+, \\ U\big|_{\mathbb{R}^n} = f. \end{cases}$$

Then for each $0 < \gamma < 1$, if

$$f \in H^\gamma(R^n) = W_0^{\gamma,2}(\mathbb{R}^n),$$

we have

$$C_{n,\gamma} \int_{\mathbb{R}^n} \int_{y>0} |\nabla U|^2 y^a \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^n} |\xi|^{2\gamma} |\widehat{f}(\xi)|^2 \mathrm{d}\xi = \int_{\mathbb{R}^n} (-\Delta)^\gamma f \cdot f \,\mathrm{d}x, \tag{2.1}$$

where $C_{n,\gamma}$ is some normalization constant. This energy equality (2.1) *implies that*

$$(-\Delta_x)^\gamma f = C_{n,\gamma} \lim_{y\to 0} y^a \frac{\partial U}{\partial n}\Big|_{y=0}. \tag{2.2}$$

This theorem is an extremely useful tool in the study of non-local operators. It has many important applications to free-boundary problems, the study of non-local minimal surfaces, etc., which we will not survey here.

3 Extension Theorem of Order $\gamma > 1$ on Euclidean Space

We now generalize the energy equality of Caffarelli and Silvestre to show that the fractional Laplacian of any positive, non-integer order can be represented as a higher-order Neumann derivative of an extended function U, where U satisfies a higher-order elliptic equation.

To illustrate the technique, we first show that in the case $1 < \gamma < 2$, the fractional Laplacian $(-\Delta_x)^\gamma$ can still be represented as a suitable Neumann derivative for the solution of a higher order equation, and subsequently we generalize this to all positive, non-integer values of γ. We remark that in Section 4 below, we explain that the extension has an interesting interpretation in terms of scattering theory, and, in particular that the following equation

$$(\Delta_{x,y}U) + \frac{a}{y}U_y = 0 \tag{3.1}$$

(where $a = 1 - 2\gamma$) holds for the extended function U and for all non-integer γ (see [7]).

3.1 The model case: $\gamma = \frac{3}{2}$

First we discuss the extension in a special case, which illustrates the main point of the argument without the complexity of notation we need for more general cases. In what follows, $\gamma = \frac{3}{2}$ and $a = 1 - 2\gamma = -2$.

In this case, the equation (3.1) takes the form

$$\Delta U - \frac{2}{y}U_y = 0 \quad \text{on } \mathbb{R}^{n+1}_+, \tag{3.2}$$

where Δ denotes $\Delta_{x,y}$ in (3.2) as well as in the rest of Section 3. The first observation is that (3.2) implies that

$$(-\Delta)^2 U = 0 \quad \text{on } \mathbb{R}^{n+1}_+. \tag{3.3}$$

We have the following theorem.

Theorem 3.1 *Function $U \in W^{2,2}(\mathbb{R}^{n+1}_+)$ satisfies the equation*

$$\Delta^2 U(x,y) = 0 \tag{3.4}$$

on the upper half space for $(x,y) \in \mathbb{R}^n \times \mathbb{R}_+$, where y is the special direction, and satisfies the boundary conditions

$$U(x,0) = f(x),$$
$$U_y(x,0) = 0$$

along $\{y = 0\}$, where $f(x)$ is some function defined on $H^{\frac{3}{2}}(\mathbb{R}^n)$.

We have the result that

$$(-\Delta_x)^{\frac{3}{2}} f(x) = C_{n,\frac{3}{2}} \frac{\partial}{\partial y}\Delta U(x,0). \tag{3.5}$$

More specifically,

$$\int_{\mathbb{R}^n} |\xi|^3 |\widehat{f}(\xi)|^2 \mathrm{d}\xi = C_{n,\frac{3}{2}} \int_{\mathbb{R}^{n+1}_+} |\Delta U(x,y)|^2 \mathrm{d}x\mathrm{d}y. \tag{3.6}$$

Proof Taking the Fourier transform in the x variable only on the energy term ΔU, we get

$$-|\xi|^2\widehat{U}(\xi,y)+\widehat{U}_{yy}(\xi,y).$$

So minimizing the energy corresponds to minimizing the integral

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}|-|\xi|^2\widehat{U}(\xi,y)+\widehat{U}_{yy}(\xi,y)|^2\mathrm{d}\xi\mathrm{d}y.$$

Integrating by parts, we see that for each value of ξ, the minimizer $\widehat{U}$ solves the ODE

$$|\xi|^4\widehat{U}-2|\xi|^2\widehat{U}_{yy}+\widehat{U}_{yyyy}=0.$$

Let $\phi\in W^{2,2}(\mathbb{R}_+)$ be the minimizer of the functional

$$J(\phi)=\int_{\mathbb{R}_+}(\phi''(y)-\phi(y))^2\mathrm{d}y$$

among functions satisfying the conditions $\phi(0)=1,\ \phi'(0)=0$. Thus ϕ solves the ODE

$$\phi(y)-2\phi''(y)+\phi''''(y)=0$$

with appropriate boundary conditions, and we see that $\widehat{U}(\xi,y)=\widehat{f}(\xi)\phi(|\xi|y)$ is a good representation for $\widehat{U}$.

By calculating, we see that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+}(\Delta U)^2\mathrm{d}x\mathrm{d}y&=C_n\int|-|\xi|^2\widehat{U}+\widehat{U}_{yy}|^2\mathrm{d}\xi\mathrm{d}y\\&=C_n\int|-|\xi|^2\widehat{f}(\xi)\phi(|\xi|y)+|\xi|^2\widehat{f}(\xi)\phi''(|\xi|y)|^2\mathrm{d}\xi\mathrm{d}y\\&=C_n\int|\xi|^4|\widehat{f}(\xi)|^2(-\phi(\overline{y})+\phi''(\overline{y}))^2\frac{\mathrm{d}\overline{y}}{|\xi|}\mathrm{d}\xi\\&=C_nJ(\phi)\int|\xi|^3|\widehat{f}(\xi)|^2\mathrm{d}\xi,\end{aligned}$$

and hence the energies are identical up to a constant.

The Euler-Lagrange equation for the left-hand side above is simply the bi-Laplace equation, while for the right-hand side it is the fractional harmonic equation of order γ, and the rest of the result follows.

Remark 3.1 A less obvious fact is that for a given f in $H^{\frac{3}{2}}(\mathbb{R}^n)$, a solution U satisfies (3.2) with $U(x,0)=f(x)$ on $\mathbb{R}^n$ if and only if it satisfies the equation (3.3) and

$$\begin{aligned}U(x,0)&=f(x),\\U_y(x,0)&=0.\end{aligned}$$

3.2 The cases $1 < \gamma < 2$

In these cases, the argument is precisely analogous to the previous section, except that, like Caffarelli and Silvestre, we shall use a weighted seminorm. To be precise, we attach the weighted measure $y^b \mathrm{d}y\mathrm{d}x$ to our Sobolev spaces, and consider energy minimizers with respect to this measure on the upper half space of an appropriate energy. Here, we take $b = 3 - 2\gamma$.

To construct the appropriate energy in this space, we introduce the following operator, which is a variant of the Laplacian adapted to the measure, whose virtue is that in the weighted space it behaves under integration by parts just as the regular Laplacian does in an unweighted space. Setting

$$\Delta_b U = \Delta U + \frac{b}{y} U_y$$

gives us the desired relationship as follows:

$$\int_{\mathbb{R}_+^{n+1}} (\nabla\Phi \cdot \nabla\Psi) y^b \mathrm{d}y\mathrm{d}x = -\int_{\mathbb{R}^n} \Phi \lim_{y\to 0} \Big(y^b \frac{\partial \Psi}{\partial y} \Big) \mathrm{d}x - \int_{\mathbb{R}_+^{n+1}} \Phi(\Delta_b \Psi) y^b \mathrm{d}y\mathrm{d}x.$$

Clearly, the appropriate 2nd-order seminorm for our space is

$$\int_{\mathbb{R}_+^{n+1}} y^b |\Delta_b U|^2 \mathrm{d}y\mathrm{d}x.$$

Our space will be equipped with the norm

$$\|U\|^2_{W^{2,2}(\mathbb{R}_+^{n+1}, y^b)} = \|y^{\frac{b}{2}} \Delta_b U\|^2_{L^2(\mathbb{R}_+^{n+1})} + \|y^{\frac{b}{2}} \nabla U\|^2_{L^2(\mathbb{R}_+^{n+1})} + \|y^{\frac{b}{2}} U\|^2_{L^2(\mathbb{R}_+^{n+1})}.$$

We now observe that if a function U satisfies (3.1) on $\mathbb{R}_+^{n+1}$ with $a = 1 - 2\gamma$, then it satisfies

$$(-\Delta_b)^2 U = 0 \tag{3.7}$$

on $\mathbb{R}_+^{n+1}$, where $b = 3 - 2\gamma$.

Our main result is as follows.

Theorem 3.2 *Functions $U \in W^{2,2}(\mathbb{R}_+^{n+1}, y^b)$ satisfy the equation*

$$\Delta_b^2 U(x,y) = 0 \tag{3.8}$$

on the upper half space for $(x,y) \in \mathbb{R}^n \times \mathbb{R}_+$, where y is the special direction, and satisfy the boundary conditions

$$U(x,0) = f(x),$$
$$\lim_{y\to 0} y^b U_y(x,0) = 0$$

along $\{y = 0\}$, where $f(x)$ is some function defined on $H^\gamma(\mathbb{R}^n)$. We have the result that

$$(-\Delta_x)^\gamma f(x) = C_{n,\gamma} \lim_{y\to 0} y^b \frac{\partial}{\partial y} \Delta_b U(x,y). \tag{3.9}$$

Specifically,

$$\int_{\mathbb{R}^n} |\xi|^{2\gamma} |\widehat{f}(\xi)|^2 \mathrm{d}\xi = C_{n,\gamma} \int_{\mathbb{R}_+^{n+1}} y^b |\Delta_b U(x,y)|^2 \mathrm{d}x\mathrm{d}y. \tag{3.10}$$

Proof Existence and uniqueness of a solution is guaranteed by the usual considerations. Taking the Fourier transform in the x variable only on the equation $\Delta_b^2 U = 0$, we get

$$|\xi|^4 \widehat{U} - \Big(\frac{2b}{y}|\xi|^2 + \frac{b(b-2)}{y^3}\Big)\widehat{U}_y + \Big(-2|\xi|^2 + \frac{b(b-2)}{y^2}\Big)\widehat{U}_{yy} + \frac{2b}{y}\widehat{U}_{yyy} + \widehat{U}_{yyyy} = 0,$$

which is a 4-th order ODE in y for each value of ξ. Let $\phi \in W^{2,2}(\mathbb{R}_+, y^b)$ be the minimizer of the functional

$$J(\phi) = \int_{\mathbb{R}_+} y^b \Big(\phi''(y) + \frac{b}{y}\phi'(y) - \phi(y)\Big)^2 \mathrm{d}y$$

among functions satisfying the conditions $\phi(0) = 1,\ \phi'(0) = 0$. Thus ϕ solves the ODE

$$\phi - \Big(\frac{2b}{y} + \frac{b(b-2)}{y^3}\Big)\phi' + \Big(-2 + \frac{b(b-2)}{y^2}\Big)\phi'' + \frac{2b}{y}\phi''' + \phi'''' = 0$$

with appropriate boundary conditions, and we see that $\widehat{U}(\xi, y) = \widehat{f}(\xi)\phi(|\xi|y)$ is a good representation for $\widehat{U}$.

By calculating, we see that

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} y^b (\Delta_b U)^2 \mathrm{d}x\mathrm{d}y &= C_n \int \Big| -|\xi|^2 \widehat{U} + \frac{b}{y}\widehat{U}_y + \widehat{U}_{yy}\Big|^2 \mathrm{d}\xi y^b \mathrm{d}y \\
&= C_n \int \Big| -|\xi|^2 \widehat{f}(\xi)\phi(|\xi|y) + \frac{b|\xi|}{y}\widehat{f}(\xi)\phi'(|\xi|y) + |\xi|^2 \widehat{f}(\xi)\phi''(|\xi|y)\Big|^2 \mathrm{d}\xi y^b \mathrm{d}y \\
&= C_n \int |\xi|^4 |\widehat{f}(\xi)|^2 \Big(-\phi(\overline{y}) - \frac{b}{\overline{y}}\phi'(\overline{y}) + \phi''(\overline{y})\Big)^2 \frac{\overline{y}^b \mathrm{d}\overline{y}}{|\xi|^{b+1}} \mathrm{d}\xi \\
&= C_n J(\phi) \int |\xi|^{2\gamma} |\widehat{f}(\xi)|^2 \mathrm{d}\xi,
\end{aligned}$$

and hence the energies are identical up to a constant.

The Euler-Lagrange equation for the left-hand side above is simply the bi-Laplace equation, while for the right-hand side it is the fractional harmonic equation of order γ, and the rest of the result follows.

Remark 3.2 It turns out that for a given function f in $H^\gamma(\mathbb{R}^n)$, a solution U satisfies (3.1) with $U(x, 0) = f(x)$ on $\mathbb{R}^n$ if and only if it satisfies the equation (3.7) and

$$\begin{aligned}
&U(x, 0) = f(x), \\
&\lim_{y \to 0} y^b U_y(x, 0) = 0.
\end{aligned}$$

This fact can be compared to Remark 5.5, which follows from a general fact in scattering theory.

3.3 The general case

The general case follows on a similar theme, taking progressively higher powers of the weighted Laplacian Δ_b. Setting our boundary conditions, we take our cue from [7], whence we learn that, when $\gamma < \frac{n}{2}$ the extension function satisfies

$$\Delta U + \frac{1 - 2\gamma}{y} U_y = 0$$

and furthermore that, if $m < \gamma < m+1$, U has a series expansion in y that has only even integer powers, up to the power $y^{2\gamma}$. A similar behaviour holds for all non-integer value of γ, hence we set boundary conditions for the y-derivatives of U in the following result.

Theorem 3.3 *Let $\gamma > 0$ be some non-integer, positive power of the Laplacian. Let $m < \gamma < m+1$, or $m = [\gamma]$, and $b(\gamma) = 2m+1-2\gamma$. Assume that $U \in W^{m+1,2}(\mathbb{R}^{n+1}_+, y^b)$ satisfying the equation*

$$\Delta_b^{m+1} U(x,y) = 0 \tag{3.11}$$

on the upper half space for $(x,y) \in \mathbb{R}^n \times \mathbb{R}_+$, where y is the special direction, and the boundary conditions are that $U(x,0) = f(x)$ along $\{y=0\}$, and, furthermore, that for every positive odd integer $2k+1 < m+1$, we have $\lim\limits_{y\to 0} y^b \frac{\partial^{2k+1}U}{\partial y^{2k+1}}(x,0) = 0$, where $f(x)$ is some function defined on $H^\gamma(\mathbb{R}^n)$. For even integers, we specify the relationship

$$\frac{\partial^{2k}U}{\partial y^{2k}}(x,0) = (\Delta_x^k U(x,0)) \prod_{j=1}^{k} \frac{1}{2\gamma - 4(j-1)}.$$

Then we have the result that

$$(-\Delta_x)^\gamma f(x) = C_{n,\gamma} \lim_{y\to 0} y^b \frac{\partial}{\partial y} \Delta_b^m U(x,y). \tag{3.12}$$

Specifically, if m is odd,

$$\int_{\mathbb{R}^n} |\xi|^{2\gamma} |\widehat{f}(\xi)|^2 \mathrm{d}\xi = C_{n,\gamma} \int_{\mathbb{R}^{n+1}_+} y^b |\Delta_b^{\frac{m+1}{2}} U(x,y)|^2 \mathrm{d}x\mathrm{d}y, \tag{3.13}$$

and if m is even,

$$\int_{\mathbb{R}^n} |\xi|^{2\gamma} |\widehat{f}(\xi)|^2 \mathrm{d}\xi = C_{n,\gamma} \int_{\mathbb{R}^{n+1}_+} y^b |\nabla \Delta_b^{\frac{m}{2}} U(x,y)|^2 \mathrm{d}x\mathrm{d}y. \tag{3.14}$$

We refer the readers to [26] for a complete proof of this theorem.

Remark 3.3 Having carefully set the boundary conditions to coincide with the function U from scattering theory (see [7]), as we explain in the section below, it is no surprise that our energy minimizer, satisfying the same equation as the U of the scattering theory, would be exactly the same function by the uniqueness of energy minimizers.

4 Fractional GJMS Operators

In this section, we first briefly describe the background material in the work of Graham-Zworski [16] and the notion of the fractional GJMS operator $P_{2\gamma}$; we then illustrate in Theorem 4.1 that in the case when the fractional Laplacian operator $(-\Delta)^\gamma$ is defined on the Euclidean space $\mathbb{R}^n$ and $\gamma \in (0,1)$, the operator agrees with $P_{2\gamma}$ on the hyperbolic space; and we describe the fact that this identification can be extended to more general exponents $0 < \gamma \leq \frac{n}{2}$.

4.1 Background, definitions

One of the important progress in conformal geometry is a discovery of a class of conformal covariant operators by Graham-Jenne-Mason-Sparling in 1992 (see [12]), called as GJMS operators. This is a class of differential operators P_{2k} for integers k defined on closed Riemannian manifolds (M^n, g) of dimension n with $2k \le n$,

$$(P_{2k})_g = (-\Delta)_g^k + \text{lower order terms}.$$

The most important property of P_{2k} is the conformal covariant property. This means that when we change the metric g to a metric $\widehat{g}$ conformal to g, say $\widehat{g} = v^{\frac{4}{n-2k}} g$ for some positive smooth function v defined on M. Then

$$(P_{2k})_{\widehat{g}}(\phi) = v^{-\frac{n+2k}{n-2k}} (P_{2k})_g(v\phi) \tag{4.1}$$

for all smooth functions ϕ defined on M (we now skip the referring to the metric g when it is fixed).

When $k = 1$, P_2 is the famous conformal Laplacian or Yamabe operator,

$$P_2 = -\Delta + \frac{n-2}{4(n-1)} R,$$

where R is the scalar curvature of the metric. When $k = 2$, P_4 was independently discovered by Paneitz [25], we now call it the Paneitz operator. To describe this operator and its associated curvature, let A denote the Schouten tensor,

$$A = \frac{1}{n-2}\Big(\mathrm{Ric} - \frac{1}{2(n-1)} \mathrm{R} g\Big), \tag{4.2}$$

where Ric is the Ricci tensor and R is the scalar curvature of the metric g. The 4-th order Paneitz operator is defined as

$$P_4 = (-\Delta)^2 + \delta\Big(4A - \frac{n-2}{2(n-1)} \mathrm{R}\Big) d + \frac{n-4}{2} Q_4, \tag{4.3}$$

where Q_4 is a fourth order curvature,

$$Q_4 = -\Delta \sigma_1(A) + 4\sigma_2(A) + \frac{n-4}{2} \sigma_1(A)^2, \tag{4.4}$$

where $\sigma_k(A)$ denote the k-th symmetric function of the eigenvalues of A.

In [16], Graham and Zworski linked the operators P_{2k} to scattering operator evaluated at its poles on conformally compact Einstein manifolds; and in this way, introduced the class of fractional GJMS operators $P_{2\gamma}$ for $0 < 2\gamma \le n$ and γ not an integer, acting on the conformal infinity of the manifolds. We now briefly recall their work.

Let M be a compact manifold of dimension n with a metric g. Let $\overline{X}^{n+1}$ be a compact manifold of dimension $n+1$ with boundary M, and denote by X the interior of $\overline{X}$. A function ρ is a defining function of ∂X in X if

$$\rho > 0 \quad \text{in } X, \quad \rho = 0 \quad \text{on } \partial X, \quad d\rho \neq 0 \quad \text{on } \partial X.$$

We say that g^+ is a conformally compact (c.c.) metric on X with conformal infinity $(M,[g])$ if there exists a defining function ρ such that the manifold $(\overline{X},\overline{g})$ is compact for $\overline{g}=\rho^2 g^+$, and $\overline{g}|_M\in[g]$. If, in addition, (X^{n+1},g^+) is a conformally compact manifold and $\mathrm{Ric}[g^+]=-ng^+$, then we call (X^{n+1},g^+) a conformally compact Einstein manifold.

Given a conformally compact Einstein manifold (X^{n+1},g^+) and a representative g in $[g]$ on the conformal infinity M, there is a uniquely defining function ρ such that, on $M\times(0,\delta)$ in X, g^+ has the normal form $g^+=\rho^{-2}(d\rho^2+g_\rho)$ where g_ρ is a one parameter family of metrics on M satisfying $g_\rho|_M=g$. Moreover, g_ρ has an asymptotic expansion which contains only even powers of ρ, at least up to degree n.

By the well-known works of Mazzeo-Melrose [23] and Graham-Zworski [16], given $f\in\mathbb{C}^\infty(M)$ and $s\in\mathbb{C}$, the eigenvalue problem

$$-\Delta_{g^+}u-s(n-s)u=0\quad\text{in } X \tag{4.5}$$

has a solution of the form

$$u=F\rho^{n-s}+H\rho^s,\quad F,H\in\mathcal{C}^\infty(X),\quad F|_{\rho=0}=f \tag{4.6}$$

for all $s\in\mathbb{C}$ unless $s(n-s)$ belongs to the pure point spectrum of $-\Delta_{g^+}$. Now, the scattering operator on M is defined as $S(s)f=H|_M$, it is a meromorphic family of pseudo-differential operators in $\mathrm{Re}(s)>\frac{n}{2}$. The values $s=\frac{n}{2},\frac{n}{2}+1,\frac{n}{2}+2,\cdots$ are simple poles of finite rank, these are known as the trivial poles; $S(s)$ may have other poles. However, for the rest of the paper, we assume that we are not in those exceptional cases.

We define the conformally covariant fractional powers of the Laplacian as follows: For $s=\frac{n}{2}+\gamma,\ \gamma\in\left(0,\frac{n}{2}\right],\ \gamma\notin\mathbb{N}$, we set

$$P_{2\gamma}[g^+,g]:=d_\gamma S\Big(\frac{n}{2}+\gamma\Big),\quad d_\gamma=2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)}. \tag{4.7}$$

With this choice of multiplicative factor, the principal symbol of $P_{2\gamma}$ is exactly the principal symbol of the fractional Laplacian $(-\Delta_g)^\gamma$, precisely, it is $|\xi|^{2\gamma}$. We thus have that $P_{2\gamma}\in(-\Delta_g)^\gamma+\Psi_{\gamma-1}$, where we denote by Ψ_m to be the set of pseudo-differential operators on M of order m.

The operators $P_{2\gamma}[g^+,g]$ satisfy an important conformal covariance property (see [16]). Indeed, for a conformal change of metric

$$g_v=v^{\frac{4}{n-2\gamma}}g,\quad v>0, \tag{4.8}$$

we have that

$$P_{2\gamma}[g^+,g_v]\phi=v^{-\frac{n+2\gamma}{n-2\gamma}}P_{2\gamma}[g^+,\widehat{g}](v\phi) \tag{4.9}$$

for all smooth functions ϕ, which is a generation of (4.1) for the class of the GJMS operators when γ are integers. We sometimes just write the operator as $P_{2\gamma}$ for simplicity.

When $2\gamma \not\equiv n$, we define the $Q_{2\gamma}$ curvature of the metric associated to the functional $P_{2\gamma}$, to be

$$\frac{n-2\gamma}{2} Q_{2\gamma}[g^+, g] := P_{2\gamma}[g^+, g](1). \tag{4.10}$$

In particular, for a change of metric as (4.8), we obtain the equation for the $Q_2\gamma$ curvature

$$P_{2\gamma}[g^+, g](v) = \frac{n-2\gamma}{2} v^{\frac{n+2\gamma}{n-2\gamma}} Q_{2\gamma}[g^+, g_v].$$

When γ is an integer, say $\gamma = k$ ($k \in \mathbb{N}$), a careful study of the poles of $S(s)$ allows to define P_{2k}. Indeed,

$$\mathrm{Res}_{s=\frac{n}{2}+k} S(s) = c_k P_{2k}, \quad c_k = (-1)^k [2^{2k} k!(k-1)!]^{-1}.$$

These are the conformally invariant powers of the Laplacian constructed by Graham-Jenne-Mason-Sparling [12] which we have mentioned at the beginning of the section.

We remark that when n is an even integer, and $2\gamma = n$, Q_n is also defined via a "dimension continuation" method by Tom Branson [3]. On compact surface, Q_2 is the Gaussian curvature. On manifold of dimension 4, Q_4 is the famous Q-curvature which has been explicitly written down by Branson in the formula (4.4).

4.2 The extension theorem on the hyperbolic space

The main observation we make in this section is that, in the case $X = \mathbb{R}^{n+1}_+$ and $M = \mathbb{R}^n$ with coordinates $x \in \mathbb{R}^n$, $y > 0$, endowed the hyperbolic metric $g_{\mathbb{H}} = \frac{\mathrm{d}y^2 + |\mathrm{d}x|^2}{y^2}$, the scattering operator $P_{2\gamma}$ is nothing but the Caffarelli-Silvestre extension problem for the fractional Laplacian when $\gamma \in (0,1)$ as stated in Section 2 of this paper.

Theorem 4.1 (see [7]) *Fix $\gamma \in (0,1)$ and f a smooth function defined on $\mathbb{R}^n$. If U is a solution of the extension problem* (3.1)*, then $u = y^{n-s}U$ is a solution of the eigenvalue problem* (4.5) *for $s = \frac{n}{2} + \gamma$, and moreover,*

$$P_{2\gamma} f = \frac{d_\gamma}{2\gamma} \lim_{y \to 0} (y^a \partial_y U) = (-\Delta_x)^\gamma f, \tag{4.11}$$

where $a = 1 - 2\gamma$, $P_{2\gamma} := P_{2\gamma}[g_{\mathbb{H}}, |\mathrm{d}x|^2]$, and the constant d_γ is defined in (4.7)*.*

Proof Fix f on $\mathbb{R}^n$ and let u be a solution of the scattering problem

$$-\Delta_{\mathbb{H}} u - s(n-s)u = 0 \quad \text{in } X. \tag{4.12}$$

We know that u can be written as

$$u = y^{n-s} F + y^s H, \tag{4.13}$$

where $F|_{y=0} = f$ and $S(s)f = h$ for $h = H|_{y=0}$. Moreover,

$$F(x,y) = f(x) + f_2(x) y^2 + o(y^2) \quad \text{and} \quad H(x,y) = h(x) + h_2(x) y^2 + o(y^2). \tag{4.14}$$

On the other hand, the conformal Laplacian operator for a Riemannian metric g in a manifold X of dimension $d = n+1$ is defined as

$$(P_2)_g = -\Delta_g + \frac{d-2}{4(d-1)} R_g.$$

For the hyperbolic metric, $R_{g_{\mathbb{H}}} = -n(n+1)$, so that

$$(P_2)_{g_{\mathbb{H}}} = -\Delta_{g_{\mathbb{H}}} - \tfrac{n^2-1}{4}. \tag{4.15}$$

Then, from (4.12), we can compute

$$0 = -\Delta_{g_{\mathbb{H}}} u - s(n-s)u = (P_2)_{g_{\mathbb{H}}} u + (\gamma^2 - \tfrac{1}{4})u = y^{\frac{n+3}{2}} (P_2)_{g_e} (y^{-\frac{n-1}{2}} u) + (\gamma^2 - \tfrac{1}{4})u, \tag{4.16}$$

where in the last equality we have used the conformal covariant property of the conformal Laplacian for the change of metric $g_e :=: y^2 g_{\mathbb{H}}$,

$$(P_2) g_{\mathbb{H}}(\psi) = y^{\frac{n+3}{2}} (P_2)_{g_e} (y^{-\frac{n-1}{2}} \psi). \tag{4.17}$$

Next, we change $u = y^{n-s} U$, and note that

$$(P_2)_{g_e} = -\Delta = -\Delta_x - \partial_{yy}, \tag{4.18}$$

so it follows that

$$\begin{aligned} &(P_2)_{g_e} (y^{\frac{n+1}{2} - s} U) \\ &= -y^{\frac{n+1}{2} - s} \Big[\Delta_x U + \partial_{yy} U + \frac{a}{y} \partial_y U + \Big(\frac{n+1}{2} - s \Big) \Big(\frac{n+1}{2} - s - 1 \Big) \frac{U}{y^2} \Big]. \end{aligned} \tag{4.19}$$

Substituting (4.19) into (4.16), we observe that with the choice of $s = \frac{n}{2} + \gamma$ and $a = 1 - 2\gamma$ we arrive at

$$\Delta_x U + \partial_{yy} U + \frac{a}{y} \partial_y U = 0,$$

as we wished.

For the second part of the lemma, note that

$$P_\gamma f = d_\gamma S\Big(\frac{n}{2} + \gamma \Big) = d_\gamma \, h, \tag{4.20}$$

where h is given in (4.14). On the other hand, we also have that

$$U = y^{s-n} u = F + y^{2s-n} H,$$

and thus, looking at the orders of y in (4.14), we can conclude that the limit

$$\lim_{y \to 0} y^a \partial_y U \tag{4.21}$$

exists and equals h times the constant 2γ. The lemma is proven by comparing (4.21), together with (4.20), with the Caffarelli-Silvestre construction for the fractional Laplacian as given in (2.2).

One advantage of identifying the operator $(-\Delta_x)^\gamma$ on $\mathbb{R}^n$ this way is that, above result can be generalized to any $\gamma \leq \frac{n}{2}$. That is as follows.

Theorem 4.2 *For any $\gamma \in (0, \frac{n}{2}]\backslash\mathbb{N}$, we have that*

$$P_{2\gamma}[g_{\mathbb{H}}, |\mathrm{d}x|^2] = (-\Delta_x)^\gamma,$$

where the fractional conformal Laplacian $P_{2\gamma}$ on $\mathbb{R}^n$ is defined as in (4.7).

Above theorem can be established by induction on the integer $k = [\gamma]$, the techniques can also be applied to identity general fractional GJMS operators as boundary operators on compactified Einstein manifolds. We refer the readers to the work in [7].

5 Extension Theorem on Conformal Compact Einstein Manifolds

We now return to the setting of conformal Einstein manifolds and the task of generalizing the extension Theorems in Sections 4.1 and 3.3 for functions defined on the "flat" space of $(\mathbb{R}^{n+1}_+, \mathbb{R}^n)$ to the "curved" case of a conformal compact Einstein manifold and its conformal infinity boundary.

We start with a general lemma (see [7, Lemma 4.1]).

Lemma 5.1 *Let $(X = X^{n+1}, g^+)$ be any conformally compact Einstein manifold with boundary M. For any defining function ρ of M in X, not necessarily geodesic, the equation*

$$-\Delta_{g^+}u - s(n-s)u = 0 \quad \textit{in } (X, g^+) \tag{5.1}$$

is equivalent to

$$-\mathrm{div}\,(\rho^a \nabla U) + E(\rho)U = 0 \quad \textit{in } (X, \overline{g}), \tag{5.2}$$

where $\overline{g} = \rho^2 g^+,\ U = \rho^{s-n}u$ and the derivatives in (5.2) *are taken with respect to the metric $\overline{g}$. The lower order term is given by*

$$E(\rho) = -\Delta_{\overline{g}}(\rho^{\frac{a}{2}})\rho^{\frac{a}{2}} + \Big(\gamma^2 - \frac{1}{4}\Big)\rho^{-2+a} + \frac{n-1}{4n}R_{\overline{g}}\rho^a. \tag{5.3}$$

Here we denote $s = \frac{n}{2} + \gamma,\ a = 1 - 2\gamma$.

Remark 5.1 The expression of $E(\rho)$ can be simplified and rewritten into a form with a geometric meaning (see [14]). See also the formula (5.17) in Remark 5.4.

Remark 5.2 For the model case $X = \mathbb{R}^{n+1}_+,\ M = \mathbb{R}^n,\ g^+ = \frac{\mathrm{d}y^2 + |\mathrm{d}x|^2}{y^2}$, with the defining function $y > 0$, $\overline{g} = \mathrm{d}y^2 + |\mathrm{d}x|^2$, it automatically follows from (5.3) that

$$E(y) \equiv 0.$$

The lemma follows from the conformal covariant property of the conformal Laplacian operators, using ideas similar as in the proof of Theorem 4.1.

In view of the equation (5.2), one can integrate and obtain an extension theorem similar to the extension theorem of Caffarelli-Silvestre for the cases $0 < \gamma < 1$.

In the following, we explain the extension theorem for the special case $\gamma = \frac{3}{2}$.

5.1 The case when $\gamma = \frac{3}{2}$

When $\gamma = \frac{3}{2}$, $s = \frac{n}{2} + \frac{3}{2}$, the equation (5.1) takes the form

$$-\Delta_{g^+} u - \frac{n^2-9}{4} u = 0 \quad \text{in } (X, g^+). \tag{5.4}$$

Our first observation is that in this case, as g^+ is an Einstein manifold with $\mathrm{Ric}_{g^+} = -ng^+$, its 4th-order Paneitz operator is

$$(P_4)_{g^+} = \Big(-\Delta_{g^+} - \frac{n^2-1}{4}\Big)\Big(-\Delta_{g^+} - \frac{n^2-9}{4}\Big). \tag{5.5}$$

Thus the solution of the second order equation (5.4) satisfies also the 4-th order equation

$$(P_4)_{g^+} u = 0. \tag{5.6}$$

We now translate this to the corresponding conformal compactified manifold $(X^{n+1}, \overline{g})$, where $\overline{g} = \rho^2 g^+$. In this case, denoting $U = \rho^{s-n} u = \rho^{\frac{3}{2}-\frac{n}{2}} u$, we notice that in this case the conformal covariant property of the Paneitz operator P_4 states exactly that

$$(P_4)_{\overline{g}}(U) = (\rho)^{-\frac{n+5}{2}} (P_4)_{g^+}(\rho^{\frac{n-3}{2}} U) = (\rho)^{-\frac{n+5}{2}} (P_4)_{g^+}(u) = 0. \tag{5.7}$$

We now make the second observation that, denoting by f the Dirichlet data of the Poisson equation (5.4), it follows from the asymptotic expansion of u (4.14) that

$$U = (f + f_2\rho^2 + f_4\rho^4 + \cdots) + \rho^3(h + h_2\rho^2 + \cdots), \tag{5.8}$$

where $h = P_3(f)$ is the scattering matrix operating on f. Thus in particular we have $\frac{\partial U}{\partial \rho}|_{\rho=0} = 0$. Combining above observations, we reach the conclusion which is a complete analogue of Remark 3.1 on the flat cases.

Remark 5.3 u satisfies the second order equation (5.4) with Dirichlet data if and only if U satisfies the 4th order equation (5.7) with

$$U|_{\partial X} = f,$$
$$\frac{\partial U}{\partial \rho}\Big|_{\partial X} = 0.$$

5.2 Right choice of ρ, the adapted metrics g^*, when $n > 3$

In the statement of Lemma 5.1 above, ρ can be any geodesic distance function. But the expression of the term $E(\rho)$ in (5.3) for such general ρ is complicated and the geometric content not clear. In order to remedy this, in [6–7, Section 6] we chose instead a preferred distance function $\rho = \rho^*$, and called the resulting compactified metric $g^* = (\rho^*)^2 g^+$ the adapted metric. With this adapted metric, we then derive a useful extension theorem to study the corresponding $P_{2\gamma}$ operators for $0 < \gamma < 2$ for general conformal compact Einstein manifolds. The choice of ρ^* in general is more complicated, here we just explain the choice when $\gamma = \frac{3}{2}$.

We first recall an important result of Lee in the subject.

Theorem 5.1 (see [21]) *Assume that the Yamabe class of the conformal infinity of* (X, g^+) *is positive. Then the first eigenvalue of* $(-\Delta_{g^+})$ *is greater than or equal to* $\frac{n^2}{4}$.

On a given conformal compact Einstein manifold (X, g^+), we denote by v the solution of the Poisson equation (5.1) when $s = \frac{n}{2} + \gamma = \frac{n}{2} + \frac{3}{2}$ with Dirichlet data $f \equiv 1$. Note that when $n > 3$, we have $n > s$.

Lemma 5.2 *Under the assumption that the scalar curvature* $\mathrm{R}(\partial X, g) > 0,\ n > 3$ *then*

(a) $v > 0$ *on* X.

(b) *Denote* $\rho^* = v^{\frac{1}{n-s}},\ g^* = (\rho^*)^2 g^+$, *then* $R_{g^*}|_{\partial X} = c_\gamma \mathrm{R}(\partial X, g) > 0$, *where* c_γ *is a positive constant when* $\gamma > 1$.

(c) R_{g^*} *is positive on* X.

Proof (a) follows directly from the theorem of Lee cited above. (b) follows by a straightforward computation. (c) also follows from the theorem of Lee, but quite indirectly from the method of the proof plus some method of continuity argument. Interested readers are referred to Proposition 6.4 in [6] for a complete argument.

We now state other good properties of the g^* metric.

Lemma 5.3 *On the setting as above, we have*

(a) $E(\rho^*) = 0$, *where* E *is defined as in* (5.3).

(b) $(Q_4)_{g^*} \equiv 0$ *on* X.

Proof (a) This fact was first pointed out in [7, Lemma 4.2]. To see this, one observe that, in turns of the g^+ metric, the expression of $E(\rho)$ can be rewritten as

$$E(\rho) = -\Delta_{g^+}(\rho^{\frac{n-1+a}{2}})\rho^{\frac{-n-3+a}{2}} - \Big(\frac{n^2}{4} - \gamma^2\Big)\rho^{-2+a}. \tag{5.9}$$

Here we denote $s = \frac{n}{2} + \gamma,\ a = 1 - 2\gamma$. Thus if we choose $\rho = \rho^*$, it follows from the Poisson equation (5.1) satisfied by v and the definition of ρ^* that $E(\rho^*) \equiv 0$.

(b) As $n > 3$, we have $\frac{1}{n-3} Q_4 = P_4(1)$ in general. In particular, we have

$$\begin{aligned}(Q_4)_{g^*} &= (n-3)(P_4)_{g^*}(1) = (n-3)(\rho^*)^{-\frac{n+5}{2}}(P_4)_{g^+}((\rho^*)^{\frac{n-3}{2}}) \\ &= (n-3)(\rho^*)^{-\frac{n+5}{2}}(P_4)_{g^+}(v) = 0.\end{aligned} \tag{5.10}$$

The last line follows from (5.7).

5.3 Right choice of ρ, the adapted metric g^*, when $n = 3$

We now indicate the modification needed in above arguement when $n = 3$ and $\gamma = \frac{3}{2}$ and $s = \frac{n}{2} + \frac{3}{2} = 3$.

In this case, the metric g^* has appeared in the literature before in the work of Fefferman-Graham [10], in their study of the conformal invariant quantity $\int Q_n$ which appears as the coefficient of the $L\log L$ term of the volume expansion of a conformally compact Poincaré-Einstein manifold of dimension $(n+1)$ when n is even. For that reason, we call this metric the Fefferman-Graham metric.

It is defined as follows. On a conformal compact Einstein manifold (or more general asymptotically hyperbolic manifolds) (X^{n+1}, g^+), for each $s \neq n$, denote v_s the solution of the Poisson equation 5.1, with Dirichlet data $f \equiv 1$, then define $w = -\frac{\mathrm{d}}{\mathrm{d}s}|_{s=n} v_s$ and $g^* = \mathrm{e}^{2w} g^+$. It turns out w satisfies the PDE

$$-\Delta_{g^+} w = n \quad \text{on } X. \tag{5.11}$$

Fefferman-Graham metrics satisfy all the properties listed in Lemmas 5.2–5.3 above. In fact, our choice of the g^* metrics in the general cases is inspired by this case (and also the metric constructed by Lee in [21]). The proof of these lemmas is essentially the same as the case when $s \neq n$, but with modifications. Here we present the proof of the property (b) in Lemma 5.3, which has played an important role in the earlier work of [27].

When $s = n = 3$, $(P_4)_{g^*}(1) \equiv 0$, but it does not indentify $(Q_4)_{g^*}$. We can instead use the functional property of the P_4 operators, that is

$$(P_4)_{g^+} w + (Q_4)_{g^+} = (Q_4)_{g^*} \mathrm{e}^{4w}. \tag{5.12}$$

We can now examine to see by the formula of $(P_4)_{g^+}$ in (5.5) and (5.11), that $(P_4)_{g^+} w = -6$, while it follows from the definition (4.4) of Q_4 that $Q_4(g^+) = 6$; and thus $Q_4(g^*) = 0$.

5.4 Extension theorem when $\gamma = \frac{3}{2}$

We finally are ready to state the extension theorem for when $\gamma = \frac{3}{2}$ on conformal compact Einstein manifolds.

Theorem 5.2 (see [6]) *Let (X^{n+1}, M^n, g^+) be a Poincaré–Einstein manifold and fix a representative g of the conformal boundary, and let (X^{n+1}, g^*) be the adapted metric space as defined above. Then for each $f \in C^\infty(M)$, the solution U of the boundary value problem*

$$\begin{cases} (P_4)_{g^*}(U) = 0 & \text{in } X^{n+1}, \\ U = f & \text{on } M, \\ \dfrac{\partial U}{\partial n_{g^*}} = 0 & \text{on } M \end{cases} \tag{5.13}$$

is such that

$$P_3 f = \frac{n-3}{2} Q_3 f + \frac{d_\gamma}{8\gamma(\gamma-1)} \lim_{\rho^* \to 0} \frac{\partial}{\partial \rho^*} \Delta_{g^*} U. \tag{5.14}$$

We define the energy of U with respect to the Paneitz operator $(P_4)_{g^*}$ as the integral quantity obtained by dropping the boundary terms when integrating by parts of the integral

$$\int_X \{(P_4)_{g^*} U\} U \mathrm{d}V \mathrm{vol}_{g^*},$$

i.e.,

$$E_4(g^*)[U] = \int_X (\Delta_{g^*} U)^2 - \Big(4A_{g^*} - \frac{n-1}{2n} R_{g^*} g^*\Big) \langle \nabla_{g^*} U, \nabla_{g^*} U \rangle \mathrm{d}\,\mathrm{vol}_{g^*}. \tag{5.15}$$

For all smooth function f defined on M, we have the identity

$$\frac{1}{2} E_4(g^*)[U_f] = \int_M (P_3 f) f \mathrm{d}\sigma_{g^*} - \frac{n-3}{2} \int_M Q_3(g^*) f^2 \mathrm{d}\sigma_{g^*}. \tag{5.16}$$

5.5 Extension theorem when $1 < \gamma < 2$

We now describe the work in [6], where we have generalized the extension theorem $\big($described above for the case $\gamma = \frac{3}{2}\big)$ also for all $1 < \gamma < 2$. The key step which enables us to do so is to adopt the concept of "metric space with measure" to rewrite the Poisson equation (5.1). We now briefly introduce the notion. For a more detail description of the topic, the reader is referred to Section 3 in [6].

Take a number $m \in \mathbb{R}$, ϕ a function defined on (X^{n+1}, g), and (F, h) a metric space of dimension m; on the metric measure space $(X, g, \mathrm{e}^{-\phi} dv_g)$, denote $P^m_{2k,\phi}$, the GJMS operators on the warped product space

$$(X \times_{\mathrm{e}^{-\phi}} F^m, g \oplus \mathrm{e}^{-\frac{2\phi}{m}} h)$$

restricted to functions on X, and denote R^m_ϕ the scalar curvature, Ric^m_ϕ the Ricci curvature of the product metric induced on X.

Remark 5.4 (1) In this setting, for all m, the role of Δ is replaced by $\Delta_\phi := \Delta - \nabla\phi\nabla$; when $m = \infty$, $\mathrm{Ric}^m_\phi = \mathrm{Ric} + \nabla^2\phi$ (Bakry-Emery Ricci tensor). (2) The precise relation between $E(\rho)$ defined in (5.3) and R^m_ϕ, using $\rho^m = \mathrm{e}^{-\phi}$ and $m = 1 - 2\gamma$, is

$$E(\rho) = \frac{m + n - 1}{4(m + n)} \rho^m R^m_\phi. \tag{5.17}$$

We now make two key observations.

(1) On $(X^{n+1}, \partial X, g^+)$, a conformal compact Einstein manifold, with $\mathrm{Ric}_{g^+} = -ng^+$, when $s = \frac{n}{2} + \gamma$, $g = \rho^2 g^+$, ρ is any defining function, consider the solution u of the Possion equation (5.1)

$$(*)_s - \Delta_{g+} u - s(n - s)u = 0 \quad \text{on } X.$$

We first notice that when $\gamma = \frac{1}{2}$, $(*)_{\frac{n}{2}+\frac{1}{2}}$ is just the conformal Laplace $(P_2)_(g^+)u = 0$, hence, if we denote $U = \rho^{s-n} u$, the equation is equivalent to

$$(P_2)_g U = 0 \quad \text{on } X.$$

We can then verify that for any $\gamma > 0$, in general, $(*)_s$ is equivalent to the PDE

$$(P^m_{2,\phi})_g U = 0 \quad \text{on } X,$$

where $U = \rho^{s-n} u$, and (F^m, h) is chosen to be the sphere with $\mathrm{Ric}_h = (m - 1)h$, and $g = \rho^2 g^+$, $m = 1 - 2\gamma$ and $\mathrm{e}^{-\phi} = \rho^m$.

(2) When $1 < \gamma < 2$, then u satisfying $(*)_s$ implies that it also satisfies the equation on X,

$$(-\Delta_{g^+} - (s - 2)(n - (s - 2))) \circ (-\Delta_{g^+} - s(n - s))u = 0. \tag{5.18}$$

(5.18) turns out to be equivalent to

$$P^m_{4,\phi_2} U = 0 \quad \text{on } X, \tag{5.19}$$

where $m = 3 - 2\gamma$ and $\mathrm{e}^{-\phi_2} = \rho^m$.

We remark that when $\gamma = \frac{3}{2}$, $m = 0$, $\phi_2 = 0$, thus in this case $P^m_{4,\phi_2} = P_4$ on X as we have claimed earlier in (5.4) and (5.6).

With this notation, and choice of the adapted metric g^* as before, we can then generalize the extension theorem to all $1 < \gamma < 2$.

Theorem 5.3 (see [6]) *Let (X^{n+1}, M^n, g_+) be a Poincaré-Einstein manifold and let $\gamma \in (1, 2)$ if $n \geq 4$ and $\gamma \in \left(1, \frac{3}{2}\right]$ if $n = 3$. Suppose that h is a representative of the conformal boundary with positive scalar curvature. Set $m = 3 - 2\gamma$ and let (X^{n+1}, g^*, ρ^m) be the adapted smooth metric measure space. Let $f \in C^\infty(M)$ and let U be the solution to* (5.19) *with Dirichlet data f. Then*

$$\int_M P_{2\gamma} f\, f \,\mathrm{d}\,\mathrm{vol}_h = \frac{n-2\gamma}{2} \int_M Q_{2\gamma} f^2 \,\mathrm{d}\,\mathrm{vol}_h + \frac{d_\gamma}{8\gamma(\gamma-1)} \int_X \Big[(\Delta_\phi U)^2 - \Big(4A_{g^*} \frac{(m+n-1)}{2n} (R^\phi_m)_{g^*}\Big)(\nabla U, \nabla U)\Big] \rho^m \,\mathrm{d}\,\mathrm{vol}_{g^*}.$$

For comparison purposes, we add a remark.

Remark 5.5 It follows from the scattering theory described above, for all $1 < \gamma < 2$, u satisfies the Poisson equation (5.1) with Dirichlet data f for $s = \frac{n}{2} + \gamma$ if and only if it satisfies the fourth order PDE (5.18) on the conformal compact Einstein manifold X^{n+1} with the Dirichlet data f and satisfies the Neumann type condition $\lim_{\rho \to 0} \rho^m \frac{\partial U}{\partial \rho} = 0$, where $U = \rho^{s-n} u$ and $m = 3 - 2\gamma$.

In [6], above extension theorems are applied to study the positivity property and strong maximum principle of the boundary operators $P_{2\gamma}$, which in turn is an extension of the earlier works of [15] and the exploding recent works of [11–13, 17–19] on the study of corresponding properties of P_4 operators on closed manifolds. We refer the readers to the recent lecture notes of [20] on a comprehensive study of the topic.

6 An Application: Sharp Sobolev Trace Inequalities of Order 4 on Model Domains

In [1], we derived sharp Sobolev trace inequalities on the model domains $(B^d, S^{d-1}, |\mathrm{d}x|^2)$ by applying the extension Theorem 5.2 in Section 5 above. Here we summarize the approach.

We start by recalling the classical Sobolev trace inequality of order 2 on $(B^d, S^{d-1}, |\mathrm{d}x|^2)$ when $d > 2$

$$\frac{d-2}{2} \mathrm{vol}(S^{d-1})^{\frac{1}{d-1}} \Big(\oint_{S^{d-1}} |f|^{\frac{2(d-1)}{d-2}} \mathrm{d}\sigma \Big)^{\frac{d-2}{d-1}} \leq \int_{B^d} |\nabla f(x)|^2 \mathrm{d}x + \frac{d-2}{2} \oint_{S^{d-1}} f^2 \mathrm{d}\sigma, \tag{6.1}$$

where u is any smooth extension of f.

We remark that inequality (6.1) was derived by Escobar [9] and was applied to study the Yamabe problem on manifolds with boundary.

When $d = 2$, the Sobolev trace inequality becomes the classical Lebedev-Milin [22] inequality

$$\log\Big(\frac{1}{\pi}\oint_{S^1} \mathrm{e}^f \mathrm{d}\sigma\Big) \le \frac{1}{4\pi}\int_D |\nabla f|^2 \mathrm{d}x + \frac{1}{\pi}\oint_{S^1} f \mathrm{d}\sigma. \tag{6.2}$$

The Lebedev-Milin inequality (6.2) has been used in a wide variety of problems in classical analysis, including the Bieberbach conjecture [2] and by Osgood-Phillips-Sarnak [24] in the study of the compactness of isospectral planar domains.

We now state two sharp Sobolev trace inequalities of order 4 we have obtained on $(B^d, S^{d-1}, |\mathrm{d}x|^2)$.

Theorem 6.1 *Let $f \in C^\infty(S^{d-1})$ for $d \ge 5$, and let v be a smooth extension of f to the unit ball satisfying $\frac{\partial}{\partial n} v|_{\partial B^d} = -\frac{(d-4)}{4} f$. Then we have*

$$\begin{aligned} &c_d(\mathrm{vol}(S^{d-1}))^{\frac{3}{d-1}}\Big(\oint_{S^{d-1}} |f|^{\frac{2(d-1)}{d-4}} \mathrm{d}\sigma\Big)^{\frac{d-4}{d-1}} \\ &\le \int_{B^d} (\Delta_{g_0} v)^2 \mathrm{d}x + 2\oint_{S^{d-1}} |\widetilde{\nabla} f|^2 \mathrm{d}\sigma + b_d \oint_{S^{d-1}} |f|^2 \mathrm{d}\sigma, \end{aligned} \tag{6.3}$$

where $c_d = \frac{d(d-2)(d-4)}{4}$, $b_d = \frac{d(d-4)}{2}$ and $\widetilde{\nabla} f$ is the gradient of f with respect to the round metric $g_{S^{d-1}}$. The equality holds for any $f(\xi) = c|1 - \langle z_0, \xi\rangle|^{\frac{4-d}{4}}$ with $\xi \in S^{d-1}$, $|z_0| < 1$, where c is a constant and v is a bi-harmonic extension of f satisfying the Neumann boundary condition. When $f \equiv 1$, $v = 1 + \frac{d-4}{4}(1 - |x|^2)$.

The following is the analogue of Lebedev-Milin inequality of order 4 on $(B^d, S^{d-1}, |\mathrm{d}x|^2)$.

Theorem 6.2 *Let $f \in C^\infty(S^3)$ and let v be a C^∞ extension of f to the ball B^4 satisfying $\frac{\partial}{\partial n} v|_{\partial B^4} = 0$. Then we have*

$$\log\Big(\frac{1}{2\pi^2}\oint_{S^3} \mathrm{e}^{3(f-\overline{f})} \mathrm{d}\sigma\Big) \le \frac{3}{16\pi^2}\int_{B^4} (\Delta_{g_0} v)^2 \mathrm{d}x + \frac{3}{8\pi^2}\oint_{S^3} |\widetilde{\nabla} f|^2 \mathrm{d}\sigma. \tag{6.4}$$

Again, the equality holds for any $f(\xi) = \log|1 - \langle z_0, \xi\rangle| + c$ with $\xi \in S^{d-1}$, $|z_0| < 1$, where c is a constant and v is a bi-harmonic extension of f satisfying the Neumann boundary condition.

The main idea in the proof of Theorems 6.1–6.2 above is first to establish the inequalities in the g^* metric using the extension theorem, then apply the conformal covariant property of the P_4 operator to transform the inequalities back from g^* back to $g_0 = |\mathrm{d}x|^2$. Our proof also relies on some sharp higher order Sobolev inequality on the spheres derived much earlier by Beckner [2]. It turns out in these model cases, we can explicitly compute the g^* metric.

Lemma 6.1 *On the model case (B^d, S^{d-1}, g_H), where $g_H = \rho^{-2}|\mathrm{d}x|^2$ and $\rho = \frac{1-|x|^2}{2}$, we have that*

(i) *when $d \ge 5$, $g^* = (\psi)^{\frac{4}{d-4}}|\mathrm{d}x|^2$, where*

$$\psi = 1 + \frac{d-4}{2}\rho;$$

(ii) *when $d = 4$, $g^* = \mathrm{e}^{2w} g_H = \mathrm{e}^{2\rho}|\mathrm{d}x|^2 = \mathrm{e}^{1-|x|^2}|\mathrm{d}x|^2$, where w is the solution of the partial differential equation $-\Delta_{g_{\mathbb{H}}} w = 3$ on B^4.*

We remark that when one applies the same scheme as above to deal with Sobolev trace inequalities of order 2 on (B^d, S^{d-1}), it turns out $g^* = |\mathrm{d}x|^2$, thus we recover the classical inequalities (6.1)–(6.2) above. Thus we believe that the metric g^* is the "natural" metric for the 4th order problem.

Above inequalities has been generalized to much general settings, with the introductions of new classes of fractional order boundary operators with conformally covariant property in the most recent works of Jeffrey Case [4–5].

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