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Homology Groups of Simplicial Complements*

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Abstract This paper deals with homology groups induced by the exterior algebra generated by the simplicial compliment of a simplicial complex K. By using Čech homology and Alexander duality, the authors prove that there is a duality between these homology groups and the simplicial homology groups of K.

Keywords Stanley-Reisner ring, Simplicial complement, Barycentric subdivision, Inflation complex
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1 Introduction

Throughout this paper, $\mathbf{k}$ is a field or an integer ring $\mathbb{Z}$. $\mathbf{k}[m] = \mathbf{k}[v_1, \cdots, v_m]$ is the graded polynomial algebra on m variables, and $\deg(v_i) = 2$. The face ring (also known as the Stanley-Reisner ring) of a simplicial complex K on $[m]$ is the quotient ring

$$\mathbf{k}(K) = \mathbf{k}[m]/\mathcal{I}_K,$$

where $\mathcal{I}_K$ is the ideal generated by those square free monomials $v_{i_1} \cdots v_{i_s}$ for which $\{i_1, \cdots, i_s\}$ is not a simplex in K.

For any simple polytope P^n, Davis and Januszkiewicz introduced a T^m-manifold $\mathcal{Z}_P$ with an orbit space P^n in [5]. After that, Buchstaber and Panov generalized this definition to any simplicial complex K with vertices $[m] = \{1, \cdots, m\}$, and named it the moment-angle complex $\big($i.e., the moment-angle complex $\mathcal{Z}_K = \bigcup\limits_{\sigma \in K} D(\sigma)$, where $D(\sigma) = Y_1 \times Y_2 \times \cdots \times Y_m$, $Y_i = D^2$ if $i \in \sigma$ and $Y_i = S^1$ if $i \notin \sigma\big)$.

The following theorem is proved by Buchstaber and Panov [3] for the case over a field by using Eilenberg-Moore spectral sequence, and by [1] for the general case.

Theorem 1.1 (see [7, Theorem 4.7]) *Let K be a simplicial complex. Then the following isomorphism of algebras holds:*

$$H^*(\mathcal{Z}_K; \mathbf{k}) = \mathrm{Tor}_*^{\mathbf{k}[\mathbf{x}]}(\mathbf{k}(K), \mathbf{k}).$$

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In their proof, they proved that $H^*(\mathcal{Z}_K;\mathbf{k}) = \widetilde{H}_*[\Lambda[\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_m]\otimes\mathbf{k}[\mathbf{x}],\,\mathrm{d}]$ first. Then they used the Koszul resolution on $\mathbf{k}$ to get

$$\mathrm{Tor}_*^{\mathbf{k}[\mathbf{x}]}(\mathbf{k},\mathbf{k}(K)) = \widetilde{H}_*[\Lambda[\mathbf{u}_1,\mathbf{u}_2,\cdots,\mathbf{u}_m]\otimes\mathbf{k}[\mathbf{x}],\,\mathrm{d}].$$

Since $\mathrm{Tor}_*^{\mathbf{k}[\mathbf{x}]}(\mathbf{k},\ \mathbf{k}(K))$ has a natural $\mathbb{Z}\oplus\mathbb{Z}^m$-bigrade, the bigraded cohomology ring can be decomposed as follows:

$$H^*(\mathcal{Z}_K;\mathbf{k}) = \mathrm{Tor}_*^{\mathbf{k}[\mathbf{x}]}(\mathbf{k},\mathbf{k}(K)) = \bigoplus_{i\geq 0}\bigoplus_{I\subseteq[m]}\mathrm{Tor}_{i,\ I}^{\mathbf{k}[\mathbf{x}]}(\mathbf{k},\mathbf{k}(K)).$$

Hochster gave a combinatorial description of the Tor-groups $\mathrm{Tor}_{i,*}^{\mathbf{k}[\mathbf{x}]}(\mathbf{k},\mathbf{k}(K))$.

Theorem 1.2 (see [6])

$$\mathrm{Tor}_{i,*}^{\mathbf{k}[\mathbf{x}]}(\mathbf{k},\mathbf{k}(K)) = \bigoplus_{I\subseteq[m]}\widetilde{H}^{|I|-i-1}(K_I;\mathbf{k}),$$

where $K_I = \{\omega\subseteq I|\omega\in K\}$, *and* $\widetilde{H}^{-1}(\varnothing;\mathbf{k}) = \mathbf{k}$.

Then in [4] they developed a more precise description.

Theorem 1.3 (see [4, Theorem 3.2.9])

$$\mathrm{Tor}_{i,\ I}^{\mathbf{k}[\mathbf{x}]}(\mathbf{k},\mathbf{k}(K)) = \widetilde{H}^{|I|-i-1}(K_I;\mathbf{k}),$$

where $\widetilde{H}^{-1}(\varnothing;\mathbf{k}) = \mathbf{k}$.

Recently, Zheng and Wang has proposed another way to compute $\mathrm{Tor}_*^{\mathbf{k}[\mathbf{x}]}(\mathbf{k}(K),\mathbf{k})$ by using Taylor resolution on Stanley-Reisner ring $\mathbf{k}(K)$ in [8]. This method was presented firstly by Yuzvinsky in [9].

They defined the simplicial complement P of a simplicial complex K as below.

Definition 1.1 (Missing Face and Simplicial Complement) *Let K be a simplicial complex on the set $[m]$ as above. A missing face of K is the subset $\tau\subseteq[m]$, where $\tau\notin K$ and every proper subset of τ is a simplex of K.*

A simplicial complement P is a subset of all non-faces of K containing all missing faces.

The Stanley-Reisner ideal $\mathcal{I}_P$ is the homogeneous ideal generated by all square-free monomials $\mathbf{x}_\tau = x_{i_1}x_{i_2}\cdots x_{i_s}$, where $\tau = \{i_1,\cdots,i_s\}\in P$. Obviously, for any two simplicial complements P and P' of the complex K, $\mathcal{I}_P = \mathcal{I}_{P'} = \mathcal{I}_K$.

Then one can define exterior algebra $\Lambda^*[P]$ generated by all faces of the simplicial complement P. For any monomial $\mathbf{u} = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_s}$, define the total set $S_{\mathbf{u}} = \tau_{i_1}\cup\tau_{i_2}\cup\cdots\cup\tau_{i_s}$. So $\Lambda^*[P]$ has a natural $\mathbb{Z}\oplus\mathbb{Z}^m$-bigrade, which means

$$\Lambda^*[P] = \bigoplus_{i\in N}\bigoplus_{I\subseteq[m]}\Lambda^{i,I}[P],$$

where $\Lambda^{i,I}[P]$ is generated by the monomial $\mathbf{u}$ satisfying $S_{\mathbf{u}} = I$ and the degree of monomial $\mathbf{u}$ is i.

Theorem 1.4 (see [8]) *Let K be a simplicial complex on the set $[m]$, and let P be one of the simplicial compliments of K. Give a differential $d : \Lambda^r[P] \longrightarrow \Lambda^{r-1}[P]$, generated by*

$$d(\mathbf{u}) = \sum_{s=1}^{r} (-1)^{s+1} \partial_s \mathbf{u} \cdot \delta_{\partial_s \mathbf{u}},$$

where $\partial_s \mathbf{u} = \tau_{i_1} \cdots \widehat{\tau}_{i_s} \cdots \tau_{i_r}$, and $\delta_{\partial_s \mathbf{u}} = 1$ if $S_{\mathbf{u}} = S_{\partial_s \mathbf{u}}$; otherwise $\delta_{\partial_s \mathbf{u}} = 0$. The differential d keeps the second grade. Then

$$\mathrm{Tor}_{i,I}^{\mathbf{k}[x]}(\mathbf{k}(K), \mathbf{k}) = H_i(\Lambda^{i,I}[P], d).$$

Remark 1.1 Let K be a simplicial complex and P be one of its simplicial complements. By Theorem 1.4, we know that the homology group $H_i(\Lambda^{i,I}[P], d)$ of simplicial complement P is not related to the choice of P. It just depends on the simplicial complex K. So if we fix the second degree by the set of all vertices $[m]$, then we can get a homology group which just depends on the simplicial complex K. We call it homology group of simplicial complements.

In this paper, we will first give the geometric description of the new differential d on $\Lambda^{*,[m]}[P]$. And the following theorem is proved by using the simplicial Alexander duality.

Theorem 1.5 *For any simplicial complex K on the set $[m]$, let P be one of the simplicial compliments of K. Then we have the following group isomorphism:*

$$H_i(\Lambda^{*,[m]}[P], d) = \widetilde{H}^{m-i-1}(K; \mathbf{k}),$$

where we assume $H_{-1}(\Lambda[\varnothing], d) = k$.

It is easy to check that $P_I = \{\tau \subseteq I \mid \tau \in P\}$ is a simplicial complement of K_I, where $K_I = \{\omega \subseteq I \mid \omega \in K\}$. So we have following corollary.

Corollary 1.1

$$H_i(\Lambda^{*,I}[P], d) = \widetilde{H}^{\,|I|-i-1}(K_I; \mathbf{k}).$$

Remark 1.2 Consider the following commutative diagram:

$$\begin{tikzcd}
H_i(\Lambda^{*,I}[P], d) \arrow[r, "\cong", "\phi"'] \arrow[d, "\cong"', "\psi"] & \mathrm{Tor}_{i,\ I}^{\mathbf{k}[\mathbf{x}]}(\mathbf{k}(K), \mathbf{k}) \arrow[d, "\cong"', "\eta"] \\
\widetilde{H}^{\,|I|-i-1}(K_I; \mathbf{k}) \arrow[r, "\cong", "\zeta"'] & \mathrm{Tor}_{i,\ I}^{\mathbf{k}[\mathbf{x}]}(\mathbf{k}, \mathbf{k}(K))
\end{tikzcd}$$

The isomorphisms ϕ, ψ and η come from Theorem 1.4, Corollary 1.1 and a classical result in homological algebra theory respectively, and $\zeta = \eta\phi\psi^{-1}$ is also an isomorphism. Thus we give a new proof of the Hochster theorem.

2 Geometric Description of the Differential d

If K is a simplex, the theorem is trivial. So in this paper, we assume that K is a simplicial complex on the set $[m]$, but not a simplex. Denote

$$P_0 = 2^{[m]} - K - [m] = \{\tau_1, \tau_2, \cdots, \tau_s\}.$$

P_0 is obviously one of the simplicial complements of the complex K. For any $\tau_i \in P_0$, we have simplicial complex $\mathrm{star}_{\partial\Delta^{m-1}}\tau_i = \{\tau \in \partial\Delta^{m-1} \mid \tau \cup \tau_i \in \partial\Delta^{m-1}\}$. Clearly, the $\mathrm{star}_{\partial\Delta^{m-1}}\tau_i$ is a triangulation of D^{m-2}. We denote by $U_i = \mathrm{Int}|\mathrm{star}_{\partial\Delta^{m-1}}\tau_i|$ the interior of the geometric realization of the complex $\mathrm{star}_{\partial\Delta^{m-1}}\tau_i$.

Proposition 2.1 $\mathbf{U} = \{U_i\}_{i=1,2,\cdots,s}$ *is an open cover of the topological space* $U(K)$*, where* $U(K) = |\partial\Delta^{m-1}|\backslash|K|$.

Proof If $x \in |\partial\Delta^{m-1}|\backslash|K|$, x must be an interior point of some simplex of $\partial\Delta^{m-1}$. Since $x \notin K$, there is a simplex $\tau \in P_0$ satisfying $x \in \mathrm{Int}|\tau|$. In other words, $x \in \mathrm{Int}|\mathrm{star}_{\partial\Delta^{m-1}}\tau|$, where $\tau \in P_0$.

Definition 2.1 (The Nerve and the Čech Homology of an Open Cover) *For any topological space* X*, let* $\mathbf{U} = \{U_i\}_{i=1,2,\cdots,s}$ *be an open cover of the space* X*. To every open set* U_i*, we assign a vertex* i*. If* $U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_r} \neq \varnothing$*, we get a simplex* $(i_1, i_2, \cdots, i_r)$*. Then we get a complex called the nerve of* $\mathbf{U}$*, denoted by* $\mathcal{N}(\mathbf{U})$*, where*

$$\mathcal{N}(\mathbf{U}) = \{(i_1, i_2 \cdots, i_r) \subseteq [s] \mid U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_r} \neq \varnothing\}.$$

Define $\check{H}_*(X; \mathbf{U}; \mathbf{k}) = \widetilde{H}_*(\mathcal{N}(\mathbf{U}); \mathbf{k})$*, called the reduced Čech homology groups of an open cover* $\mathbf{U}$.

Theorem 2.1 *Let* K *be a simplicial complex on* $[m]$*,* $P_0 = \{\tau_1, \tau_2, \cdots, \tau_s\}$ *be defined as above. By Proposition* 2.1*,* $\mathbf{U} = \{U_i\}_{i=1,2,\cdots,s}$ *forms an open cover of the topological space* $U(K)$*. Then the homology groups* $H_*(\Lambda^{*,[m]}[P_0],\ d)$ *is exactly the reduced Čech homology groups of the open cover* $\mathbf{U}$*. Precisely, we have the following isomorphisms:*

$$H_n(\Lambda^{*,[m]}[P_0], d) = \check{H}_{n-2}(U(K); \mathbf{U}; \mathbf{k}).$$

Before proving Theorem 2.3, we are going to work on the following lemma first.

Lemma 2.1 *All notations are as above,* $i, j = 1, 2, \cdots, s$*. Then*
(1) *if* $\tau_i \cup \tau_j \neq [m]$*, then*

$$(\mathrm{star}_{\partial\Delta^{m-1}}\tau_i) \cap (\mathrm{star}_{\partial\Delta^{m-1}}\tau_j) = \mathrm{star}_{\partial\Delta^{m-1}\tau_i\cup\tau_j};$$

(2) $U_i \cap U_j \neq \varnothing \Leftrightarrow \tau_i \cup \tau_j \neq [m]$.

Proof (1) Obviously holds, by definition.
(2) If $\tau = \tau_i \cup \tau_j \subsetneq [m]$, then $\mathrm{star}_{\partial\Delta^{m-1}\tau} \subset \mathrm{star}_{\partial\Delta^{m-1}}\tau_i$, since $\tau \subset \tau_i$. So

$$\mathrm{Int}|\mathrm{star}_{\partial\Delta^{m-1}}\tau| \subset \mathrm{Int}|\mathrm{star}_{\partial\Delta^{m-1}}\tau_i|.$$

Similarly, $\mathrm{Int}|\mathrm{star}_{\partial\Delta^{m-1}}\tau| \subset \mathrm{Int}|\mathrm{star}_{\partial\Delta^{m-1}}\tau_j|$. Since $\tau \neq [m]$, $\mathrm{Int}|\mathrm{star}_{\partial\Delta^{m-1}}\tau| \neq \varnothing$, and then $U_i \cap U_j \neq \varnothing$.

On the other hand, if $U_i \cap U_j \neq \varnothing$, then from (1)

$$(\mathrm{star}_{\partial\Delta^{m-1}}\tau_i) \cap (\mathrm{star}_{\partial\Delta^{m-1}}\tau_j) = \mathrm{star}_{\partial\Delta^{m-1}}\tau_i \cup \tau_j \neq \varnothing.$$

Thus $\tau_i \cup \tau_j \neq [m]$.

Proof of Theorem 2.1 By Proposition 2.1 and Definition 2.2, we know $\mathbf{U} = \{U_i\}_{i=1,2,\cdots,s}$ forms an open cover of the topological space $U(K)$. And the nerve of the cover $\mathbf{U}$ is the complex

$$\mathcal{N}(\mathbf{U}) = \{(i_1, i_2 \cdots, i_r) \subseteq [s] \mid U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_r} \neq \varnothing\}.$$

Lemma 2.4 shows that $U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_r} \neq \varnothing$ if and only if $\tau_{i_1} \cup \tau_{i_2} \cdots \cup \tau_{i_r} \neq [m]$.

So the nerve complex can be written as

$$\mathcal{N}(\mathbf{U}) = \{(i_1, i_2 \cdots, i_r) \subseteq [s] \mid \tau_{i_1} \cup \tau_{i_2} \cdots \cup \tau_{i_r} \neq [m]\}.$$

Let $\Lambda^*[P_0]$ be the exterior algebra generated by $\{\tau_1, \tau_2, \cdots, \tau_s\}$. We define another differential $\partial : \Lambda^r[P_0] \to \Lambda^{r-1}[P_0]$ by

$$\partial(\mathbf{u}) = \sum_{s=1}^{r} (-1)^{s+1} \partial_s \mathbf{u},$$

where $\partial_s \mathbf{u} = \tau_{i_1} \cdots \widehat{\tau}_{i_s} \cdots \tau_{i_r}$ for any monomial $\mathbf{u} = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_s}$.

We define a map

$$\Phi : \quad \widetilde{C}_*(\mathcal{N}(\mathbf{U}), \mathbf{k}) \longrightarrow \Lambda^{*+1}[P_0],$$

generated by $\Phi((i_1, i_2 \cdots, i_r)) := \tau_{i_1} \tau_{i_2} \cdots \tau_{i_r} \in \Lambda^r[P]$, where $(i_1, i_2, \cdots, i_r)$ is an $(r-1)$-simplex of $\mathcal{N}(\mathbf{U})$. Obviously, Φ is a monomorphism.

Then we get a short exact sequence of the chain complexes,

$$0 \to (\widetilde{C}_*(\mathcal{N}(\mathbf{U}), \mathbf{k}), \partial) \to (\Lambda^{*+1}[P_0], \partial) \to (\Lambda^{*+1}[P_0]/\widetilde{C}_*(\mathcal{N}(\mathbf{U}), \mathbf{k}), d') \to 0,$$

where $\Lambda^{*+1}[P_0]/\widetilde{C}_*(\mathcal{N}(\mathbf{U}), \mathbf{k})$ is generated by all monomials $\mathbf{u} \in \Lambda^{*,[m]}[P_0]$ (i.e., $S_{\mathbf{u}} = [m]$). The differential d' is induced by ∂.

It is easy to see that there is a chain isomorphism

$$(\Lambda^{*+1}[P_0]/\widetilde{C}_*(\mathcal{N}(\mathbf{U}), \mathbf{k}), d') \cong (\Lambda^{*,[m]}[P_0], d),$$

where $(\Lambda^{*,[m]}[P_0], d)$ is as in Theorem 1.4.

Since $(\Lambda^*[P_0], \partial)$ is isomorphic to the chain complex of the simplex with $s+1$ vertices, clearly $\widetilde{H}_*(\Lambda^*[P_0], \partial) = 0$, and from the long exact sequence induced by the short exact sequence above, we get that

$$H_n(\Lambda^{*,[m]}[P_0], d) = \widetilde{H}_{n-2}(\mathcal{N}(\mathbf{U}); \mathbf{k}) = \check{H}_{n-2}(\,U(K); \mathbf{U}; \mathbf{k}).$$

3 Barycentric Subdivision and Inflation Complex

Let K be a simplicial complex on the set $[m]$ as above. Here come two new complexes constructed from K.

Definition 3.1 (Barycentric Subdivision and Inflation Complex) *The barycentric subdivision of the simplicial complex K is a simplicial complex K' on the set $\{\sigma \in K\}$, where*

$$K' = \{(\sigma_0, \sigma_1, \cdots, \sigma_n) \mid \sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n;\ \sigma_i \in K,\ i = 0, 1, \cdots, n\}.$$

The inflation complex of the complex K is also a simplicial complex $\mathcal{F}(K)$ on the set $\{\sigma \in K\}$, where

$$\mathcal{F}(K) = \{(\sigma_0, \sigma_1, \cdots, \sigma_n) \mid \sigma_0 \cap \sigma_1 \cap \cdots \cap \sigma_n \neq \varnothing;\ \sigma_i \in K,\ i = 0, 1, \cdots, n\}.$$

Remark 3.1 The barycentric subdivision K' and inflation complex $\mathcal{F}(K)$ of the same complex K are both the complexes on the set $\{\sigma \in K\}$. For a simple $x(\sigma_0, \sigma_1, \cdots, \sigma_n)$ of K', it is clear that $\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n$, which means $\sigma_0 \cap \sigma_1 \cap \cdots \cap \sigma_n \neq \varnothing$. So $(\sigma_0, \sigma_1, \cdots, \sigma_n)$ is a simplex in $\mathcal{F}(K)$. Thus the barycentric subdivision K' is a subcomplex of the inflation complex $\mathcal{F}(K)$.

Definition 3.2 *Let K be a simplicial complex. For any sunbclomlex $L \subset K$, define the (closed) combinatorial neighborhood $U_K(L)$ of L in K by $U_K(L) = \bigcup_{\sigma \in L} \mathrm{star}_K \sigma$.*

Lemma 3.1 *Let K be a simplicial complex on $[m]$. Then the geometric realization of the barycentric subdivision K' is a deformation retract of the geometric realization of the inflation complex $\mathcal{F}(K)$.*

Before proving Lemma 3.1, we need the following statement coming from homotopy theory.

Statement A Given that a pair (X, A) satisfies the homotopy extension property, if the inclusion $A \hookrightarrow X$ is a homotopy equivalence, then A is a deformation retraction of X.

Proof of Lemma 3.1 In this proof, we do not distinguish simplicial complexes and their geometric realizations.

We prove this by induction on the number l of simplices of K. If $l = 1$, the lemma is clearly true. For the induction step, choose a maximal simplex τ of K. Then $K_0 = K \setminus \tau$ is a simplicial complex. Let $L = \partial\tau = \{\sigma \mid \sigma \subsetneq \tau\}$. Clearly L' is a subcomplex of K_0'. There is a deformation retraction $r' : U_{K_0'}(L') \to L'$ corresponding to the vertex set map $\sigma \mapsto \sigma \cap \tau$ (easy to verify that this map is simplicial).

Meanwhile, define a subcomplex $\mathcal{L}$ of $\mathcal{F}(K_0)$ by

$$\mathcal{L} = \{(\sigma_0, \sigma_1, \cdots, \sigma_i) \in \mathcal{F}(K_0) \mid \sigma_0 \cap \sigma_1 \cap \cdots \cap \sigma_i \cap \tau \neq \varnothing\}.$$

Similarly, there is a deformation retraction $r'' : \mathcal{L} \to \mathcal{F}(L)$ corresponding to the vertex set map: $\sigma \mapsto \sigma \cap \tau$. Since $\mathcal{F}(L) \simeq L'$ by induction, the two deformation retractions give $\mathcal{L} \simeq U_{K_0'}(L')$, and then by statement A, $U_{K_0'}(L')$ is a deformation retraction of $\mathcal{L}$. It is easy to see that $U_{K_0'}(L') = K_0' \cap \mathcal{L}$. So there is a deformation retraction

$$r_1 : \ K_0' \bigcup_{U_{K_0'}(L')} \mathcal{L} \to K_0',$$

which satisfies $r_1(\mathcal{L}) = U_{K_0'}(L')$. Since $K_0' \simeq \mathcal{F}(K_0)$ by induction and $K_0' \bigcup_{U_{K_0'}(L')} \mathcal{L}$ is a subcomplex of $\mathcal{F}(K_0)$, applying statement A again, we get a deformation retraction:

$$r_2 : \mathcal{F}(K_0) \to K_0' \bigcup_{U_{K_0'}(L')} \mathcal{L}.$$

The composition $r_1 \circ r_2$ is a deformation retraction from $\mathcal{F}(K_0)$ to K_0' which satisfies $r_1 \circ r_2(\mathcal{L}) = U_{K_0'}(L')$.

From the definition of K_0 and $\mathcal{L}$, we have $K' = K_0' \bigcup_{L'} \mathrm{cone}\, L'$ and $\mathcal{F}(K) = \mathcal{F}(K_0) \bigcup_{\mathcal{L}} \mathrm{cone}\, \mathcal{L}$. So $r_1 \circ r_2$ can be naturally extended to a deformation retraction

$$r_0 : \mathcal{F}(K) \to K_0' \bigcup_{U_{K_0'}(L')} \mathrm{cone}\, U_{K_0'}(L').$$

Note that $U_{K_0'}(L')\bigcup_{L'}\operatorname{cone} L'$ is a subcomplex of $\operatorname{cone} U_{K_0'}(L')$ and they are both contractible spaces $\big(r'$ extends to a deformation retraction from $U_{K_0'}(L')\bigcup_{L'}\operatorname{cone} L'$ to $\operatorname{cone} L'\big)$. Then by applying statement A again, there is a deformation retraction from $\operatorname{cone} U_{K_0'}(L')$ to

$$U_{K_0'}(L')\bigcup_{L'}\operatorname{cone} L',$$

which can be extended to a deformation retraction

$$r : K_0' \bigcup_{U_{K_0'}(L')} \operatorname{cone} U_{K_0'}(L') \to K'.$$

Thus the composition $r \circ r_0$ is the desired deformation retraction and the induction step is finished.

Remark 3.2 By Lemma 3.1, we have the following isomorphisms of homology groups (reduced or unreduced):

$$\begin{aligned} i_* : \quad & H_*(K';\mathbf{k}) \longrightarrow H_*(\mathcal{F}(K);\mathbf{k}),\\ i_* : \quad & \widetilde{H}_*(K';\mathbf{k}) \longrightarrow \widetilde{H}_*(\mathcal{F}(K);\mathbf{k}). \end{aligned}$$

4 Proof of Theorem 1.5

Following the definitions in [4, 7], we have Alexander dual simplicial complex of a complex and simplicial Alexander duality theorem.

Definition 4.1 (Alexander Dual Simplicial Complex) *Let K be a simplicial complex on $[m]$, but not the simplex Δ^{m-1}. The Alexander dual simplicial complex is defined as*

$$\widehat{K} := \{\sigma \subset [m] \mid [m]\backslash\sigma \notin K\}.$$

Theorem 4.1 (Simplicial Alexander Duality, see [4]) *Let K be a simplicial complex on $[m]$, but not Δ^{m-1}. Then the following duality holds:*

$$\widetilde{H}_j(\widehat{K};\mathbf{k}) \cong \widetilde{H}^{m-3-j}(K;\mathbf{k}),$$

where $-1 \le j \le m-2$ and we use the agreement $\widetilde{H}_{-1}(\varnothing) = \widetilde{H}^{-1}(\varnothing) = \mathbf{k}$.

Remark 4.1 As before, we use the notation $P_0 = 2^{[m]} - K - [m]$ to denote a simplicial complement of K.

The inflation complex of the dual complex $\widehat{K}$ is the complex on the set $\{\sigma \mid [m]\backslash\sigma \in P_0\}$, i.e.,

$$\mathcal{F}(\widehat{K}) = \{(\sigma_1, \sigma_2, \cdots, \sigma_i) \mid [m]\backslash\sigma_j \notin K, \sigma_1 \cap \sigma_2 \cap \cdots \cap \sigma_i \neq \varnothing\}.$$

There is a one-to-one map from $\{\sigma \neq [m] \mid [m] \setminus \sigma \notin K\}$ to P_0 ($\sigma \to [m] \setminus \sigma$). Moreover, it is easy to check that $\sigma_1 \cap \sigma_2 \cap \cdots \cap \sigma_i \neq \varnothing$, $[m]\backslash\sigma_j \in P_0$ for $j = 1, \cdots, i$, if and only if $\tau_1 \cup \tau_2 \cup \cdots \cup \tau_i \neq [m]$, where $\tau_j = [m]\backslash\sigma_j \in P_0$, $j = 1, \cdots, i$.

So the inflation complex of the dual complex $\widehat{K}$ is isomorphic to the complex on P_0, which is also denoted by $\mathcal{F}(\widehat{K})$:

$$\mathcal{F}(\widehat{K}) = \{(\tau_1, \tau_2, \cdots, \tau_i) \mid \tau_j \in P_0; \tau_1 \cup \tau_2 \cup \cdots \cup \tau_i \neq [m]\}.$$

We recall the proof of Theorem 2.3. If we assume $P_0 = \{\tau_1, \tau_2, \cdots, \tau_s\}$, $\mathbf{U} = \{U_i\}_{i=1,2,\cdots,s}$ would form an open cover of the topological space $U(K)$. The nerve complex is the complex below:

$$\mathcal{N}(\mathbf{U}) = \{(i_1, i_2, \cdots, i_r) \mid \tau_{i_1} \cup \tau_{i_2} \cdots \cup \tau_{i_r} \neq [m]\}.$$

It is obvious that the inflation complex $\mathcal{F}(\widehat{K})$ is isomorphic to the nerve complex $\mathcal{N}(\mathbf{U})$.

Now we can finish the proof of Theorem 1.5.

Proof of Theorem 1.5 By Theorem 2.3, we have

$$H_i(\Lambda^{*,[m]}[P], d) \cong \widetilde{H}_{i-2}(\mathcal{N}(\mathbf{U}); \mathbf{k}).$$

Remark 4.3 tells us $\mathcal{F}(\widehat{K}) \cong \mathcal{N}(\mathbf{U})$. Then by Lemma 3.1, we have

$$\widetilde{H}_i(\mathcal{N}(\mathbf{U}); \mathbf{k}) \cong \widetilde{H}_i(\mathcal{F}(\widehat{K}); \mathbf{k}) \cong \widetilde{H}_i(\widehat{K}'; \mathbf{k}).$$

Combining with the simplicial Alexander duality

$$\widetilde{H}_i(\widehat{K}; \mathbf{k}) \cong \widetilde{H}^{m-3-i}(K; \mathbf{k}),$$

we get the final result

$$H_i(\Lambda^{*,[m]}[P], d) \cong \widetilde{H}^{m-i-1}(K; \mathbf{k}).$$

Remark 4.2 Proposition 2.1 told us that $\mathbf{U} = \{U_i\}_{i=1,2,\cdots,s}$ is an open cover of the topological space $U(K)$, where $U(K) = |\partial\Delta^{m-1}| \backslash |K|$. In [2, Corollary 13.3], there is a theory that if the cover $\mathbf{U}$ of the topological space X is good enough, then the Čech homology of this cover is exactly the homology of the space X. Here "good" means that for each simplex $\sigma = (i_1, i_2, \cdots, i_n) \in \mathcal{N}(\mathbf{U})$, $\widetilde{H}_*(U_\sigma) = 0$, where $U_\sigma = U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_n}$. Luckily, it is easy to prove that the open cover $\mathbf{U}$ given by any simplicial complement of the complex P is "good". It will give us another proof of Theorem 1.5, combining with the geometric Alexander duality theorem.

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