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BSDEs with Jumps and Path-Dependent Parabolic Integro-differential Equations*

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Abstract This paper deals with backward stochastic differential equations with jumps, whose data (the terminal condition and coefficient) are given functions of jump-diffusion process paths. The author introduces a type of nonlinear path-dependent parabolic integro-differential equations, and then obtains a new type of nonlinear Feynman-Kac formula related to such BSDEs with jumps under some regularity conditions.

Keywords Backward stochastic differential equations, Jump-diffusion processes, Itô integral and Itô calculus, Path-dependent parabolic integro-differential equations
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1 Introduction

Linear backward stochastic differential equations (BSDEs for short) were introduced by Bismut [2] in 1973. Pardoux and Peng [16] established the existence and uniqueness theorem for nonlinear BSDEs under a standard Lipschitz condition in 1990. Then, Peng [18–19] and Pardoux and Peng [17] introduced the nonlinear Feynman-Kac formula, which provides a probabilistic representation for a wide class of semilinear partial differential equations (see also [13]). Since then, especially after the publication of the paper [11], in which the applications of BSDEs in finance were discussed, the theory of BSDE has received wide attention for both theoretical research and applications.

Recently, Dupire [7] introduced a new functional Itô's formula, which non-trivially generalized the classical one through a new notion-path derivative (see [4–6] for more general and systematic research). It extends the Itô stochastic calculus to functionals of a given process. It provides an excellent tool for the study of path-dependence. In fact, he showed that a smooth path functional solves a linear path-dependent PDE if its composition with a Brownian motion generates a martingale, which provided a functional extension of the classical Feynman-Kac formula. Moreover, by virtue of the BSDE approach, we obtained the existence and uniqueness of the smooth solution to the semilinear path-dependent PDE (see [21]). These methods are mainly based on stochastic calculus.

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The aim of this paper is to generalize the above results to the case of BSDEs with both Brownian motion and a Poisson random measure. Consider the following BSDE with jumps:

$$\begin{cases} -\mathrm{d}Y(t) = f(t, X, Y(t), Z(t), K(t))\,\mathrm{d}t - Z(t)\,\mathrm{d}B(t) - \displaystyle\int_{\mathbb{E}} K(t, e)\widetilde{\mu}(\mathrm{d}t, \mathrm{d}e), \quad t \in [0, T], \\ Y(T) = \Phi(X), \end{cases} \tag{1.1}$$

where X is a d-dimensional diffusion satisfying the SDE

$$X(t) = x + \int_0^t b(X(r))\mathrm{d}r + \int_0^t \sigma(X(r))\mathrm{d}B(r) + \int_0^t \int_{\mathbb{E}} \beta(X(r-), e)\widetilde{\mu}(\mathrm{d}r, \mathrm{d}e), \tag{1.2}$$

in which $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$, $\beta : \mathbb{R}^d \times \mathbb{E} \mapsto \mathbb{R}^d$ are some measurable functions, and $f : \Lambda \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times L^2(\mathbb{E}, \mathcal{E}, \lambda; \mathbb{R}^n) \mapsto \mathbb{R}^n$ is a non-anticipative functional with respect to X. Note that (1.1) is "non-Markovian". We will prove that under certain smooth assumptions (see Section 4) the solution $(Y(t), Z(t), K(t))$ to (1.1) solves the following type of PDE, which is said to be path-dependent parabolic integro-differential equations (PIDEs for short). For each $l \in \{1, \cdots, n\}$,

$$\begin{aligned} &D_t u_l(\gamma_t) + \mathcal{L} u_l(\gamma_t) + f_l(\gamma_t, u(\gamma_t), D_x u(\gamma_t)\sigma(\gamma_t(t)), u(\gamma_t^{\beta(\gamma_t(t), e)}) - u(\gamma_t)) = 0, \\ &u(\gamma) = \Phi(\gamma), \end{aligned} \tag{1.3}$$

where the derivative is the Dupire's path derivative (see Section 2.1). More specifically, the path-function $u(t, X(s)_{0\le s\le t}) := Y(t, \omega)$ is the unique $\mathbb{C}^{1,2}_{l,\mathrm{lip}}$-solution to the path-dependent PIDEs (1.3). We refer to Buckdahn-Pardoux [3] for the Markovian case when both Φ and f are functions of the forward diffusion. The results of this paper non-trivially generalize the ones of [3] (see also [1]) for the path-dependent situation.

The paper is organized as follows. In Section 2, we present some existing results in the theory of functional Itô's formula and BSDEs that we will use in this paper. In Section 3, we state the nonlinear Feynman-Kac formula for the "discrete functional" form. Then, in Subsections 4.1–4.2, we first establish some estimates and regularity results for the solution to BSDEs with path. Finally, in Subsection 4.3, we obtain our main results, i.e., Theorems 4.4 and 4.5, which provide a one to one correspondence between BSDEs and the path-dependent PIDEs.

When the coefficients of BSDE are only Lipschitz functions, we usually can not obtain the smooth results given in this paper, and therefore a new type of viscosity solutions is required. In the Brownian motion case, we refer to [20] for the corresponding comparison theorem. Moreover, [8] introduced a different stochastic approach to derive a maximum principle for semilinear path-dependent partial differential equations. For a recent account and development of this theory, we refer the readers to [9–10].

2 Preliminaries

2.1 Functional Itô's formula

The following notations are mainly from Dupire [7].

Let $T > 0$ be fixed. For each $t \in [0, T]$, we denote by Λ_t the set of càdlàg $\mathbb{R}^d$-valued functions on $[0, t]$. For each $\gamma \in \Lambda_T$, the value of γ at time $s \in [0, T]$ is denoted by $\gamma(s)$. Thus

$\gamma = \gamma(s)_{0\le s\le T}$ is a càdlàg process on $[0,T]$ and its value at time s is $\gamma(s)$. The path of γ up to time t is denoted by γ_t, i.e., $\gamma_t = \gamma(s)_{0\le s\le t} \in \Lambda_t$. Denote $\Lambda = \bigcup_{t\in[0,T]} \Lambda_t$. We sometimes specifically write

$$\gamma_t = \gamma(s)_{0\le s\le t} = (\gamma(s)_{0\le s<t}, \gamma(t))$$

to indicate the terminal position $\gamma(t)$ of γ_t, which often plays a special role in this framework. For each $\gamma_t \in \Lambda$ and $x \in \mathbb{R}^d$, we denote by $\gamma_t(s)$ the value of γ_t at $s \in [0,t]$ and $\gamma_t^x := (\gamma_t(s)_{0\le s<t}, \gamma_t(t)+x)$, which is also an element of Λ_t. Analogously, we can define $X(t)$ and X_t for a process X.

Now consider the function u of path, i.e., $u : \Lambda \mapsto \mathbb{R}$. This function $u = u(\gamma_t)_{\gamma_t\in\Lambda}$ can also be regarded as a family of real valued functions

$$u(\gamma_t) = u(t, \gamma_t(s)_{0\le s\le t}) = u(t, \gamma_t(s)_{0\le s<t}, \gamma_t(t)), \quad \gamma_t \in \Lambda_t,\ t \in [0,T].$$

We also denote $u(\gamma_t^x) := u(t, \gamma_t(s)_{0\le s<t}, \gamma_t(t)+x)$ for $\gamma_t \in \Lambda_t$, $x \in \mathbb{R}^d$.

We introduce the distance on Λ. Let $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the inner product and the norm in $\mathbb{R}^d$. For each $0 \le t \le \bar{t} \le T$ and $\gamma_t, \overline{\gamma}_{\bar{t}} \in \Lambda$, we denote

$$\begin{aligned} \|\gamma_t\| &:= \sup_{s\in[0,t]} |\gamma_t(s)|, \\ d_\infty(\gamma_t, \overline{\gamma}_{\bar{t}}) &:= \sup_{s\in[0,t\vee\bar{t}]} |\gamma_t(s\wedge t) - \overline{\gamma}_{\bar{t}}(s\wedge\bar{t})| + |t-\bar{t}|^{\frac{1}{2}}. \end{aligned}$$

It is obvious that Λ_t is a Banach space with respect to $\|\cdot\|$. Since Λ is not a linear space, d_∞ is not a norm.

Definition 2.1 (Continuous) *A function $u : \Lambda \mapsto \mathbb{R}$ is said to be Λ-continuous at $\gamma_t \in \Lambda$, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for each $\overline{\gamma}_{\bar{t}} \in \Lambda$ with $d_\infty(\gamma_t, \overline{\gamma}_{\bar{t}}) < \delta$, we have $|u(\gamma_t) - u(\overline{\gamma}_{\bar{t}})| < \varepsilon$. u is said to be Λ-continuous if it is Λ-continuous at each $\gamma_t \in \Lambda$.*

Remark 2.1 In our framework, we often regard $u(\gamma_t^x)$ as a function of t, γ_t and x, i.e., $u(\gamma_t^x) = u(t, \gamma_t(s)_{0\le s<t}, \gamma_t(t)+x)$. Thus, for a fixed $\gamma_t \in \Lambda$, $u(\gamma_t^x)$ is regarded as a function of $(t,x) \in [0,T] \times \mathbb{R}^d$.

Definition 2.2 *Given $u : \Lambda \mapsto \mathbb{R}$ and $\gamma_t \in \Lambda$, if there exists $p \in \mathbb{R}^d$, such that*

$$u(\gamma_t^x) = u(\gamma_t) + \langle p, x\rangle + o(|x|) \quad \textit{as } x \to 0,\ x \in \mathbb{R}^d, \tag{2.1}$$

then we say that u is (vertically) differentiable at γ_t and denote $D_x u(\gamma_t) = p$. u is said to be vertically differentiable in Λ if $D_x u(\gamma_t)$ exists for each $\gamma_t \in \Lambda$. We can similarly define the Hessian $D_{xx} u(\gamma_t)$. It is an $\mathbb{S}(d)$-valued function defined on Λ, where $\mathbb{S}(d)$ is the space of all $d \times d$ symmetric matrices.

For each $\gamma_t \in \Lambda$, we denote

$$\gamma_{t,s}(r) = \gamma_t(r)\mathbf{1}_{[0,t)}(r) + \gamma_t(t)\mathbf{1}_{[t,s]}(r), \quad r \in [0,s].$$

It is clear that $\gamma_{t,s} \in \Lambda_s$.

Definition 2.3 *For a given $\gamma_t \in \Lambda$, if we have*

$$u(\gamma_{t,s}) = u(\gamma_t) + a(s-t) + o(|s-t|), \quad \text{as } s \to t, \ s \geq t, \tag{2.2}$$

then we say that $u(\gamma_t)$ is (horizontally) differentiable in t at γ_t and denote $D_t u(\gamma_t) = a$. u is said to be horizontally differentiable in Λ if $D_t u(\gamma_t)$ exists for each $\gamma_t \in \Lambda$.

Definition 2.4 *Define $\mathbb{C}^{j,k}(\Lambda)$ as the set of functions u defined on Λ, which are j times horizontally and k times vertically differentiable in Λ, such that all these derivatives are Λ-continuous.*

Definition 2.5 *Function u is said to have the horizontal local Lipschitz property if and only if*

$$d_\infty(\gamma_t, \overline{\gamma}_{t_1}) < \eta \Rightarrow |u(\overline{\gamma}_{t_1,t_2}) - u(\overline{\gamma}_{t_1})| < C(t_2 - t_1),$$
$$\forall \gamma_t \in \Lambda, \ \exists C \geq 0, \ \eta \geq 0, \ \forall t_1 < t_2 \leq T, \ \forall \overline{\gamma}_{t_1} \in \Lambda.$$

Definition 2.6 *u is said to be in $\mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$, if $u \in \mathbb{C}^{1,2}(\Lambda)$ and for $\varphi = u, D_t u, D_x u, D_{xx} u$, we have*

$$|\varphi(\gamma_t) - \varphi(\overline{\gamma}_{\overline{t}})| \leq C(1 + \|\gamma_t\|^k + \|\overline{\gamma}_{\overline{t}}\|^k) d_\infty(\gamma_t, \overline{\gamma}_{\overline{t}}) \quad \text{for each } \gamma_t, \overline{\gamma}_{\overline{t}} \in \Lambda,$$

where C and k are some constants depending only on φ.

Example 2.1 If $u(\gamma_t) = f(t, \gamma_t(t))$ with $f \in C^{1,1}([0,T[\times\mathbb{R})$, then

$$D_t u(\gamma_t) = \partial_t f(t, \gamma_t(t)), \quad D_x u(\gamma_t) = \partial_x f(t, \gamma_t(t)),$$

which are the classic derivatives. In general, these derivatives also satisfy the classic properties: linearity, the product rule and the chain rule.

The functional Itô formula for continuous martingale was firstly obtained by Dupire [7], and then generalized by Cont and Fournié [4] to more general formulation.

Theorem 2.1 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a probability space. X is a semimartingale and u is in $\mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$. If $D_x u$ has the horizontal local Lipschitz property, then for any $t \in [0,T[$:*

$$\begin{aligned} u(X_t) - u(X_0) = &\int_0^t D_s u(X_{s-}) \mathrm{d}s + \int_0^t D_x u(X_{s-}) \mathrm{d}X(s) + \frac{1}{2}\int_0^t \mathrm{tr}[D_{xx} u(X_{s-}) \mathrm{d}[X]^c(s)] \\ &+ \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} [u(X^z_{s-}) - u(X_{s-}) - D_x u(X_{s-}) z] \mu(\mathrm{d}t, \mathrm{d}z), \quad a.s. \end{aligned}$$

Remark 2.2 If $u \in \mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$, the horizontal local Lipschitz property for $D_x u$ does not hold in general.

In this paper, we will use the following functional Itô formula.

Theorem 2.2 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a probability space. $X = M + A$ is a semimartingale, where M is a continuous local martingale and A is a finite variation process. If*

$u \in \mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$, *then for any* $t \in [0, T[$,

$$u(X_t) - u(X_0) = \int_0^t D_s u(X_{s-})\mathrm{d}s + \int_0^t D_x u(X_{s-})\mathrm{d}X(s) + \frac{1}{2}\int_0^t \mathrm{tr}[D_{xx}u(X_{s-})\mathrm{d}[X]^c(s)] + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} [u(X^z_{s-}) - u(X_{s-}) - D_x u(X_{s-})z]\mu(\mathrm{d}t, \mathrm{d}z), \quad a.s.$$

We give the sketch of proof of Theorem 2.2, which is essentially from Cont and Fournié [4].

Proof Without loss of generality, we assume that X and $\sum_{s\le T} |\Delta X(s)|$ are bounded. Otherwise, for each p, denote $\tau_p := \inf\big\{s \ge 0 : |X(s)| \ge p$ or $\sum_{r\le s} |\Delta X(r)| \ge p\big\}$ and consider a process X^{τ_p}.

Let us introduce a sequence of random subdivisions of $[0, t]$, and define the following sequence of stopping times:

$$\sigma_0^n = 0, \quad \sigma_i^n = \inf\Big\{s \ge \sigma_{i-1}^n : |\Delta X(s)| \ge \frac{1}{n} \text{ or } \ 2^n s \in \mathbb{N}\Big\} \wedge t.$$

Then $\sup\big\{|X(u) - X(\sigma_i^n)| + \frac{t}{2^n},\ i \le 2^n,\ u \in [\sigma_i^n, \sigma_{i+1}^n)\big\}$ tends to 0 as $n \to \infty$. We set $X^n(s) = \sum_{i=0}^{\infty} X(\sigma_{i+1}^n -)\mathbf{1}_{[\sigma_i^n, \sigma_{i+1}^n)}(s) + X(t)\mathbf{1}_{\{t\}}(t)$.

Recall that $u \in \mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$ and

$$\sum_i |\sigma_{i+1}^n - \sigma_i^n|^{\frac{1}{2}} |\Delta X(\sigma_i^n)| \le C\frac{1}{2^{\frac{n}{2}}},$$

(note that (56) also converges to 0 in [4]), and then using the same method as in [4], one can get

$$u(X_t) - u(X_0) = \int_0^t D_s u(X_{s-})\mathrm{d}s + \int_0^t D_x u(X_{s-})\mathrm{d}X(s) + \frac{1}{2}\int_0^t \mathrm{tr}[D_{xx}u(X_{s-})\mathrm{d}[X]^c(s)] + \int_0^t \int_{\mathbb{R}^d\setminus\{0\}} [u(X^z_{s-}) - u(X_{s-}) - D_x u(X_{s-})z]\mu(\mathrm{d}t, \mathrm{d}z), \quad a.s.,$$

which completes the proof.

2.2 BSDEs

Let $(\Omega, \mathcal{F}, P)$ be a completed probability space. The filtration $(\mathcal{F}_t)_{0\le t\le T}$ is generated by the following two mutually independent stochastic processes defined on $(\Omega, \mathcal{F}, P)$, and augmented by all P-null sets:

(1) A d-dimensional standard Wiener process $\{B(t)\}_{t\ge 0}$.

(2) A Poisson random measure μ on $\mathbb{R}_+ \times \mathbb{E}$, where $\mathbb{E} := \mathbb{R}^d\setminus\{0\}$ is equipped with its Borel field $\mathcal{E}$, with a compensator $\nu(\mathrm{d}t, \mathrm{d}e) = \mathrm{d}t\lambda(\mathrm{d}e)$, such that $\{\widetilde{\mu}([0, t] \times A) = (\mu - \nu)([0, t] \times A)\}_{t\ge 0}$ is a martingale for all $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. λ is assumed to be a σ-finite measure on $(\mathbb{E}, \mathcal{E})$ satisfying

$$\int_{\mathbb{E}} (1 \wedge |e|)\lambda(\mathrm{d}e) < \infty. \tag{2.3}$$

It is also a right continuous filtration.

Remark 2.3 We assume that the Lévy measure λ satisfies (2.3) instead of $\int_{\mathbb{E}}(1\wedge|e|^2)\lambda(\mathrm{d}e) < \infty$, and then the jump-diffusion process X in the sequel satisfies the conditions of the functional Itô formula (2.2).

We also introduce the following spaces of processes which will be used frequently in the sequel:

$$\begin{aligned}
L^p(\mathcal{F}_T;\mathbb{R}^n) &:= \{\text{the } \mathbb{R}^n\text{-valued } \mathcal{F}_T\text{-measurable random variable } \xi : E[|\xi|^p] < \infty\},\\
\mathcal{S}^p(0,T;\mathbb{R}^n) &:= \Big\{\text{the } \mathbb{R}^n\text{-valued } \mathcal{F}_t\text{-adapted càdlàg process } Y : E\Big[\sup_{0\le t\le T}|Y(t)|^p\Big] < \infty\Big\},\\
\mathcal{H}^p(0,T;\mathbb{R}^{n\times d}) &:= \Big\{\text{the } \mathbb{R}^{n\times d}\text{-valued } \mathcal{F}_t\text{-predictable measurable process } Z :\\
&\qquad E\Big[\Big(\int_0^T |Z(t)|^2\mathrm{d}t\Big)^{\frac{p}{2}}\Big] < \infty\Big\},\\
\mathcal{K}^p_\lambda(0,T;\mathbb{R}^n) &:= \Big\{\text{mappings } K : \Omega\times[0,T]\times\mathbb{E}\mapsto\mathbb{R}^n\mathcal{P}\times\mathcal{E} \text{ measurable:}\\
&\qquad E\Big[\Big(\int_0^T\int_{\mathbb{E}}|K(t,e)|^2\lambda(\mathrm{d}e)\mathrm{d}t\Big)^{\frac{p}{2}}\Big] < \infty\Big\},
\end{aligned}$$

Finally, we define $\Sigma := (Y,Z,K)\in\mathcal{B}^p$ with

$$\|\Sigma\|^p_{\mathcal{B}^p} = E\Big[\sup_{0\le t\le T}|Y(t)|^p + \Big(\int_0^T |Z(t)|^2\mathrm{d}t\Big)^{\frac{p}{2}} + \Big(\int_0^T\int_{\mathbb{E}}|K(t,e)|^2\lambda(\mathrm{d}e)\mathrm{d}t\Big)^{\frac{p}{2}}\Big].$$

Let us consider a function $f : \Omega\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}^n)\mapsto\mathbb{R}^n$, which is $\mathcal{P}$-measurable for each $(y,z,k)\in\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}^n)$. For the function f, we will make the following assumptions:

(A1) $f(\cdot,0,0,0)\in\mathcal{H}^2(0,T;\mathbb{R}^n)$.

(A2) There exists a constant $C\ge 0$, such that for all $t\in[0,T]$, $y,\overline{y}\in\mathbb{R}^n$, $z,\overline{z}\in\mathbb{R}^{n\times d}$, $k,\overline{k}\in L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}^n)$, P-a.s.

$$|f(t,y,z,k)-f(t,\overline{y},\overline{z},\overline{k})|\le C(|y-\overline{y}|+|z-\overline{z}|+\|k-\overline{k}\|_{\mathbb{E}}).$$

The following result on BSDEs with jumps is by now well-known, and for its proof the readers are referred to Lemma 2.4 in [22] or Theorem 2.1 in [1].

Lemma 2.1 *Let f satisfy the conditions* (A1)–(A2), *and then for each $\xi\in L^2(\mathcal{F}_T;\mathbb{R}^n)$, the BSDE with jump*

$$\begin{aligned}
Y(t) = \xi + \int_t^T f(s,Y(s),Z(s),K(s))\mathrm{d}s - \int_t^T Z(s)\mathrm{d}B(s)\\
- \int_t^T\int_{\mathbb{E}} K(s,e)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}e),\quad 0\le t\le T
\end{aligned}\tag{2.4}$$

has a unique adapted solution

$$(Y(t),Z(t),K(t))_{0\le t\le T}\in\mathcal{B}^2.$$

We have the following comparison theorem for solutions to (2.4) (see Proposition 2.6 in [1]).

Lemma 2.2 (Comparison Theorem) *Let $h : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$ be $\mathcal{P} \times \mathcal{B} \times \mathcal{B}_d \times \mathcal{B}$-measurable and satisfy that for any $y, \overline{y} \in \mathbb{R}$, $z, \overline{z} \in \mathbb{R}^d$, $q, \overline{q} \in \mathbb{R}$, and $t \in [0,T]$, there exists some constant $K > 0$, such that*

(i) $E\big[\int_0^T |h(t,0,0,0)|^2 \mathrm{d}t\big] < \infty$,

(ii) $|h(t,y,z,q) - h(t,\overline{y},\overline{z},\overline{q})| \le K(|y-\overline{y}| + |z-\overline{z}| + |q-\overline{q}|)$,

(iii) $q \mapsto h(t,y,z,q)$ *is non-decreasing.*

Furthermore, let $l : \Omega \times [0,T] \times \mathbb{E} \mapsto \mathbb{R}$ be $\mathcal{P} \times \mathcal{B}(\mathbb{E})$ measurable and satisfy

$$0 \le l(t,e) \le K(1 \wedge |e|), \quad e \in \mathbb{E}.$$

Set

$$f^1(t,\omega,y,z,k) := h\Big(t,\omega,y,z,\int_{\mathbb{E}} k(e) l(t,\omega,e)\lambda(\mathrm{d}e)\Big),$$
$$(t,\omega,y,z,k) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}).$$

Given $\xi^1, \xi^2 \in L^2(\mathcal{F}_T;\mathbb{R})$, we have that f^2 satisfies (A1)–(A2). *Denote by (Y^1,Z^1,K^1) and (Y^2,Z^2,K^2) the solutions to the BSDE with the data (ξ^1,f^1) and (ξ^2,f^2), respectively. Then we have the following result: If $\xi^1 \ge \xi^2$ and $f^1(t,y,z,k) \ge f^2(t,y,z,k)$, a.s., a.e. for any $(y,z,k) \in \mathbb{R} \times \mathbb{R}^d \times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R})$, then $Y^1(t) \ge Y^2(t)$, a.s., for all $t \in [0,T]$.*

For each $i \in \{1,2\}$, the drivers g^i are given by

$$g^i(s,y,z,u) = f(s,y,z,u) + \varphi^i(s), \quad \mathrm{d}s \times \mathrm{d}P\text{-}a.e.,$$

where $\varphi^i \in \mathcal{H}^2(0,T;\mathbb{R}^n)$ and f satisfies the assumptions (A1)–(A2). The following lemma is due to Buckdahn-Pardoux [3].

Lemma 2.3 *Let $\xi^i \in L^2(\mathcal{F}_T;\mathbb{R}^n)$. Then the solution (Y^i,Z^i,K^i) to the BSDE* (2.4) *with the data (ξ^i,g^i) satisfies the following estimate: For any $p \ge 2$, there exists C_p depending on T and p, such that*

$$\|\Sigma^1 - \Sigma^2\|^p_{\mathcal{B}^p} \le E\Big[|\xi^1 - \xi^2|^p + \Big(\int_0^T |\varphi^1(s) - \varphi^2(s)|\mathrm{d}s\Big)^p\Big].$$

Remark 2.4 Note that in Lemma 2.3, we assume only that $\xi^i \in L^2(\mathcal{F}_T)$ and the process $\varphi^i \in \mathcal{H}^2(t,T;\mathbb{R}^n)$ to guarantee the solvability of the BSDE. However, if $p \ge 2$ and the right-hand side is ∞, the estimate obviously holds.

3 Nonlinear Feynman-Kac Formula for "Discrete Functional" Form

$C^n(\mathbb{R}^p;\mathbb{R}^q), C^n_{b,\mathrm{lip}}(\mathbb{R}^p;\mathbb{R}^q), C^n_{l,\mathrm{lip}}(\mathbb{R}^p;\mathbb{R}^q)$ will denote respectively the set of functions of class C^n from $\mathbb{R}^p$ into $\mathbb{R}^q$, the set of those functions of class $C^n(\mathbb{R}^p;\mathbb{R}^q)$ whose partial derivatives of order less than or equal to n are bounded Lipschitz continuous functions, and the set of those functions of class $C^n(\mathbb{R}^p;\mathbb{R}^q)$, which together with all their partial derivatives of order less than or equal to n are in $C_{l,\mathrm{lip}}(\mathbb{R}^p;\mathbb{R}^q)$, where $C_{l,\mathrm{lip}}(\mathbb{R}^p;\mathbb{R}^q)$ is the space of all $\mathbb{R}^q$-valued continuous functions φ defined on $\mathbb{R}^p$, such that

$$|\varphi(x) - \varphi(y)| \le C(1 + |x|^k + |y|^k)|x-y|, \quad \forall x,y \in \mathbb{R}^p.$$

Here C and k are some constants depending only on φ.

In this section, we shall study the nonlinear Feynman-Kac formula when the BSDEs with jumps are of the "discrete functional" form. We refer to [12] for the BSDEs case (see also [14–15]). Consider the following (discrete) functional-type BSDEs defined on an arbitrary interval $[t,T]\subset[0,T]$: For each $s\in[t,T]$,

$$\begin{cases} X(s)=x+\int_t^s b(X(r))\mathrm{d}r+\int_t^s\sigma(X(r))\mathrm{d}B(r)+\int_t^s\int_{\mathbb{E}}\beta(X(r-),e)\widetilde{\mu}(\mathrm{d}r,\mathrm{d}e),\\ Y(s)=g(X(t_1),\cdots,X(t_N))\\ \qquad +\int_s^T f(s,X(t_1\wedge r),\cdots,X(t_N\wedge r),Y(r),Z(r),K(r))\mathrm{d}r\\ \qquad -\int_s^T Z(r)\mathrm{d}B(r)-\int_s^T\int_{\mathbb{E}}K(r,e)\widetilde{\mu}(\mathrm{d}r,\mathrm{d}e),\end{cases}\tag{3.1}$$

where $0=t_0\le t_1\le\cdots\le t_N=T$ is a given partition on $[0,T]$. We denote any solution to (3.1), whenever it exists, by $(X^{t,x},Y^{t,x},Z^{t,x},K^{t,x})$ to indicate its dependence on the initial data (t,x). For convenience, for any $x=(x_1,\cdots,x_N)\in\mathbb{R}^{Nd}$ and $k=1,\cdots,N$, set

$$\begin{aligned} x^{(k)}&=(x_1,x_2,\cdots,x_k)\in\mathbb{R}^{kd},\quad x^{(k,N)}=(x_k,\cdots,x_{N)})\in\mathbb{R}^{(N-k+1)d},\\ X_t^{(k)}&=(X^{0,x}(t_1\wedge t),X^{0,x}(t_2\wedge t)-X^{0,x}(t_1\wedge t),\cdots,X^{0,x}(t_k\wedge t)-X^{0,x}(t_{k-1}\wedge t)),\\ X_t^{(k,N)}&=(X^{0,x}(t_k\wedge t)-X^{0,x}(t_{k-1}\wedge t),\cdots,X^{0,x}(t_N\wedge t)-X^{0,x}(t_{N-1}\wedge t)).\end{aligned}$$

In particular, denote

$$\begin{aligned} X^{(k)}&=(X^{0,x}(t_1),\cdots,X^{0,x}(t_k)-X^{0,x}(t_{k-1})),\\ X^{(k,N)}&=(X^{0,x}(t_k)-X^{0,x}(t_{k-1}),\cdots,X^{0,x}(t_N)-X^{0,x}(t_{N-1})).\end{aligned}$$

Then (3.1) can be rewritten as

$$\begin{aligned} Y(t)=g(X^{(N)})&+\int_t^T f(s,X_s^{(N)},Y(s),Z(s),K(s))\mathrm{d}s\\ &-\int_t^T Z(s)\mathrm{d}B(s)-\int_t^T\int_{\mathbb{E}}K(s,e)\widetilde{\mu}(\mathrm{d}s,\mathrm{d}e).\end{aligned}$$

For each $k=N,N-1,\cdots,1$, consider a sequence of semilinear PIDEs with parameters, defined recursively in a "backward" manner as follows: First, fix $x^{(N-1)}$ as a parameter, and define

$$u^{N+1}(T,x^{(N-1)},x,0)=g(x^{(N-1)},x),\quad x\in\mathbb{R}^d.$$

Next, for each $k=N,N-1,\cdots,1$, we fix $x^{(k-1)}$ as a parameter, and consider the following PIDEs: For each $(t,x)\in[t_{k-1},t_k)\times\mathbb{R}^d$,

$$\begin{cases} \frac{\partial u_i^k}{\partial t}(t,x^{(k-1)},x)+\mathcal{L}u_i^k(t,x^{(k-1)},x)+f_i\big(t,x^{(k-1)},x,0,\cdots,0,u^k(t,x^{(k-1)},x),\\ \quad\partial_x u^k(t,x^{(k-1)},x)\sigma(x),u^k(t,x^{(k-1)},x+\beta(x,\cdot))-u^k(t,x^{(k-1)},x)\big)=0,\\ u_i^k(t_k,x^{(k-1)},x)=u_i^{k+1}(t_k,x^{(k-1)},x,0),\quad i=1,\cdots,n.\end{cases}\tag{3.2}$$

For $\varphi\in C^2(\mathbb{R}^d;\mathbb{R})$, the operator $\mathcal{L}$ is given by

$$\mathcal{L}\varphi(t,x)=\frac{1}{2}\mathrm{tr}[\sigma\sigma^{\mathrm{T}}(x)\partial_{xx}^2\varphi(x)]+\partial_x\varphi(x)b(x)+\int_{\mathbb{E}}[\varphi(x+\beta(x,e))-\varphi(x)-\partial_x\varphi(x)\beta(x,e)]\lambda(\mathrm{de}).$$

Now suppose that all PIDEs have classical solutions which are denoted by $u_i^k,\ i=1,\cdots,n,\ k=N,N-1,\cdots,1$. For convenience, set

$$v^k(t,x^{(k-1)},x)=\partial_x u^k(t,x^{(k-1)},x),\quad k=N,N-1,\cdots,1.$$

Finally, for $t\in[0,T]$, if $t\in(t_{k-1},t_k]$, $k=N,N-1,\cdots,2$ or $t\in[t_0,t_1]$, denote

$$\begin{aligned}&Y(t):=u^k(t,X_t^{(k)}),\quad Z(t):=v^k(t,X_{t-}^{(k)})\sigma(X(t-)),\\&K(t,e):=u^k(t,X_t^{(k-1)},X(t-)+\beta(X(t-),e)-X(t_{k-1}))-u(t,X_{t-}^{(k)}).\end{aligned}\tag{3.3}$$

Then, we have the following nonlinear Feynman-Kac formula.

Theorem 3.1 *Assume that all PIDEs in* (3.2) *have classical solutions whose derivatives are of polynomial growth. Then, the process* (Y,Z,K) *defined by* (3.3) *solves BSDE* (3.1) *on* $[0,T]$.

Proof We shall check the case $t\in[t_{N-1},t_N]$, and the other cases can be argued in the same way.

For $t\in(t_{N-1},t_N]$,

$$\begin{aligned}Y(t)&=u^N(t,X^{(N-1)},X(t)-X(t_{N-1})),\\Z(t)&=\partial_x u^N(t,X^{(N-1)},X(t-)-X(t_{N-1}))\sigma(X(t-)),\\K(t,e)&=u(t,X^{(N-1)},X(t-)+\beta(X(t-),e)-X(t_{N-1}))-u(t,X^{(N-1)},X(t-)-X(t_{N-1})).\end{aligned}$$

Applying the Itô's formula and by the definition of u^N, we deduce that

$$\begin{aligned}&\mathrm{d}u^N(t,X^{(N-1)},X(t)-X(t_{N-1}))\\&=(\partial_t u^N(t,X^{(N-1)},X(t-)-X(t_{N-1}))+\mathcal{L}u^N(t,X^{(N-1)},X(t-)-X(t_{N-1})))\mathrm{d}t\\&\quad+\partial_x u^N(t,X^{(N-1)},X(t-)-X(t_{N-1}))\sigma(X(t-))\mathrm{d}B(t)\\&\quad+\int_{\mathbb{E}}(u(t,X^{(N-1)},X(t-)+\beta(X(t-),e)-X(t_{N-1}))\\&\quad-u(t,X^{(N-1)},X(t-)-X(t_{N-1})))\widetilde{u}(\mathrm{d}t,\mathrm{d}e).\end{aligned}$$

From (3.2), we obtain that (Y,Z,K) solves the BSDE on $(t_{N-1},T]$. Note that at $t=t_{N-1}$,

$$\begin{aligned}Y(t_{N-1},\omega)&=u^{N-1}(t_{N-1},X^{(N-2)}(\omega),X(t_{N-1},\omega)-X(t_{N-2},\omega))\\&=u^N(t_{N-1},X^{(N-2)}(\omega),X(t_{N-1},\omega)-X(t_{N-2},\omega),0)\\&=u^N(t_{N-1},X^{(N-1)}(\omega),0).\end{aligned}$$

From the definitions of the functions u^{N-1} and v^{N-1}, we can similarly prove that (Y,Z,K) solves the BSDE on $(t_{N-2},t_{N-1}]$. Continuing this way for N steps, the proof is completed.

We should note that various assumptions can be made to guarantee the existence and uniqueness of the classical solution to the system of PIDEs, as well as the adapted solution to the BSDE (3.1). In particular, by Theorem 4.1 in [3], we have the following lemma.

Lemma 3.1 *Let $b \in C^2_{b,\mathrm{lip}}(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in C^2_{b,\mathrm{lip}}(\mathbb{R}^d;\mathbb{R}^{d\times d})$, $\beta : \mathbb{R}^d \times \mathbb{E} \mapsto \mathbb{R}^d$ be measurable. For all $e \in \mathbb{E}$, $\beta(\cdot, e) \in C^3_b(\mathbb{R}^d;\mathbb{R}^d)$, there exists a constant $K \geq 0$, such that*

$$|\beta(0,e)| \leq K(1 \wedge |e|),$$
$$\Big|\frac{\partial^{|k|}}{\partial x^k}\beta(x,e)\Big| \leq K(1 \wedge |e|), \quad \forall x \in \mathbb{R}^d, e \in \mathbb{E}, k \in \mathbb{N} \text{ with } 1 \leq \sum_{i=1}^d k_i \leq 3.$$

For each $i \in \{1, \cdots, N\}$, $f^i(s, x_1, \cdots, x_N, y, z, k) \in C^{0,0,2}([0,T] \times \mathbb{R}^{(N-1)d} \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times L^2(\mathbb{E}, \mathcal{E}, \lambda; \mathbb{R}^n); \mathbb{R}^n)$ and

$$f^i(s, x_1, \cdots, x_i, \cdots, x_N, y, z, k) = f(s, x_1, \cdots, x_N, \cdots, x_i, y, z, k), \quad \forall x_i \in \mathbb{R}^d.$$

Moreover, for each $s \in [0,T]$, $f^i(s, x_1, \cdots, x_N, y, z, k) \in C^{0,2}_{l,\mathrm{lip}}(\mathbb{R}^{(N-1)d} \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times L^2(\mathbb{E}, \mathcal{E}, \lambda; \mathbb{R}^n); \mathbb{R}^n)$. Functions $\partial_y f^i(t,\cdot)$, $\partial_z f^i(t,\cdot)$, $\partial_k f^i(t,\cdot)$ are bounded Lipschitz functions, and so are their derivatives of order one with respect to x_N, y, z, k. Furthermore, all their Lipschitz coefficients are uniformly bounded. If for each $i \in \{1, \cdots, N\}$, $g^i(x_1, \cdots, x_N) \in C^{0,2}_{l,\mathrm{lip}}(\mathbb{R}^{(N-1)d} \times \mathbb{R}^d; \mathbb{R}^n)$ and

$$g^i(x_1, \cdots, x_i, \cdots, x_N) = g(x_1, \cdots, x_N, \cdots, x_i), \quad \forall x_i \in \mathbb{R}^d,$$

then all PDEs in (3.2) *have classical solutions.*

Note that when we say the (Frechet) derivative w.r.t k is bounded, we mean that its norm in $L^2(\mathbb{E}, \mathcal{E}, \lambda; \mathbb{R}^n)$ is bounded. In the sequel, we always assume that b, σ, β satisfy the above conditions.

Remark 3.1 From (3.3), the process Z is left continuous with right limit (LCRL for short). Since the Brownian motion is continuous, we can also define Z by $Z(t) := v^k(t, X_t^{(k)})\sigma(X(t))$ in (3.3).

4 Nonlinear Feynman-Kac Formula for Functional Form

In this section, we will generalize the nonlinear Feynman-Kac formula for the path-dependent situation. The following directional derivatives will be used frequently in the sequel.

Definition 4.1 *Suppose that Φ is an $\mathbb{R}^n$-valued function on Λ_T. Then Φ is said to be in $C^2(\Lambda_T;\mathbb{R}^n)$, if it is twice continuously Frechet differentiable at each $\gamma \in \Lambda_T$. Φ is said to be in $C^2_{l,\mathrm{lip}}(\Lambda_T;\mathbb{R}^n)$ if $\Phi \in C^2(\Lambda_T;\mathbb{R}^n)$ and there exist some constants $C \geq 0$ and $k \geq 0$ depending only on Φ such that for each $s \leq t \in [0,T]$, $\gamma, \overline{\gamma} \in \Lambda_T$,*

(i) $|\Phi'_x(\gamma)\mathbf{1}_{[s,t)}\| + \|\Phi''_{xx}(\gamma)\mathbf{1}_{[s,t)}\| \leq C(1 + \|\gamma\|^k + \|\overline{\gamma}\|^k)(t-s)$,

(ii) $|\Phi(\gamma) - \Phi(\overline{\gamma})| \leq C(1 + \|\gamma\|^k + \|\overline{\gamma}\|^k)\int_0^T |\gamma(s) - \overline{\gamma}(s)|\mathrm{d}s$,

(iii) $|\Psi(\gamma) - \Psi(\overline{\gamma})| \leq C(1 + \|\gamma\|^k + \|\overline{\gamma}\|^k)\|\gamma - \overline{\gamma}\|$

with $\Psi = \Phi'_x, \Phi''_{xx}$. Analogously, for each $t \in [0,T]$, we can define $C^2(\Lambda_t;\mathbb{R}^n)$, $C^2_{l,\mathrm{lip}}(\Lambda_t;\mathbb{R}^n)$, $C^1_{l,\mathrm{lip}}(\Lambda_t;\mathbb{R}^n)$, $C_{l,\mathrm{lip}}(\Lambda_t;\mathbb{R}^n)$.

Remark 4.1 Since Λ_T is a Banach space with respect to the uniform norm, for each γ, $\Phi'_x(\gamma)$ is a $\Lambda_T \mapsto \mathbb{R}^n$ bounded linear map, and $\Phi''_{xx}(\gamma)$ is a $\Lambda_T \times \Lambda_T \mapsto \mathbb{R}^n$ bounded linear map.

Example 4.1 If $\Phi(\gamma)=\int_0^T\varphi(\gamma(s))\,\mathrm{d}s$ for some $\varphi\in C^2_{b,\mathrm{lip}}(\mathbb{R})$, then $\Phi\in C^2_{l,\mathrm{lip}}(\Lambda_T;\mathbb{R})$.

In the rest of this paper, we shall make use of the following assumptions on the generator f and the terminal Φ of our BSDE.

(H1) Φ is an $\mathbb{R}^n$-valued function on Λ_T. Moreover, $\Phi\in C^2_{l,\mathrm{lip}}(\Lambda_T;\mathbb{R}^n)$ with the Lipschitz coefficients C and k.

(H2) $f(\gamma_t,y,z,k)$ is an $\mathbb{R}^n$-valued continuous function on $\Lambda\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}^n)$. For any $\gamma_t\in\Lambda$, $(y,z,k)\mapsto f(\gamma_t,y,z,k)$ is in $C^2_{l,\mathrm{lip}}(\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}^n);\mathbb{R}^n)$, $(y,z,k)\mapsto f'_y(\gamma_t,y,z,k)$, $f'_z(\gamma_t,y,z,k)$, $f'_k(\gamma_t,y,z,k)$ are in $C^1_{b,\mathrm{lip}}(\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}^n);\mathbb{R}^n)$, $(y,z,k)\mapsto f'_x(\gamma_t,y,z,k)$ is a Lispchitz function and $(y,z,k)\mapsto f''_{xx}(\gamma_t,y,z,k)$, $f''_{xy}(\gamma_t,y,z,k)$, $f''_{xz}(\gamma_t,y,z,k)$, $f''_{xk}(\gamma_t,y,z,k)$ are in $C_{l,\mathrm{lip}}(\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}^n);\mathbb{R}^n)$ for any (y,z,k), $\gamma_t\mapsto f(\gamma_t,y,z,k)$ is in $C^2_{l,\mathrm{lip}}(\Lambda_t;\mathbb{R}^n)$, $\gamma_t\mapsto f'_y(\gamma_t,y,z,k)$, $f'_z(\gamma_t,y,z,k)$, $f'_k(\gamma_t,y,z,k)$ are in $C^1_{l,\mathrm{lip}}(\Lambda_t;\mathbb{R}^n)$, $\gamma_t\mapsto f''_{yy}(\gamma_t,y,z,k)$, $f''_{zz}(\gamma_t,y,z,k)$, $f''_{kk}(\gamma_t,y,z,k)$, $f''_{yz}(\gamma_t,y,z,k)$, $f''_{yk}(\gamma_t,y,z,k)$, $f'_{zk}(\gamma_t,y,z,k)$ are in $C_{l,\mathrm{lip}}(\Lambda_t;\mathbb{R}^n)$. They are all continuous in t, and so are their derivatives. Moreover, all their Lipschitz coefficients are uniformly bounded.

(H3) $f(\gamma_t,y,z)=\overline{f}(t,\gamma_t(t),y,z,k)$, where

$$\overline{f}\in C^{0,2}([0,T]\times\mathbb{R}^d\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}^n);\mathbb{R}^n).$$

For each $t\in[0,T]$, $\overline{f}(t,\cdot)\in C^2_{l,\mathrm{lip}}(\mathbb{R}^d\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}^n);\mathbb{R}^n)$ and $\partial_y\overline{f}(t,\cdot)$, $\partial_z\overline{f}(t,\cdot)$, $\partial_k\overline{f}(t,\cdot)$ are in $C_{b,\mathrm{lip}}(\mathbb{R}^d\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2(\mathbb{E},\mathcal{E},\lambda;\mathbb{R}^n);\mathbb{R}^n)$, and so are their derivatives of order one with respect to x,y,z,k. Moreover, all their Lipschitz coefficients are uniformly bounded.

It is obvious that the assumption (H3) implies the assumption (H2).

Assume that (H1)–(H2) hold. For any $\gamma_t\in\Lambda$, $(Y_{\gamma_t}(s),Z_{\gamma_t}(s),K_{\gamma_t}(s))_{t\le s\le T}$ is the solution to the following BSDE:

$$\begin{aligned}Y_{\gamma_t}(s)=\Phi(X^{\gamma_t})&+\int_s^T f(X_r^{\gamma_t},Y_{\gamma_t}(r),Z_{\gamma_t}(r),K_{\gamma_t}(r))\mathrm{d}r\\&-\int_s^T Z_{\gamma_t}(r)\mathrm{d}B(r)-\int_s^T\int_{\mathbb{E}}K_{\gamma_t}(r,e)\widetilde{\mu}(\mathrm{d}r,\mathrm{d}e),\end{aligned}\tag{4.1}$$

where

$$X^{\gamma_t}(u):=\gamma_t(u)\mathbf{1}_{[0,t]}(u)+X^{t,\gamma_t(t)}(u)\mathbf{1}_{(t,T]}(u).$$

By Lemma 2.1, for each $\gamma_t\in\Lambda$, (4.1) has a unique solution $(Y_{\gamma_t},Z_{\gamma_t},K_{\gamma_t})\in\mathcal{B}^2$ and $Y_{\gamma_t}(t)$ defines a deterministic mapping from Λ to $\mathbb{R}^n$.

4.1 Property of solution to the BSDE with jumps

We next establish higher-order moment estimates for the solution of the BSDE (4.1). Without loss of generality, the Lipschitz coefficients of f are also denoted by C and k. For convenience, define $Y_{\gamma_t}(s),Z_{\gamma_t}(s),K_{\gamma_t}(s)$ for any $t,s\in[0,T],\gamma_t\in\Lambda$ by $Y_{\gamma_t}(s)=Y_{\gamma_t}(s\vee t)$, while $Z_{\gamma_t}(s)=0$ and $K_{\gamma_t}(s)=0$ for $s<t$.

From Lemma 2.3 and Proposition 3.5 in [21], we deduce the following theorem.

Theorem 4.1 *For any $p\ge 2$, there exist some constants $C_p>0$ and $q>0$ depending on C,T,k,p, such that for any $t,\overline{t}\in[0,T]$, $\gamma_t,\overline{\gamma}_{\overline{t}}\in\Lambda$, $h,\overline{h}\in\mathbb{R}\backslash\{0\}$,*

(i) $\|\Sigma_{\gamma_t}-\Sigma_{\overline{\gamma}_{\overline{t}}}\|^p_{\mathcal{B}^p}\le C_p(1+\|\gamma_t\|^q+\|\overline{\gamma}_{\overline{t}}\|^q)\big[|\overline{t}-t|^{\frac{p}{2}}+\big(\int_0^T|\gamma(t\wedge s)-\overline{\gamma}(\overline{t}\wedge s)|\mathrm{d}s\big)^p\big]$,

(ii) $\|\Delta_h^i\Sigma_{\gamma_t} - \Delta_{\overline{h}}^i\Sigma_{\overline{\gamma}_{\overline{t}}}\|_{\mathcal{B}^p}^p \le C_p(1 + \|\gamma_t\|^q + \|\overline{\gamma}_{\overline{t}}\|^q + |h|^q + |\overline{h}|^q)$
$\cdot(|h|^p\mathbf{1}_{t<\overline{t}} + |\overline{h}|^p\mathbf{1}_{\overline{t}<t} + |h-\overline{h}|^p + d_\infty^p(\gamma_t, \overline{\gamma}_{\overline{t}})),$

where $\Delta_h^i\Sigma_{\gamma_t}(s) = \frac{1}{h}(Y_{\gamma_t^{he_i}}(s) - Y_{\gamma_t}(s), Z_{\gamma_t^{he_i}}(s) - Z_{\gamma_t}(s), K_{\gamma_t^{he_i}}(s) - K_{\gamma_t}(s))$ *and* $(e_1, \cdots, e_d)$ *is an orthonormal basis of* $\mathbb{R}^d$.

Now we define

$$u(\gamma_t) := Y_{\gamma_t}(t) \quad \text{for } \gamma_t \in \Lambda. \tag{4.2}$$

Theorem 4.2 *For each* $\gamma_t \in \Lambda$, $\{Y_{\gamma_t^x}(s), s \in [0,T], x \in \mathbb{R}^d\}$ *has a version which is a.e. in* $C^{0,2}([0,T] \times \mathbb{R}^d)$. *In particular,* $D_x u(\gamma_t)$, $D_{xx}u(\gamma_t)$ *exist and* $u \in \mathbb{C}_{l,\text{lip}}^{0,2}(\Lambda)$.

Proof To simplify presentation, we shall only prove the case when $n = d = 1$, as the higher-dimensional case can be treated in the same way without substantial difficulty.

Since for each $h, \overline{h} \in \mathbb{R}\backslash\{0\}$ and $k, \overline{k} \in \mathbb{R}$,

$$\|\Sigma_{\gamma_t^k} - \Sigma_{\gamma_t^{\overline{k}}}\|_{\mathcal{B}^p}^p \le C_p(1 + \|\gamma_t\|^q + |k|^q + |\overline{k}|^q)|k-\overline{k}|^p,$$
$$\|\Delta_h\Sigma_{\gamma_t^k} - \Delta_{\overline{h}}\Sigma_{\gamma_t^{\overline{k}}}\|_{\mathcal{B}^p}^p \le C_p(1 + \|\gamma_t\|^q + |h|^q + |\overline{h}|^q + |k|^q + |\overline{k}|^q)(|h-\overline{h}|^p + |k-\overline{k}|^p),$$

using the Kolmogorov's criterion, the existence of a continuous derivative of $Y_{\gamma_t^x}(s)$ with respect to x follows from the above estimates, as so is the existence of mean-square derivatives of $Z_{\gamma_t^x}(s)$ and $K_{\gamma_t^x}(s, \cdot)$ with respect to x, which is mean square continuous in x. Denote them by $(D_xY_{\gamma_t}, D_xZ_{\gamma_t}, D_xK_{\gamma_t})$.

By the definition of vertical derivatives, $D_xu(\gamma_t)$ exists. We shall prove $u(\gamma_t)$ is Λ-continuous. Putting $s = t$ in the BSDE (4.1) and taking expectation, we get

$$u(\gamma_t) = E\Big[\Phi(X^{\gamma_t}) + \int_t^T f(X_r^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r), K_{\gamma_t}(r))\,\mathrm{d}r\Big].$$

For each $\gamma_t, \overline{\gamma}_{\overline{t}} \in \Lambda$ with $\overline{t} \ge t$,

$$\begin{aligned}
&|u(\gamma_t) - u(\overline{\gamma}_{\overline{t}})| \\
&\le E[|\Phi(X^{\gamma_t}) - \Phi(X^{\overline{\gamma}_{\overline{t}}})|] + E\Big[\int_t^{\overline{t}} |f(X_r^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r), K_{\gamma_t}(r))|\,\mathrm{d}r\Big] \\
&\quad + E\Big[\int_{\overline{t}}^T |f(X_r^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r), K_{\gamma_t}(r)) - f(X_r^{\overline{\gamma}_{\overline{t}}}, Y_{\overline{\gamma}_{\overline{t}}}(r), Z_{\overline{\gamma}_{\overline{t}}}(r), K_{\gamma_{\overline{t}}}(r))|\,\mathrm{d}r\Big] \\
&\le E\Big[C(1+T)(1 + \|X^{\gamma_t}\|^k + \big\|X^{\overline{\gamma}_{\overline{t}}}\big\|^k)\|X^{\gamma_t} - X^{\overline{\gamma}_{\overline{t}}}\| \\
&\quad + 4(\overline{t}-t)^{\frac{1}{2}}\Big(\int_t^{\overline{t}} (|f(X_r^{\gamma_t}, 0,0,0)|^2 + |CY_{\gamma_t}(r)|^2 + |CZ_{\gamma_t}(r)|^2 + \|CK_{\gamma_t}(r)\|_{\mathbb{E}}^2)\,\mathrm{d}r\Big)^{\frac{1}{2}} \\
&\quad + C\int_{\overline{t}}^T (|Y_{\gamma_t}(r) - Y_{\overline{\gamma}_{\overline{t}}}(r)| + |Z_{\gamma_t}(r) - Z_{\overline{\gamma}_{\overline{t}}}(r)| + \|K_{\gamma_t}(r) - K_{\overline{\gamma}_{\overline{t}}}(r)\|_{\mathbb{E}})\,\mathrm{d}r\Big],
\end{aligned}$$

where we have used the assumptions (H1)–(H2) in the last inequality. Applying Theorem 4.1, we can find some constant C_1 depending only on C, k and T so that

$$\begin{aligned}
|u(\gamma_t) - u(\overline{\gamma}_{\overline{t}})| &\le C_1(1 + \|\gamma_t\|^k + \|\overline{\gamma}_{\overline{t}}\|^k)\Big(|\overline{t}-t|^{\frac{1}{2}} + \int_0^T |\gamma(t\wedge s) - \overline{\gamma}(\overline{t}\wedge s)|\mathrm{d}s\Big) \\
&\le C_1(1 + \|\gamma_t\|^k + \|\overline{\gamma}_{\overline{t}}\|^k)d_\infty(\gamma_t, \overline{\gamma}_{\overline{t}}).
\end{aligned}$$

By a similar argument, we have $u \in \mathbb{C}^{0,1}_{l,\mathrm{lip}}(\Lambda)$.

By Lemma 4.1, we conclude that $\sup_s |Z_{\gamma_t}(s)|$ and $\sup_s \|K_{\gamma_t}(s)\|_{\mathbb{E}}$ are in $L^p(\mathcal{F}_T)$ for any $p > 0$. Since $(D_x Y_{\gamma_t}, D_x Z_{\gamma_t}, D_x K_{\gamma_t})$ is the solution to the following linearized BSDE:

$$\begin{aligned} D_x Y_{\gamma_t}(s) = \Phi'_x(X^{\gamma_t})\nabla X^{\gamma_t} + \int_s^T \big[& f'_y(X_r^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r), K_{\gamma_t}(r)) D_x Y_{\gamma_t}(r) \\ & + f'_z(X_r^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r), K_{\gamma_t}(r)) D_x Z_{\gamma_t}(r) \\ & + f'_k(X_r^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r), K_{\gamma_t}(r)) D_x K_{\gamma_t}(r) \\ & + f'_x(X_r^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r)) \nabla X_r^{\gamma_t} \big] \,\mathrm{d}r \\ & - \int_s^T D_x Z_{\gamma_t}(r)\,\mathrm{d}B(r) - \int_s^T \int_{\mathbb{E}} D_x K_{\gamma_t}(r,e) \widetilde{\mu}(\mathrm{d}r, \mathrm{d}e), \end{aligned}$$

the existence of a continuous second-order partial derivative of $Y_{\gamma_t^x}(s)$ with respect to x is proved in a similar fashion and this completes the proof.

4.2 Path regularity of processes Z and K

In [3], when the BSDE is the state-dependent case, i.e., $f = f(t, \gamma(t), y, z, k)$ and $\Phi = \varphi(\gamma(T))$, it is shown that Z, K and Y are connected in the following sense under appropriate assumptions:

$$\begin{aligned} Z_{\gamma_t}(s) &= \partial_x u(s, X^{\gamma_t}(s-))\sigma(X^{\gamma_t}(s-)), \\ K_{\gamma_t}(s) &= u(s, X^{\gamma_t}(s-) + \beta(X^{\gamma_t}(s-))) - u(s, X^{\gamma_t}(s-)). \end{aligned}$$

In this section, we extend this result to the path-dependent case. Indeed, we have below a formula relating Z, K with Y.

Theorem 4.3 *Under assumptions* (H1)–(H2)*, for each fixed $\gamma_t \in \Lambda$, the processes $(Z_{\gamma_t}(s), K_{\gamma_t}(s))_{s\in[t,T]}$ have the following a.s. left continuous version given by*

$$\begin{aligned} Z_{\gamma_t}(s) &= D_x u(X_{s-}^{\gamma_t})\sigma(X^{\gamma_t}(s-)), \\ K_{\gamma_t}(s) &= u((X_{s-}^{\gamma_t})^{\beta(X^{\gamma_t}(s-),\cdot)}) - u(X_{s-}^{\gamma_t}). \end{aligned} \tag{4.3}$$

A direct consequence of Theorem 4.3 is the following result.

Lemma 4.1 *For each $p \geq 2$, there exist some constants C_p and q depending only on C, T, k and p, such that*

$$Z_{\gamma_t}(s) \leq C_p(1 + \|X_s^{\gamma_t}\|^q), \quad \forall s \in [t,T], \ P\text{-}a.s.$$

and

$$E\Big[\sup_{s\in[t,T]} |Z_{\gamma_t}(s)|^p\Big] \leq C_p(1 + \|\gamma_t\|^q), \quad E\Big[\sup_{s\in[t,T]} \|K_{\gamma_t}(s)\|_{\mathbb{E}}^p\Big] \leq C_p(1 + \|\gamma_t\|^q).$$

Now we give the proof of Theorem 4.3.

Proof of Theorem 4.3 To simplify presentation, we shall only prove the case when $n = d = 1$, as the higher-dimensional case can be treated in the same way without substantial difficulty. We will suppress the superscript γ_t for notational convenience.

Step 1 For each $s \in [0,T)$ and a positive integer m, we introduce a mapping $\gamma^m(\overline{\gamma}_s)$: $\Lambda_s \mapsto \Lambda_s$:

$$\gamma^m(\overline{\gamma}_s)(r) = \overline{\gamma}_s(r)\mathbf{1}_{[0,t)}(r) + \sum_{k=0}^{m-1} \overline{\gamma}_s(t^m_{k+1} \wedge s)\mathbf{1}_{[t^m_k \wedge s, t^m_{k+1} \wedge s)}(r) + \overline{\gamma}_s(s)\mathbf{1}_{\{s\}}(r),$$

where $t^m_k = t + \frac{k(T-t)}{m}$, $k = 0, 1, \cdots, m$. Denote $u^m(\overline{\gamma}_s) := u(\gamma^m(\overline{\gamma}_s))$, and then there exists some constant C, such that

$$\begin{aligned} E\Big[\sup_{s\in[t,T]} |u^m(X_s) - u(X_s)|\Big] &\le CE\Big[\Big(1 + \sup_{s\in[t,T]} |X(s)|^k\Big) \sup_{s\in[t,T]} |X_s - \gamma^m(X_s)|\Big] \\ &\le C(1 + \|\gamma_t\|^k)\frac{1}{m^{\frac{1}{4}}}. \end{aligned}$$

By Lemma 2.3, we also have $E\big[\sup_{s\in[t,T]} |u^m(X_s) - Y(s)|\big] \le C(1 + \|\gamma_t\|^k)\frac{1}{m^{\frac{1}{4}}}$. Consequently, $u(X_s) = Y(s)$.

Step 2 Denote $\Phi^m(\overline{\gamma}) := \Phi(\gamma^m(\overline{\gamma}))$ and $f^m(\overline{\gamma}_s, y, z, k) := f(\gamma^m(\overline{\gamma}_s), y, z, k)$. Then for each m, there exist some functions φ_m defined on $\Lambda_t \times \mathbb{R}^{m\times d}$ and ψ_m defined on $[t,T] \times \Lambda_t \times \mathbb{R}^{m\times d} \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times L^2(\mathbb{E}, \mathcal{E}, \lambda; \mathbb{R})$, such that

$$\begin{aligned} \Phi^m(\overline{\gamma}) &= \varphi_m(\overline{\gamma}_t, \overline{\gamma}(t^m_1) - \overline{\gamma}(t), \cdots, \overline{\gamma}(t^m_m) - \overline{\gamma}(t^n_{m-1})), \\ f^m(\overline{\gamma}_s, y, z, k) &= \psi_m(s, \overline{\gamma}_t, \overline{\gamma}_s(t^m_1 \wedge s) - \overline{\gamma}_s(t), \cdots, \overline{\gamma}_s(t^m_m \wedge s) - \overline{\gamma}_s(t^m_{n-1} \wedge s), y, z, k). \end{aligned}$$

Indeed, set

$$\begin{aligned} \overline{\varphi}_m(\overline{\gamma}_t, x_1, \cdots, x_m) &:= \Phi\Big(\Big(\overline{\gamma}_t(s)\mathbf{1}_{[0,t)}(s) + \sum_{k=1}^m x_k \mathbf{1}_{[t^m_{k-1}, t^m_k)}(s) + x_m \mathbf{1}_{\{T\}}(s)\Big)_{0\le s\le T}\Big), \\ \varphi_m(\overline{\gamma}_t, x_1, \cdots, x_m) &:= \overline{\varphi}_m\Big(\overline{\gamma}_t, \overline{\gamma}_t(t) + x_1, \overline{\gamma}_t(t) + x_1 + x_2, \cdots, \overline{\gamma}_t(t) + \sum_{i=1}^m x_i\Big). \end{aligned}$$

Recalling the assumptions (H1)–(H2), we obtain that $\varphi^i_m(\overline{\omega}_t, x_1, \cdots, x_m) \in C^{0,2}_{l,\mathrm{lip}}(\mathbb{R}^{(m-1)d} \times \mathbb{R}^d; \mathbb{R}^n)$ for each fixed $\overline{\omega}_t$ and $i \in \{1, \cdots, m\}$, where

$$\varphi^i_m(\overline{\omega}_t, x_1, \cdots, x_i, \cdots, x_m) = \varphi_m(\overline{\omega}_t, x_1, \cdots, x_m, \cdots, x_i), \quad \forall x_i \in \mathbb{R}^d.$$

In particular,

$$\partial_{x_i}\varphi_m(\overline{\gamma_t}, \overline{\gamma}(t^m_1) - \overline{\gamma}(t), \cdots, \overline{\gamma}(t^m_m) - \overline{\gamma}(t^m_{m-1})) = \Phi'_x(\gamma^m(\overline{\gamma}))\mathbf{1}_{[t^m_{i-1}, T]} \quad \text{for each } \overline{\gamma} \in \Lambda.$$

Furthermore, for each fixed $\overline{\omega}_t$ and $i \in \{1, \cdots, m\}$, $\psi^i_m(s, \overline{\omega}_t, x_1, \cdots, x_m, y, z, k) \in C^{0,0,2}([t,T] \times \mathbb{R}^{(m-1)d} \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times L^2(\mathbb{E}, \mathcal{E}, \lambda; \mathbb{R}^n); \mathbb{R}^n)$, where

$$\psi^i_m(s, \overline{\omega}_t, x_1, \cdots, x_i, \cdots, x_m, y, z, k) = \psi_m(s, \overline{\omega}_t, x_1, \cdots, x_m, \cdots, x_i, y, z, k), \quad \forall x_i \in \mathbb{R}^d.$$

For each

$$s \in [t,T], \ \psi^i_m(s, \overline{\omega}_t, x_1, \cdots, x_n, y, z) \in C^{0,2}_{l,\mathrm{lip}}(\mathbb{R}^{(m-1)d} \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times L^2(\mathbb{E}, \mathcal{E}, \lambda; \mathbb{R}^n); \mathbb{R}^n).$$

Functions $\partial_y \psi^i_m(t,\cdot)$, $\partial_z \psi^i_m(t,\cdot)$, and $\partial_k \psi^i_m(t,\cdot)$ are bounded Lipschitz functions, and so are their derivatives of order one with respect to x_m, y, z, k.

Now consider the following BSDEs, for any $\overline{t} \geq t$, $\overline{\gamma}_{\overline{t}} \in \Lambda_{\overline{t}}$,

$$Y^{(m)}_{\overline{\gamma}_{\overline{t}}}(s) = \Phi^m(X^{\overline{\gamma}_{\overline{t}}}) + \int_s^T f^m(X^{\overline{\gamma}_{\overline{t}}}_r, Y^{(m)}_{\overline{\gamma}_{\overline{t}}}(r), Z^{(m)}_{\overline{\gamma}_{\overline{t}}}(r), K^{(m)}_{\overline{\gamma}_{\overline{t}}}(r))\,\mathrm{d}r \\ - \int_s^T Z^{(m)}_{\overline{\gamma}_{\overline{t}}}(r)\,\mathrm{d}B(r) - \int_s^T \int_{\mathbb{E}} K^{(m)}_{\overline{\gamma}_{\overline{t}}}(r,e)\widetilde{\mu}(\mathrm{d}r,\mathrm{d}e).$$

Denote $u^{(m)}(\overline{\gamma}_{\overline{t}}) := Y^{(m)}_{\overline{\gamma}_{\overline{t}}}(\overline{t})$ for each $\overline{\gamma}_{\overline{t}} \in \Lambda$. By Theorems 3.1, 4.2 and Lemma 3.1 (see also Lemma 3.10 in [21]), we can get $u^{(m)}(\gamma_t) \in \mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$. Moreover, for each $s \in [t,T]$, $\overline{\gamma}_s \in \Lambda$, and $l \in \{1,\cdots,n\}$,

$$D_t u^{(m)}_l(\overline{\gamma}_s) + \mathcal{L}u^{(m)}_l(\overline{\gamma}_s) + f_l(\overline{\gamma}_s, u^{(m)}(\overline{\gamma}_s), D_x u^{(m)}(\overline{\gamma}_s)\sigma(\overline{\gamma}_s(s)), u^{(m)}(\overline{\gamma}_s^{\beta(\overline{\gamma}_s(s),e)}) \\ - u^{(m)}(\overline{\gamma}_s)) = 0. \tag{4.4}$$

In particular, for each $s \in [t,T]$,

$$D_x u^{(m)}(X_{s-})\sigma(X(s-)) = Z^{(m)}(s) \quad a.s.$$

Denote by C_0 a constant that depends only on C, T and k, which is allowed to change from line to line. From Lemma 2.3, for each $\overline{\gamma}_{\overline{t}} \in \Lambda$,

$$\begin{aligned} &|u^{(m)}(\overline{\gamma}_{\overline{t}}) - u(\overline{\gamma}_{\overline{t}})| \\ &\leq C_0 E\Big[|\Phi^m(X^{\overline{\gamma}_{\overline{t}}}) - \Phi(X^{\overline{\gamma}_{\overline{t}}})|^2 \\ &\quad + \int_0^T |f(X^{\overline{\gamma}_{\overline{t}}}_r, Y^{(m)}_{\overline{\gamma}_{\overline{t}}}(r), Z^{(m)}_{\overline{\gamma}_{\overline{t}}}(r), K^{(m)}_{\overline{\gamma}_{\overline{t}}}(r)) - f^m(X^{\overline{\gamma}_{\overline{t}}}_r, Y^{(m)}_{\overline{\gamma}_{\overline{t}}}(r), Z^{(m)}_{\overline{\gamma}_{\overline{t}}}(r), K^{(m)}_{\overline{\gamma}_{\overline{t}}}(r))|^2\,\mathrm{d}r\Big]^{\frac{1}{2}} \\ &\leq C_0 E\Big[\Big(1 + \|\overline{\gamma}_{\overline{t}}\|^k + \sup_s |X^{\overline{\gamma}_{\overline{t}}}(s)|^k\Big)^2 \|\gamma^m(X^{\overline{\gamma}_{\overline{t}}}) - X^{\overline{\gamma}_{\overline{t}}}\|^2\Big]^{\frac{1}{2}} \\ &\leq C_0(1 + \|\overline{\gamma}_{\overline{t}}\|^k)\Big(E\Big[\sup_s |X^{\overline{\gamma}_{\overline{t}}}(s) - \sum_{k=1}^{m-1} X^{\overline{\gamma}_{\overline{t}}}(t^m_{k+1})\mathbf{1}_{[t^m_k, t^m_{k+1})}(s)|^4\Big] + \|\gamma^m(\overline{\gamma}_{\overline{t}}) - \overline{\gamma}_{\overline{t}}\|^4\Big)^{\frac{1}{4}} \\ &\leq C_0(1 + \|\overline{\gamma}_{\overline{t}}\|^k)\Big(\frac{1}{m^{\frac{1}{4}}} + \|\gamma^m(\overline{\gamma}_{\overline{t}}) - \overline{\gamma}_{\overline{t}}\|\Big). \end{aligned}$$

Moreover, we also deduce that

$$|D_x u^{(m)}(\overline{\gamma}_{\overline{t}}) - D_x u(\overline{\gamma}_{\overline{t}})| \leq C_0(1 + \|\overline{\gamma}_{\overline{t}}\|^k)\Big(\frac{1}{m^{\frac{1}{4}}} + \|\gamma^m(\overline{\gamma}_{\overline{t}}) - \overline{\gamma}_{\overline{t}}\|\Big). \tag{4.5}$$

Thus, it holds that

$$\begin{aligned} &\lim_m E\Big[\sup_{s\in[t,T]} |D_x u^{(m)}(X_{s-}) - D_x u(X_{s-})|^p\Big] \\ &\leq C_0 \lim_m E\Big[\sup_{s\in[t,T]} \Big|(1 + \|X_s\|^k)\Big(\frac{1}{m^{\frac{1}{4}}} + \|\gamma^m(X_s) - X_s\|\Big)\Big|^p\Big] \\ &= 0. \end{aligned}$$

By Lemma 2.3, $\lim_m E\big[|\int_t^T |Z(u)-Z^{(m)}(u)|^2\,\mathrm{d}u|^{\frac{p}{2}}\big]=0$, which implies that

$$D_x u(X_{s-})\sigma(X(s-)) = Z(s), \quad \mathrm{d}s\times \mathrm{d}P\text{-}a.e. \text{ on } [t,T].$$

Step 3 From BSDE (4.1), we see that the compensated processes of jumps of Y are given by

$$\widetilde{\sum_{t<r\le s}\Delta Y(r)} = \int_t^s\int_{\mathbb{E}} K(r,e)\widetilde{\mu}(\mathrm{d}r,\mathrm{d}e).$$

Since $Y(s)=u(X_s)$, we also have the representation

$$\widetilde{\sum_{t<r\le s}\Delta Y(r)} = \int_t^s\int_{\mathbb{E}} (u(X_{r-}^{\beta(X(r-),e)}) - u(X_{r-}))\widetilde{\mu}(\mathrm{d}r,\mathrm{d}e). \tag{4.6}$$

We claim that the function $t\mapsto u(\gamma_{t-}^x)$ is left continuous for each $\gamma_t\in\Lambda$ and $x\in\mathbb{R}^d$. Recalling the proof of Theorem 4.2, for each $\gamma_t\in\Lambda$ and $x\in\mathbb{R}^d$, there exists some constant C_1 depending on γ_t, x, such that

$$\lim_{t_n\uparrow t}|u(\gamma_{t-}^x)-u(\gamma_{t_n-}^x)| \le C_1\lim_{t_n\uparrow t}\Big[|t_n-t|^{\frac{1}{2}}+\int_{t_n}^t|\gamma(t_n-)+x-\gamma(s)|\mathrm{d}s+\int_t^T|\gamma(t_n-)-\gamma(t-)|\mathrm{d}s\Big]=0.$$

Thus the integrand in (4.6) is predictable, and from the uniqueness of the integral representation, we obtain the desired result.

4.3 Path-dependent parabolic integro-differential equations

We now establish the relationship between our BSDE and the following path-dependent parabolic integro-differential equation:

$$\begin{cases}\forall l\in\{1,\cdots,n\},\ \gamma_t\in\Lambda,\ t\in[0,T),\\ D_t u_l(\gamma_t)+\mathcal{L}u_l(\gamma_t)+f_l(\gamma_t,u(\gamma_t),D_x u(\gamma_t)\sigma(\gamma_t(t)),u(\gamma_t^{\beta(\gamma_t(t),e)})-u(\gamma_t))=0,\\ u(\gamma_T)=\Phi(\gamma_T),\quad \gamma_T\in\Lambda_T,\end{cases} \tag{4.7}$$

where $u=(u_1,\cdots,u_n):\Lambda\mapsto\mathbb{R}^n$ is a function on Λ and

$$\begin{aligned}\mathcal{L}u_l(\gamma_t) := {} & D_x u_l(\gamma_t) b(\gamma_t(t)) + \frac{1}{2}\mathrm{tr}[(\sigma\sigma^{\mathrm{T}})(\gamma_t(t))D_{xx}u_l(\gamma_t)]\\ & + \int_{\mathbb{E}}[u_l(\gamma_t^{\beta(\gamma_t(t),e)})-u_l(\gamma_t)-D_x u_l(\gamma_t)\beta(\gamma_t(t),e)]\lambda(\mathrm{d}e).\end{aligned}$$

We immediately obtain the following theorem.

Theorem 4.4 *Assume that assumptions* (H1)–(H2) *hold, and let* $u\in\mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$ *be a solution to* (4.7). *Then for each* $\gamma_t\in\Lambda$*, we have* $u(\gamma_t)=Y_{\gamma_t}(t)$*, where* $(Y_{\gamma_t}(s),Z_{\gamma_t}(s),K_{\gamma_t}(s))_{t\le s\le T}$ *is the unique solution to the BSDE* (4.1). *Consequently, the path-dependent PDE* (4.7) *has at most one* $\mathbb{C}^{1,2}_{l,\mathrm{lip}}$*-solution.*

Proof We again suppress the superscript γ_t for notational convenience. Applying the functional Itô's formula (2.2) to $u(X_s)$ on $s \in [t,T)$, for each $l \in \{1, \cdots, n\}$,

$$\begin{aligned} \mathrm{d}u_l(X_s) &= (D_s u_l(X_{s-}) + \frac{1}{2}\mathrm{tr}[\sigma\sigma^{\mathrm{T}} D_{xx} u_l(X_{s-})])\mathrm{d}s + D_x u_l(X_{s-})\mathrm{d}X(s) \\ &\quad + \int_{\mathbb{E}} [u_l(X_{s-}^{\beta(X(s-),e)}) - u_l(X_{s-}) - D_x u_l(X_{s-})\beta(X(s-),e)]\mu(\mathrm{d}s,\mathrm{d}e) \\ &= (D_s u_l(X_{s-}) + D_x u_l(X_{s-}) b(X(s-)) + \frac{1}{2}\mathrm{tr}[(\sigma\sigma^{\mathrm{T}})(X(s-)) D_{xx} u_l(X_{s-})])\mathrm{d}s \\ &\quad + D_x u_l(X_{s-})\sigma(X(s-))\mathrm{d}B(s) + \int_{\mathbb{E}} [u_l(X_{s-}^{\beta(X(s-),e)}) - u_l(X_{s-})]\widetilde{\mu}(\mathrm{d}s,\mathrm{d}e) \\ &\quad + \int_{\mathbb{E}} [u_l(X_{s-}^{\beta(X(s-),e)}) - u_l(X_{s-}) - D_x u_l(X_{s-})\beta(X(s-),e)]\lambda(\mathrm{d}e)\mathrm{d}s. \end{aligned}$$

Since u solves PDE (4.7), we have that

$$\begin{aligned} -\mathrm{d}u_l(X_s) &= f_l(X_s, u(X_s), D_x u(X_{s-})\sigma(X(s-)), u(X_{s-}^{\beta(X(s-),e)}) - u(X_{s-}))\mathrm{d}s \\ &\quad - D_x u_l(X_{s-})\sigma(X(s-))\mathrm{d}B(s) - \int_{\mathbb{E}} [u_l(X_{s-}^{\beta(X(s-),e)}) - u_l(X_{s-})]\widetilde{\mu}(\mathrm{d}t,\mathrm{d}e). \end{aligned}$$

Recalling $u(X_T) = \Phi(X_T)$ and $u \in \mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$, $(Y(s), Z(s), K(s)) = (u(X_s), D_x u(X_{s-})\sigma(X(s-)),$ $u(X_{s-}^{\beta(X(s-),e)}) - u(X_{s-}))$ is the unique solution to the BSDE (4.1). In particular, $u(\gamma_t) = Y_{\gamma_t}(t)$. This completes the proof.

By Theorem 4.4 and Lemma 2.2, we have the following comparison theorem of path-dependent PIDE.

Lemma 4.2 *Let $n = 1$. We assume that $f = f_i$, $\Phi = \Phi_i$, $i = 1, 2$ satisfy the same assumptions as in Lemma* 2.2 *and Theorem* 4.4*. Moreover,*

(i) $f_1(\gamma_t, y, z, k) \le f_2(\gamma_t, y, z, k)$, *for each* $(\gamma_t, y, z, k) \in \Lambda \times \mathbb{R} \times \mathbb{R}^d \times L^2(\mathbb{E}, \mathcal{E}, \lambda; \mathbb{R})$;

(ii) $\Phi_1(\gamma_T) \le \Phi_2(\gamma_T)$ *for each* $\gamma_T \in \Lambda_T$.

If $u_i \in \mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$ is the solution to (4.7) *associated with* $(f, \Phi) = (f_i, \Phi_i)$, $i = 1, 2$, *respectively, then for each $\gamma_t \in \Lambda$, $u_1(\gamma_t) \le u_2(\gamma_t)$.*

We are now in a position to prove the converse to the above result.

Theorem 4.5 *Under assumptions* (H1)–(H2)*, the function u defined in* (4.2) *is the unique* $\mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$*-solution to the path-dependent PIDE* (4.7).

Proof Let $\delta > 0$ be such that $t + \delta \le T$. We again suppress the superscript γ_t for notational convenience. Hence

$$u(\gamma_{t,t+\delta}) - u(\gamma_t) = u(\gamma_{t,t+\delta}) - u(X_{t+\delta}) + u(X_{t+\delta}) - u(\gamma_t). \tag{4.8}$$

By (4.8) and the proof of Theorem 4.3, we obtain a.s. (choosing a subsequence if necessary)

$$\begin{aligned} u(\gamma_{t,t+\delta}) - u(\gamma_t) &= \lim_{n\to\infty} [u^{(m)}(\gamma_{t,t+\delta}) - u^{(m)}(X_{t+\delta})] - \int_t^{t+\delta} f(X_s, Y(s), Z(s), K(s))\,\mathrm{d}s \\ &\quad + \int_t^{t+\delta} Z(s)\,\mathrm{d}B(s) + \int_t^{t+\delta}\int_{\mathbb{E}} K(s)\widetilde{\mu}(\mathrm{d}t,\mathrm{d}e). \end{aligned}$$

Moreover, for each $\overline{\gamma}_{\overline{t}} \in \Lambda^d$ with $\overline{t} \geq t$,

$$|D_{xx}u^{(m)}(\overline{\gamma}_{\overline{t}}) - D_{xx}u(\overline{\gamma}_{\overline{t}})| \leq C_0(1 + \|\overline{\gamma}_{\overline{t}}\|^k)\Big(\frac{1}{m^{\frac{1}{4}}} + \|\gamma^m(\overline{\gamma}_{\overline{t}}) - \overline{\gamma}_{\overline{t}}\|\Big). \tag{4.9}$$

Now applying the Itô's formula, we deduce that

$$\begin{aligned}
& u_l^{(m)}(\gamma_{t,t+\delta}) - u_l^{(m)}(X_{t+\delta}) \\
= & \int_t^{t+\delta} D_s u_l^{(m)}(\gamma_{t,s})\,\mathrm{d}s - \int_t^{t+\delta} D_s u_l^{(m)}(X_s)\,\mathrm{d}s - \int_t^{t+\delta} D_x u_l^{(m)}(X_s)\sigma(X(s))\,\mathrm{d}B(s) \\
& - \int_t^{t+\delta} \mathcal{L}u_l^{(m)}(X_s)\,\mathrm{d}s - \int_t^{t+\delta}\int_{\mathbb{E}} [u_l^{(m)}(X_{s-}^{\beta(X(s-),e)}) - u_l^{(m)}(X_{s-})]\widetilde{\mu}(\mathrm{d}t,\mathrm{d}e).
\end{aligned}$$

Thus by (4.5), (4.9) and the dominated convergence theorem, we have

$$\begin{aligned}
& u_l(\gamma_{t,t+\delta}) - u_l(\gamma_t) \\
= & -\int_t^{t+\delta} D_x u_l(X_s)\sigma(X(s))\,\mathrm{d}B(s) - \int_t^{t+\delta} [\mathcal{L}u_l(X_s) + f_l(X_s, Y(s), Z(s), K(s))]\,\mathrm{d}s \\
& - \int_t^{t+\delta}\int_{\mathbb{E}} [u_l(X_{s-}^{\beta(X(s-),e)}) - u_l(X_{s-})]\widetilde{\mu}(\mathrm{d}t,\mathrm{d}e) + \int_t^{t+\delta} Z_l(s)\,\mathrm{d}B(s) \\
& + \int_t^{t+\delta}\int_{\mathbb{E}} K_l(s,e)\widetilde{\mu}(\mathrm{d}t,\mathrm{d}e) + \lim_{m\to\infty} C_l^m,
\end{aligned} \tag{4.10}$$

where

$$C_l^m = \int_t^{t+\delta} D_s u_l^{(m)}(\gamma_{t,s})\mathrm{d}s - \int_t^{t+\delta} D_s u_l^{(m)}(X_s)\,\mathrm{d}s.$$

Recalling (4.4), we can find some constant c depending only on C, T, γ_t and k so that

$$|D_s u^{(m)}(\gamma_{t,s}) - D_s u^{(m)}(X_s)| \leq c\Big(1 + \sup_{u\in[t,s]} |X(u)|^k\Big)\|\gamma_{t,s} - X_s\|.$$

Hence

$$|C_l^m| \leq c\delta \sup_{s\in[t,t+\delta]} (1 + |X(s) - X(t)|^k)|X(s) - \gamma_t(t)|.$$

Finally, taking expectation on both sides of (4.10) yields

$$\lim_{\delta\to 0} \frac{u_l(\gamma_{t,t+\delta}) - u_l(\gamma_t)}{\delta} = -\mathcal{L}u_l(\gamma_t) - f_l(\gamma_t, u(\gamma_t), D_x u(\gamma_t)\sigma(\gamma_t(t)), u(\gamma_t^{\beta(\gamma_t(t),e)}) - u(\gamma_t)).$$

Thus $u \in \mathbb{C}^{1,2}_{l,\mathrm{lip}}(\Lambda)$ and it satisfies (4.7).

Remark 4.2 We make assumptions (H1)–(H2). Then

$$(u(X_t), D_x u(X_t)\sigma(X(t)), u(X_{t-}^{\beta(X(t-),e)}) - u(X_{t-}))$$

is the unique solution to the BSDE (2.4).

Remark 4.3 In the case that $\Phi(\gamma) = \varphi(\gamma(T))$ for some $\varphi \in C^2_{l,\mathrm{lip}}(\mathbb{R}^d; \mathbb{R}^n)$ and f satisfies (H3), the above result is the nonlinear Feynman-Kac formula, which is given by Buckdahn-Pardoux [3].

Example 4.2 Suppose $n = d = 1$ and

$$f(t, y, z, k) = c(t)y,$$

where f satisfies (H3). In this case, the BSDE (4.1) has the explicit solution as follows:

$$Y_{\gamma_t}(s) = \Phi(X^{\gamma_t})e^{\int_s^T c(r)\,\mathrm{d}r} - \int_s^T e^{\int_s^u c(r)\mathrm{d}r} Z_{\gamma_t}(u)\,\mathrm{d}B(u) - \int_t^T \int_{\mathbb{E}} e^{\int_s^u c(r)\mathrm{d}r} K_{\gamma_t}(u)\widetilde{\mu}(\mathrm{d}u, \mathrm{d}e).$$

Given $\Phi : \Lambda_T \mapsto \mathbb{R}$,

$$\Phi(\gamma) = \int_0^T \varphi(\gamma(s))\,\mathrm{d}s$$

for some $\varphi \in C^2_{b,\mathrm{lip}}(\mathbb{R})$, then for each $\gamma_t \in \Lambda_t$,

$$u(\gamma_t) = \int_0^t \varphi(\gamma_t(s))\,\mathrm{d}s e^{\int_t^T c(r)\,\mathrm{d}r} + \int_t^T e^{\int_t^T c(r)\,\mathrm{d}r} E[\varphi(X^{t,\gamma_t(t)}(s))]\,\mathrm{d}s.$$

Using the classic Feynman-Kac formula, we deduce that

$$u^s(t, x) = E[\varphi(X^{t,x}(s))]$$

is the solution to the following parabolic integro-differential equation:

$$\begin{cases} \partial_t u^s(t, x) + \mathcal{L}u^s(t, x) = 0, & t \in [0, s), \\ u^s(s, x) = \varphi(x). \end{cases}$$

Thus

$$u(\gamma_t) = \int_0^t \varphi(\gamma_t(s))\,\mathrm{d}s e^{\int_t^T c(r)\,\mathrm{d}r} + \int_t^T e^{\int_t^T c(r)\,\mathrm{d}r} u^s(t, \gamma_t(t))\,\mathrm{d}s.$$

By the definitions of horizontal derivatives and vertical derivatives, we have

$$\begin{aligned} D_t u(\gamma_t) &= -c(t)u(\gamma_t) + e^{\int_t^T c(r)\,\mathrm{d}r} \int_t^T \partial_t u^s(t, \gamma_t(t))\,\mathrm{d}s, \\ D_x u(\gamma_t) &= e^{\int_t^T c(r)\,\mathrm{d}r} \int_t^T \partial_x u^s(t, \gamma_t(t))\,\mathrm{d}s, \\ D_{xx} u(\gamma_t) &= e^{\int_t^T c(r)\,\mathrm{d}r} \int_t^T \partial^2_{xx} u^s(t, \gamma_t(t))\,\mathrm{d}s. \end{aligned}$$

Consequently,

$$D_t u(\gamma_t) + \mathcal{L}u(\gamma_t) = -c(t)u(\gamma_t),$$

which satisfies (4.7).

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