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Finite p-Groups all of Whose Maximal Subgroups Either are Metacyclic or Have a Derived Subgroup of Order $\leq p$ *

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Abstract The groups as mentioned in the title are classified up to isomorphism. This is an answer to a question proposed by Berkovich and Janko.

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1 Introduction

A finite nonabelian p-group is said to be minimal nonabelian if its every proper subgroup is abelian. As numerous results in [1–3] show, the structure of a p-group depends essentially on its minimal nonabelian subgroups. Minimal nonabelian p-groups were classified by L. Rédei [4]. More general groups, D_1-groups, than minimal nonabelian p-groups were introduced and characterized in [5]. A finite p-group G is called D_1-group if the order of the derived subgroup of every maximal subgroup of G is at most p. On the other hand, nonmetacyclic p-groups all of whose maximal subgroups are metacyclic were classified in [6]. A natural question is: What can be said about finite p-groups all of whose maximal subgroups either are metacyclic or have a derived subgroup of order $\leq p$? This is also a question proposed by Berkovich and Janko in their joint book.

Problem 725 (cf. [2]) Study the p-groups all of whose maximal subgroups either are metacyclic or have a derived subgroup of order $\leq p$.

For convenience, the groups studied in Problem 725 is called $\mathcal{P}$-groups. Since metacyclic p-groups have been studied and classified in [7–8], respectively, hence assume that $\mathcal{P}$-groups are nonmetacyclic in this paper. We classify $\mathcal{P}$-groups up to isomorphism.

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2 Preliminaries

Let G be a finite p-group. We use $c(G)$, $\exp(G)$ and $d(G)$ to denote the nilpotency class, the exponent and the minimal number of generators of G respectively. For any positive integer s, $\Omega_s(G) = \langle a \in G \mid a^{p^s} = 1\rangle$ and $\mho_s(G) = \langle a^{p^s} \mid a \in G\rangle$. Let

$$G > G' = G_2 > G_3 > \cdots > G_{c+1} = 1$$

denote the lower central series of G, where $c = c(G)$, and G_n denote the nth term of the lower central series of a group G. We use C_{p^m}, $C_{p^m}^n$ and $H * K$ to denote the cyclic group of order p^m, the direct product of n cyclic groups of order p^m, and a central product of H and K respectively. $H \lessdot G$ denotes that H is a maximal subgroup of G. Other notations and symbols refer to [9].

Lemma 2.1 (cf. [10, Lemma 2.2]) *Assume that G is a finite nonabelian p-group. Then the following conditions are equivalent*:

(1) *G is minimal nonabelian*;

(2) *$d(G) = 2$ and $|G'| = p$*;

(3) *$d(G) = 2$ and $\Phi(G) = Z(G)$.*

Lemma 2.2 (cf. [2, Lemma 65.1]) *Assume that G is a minimal nonabelian p-group. Then G is one of the following groups*:

(1) Q_8;

(2) $\mathrm{M}_p(n, m) := \langle a, b \mid a^{p^n} = b^{p^m} = 1,\ a^b = a^{1+p^{n-1}}\rangle$, $n \geq 2$, $m \geq 1$ (*metacyclic*);

(3) $\mathrm{M}_p(n, m, 1) := \langle a, b, c \mid a^{p^n} = b^{p^m} = c^p = 1,\ [a, b] = c,\ [c, a] = [c, b] = 1\rangle$, $n \geq m \geq 1$ (*nonmetacyclic*).

Lemma 2.3 (cf. [6]) *Assume that G is a minimal nonmetacyclic p-group. Then G is isomorphic to one of the following groups*:

(1) *an elementary abelian p-group of order p^3*;

(2) *$p > 2$, a nonabelian p-group of order p^3 and $\exp(G) = p$*;

(3) *a group of order 3^4 with $c(G) = 3$: $\langle a, b, c \mid b^9 = c^3 = 1,\ [c, b] = 1,\ a^3 = b^{-3},\ [b, a] = c,\ [c, a] = b^{-3}\rangle$*;

(4) *$|G| = 16$ and $G \cong Q_8 \times C_2$ or $G \cong Q_8 * C_4$ and $G \cong \langle a, b, c \mid a^4 = 1,\ a^2 = b^2 = c^2,\ [a, b] = a^2,\ [c, a] = [c, b] = 1\rangle$*;

(5) *$|G| = 32$ and $G \cong \langle a, b, c \mid a^4 = b^4 = 1,\ c^2 = a^2b^2,\ [a, b] = b^2,\ [c, a] = a^2,\ [c, b] = 1\rangle$.*

Lemma 2.4 (cf. [6]) *A finite p-group G is metacyclic if and only if $G/\Phi(G')G_3$ is metacyclic.*

Lemma 2.5 (cf. [11, Theorem 5.12]) *Let G be a p-group.*

(1) *If $G' \cap Z_2(G)$ is cyclic, then G' is cyclic*;

(2) *If $Z(G')$ is cyclic, then G' is cyclic*;

(3) *If $Z(\Phi(G))$ is cyclic, then $\Phi(G)$ is cyclic.*

Lemma 2.6 (cf. [1, Lemma 1.1]) *If a nonabelian p-group G has an abelian maximal subgroup, then $|G| = p|G'||Z(G)|$.*

Lemma 2.7 (cf. [9, Theorem 2.4.4]) *Let p be an odd prime. Then a finite p-group G is metacyclic if and only if $\omega(G) \le 2$.*

Lemma 2.8 (cf. [9, Theorem 2.4.3]) *A two-generator 2-group G is metacyclic if and only if $d(M) \le 2$ for all maximal subgroups M of G.*

Lemma 2.9 *If G is a finite p-group with $d(G) = 2$, then $\Phi(G')G_3 \lessdot G'$.*

Proof Since $d(G) = 2$, G'/G_3 is cyclic. It follow that $G'/\Phi(G')G_3$ is cyclic. Since $\exp(G'/\Phi(G')G_3) = p$, $|G'/\Phi(G')G_3| = p$. So $\Phi(G')G_3 \lessdot G'$.

Lemma 2.10 *Let G be a p-group, $N < G'$ and $N \lhd G$. If G/N is metacyclic, then G is metacyclic.*

Proof Let $M \lessdot G'$, $N \le M$ and $M \trianglelefteq G$. Then $|(G/M)'| = |G'/M| = p$. It follows that $\Phi((G/M)') = (G/M)_3 = 1$. Thus $\Phi(G')G_3 \le M$. It follows by Lemma 2.9 that $M = \Phi(G')G_3$. Since G/N is metacyclic and $N \le M$, G/M is metacyclic. It follows from Lemma 2.4 that G is metacyclic.

Lemma 2.11 *Let G be a metacyclic p-group. Then the following conclusions are true*:
(1) *$d(G) \le 2$ and G' is cyclic.*
(2) *$G' \le \mho_1(G)$.*
(3) *If $H \le G$, then H is metacyclic.*
(4) *If $N \trianglelefteq G$, then G/N is metacyclic.*

Proof (1) is obvious.

(2) If G is abelian, then $G' \le \mho_1(G)$. If G is not abelian, then there exists $\langle a\rangle \trianglelefteq G$ such that $G/\langle a\rangle$ is cyclic. Thus $G' \le \langle a\rangle$. Since G/G' is not cyclic, $G' < \langle a\rangle$. Thus $G' \le \langle a^p\rangle \le \mho_1(G)$.

(3) Since G is metacyclic, there exists $\langle a\rangle \trianglelefteq G$ such that $G/\langle a\rangle$ is cyclic. Since $H \cap \langle a\rangle$ char $\langle a\rangle \trianglelefteq G$, $H \cap \langle a\rangle \trianglelefteq G$. Since $H/H \cap \langle a\rangle \cong H\langle a\rangle/\langle a\rangle \le G/\langle a\rangle$, $H/H \cap \langle a\rangle$ is cyclic. So H is metacyclic.

(4) Since G is metacyclic, there exists $\langle a\rangle \trianglelefteq G$ such that $G/\langle a\rangle$ is cyclic. Since $G/N\langle a\rangle \cong G/\langle a\rangle/N\langle a\rangle/\langle a\rangle$, $G/N\langle a\rangle$ is cyclic. Since $N\langle a\rangle/N \cong \langle a\rangle/N \cap \langle a\rangle$, $N\langle a\rangle/N$ is cyclic. Since $G/N/N\langle a\rangle/N \cong G/N\langle a\rangle$, G/N is metacyclic.

Lemma 2.12 *Let G be a p-group with $|G'| = p^3$. Then G' is abelian.*

Proof If G' not abelian, then $|Z(G')| = p$. By Lemma 2.5(2), G' is cyclic, a contradiction.

Lemma 2.13 *Assume that G is a finite p-group, and M_1, M_2 are two distinct maximal subgroups of G. Then $|G'| \le p|M_1'M_2'|$.*

Proof Since M_i' char $M_i \trianglelefteq G$, $M_i' \trianglelefteq G$ for $i = 1, 2$. Let $\overline{G} = G/M_1'M_2'$. Then $\overline{M}_1, \overline{M}_2 \lessdot \overline{G}$, and $\overline{M}_1$ and $\overline{M}_2$ are abelian. If $\overline{G}$ is abelian, then $|\overline{G}'| = 1$. If $\overline{G}$ is not abelian, then $Z(\overline{G}) = \overline{M}_1 \cap \overline{M}_2$. Since $\overline{M}_1 \cap \overline{M}_2$ is the second maximal subgroup of $\overline{G}$, $p|\overline{G}'| = |\overline{G}/Z(\overline{G})| = p^2$ by Lemma 2.6. Thus $|\overline{G}'| = p$.

Lemma 2.14 *Let G be a $\mathcal{P}$-group. Then*
(1) *if $H \le G$, then H is a $\mathcal{P}$-group*;
(2) *if $N \trianglelefteq G$, then G/N is a $\mathcal{P}$-group.*

Proof (1) Let $K \lessdot H$. Then there exists $M \lessdot G$ such that $K \leq M$. If $|M'| \leq p$, then $K' \leq p$. If $|M'| > p$, then M is metacyclic since G is a $\mathcal{P}$-group. By Lemma 2.11(3), K is metacyclic. So H is a $\mathcal{P}$-group.

(2) Let $M/N \lessdot G/N$. Then $M \lessdot G$. If $|M'| \leq p$, then $|(M/N)'| \leq p$. If $|M'| > p$, then M is metacyclic since G is a $\mathcal{P}$-group. By Lemma 2.11(4), M/N is metacyclic.

Lemma 2.15 *If a finite p-group G has at least one metacyclic maximal subgroup, then* $\Phi(G)$ *is metacyclic.*

Proof It is straightforward by Lemma 2.11.

Lemma 2.16 *If G is a finite p-group with* $\exp(G') = p$, *then* $c(G) = 2$ *if and only if* $\Phi(G) \leq Z(G)$.

Proof $\Leftarrow$. It is obvious.

$\Rightarrow$. Since $c(G) = 2$, $G' \leq Z(G)$. Since $\exp(G') = p$, $[x^p, y] = [x, y]^p = 1$ for all $x, y \in G$. It follows that $\mho_1(G) \leq Z(G)$. That is, $\Phi(G) = G'\mho_1(G) \leq Z(G)$.

Lemma 2.17 *Assume that G is a finite p-group. If* $G' \cong C_p^3$, *then* $d(G) = 2$ *if and only if* $G/\Phi(G')G_3 \cong \mathrm{M}_p(n, m, 1)$.

Proof $\Leftarrow$. Since $d(G/\Phi(G')G_3) = 2$ and $\Phi(G')G_3 \leq \Phi(G)$, $d(G) = 2$.

$\Rightarrow$. Since $d(G) = 2$, $d(G/\Phi(G')G_3) = 2$. On the other hand, by Lemma 2.9, we get

$$|(G/\Phi(G')G_3)'| = |G'/\Phi(G')G_3| = p.$$

It follows from Lemma 2.1 that $G/\Phi(G')G_3$ is minimal nonabelian. Since G' is nonmetacyclic, G is nonmetacyclic by Lemma 2.11. It follows from Lemma 2.4 that $G/\Phi(G')G_3$ is nonmetacyclic. So $G/\Phi(G')G_3 \cong \mathrm{M}_p(n, m, 1)$ by Lemma 2.2.

Lemma 2.18 *If G is a* $\mathcal{P}$*-group, then all the proper quotient groups of* $G/\Omega_1(G')$ *are abelian or metacyclic.*

Proof Let $H/\Omega_1(G') < G/\Omega_1(G')$. Then $H < G$. If $H/\Omega_1(G')$ is nonmetacyclic, then H is nonmetacyclic. It follows from Lemma 2.14 that $|H'| \leq p$. Hence $H' \leq \Omega_1(G')$. Moreover, $H/\Omega_1(G')$ is abelian.

Lemma 2.19 *Let G be a* $\mathcal{P}$*-group. If G has at least two nonmetacyclic maximal subgroups, then* $|G'| \leq p^3$.

Proof Let H and K be two distinct nonmetacyclic maximal subgroups of G. Then $|H'| \leq p$ and $|K'| \leq p$. Moreover, $|H'K'| \leq p^2$. It follows from Lemma 2.13 that $|G'| \leq p^3$.

3 $\mathcal{P}$-Groups with Exactly One Nonmetacyclic Maximal Subgroup

Theorem 3.1 *Let G be a p-group with exactly one nonmetacyclic maximal subgroup, where p is an odd prime. Then G is a* $\mathcal{P}$*-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups*:

(1) $C_{p^n} \times C_p \times C_p$, *where* $n \geq 2$;

(2) $\mathrm{M}_p(1, 1, 1) * C_{p^k}$, *where* $k \geq 2$;

(3) $\mathrm{M}_p(n,1)\times C_p$, *where* $n\geq 2$;

(4) $\langle a,b,c\mid a^{p^n}=b^p=c^p=1,\ [a,b]=c,\ [a,c]=[b,c]=1\rangle\cong \mathrm{M}_p(n,1,1)$, $n\geq 2$;

(5) $\langle a,b,c\mid a^{p^{n+1}}=b^p=c^p=1,\ [a,b]=c,\ [c,a]=a^{p^n},\ [c,b]=1\rangle$, *where* $n\geq 2$;

(6) $\langle a,b,c\mid a^{p^{n+1}}=b^p=c^p=1,\ [a,b]=c,\ [c,a]=1,\ [c,b]=a^{vp^n}\rangle$, *where* $n\geq 2$, $v=1$ *or a fixed square non-residue modulo* p.

Proof Obviously, $|G|\geq p^4$ and $d(G)\leq 3$. If G is abelian, then $d(G)=3$. In this case, $G\cong C_{p^n}\times C_{p^m}\times C_{p^k}$, where $n\geq m\geq k$. We claim $m=k=1$. If not, then $m\neq 1$. Thus there exist $M_1\lessdot G$ and $M_2\lessdot G$ such that $M_1\cong C_{p^n}\times C_{p^{m-1}}\times C_{p^k}$ and $M_2\cong C_{p^{n-1}}\times C_{p^m}\times C_{p^k}$ are nonmetacyclic. This contradicts the hypothesis. Thus we get the group (1).

Assume that G is not abelian. Let $N\leq G'$ with $|N|=p$ and $N\trianglelefteq G$, and $\overline{G}=G/N$.

If $\overline{G}$ is metacyclic, then, since G is nonmetacyclic, $G'=N$ by Lemma 2.10. Since $d(\overline{G})=d(G)=2$, G is minimal nonabelian by Lemma 2.1. Thus $G\cong \mathrm{M}_p(n,m,1)$ by Lemma 2.2. We claim $m=1$. If not, then let $M_1=\langle a,b^p,c\rangle$ and $M_2=\langle a^p,b,c\rangle$. It is easy to see that

$$M_1\cong C_{p^n}\times C_{p^{m-1}}\times C_p \quad\text{and}\quad M_2\cong C_{p^{n-1}}\times C_{p^m}\times C_p.$$

Hence M_1 and M_2 are two distinct nonmetacyclic maximal subgroups of G. This contradicts the hypothesis. Thus we get the group (4).

Assume that $\overline{G}$ is nonmetacyclic. Then $\overline{G}$ is minimal nonmetacyclic or $\overline{G}$ has exactly one nonmetacyclic maximal subgroup.

If $|G|=p^4$, then, by checking the classification of groups of order p^4, we get that G is one of the groups (1), (3), (4) when $n=2$ and is group (2) when $k=2$.

If $|G|=p^5$, then the theorem is true by checking the classification of groups of order p^5 given in [12].

Let $|G|\geq p^6$. Then $|\overline{G}|\geq p^5$. By the classification of minimal nonmetacyclic p-groups by Blackburn [6] and Lemma 2.3, $\overline{G}$ is not minimal nonmetacyclic. Thus $\overline{G}$ has exactly one nonmetacyclic maximal subgroup. By the induction hypothesis, $\overline{G}$ is isomorphic to one of the groups in the theorem.

Case 1 $\overline{G}\cong C_{p^n}\times C_p\times C_p$.

In this case, $d(G)=3$ and $|G'|=p$. Such groups are classified by [13, Theorem 3.1], and G is isomorphic to one of the groups $\mathrm{M}_p(n,m,1)\times C_{p^k}$, $\mathrm{M}_p(n,m,1)*C_{p^{k+1}}$ and $\mathrm{M}_p(n+1,m)\times C_{p^k}$.

If $G\cong \mathrm{M}_p(n,m,1)\times C_{p^k}$, where $C_{p^k}=\langle d\rangle$, then let $M_1=\langle a,b,c^p\rangle$ and $M_2=\langle a^p,b,c,d\rangle$. It is easy to see that

$$M_1\cong \mathrm{M}_p(n,m,1)\times C_{p^{k-1}} \quad\text{and}\quad M_2\cong C_{p^{n-1}}\times C_{p^m}\times C_{p^k}\times C_p.$$

It follows that M_1 and M_2 are two distinct nonmetacyclic maximal subgroups of G. This contradicts the hypothesis.

If $G\cong \mathrm{M}_p(n,m,1)*C_{p^{k+1}}$, then we claim $n=1$. If not, then let $M_1=\langle a,c^p,b\rangle$ and $M_2=\langle a^p,b,c\rangle$. We have

$$M_1\cong \mathrm{M}_p(n,m,1)*C_{p^k} \quad\text{and}\quad M_2\cong C_{p^{n-1}}\times C_{p^m}\times C_{p^k}.$$

It follows that M_1 and M_2 are two distinct nonmetacyclic maximal subgroups of G. This contradicts the hypothesis. Thus $n=m=1$. This is the group (2).

If $G \cong \mathrm{M}_p(n+1,m) \times C_{p^k}$, then we claim $m = k = 1$. If not, then, when $k \neq 1$, let $M_1 = \langle a, c^p, b\rangle$ and $M_2 = \langle a^p, b, c\rangle$. We have

$$M_1 \cong \mathrm{M}_p(n+1,m) \times C_{p^{k-1}} \quad \text{and} \quad M_2 \cong C_{p^n} \times C_{p^m} \times C_{p^k};$$

when $m \neq 1$, let $M_1 = \langle a, b^p, c\rangle$ and $M_2 = \langle a^p, b, c\rangle$. We have

$$M_1 \cong C_{p^{n+1}} \times C_{p^{m-1}} \times C_{p^k} \quad \text{and} \quad M_2 \cong C_{p^n} \times C_{p^m} \times C_{p^k}.$$

In either case, it follows that M_1 and M_2 are two distinct nonmetacyclic maximal subgroups of G. This contradicts the hypothesis. Thus we get the group (3).

Case 2 $\overline{G} = \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^p = \overline{b}^p = \overline{c}^{p^k} = 1,\ [\overline{a}, \overline{b}] = \overline{c}^{p^{k-1}},\ [\overline{c}, \overline{a}] = 1,\ [\overline{c}, \overline{b}] = 1\rangle$
$\cong \mathrm{M}_p(1,1,1) * C_{p^k}$.

Since $|\overline{G}| > p^4$, $k > 2$. Let $G = \langle a, b, c\rangle$. Notice that $N \leq Z(G)$. Then

$$[a, b, a] = [c^{p^{k-1}}, a] = [c, a]^{p^{k-1}} = 1.$$

In the same way, $[a, b, b] = 1$. It follows that $G_3 = 1$. Thus $c(G) = 2$. Since $b^p \in N \leq Z(G)$, $[a, b]^p = [a, b^p] = 1$. Hence $G' = \langle [a, b], N\rangle \cong C_p^2$ and $o(c) = p^k$. Let $N = \langle d\rangle$. Then

$$G = \langle a, b, c \mid a^p = d^x,\ b^p = d^y,\ c^{p^k} = 1,\ d^p = 1,\ [a, b] = c^{p^{k-1}} d^i,\ [c, a] = d^j,\ [c, b] = d^t\rangle,$$

where x, y, i, j and t are positive integers, and $p \mid j$ and $p \mid t$ have at most one to be true. It is easy to prove that $\Phi(G) = \langle c^p, d\rangle$.

We discuss the two cases: $x \equiv 0 \pmod p$ and $x \not\equiv 0 \pmod p$.

(i) $x \equiv 0 \pmod p$.

Let $M_1 = \langle a, b, c^p, d\rangle$ and $M_2 = \langle a, c, d\rangle$. Since M_1 and M_2 contain a subgroup $\langle a, c^{p^{k-1}}, d\rangle \cong C_p^3$, M_1 and M_2 are two distinct nonmetacyclic maximal subgroups of G. This contradicts the hypothesis.

(ii) $x \not\equiv 0 \pmod p$.

(iia) $y \equiv 0 \pmod p$.

By the same argument as that of (i), G has two distinct nonmetacyclic maximal subgroups, a contradiction.

(iib) $y \not\equiv 0 \pmod p$.

In this case, $b^p = a^{x^{-1}yp}$. Let $b_1 = ba^{-x^{-1}y}$. Then

$$b_1^p = 1, \quad [a, b_1] = [a, b] = c^{p^{k-1}} d^i, \quad [c, b_1] = [c, a^{-x^{-1}y}][c, b] = d^{-x^{-1}yj} d^t = d^{t_1}.$$

This is reduced to the case of (iia).

Case 3 $\overline{G} = \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^{p^n} = \overline{b}^p = \overline{c}^p = 1,\ [\overline{a}, \overline{b}] = \overline{a}^{p^{n-1}},\ [\overline{c}, \overline{a}] = 1,\ [\overline{c}, \overline{b}] = 1\rangle$
$\cong \mathrm{M}_p(n,1) \times C_p$.

By a similar argument as that of Case 2, we get that G has two distinct nonmetacyclic maximal subgroups. Hence this case does not occur. These details are omitted.

Case 4 $\overline{G} = \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^{p^n} = \overline{b}^p = \overline{c}^p = 1,\ [\overline{a}, \overline{b}] = \overline{c},\ [\overline{c}, \overline{a}] = 1,\ [\overline{c}, \overline{b}] = 1\rangle \cong \mathrm{M}_p(n,1,1)$.

Let $G = \langle a, b, c\rangle$. Since $|\overline{G}'| = p$, $|G'| = p^2$. Thus $|G_4| = 1$. Hence

$$G_3 = \langle [a, b, a], [a, b, b]\rangle = \langle [c, a], [c, b]\rangle.$$

Notice that $b^p \in N \leq Z(G)$. Thus

$$1 = [a, b^p] = [a, b]^p [a, b, b]^{\binom{p}{2}} = [a, b]^p [c, b]^{\binom{p}{2}} = [a, b]^p.$$

It follows that $G' = \langle [a, b], N \rangle \cong C_p^2$. Finite p-groups G with $G' \cong C_p^2$ and $G/N \cong \mathrm{M}_p(n, m, 1)$ are classified by [14], and such G are the groups (1), (2), (4), (5), (6), (7) and (8) with $m = 1$ in [14, Theorem 11].

If G is the group (1) or (2) in [14, Theorem 11], then it is the group (5) or (6) in this theorem.

If G is the group (4) in [14, Theorem 11], then G is minimal nonmetacyclic. This contradicts the hypothesis.

If G is the group (5) in [14, Theorem 11], then let $M_1 = \langle a, b^p, c \rangle$ and $M_2 = \langle a^p, b, c \rangle$. It is easy to verify that M_1 and M_2 contain a subgroup $\langle a^{p^{n-1}}, b^{p^m}, c \rangle \cong C_p^3$. Thus M_1 and M_2 are two distinct nonmetacyclic maximal subgroups of G. This contradicts the hypothesis.

If G is one of the groups (6), (7) and (8) in [14, Theorem 11], then it is easy to verify that G has two distinct nonmetacyclic maximal subgroups. This is also a contradiction.

Case 5 $\overline{G} = \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^{p^n} = \overline{b}^p = \overline{c}^p = 1,\ [\overline{a}, \overline{b}] = \overline{c},\ [\overline{c}, \overline{a}] = \overline{a}^{p^{n-1}},\ [\overline{c}, \overline{b}] = 1 \rangle$.

Let $G = \langle a, b \rangle$. Since $|\overline{G}'| = p^2$, $|G'| = p^3$. It follows from Lemma 2.12 that G' is abelian. Thus $G_4 = 1$ and $G_3 = \langle [c.a], [c, b] \rangle \neq 1$. Notice that $b^p, c^p \in N \leq Z(G)$. Thus

$$1 = [a, b^p] = [a, b]^p [a, b, b]^{\binom{p}{2}} = [a, b]^p [c, b]^{\binom{p}{2}} = [a, b]^p$$

and

$$1 = [a, c^p] = [a, c]^p [a, c, c]^{\binom{p}{2}} = [a, c]^p [a^{p^{n-1}}, c]^{\binom{p}{2}} = [a, c]^p.$$

It follows that $G' = \langle [a, b], [a, c], N \rangle \cong C_p^3$, and $o(a) = p^n$, $o(c) = p$. Let $N = \langle d \rangle$. Then $[a, b] = c$, $[c, a] = a^{p^{n-1}} d^j$ and $[c, b] = d^t$, where $t \not\equiv 0 \pmod p$.

If $j \equiv 0 \pmod p$, then

$$G = \langle a, b, c \mid a^{p^n} = 1,\ b^p = d^s, c^p = 1,\ d^p = 1,\ [a, b] = c,\ [c, a] = a^{p^{n-1}},\ [c, b] = d^t \rangle,$$

where s and t are positive integers. Since $\Phi(G) = \langle a^p, c, d \rangle$, we let

$$M_1 = \langle a^p, b, c, d \rangle \quad \text{and} \quad M_2 = \langle a, c, d \rangle.$$

It is easy to verify that M_1 and M_2 contain a subgroup $\langle a^{p^{n-1}}, c, d \rangle \cong C_p^3$. Hence M_1 and M_2 are two distinct nonmetacyclic maximal subgroups of G. This contradicts the hypothesis.

If $j \not\equiv 0 \pmod p$, then, by replacing a with $ab^{-t^{-1}j}$ and letting $c = [ab^{-t^{-1}j}, b]$, this is reduced to the case of $j \equiv 0 \pmod p$.

Case 6 $\overline{G} = \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^{p^n} = \overline{b}^p = \overline{c}^p = 1, [\overline{a}, \overline{b}] = \overline{c}, [\overline{c}, \overline{a}] = 1, [\overline{c}, \overline{b}] = \overline{a}^{vp^{n-1}} \rangle$, where $v = 1$ or a fixed square non-residue modulo p.

Let $G = \langle a, b \rangle$. Since $|\overline{G}'| = p^2$, $|G'| = p^3$. It follows from Lemma 2.12 that G' is abelian. Thus $G_4 = 1$ and $G_3 = \langle [c.a], [c, b] \rangle \neq 1$. Notice that $b^p, c^p \in N \leq Z(G)$. Hence

$$1 = [a, b^p] = [a, b]^p [a, b, b]^{\binom{p}{2}} = [a, b]^p [c, b]^{\binom{p}{2}} = [a, b]^p$$

and

$$1 = [b, c^p] = [b, c]^p [b, c, c]^{\binom{p}{2}} = [b, c]^p [a^{p^{n-1}}, c]^{\binom{p}{2}} = [b, c]^p.$$

It follows that $G' = \langle [a,b],[b,c],N\rangle \cong C_p^3$, and $o(a) = p^n$, $o(c) = p$. Let $N = \langle d\rangle$. Then $[a,b] = c$, $[c,a] = d^j$ and $[c,b] = a^{vp^{n-1}}d^t$. Thus

$$G = \langle a,b,c \mid a^{p^n} = 1,\ b^p = d^s,\ c^p = 1,\ d^p = 1,\ [a,b] = c,\ [c,a] = d^j,\ [c,b] = a^{vp^{n-1}}d^t\rangle,$$

where s,j,t are positive integers and $j \not\equiv 0 \pmod p$. Since $\Phi(G) = \langle a^p, c, d\rangle$, we let

$$M_1 = \langle a^p, b, c, d\rangle \quad \text{and} \quad M_2 = \langle a, c, d\rangle.$$

It is easy to verify that M_1 and M_2 contain a subgroup $\langle a^{p^{n-1}}, c, d\rangle \cong C_p^3$. Hence M_1 and M_2 are two distinct nonmetacyclic maximal subgroups of G. This contradicts the hypothesis.

We prove the groups (1)–(6) in the theorem are pairwise non-isomorphic according to $d(G) =$ 2 or 3.

If $d(G) = 2$, then G is one of the groups (1), (2) and (3). $|G'| = 1$ for the group (1), and $|G'| = p$ for the groups (2) and (3).

Let the group (2) be isomorphic to the group (3). Then for the group (3), let $a_1 = a^{i_1}b^{j_1}c^{k_1}$, $b_1 = a^{i_2}b^{j_2}c^{k_2}$ and $c_1 = a^{i_23}b^{j_3}c^{k_3}$, where $i_1,\ j_1,\ k_1,\ i_2,\ j_2,\ k_2,\ i_3,\ j_3$ and k_3 are positive integers with $(i_1,p) = 1$, $(j_2,p) = 1$ and $(k_3,p) = 1$. It follows that a_1, b_1 and c_1 satisfy the relation of the group (2). But $a_1^p = (a^{i_1}b^{j_1}c^{k_1})^p = a^{i_1}p = 1$. Hence $i_1 \equiv 0 \pmod{p^{n-1}}$, a contradiction.

If $d(G) = 3$, then G is one of the groups (4), (5) and (6). $|G'| = p$ for the group (4), and $|G'| = p^2$ for the groups (5) and (6).

Let the group (5) be isomorphic to the group (6). Then for the group (5), let $a_1 = a^{i_1}b^{j_1}c^{k_1}$ and $b_1 = a^{i_2}b^{j_2}c^{k_2}$, where $i_1,\ j_1,\ k_1,\ i_2,\ j_2$ and k_2 are positive integers with $(i_1,p) = 1$ and $(j_2,p) = 1$. Let $[a_1,b_1] = c_1$. Then a_1, b_1 and c_1 satisfy the relation of the group (6). Since $b_1^p = 1$, $i_2 \equiv 0 \pmod{p^n}$. On the other hand, $[c_1,a_1] = [a_1,b_1,a_1] = 1 = a_1^{vp^n} = a^{i_1vp^n}$, a contradiction.

Finally, we prove that the groups (1)–(6) in the theorem satisfy the hypothesis by taking the group (3) for example. In this case,

$$G \cong \mathrm{M}_p(n,1) \times C_p = \langle a,b,c \mid a^{p^n} = b^p = c^p = 1,\ [a,b] = a^{p^{n-1}},\ [c,a] = 1,\ [c,b] = 1\rangle.$$

Obviously, $\Phi(G) = \langle a^p\rangle$, and $M_1 = \langle a^p, b, c\rangle$, $M_2 = \langle ab^i, a^p, c\rangle$ and $M_3 = \langle ac^j, bc^t, a^p\rangle$ are all maximal subgroups of G, where $0 \le i,j,t < p$. Since M_1 contains a subgroup which is isomorphic to C_p^3, M_1 is nonmetacyclic. Obviously, $|M_1| \le p$. It is easy to check that $\mho_1(M_2) = \Phi(M_2)$, $\mho_1(M_3) = \Phi(M_3)$ and $d(M_1) = d(M_2) = 2$. So $\omega(M_2) \le 2$ and $\omega(M_3) \le 2$. By Lemma 2.7, M_2 and M_3 are metacyclic. So G satisfies the hypothesis.

Due to the classification of finite 2-groups with exactly one nonmetacyclic maximal subgroup by Z. Janko [15], it is enough to check that those groups in [15] are $\mathcal{P}$-groups. The following theorem lists the results and the proof is omitted.

Theorem 3.2 *Let G be a 2-group with exactly one nonmetacyclic maximal subgroup. Then G is a $\mathcal{P}$-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups*:

(I) $d(G) = 3$.

(1) $C_{2^n} \times C_2 \times C_2$, *where* $n \ge 2$;

(2) $\mathrm{M}_2(n,1)\times C_2$, *where* $n\geq 3$;

(3) $Q_8 * C_{2^n}$, *where* $n\geq 3$;

(4) $Q_8\times C_{2^n}$, *where* $n\geq 2$;

(5) $\langle a,b,c \mid a^4=b^4=c^{2^n}=1,\ a^2=b^2,\ [a,c]=1,\ [c,b]=c^{2^{n-1}},\ [a,b]=a^2\rangle\cong Q_8C_{2^n}$, *where* $n\geq 3$.

(II) $d(G)=2$.

(6) $\mathrm{M}_2(n,1,1)$, *where* $n\geq 2$;

(7) $\langle a,b,c \mid a^{2^n}=b^2=c^2=1,\ [c,a]=b,\ [b,c]=1,\ [b,a]=a^{2^{n-1}}\rangle$, *where* $n\geq 2$;

(8a) $\langle a,x \mid a^{2^m}\in\langle v\rangle,\ x^2\in\langle v^{2^{n-1}}\rangle,\ v^{2^n}=1,\ [a,x]=v,\ [v,x]\in\langle v^{2^{n-1}}\rangle,\ [v,a^2]=1,\ [a^2,x]=1,\ [v,a]=v^{-2}\rangle$, *where* $m\geq 2$ *and* $n\geq 2$;

(8b) $\langle a,x \mid a^{2^m}\in\langle v\rangle,\ x^2\in\langle v^{2^{n-1}}\rangle,\ v^{2^n}=1,\ [a,x]=v,\ [v,x]\in\langle v^{2^{n-1}}\rangle,\ [v,a^2]=1,\ [a^2,x]=v^{2^{n-1}},\ [v,a]=v^x,\ 2^{n-1}|s+2\rangle$, *where* $m\geq 2$ *and* $n\geq 2$;

(8c) $\langle a,x \mid a^{2^m},x^2\in\langle v,b\rangle,\ v^2=b^2=[v,b]=1,\ [a,x]=v,\ [v,a]=b,\ [b,a]=[b,x]=1,\ [v,x]=z^t\in\langle v,b\rangle\cap Z(G),\ t=0,1\rangle$, *where* $m\geq 2$;

(8d) $\langle a,x \mid a^{2^m}\in\langle v,b\rangle,\ x^2\in\langle v^2,\ b\rangle,\ v^4=b^2=1,\ [a,x]=v,\ [v,a]=b,\ [v,x]=v^2b,\ [b,a]=[b,x]=[v,b]=1\rangle$, *where* $m\geq 2$;

(9) $\langle a,b,c \mid a^{2^2}=1,\ b^2,c^{2^m}\in\langle a^2\rangle,\ [a,b]=[a,c]=a^2,\ [c,b]=a\rangle$, *where* $m\geq 2$;

(10) $\langle a,b \mid a^8=b^4=d^2=1,\ a^4=b^2=c,\ d^2=c^2=1,\ [a,b]=d,\ [d,a]=c,\ [d,c]=1\rangle$.

4 $\mathcal{P}$-Groups with at Least Two Nonmetacyclic Maximal Subgroups

For convenience of statement, we introduce the following.

Assumption 4.1 *A finite p-group G has at least two nonmetacyclic maximal subgroups and a metacyclic maximal subgroup.*

Theorem 4.1 *If G is not a D_1-group, then G is a $\mathcal{P}$-group satisfying Assumption* 4.1 *if and only if G is isomorphic to one of the following non-isomorphic groups*:

(I) $c(G)=2$.

(1) $\langle a,b,c \mid a^{p^{n+2}}=b^{p^m}=c^p=1,\ [a,b]=a^{p^n},\ [a,c]=1,\ [b,c]=1\rangle$, $m\geq 2$;

(2) $\langle a,b,c \mid a^{p^{n+2}}=b^{p^m}=c^p=1,\ [a,b]=a^{p^n},\ [a,c]=a^{p^{n+1}},\ [b,c]=1\rangle$, $m\geq 3$, $n\geq 2$;

(3) $\langle a,b,c \mid a^{p^{n+2}}=b^{p^m}=c^p=1,\ [a,b]=a^{p^n},\ [a,c]=1,\ [b,c]=a^{p^{n+1}}\rangle$, $m\geq 2$, $n\geq 2$;

(4) $\langle a,b,c \mid a^{p^{n+2}}=b^{p^{m+1}}=c^p=1,\ [a,b]=a^{p^n},\ [a,c]=1,\ [b,c]=1,\ b^{p^m}=a^{p^{n+1}}\rangle$, $n+1<m$, $n\geq 2$;

(5) $\langle a,b,c \mid a^{p^{n+2}}=b^{p^{m+1}}=c^p=1,\ [a,b]=a^{p^n},\ [a,c]=a^{p^{n+1}},\ [b,c]=1,\ b^{p^m}=a^{p^{n+1}}\rangle$, $n+1\leq m$, $n\geq 2$;

(6) $\langle a,b,c \mid a^{p^{n+2}}=b^{p^{m+1}}=c^p=1,\ [a,b]=a^{p^n},\ [a,c]=1,\ [b,c]=a^{p^{n+1}},\ b^{p^m}=a^{p^{n+1}}\rangle$, $n+1<m$, $n\geq 2$;

(II) $c(G)=3$.

(7) $\langle a,b,c \mid a^{p^3}=b^{p^m}=c^p=1,\ [a,b]=a^p,\ [a,c]=1,\ [b,c]=1\rangle$, *if* $p>2$, *then* $m\geq 2$; *if* $p=2$, *then* $m\geq 1$;

(8) $\langle a,b,c \mid a^8=b^{2^m}=c^2=1,\ [a,b]=a^6,\ [a,c]=1,\ [b,c]=1\rangle$, $m\geq 1$;

(9) $\langle a,b,c \mid a^{p^3}=b^{p^m}=c^p=1,\ [a,b]=a^p,\ [a,c]=a^{p^2},\ [b,c]=1\rangle$, $m\geq 3$ *if* $p>2$; $m\geq 1$ *if* $p=2$;

(10) $\langle a,b,c \mid a^{p^3}=b^{p^m}=c^p=1,\ [a,b]=a^p,\ [a,c]=1,\ [b,c]=a^{p^2}\rangle$, $m\geq 2$ *if* $p>2$; $m\geq 1$ *if* $p=2$;

(11) $\langle a,b,c \mid a^{p^3}=b^{p^{m+1}}=c^p=1,\ [a,b]=a^p,\ [a,c]=1,\ [b,c]=1,\ b^{p^m}=a^{p^2}\rangle$, $m\geq 3$ *if* $p>2$; $m\geq 2$ *if* $p=2$;

(12) $\langle a,b,c \mid a^8=b^4=c^2=1,\ [a,b]=a^6,\ [a,c]=1,\ [b,c]=1,\ b^2=a^4\rangle$;

(13) $\langle a,b,c \mid a^{p^3}=b^{p^{m+1}}=c^p=1,\ [a,b]=a^p,\ [a,c]=a^{p^2},\ [b,c]=1,\ b^{p^m}=a^{p^2}\rangle$, $m\geq 3$ *if* $p>2$; $m\geq 2$ *if* $p=2$.

Proof Let M be a metacyclic maximal subgroup of G. Thus $d(G)\leq 3$. By Lemma 2.19, $|G'|\leq p^3$. Moreover, we claim $|G'|\geq p^2$. If not, then $|G'|\leq p$. It follows from [5, Theorem 3.1] that G is a D_1-group. This contradicts the hypothesis. So $|G'|=p^2$ or p^3.

Case 1 $|G'|=p^3$.

By Lemma 2.12, G' is abelian. Let M_1 and M_2 be two distinct nonmetacyclic maximal subgroups of G. It follows from $|G'|=p^3$ and Lemma 2.13 that $|M_1'|=|M_2'|=p$ and $M_1'\cap M_2'=1$. Thus $M_1'M_2'\cong C_p^2$ and $M_1'M_2'\leq Z(G)$.

If $G'\cong C_{p^3}$, then $|M_1'|=|M_2'|=p$. Hence $M_1'\cap M_2'=1$, a contradiction.

If $G'\cong C_p^3$, then, since G is a $\mathcal{P}$-group and not a D_1-group, there exists $M\lessdot G$ such that $|M'|>p$ and M is metacyclic. It follows from Lemma 2.11 that M' is cyclic. Since $M'\leq G'$, $M'\cong C_p^2$ or $M'\cong C_p^3$, a contradiction.

Assume $G'\cong C_{p^2}\times C_p$. Since G is a $\mathcal{P}$-group and not a D_1-group, there exists $M\lessdot G$ such that $|M'|>p$ and M is metacyclic.

If $d(G)=2$, then let $\overline{G}=G/M_1'M_2'$. We get that $\overline{G}$ is minimal nonabelian. Thus $M'=M_1'M_2'\cong C_p\times C_p$. It follows from Lemma 2.11 that M' is cyclic. This is a contradiction.

Assume $d(G)=3$. Since G' is abelian, $|\Omega_1(G')|=|G'/\mho_1(G')|=|G'/\Phi(G')|=p^2$. Since $M_1'M_2'\leq\Omega_1(G')$, $\Omega_1(G')=M_1'M_2'\leq Z(G)$. Let $\overline{G}=G/\Omega_1(G')$. Then $d(\overline{G})=3$ and $|\overline{G}'|=p$. Thus $\overline{G}$ is isomorphic to one of the groups in [13, Theorem 3.1].

If $\overline{G}$ is isomorphic to the group (1) in [13, Theorem 3.1], that is,

$$\overline{G}\cong\langle \overline{a},\overline{b},\overline{c}\mid \overline{a}^4=\overline{c}^{2^k}=1,\ [\overline{a},\overline{b}]=\overline{a}^2=\overline{b}^2,\ [\overline{c},\overline{a}]=1,\ [\overline{c},\overline{b}]=1\rangle\cong Q_8\times C_{2^k},$$

then $G'=\langle [a,b] \text{ and } \Omega_1(G')\rangle\cong C_4\times C_2$. Notice that $\Omega_1(G')\cong C_2^2$. Thus $o[a,b]=4$. Since $[a,b]\equiv a^2\equiv b^2 \pmod{\Omega_1(G')}$, $1=[a^2,b]=[a,b]^2[a,b,a]=[a,b]^2$. This is a contradiction.

If $\overline{G}$ is isomorphic to the group (2) in [13, Theorem 3.1], that is,

$$\overline{G}\cong\langle \overline{a},\overline{b},\overline{c}\mid \overline{a}^4=\overline{c}^{2^{k+1}}=1,\ [\overline{a},\overline{b}]=\overline{a}^2=\overline{b}^2=\overline{c}^{2^k},\ [\overline{c},\overline{a}]=1,\ [\overline{c},\overline{b}]=1\rangle\cong Q_8*C_{2^{k+1}},$$

then, by the same argument as that of the above paragraph, a contradiction occurs.

If $\overline{G}$ is isomorphic to the group (3) in [13, Theorem 3.1], that is,

$$\begin{aligned}\overline{G}&\cong\langle \overline{a},\overline{b},\overline{c},\overline{d}\mid \overline{a}^{p^n}=\overline{b}^{p^m}=\overline{c}^{p^k}=\overline{d}^p=1,\ [\overline{a},\overline{b}]=\overline{d},\ [\overline{c},\overline{a}]=1,\ [\overline{c},\overline{b}]=1\rangle\\&\cong \mathrm{M}_p(n,m,1)\times C_{p^k},\end{aligned}$$

then $\Phi(\overline{G})=\langle\overline{a}^p,\overline{b}^p \text{ and } \overline{c}^p,\overline{d}\rangle$. Thus $\overline{M}=\langle\overline{a},\overline{b},\overline{c}^p,\overline{d}\rangle\lessdot\overline{G}$ and $|\overline{M}'|=p$. If $k>1$, then $\overline{M}\cong\mathrm{M}_p(n,m,1)\times C_{p^{k-1}}$. If $k=1$, then $\overline{M}\cong\mathrm{M}_p(n,m,1)$. It follows from Lemma 2.18 that G is not a $\mathcal{P}$-group. This contradicts the hypothesis.

If $\overline{G}$ is isomorphic to the group (4) in [13, Theorem 3.1], that is,

$$\begin{aligned}\overline{G} &\cong \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^{p^n} = \overline{b}^{p^m} = \overline{c}^{p^{k+1}} = 1,\ [\overline{a}, \overline{b}] = \overline{c}^{p^k},\ [\overline{c}, \overline{a}] = 1,\ [\overline{c}, \overline{b}] = 1\rangle \\ &\cong \mathrm{M}_p(n, m, 1) * C_{p^{k+1}},\end{aligned}$$

then, by the same argument as that of the group (3), we get that G is not a $\mathcal{P}$-group.

If $\overline{G}$ is isomorphic to the group (5) in [13, Theorem 3.1], that is,

$$\begin{aligned}\overline{G} &\cong \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^{p^{n+1}} = \overline{b}^{p^m} = \overline{c}^{p^k} = 1,\ [\overline{a}, \overline{b}] = \overline{a}^{p^n},\ [\overline{c}, \overline{a}] = 1,\ [\overline{c}, \overline{b}] = 1\rangle \\ &\cong \mathrm{M}_p(n+1, m) \times C_{p^k},\end{aligned}$$

then $\Phi(\overline{G}) = \langle \overline{a}^p, \overline{b}^p, \overline{c}^p\rangle$. Let $G = \langle a, b, c\rangle$. Since $|\overline{G}'| = p$, $G_4 = 1$. Thus

$$G_3 = \langle [a, b, b]\rangle \quad \text{and} \quad G' = \langle [a, b],\ [c, a],\ [c, b],\ [a, b, b]\rangle.$$

Notice that $\Omega_1(G') \cong C_p^2$. Thus $o([a, b]) = p^2$, and $[a, c]$, $[b, c]$ and $[a, b, a]$ can not belong to $\langle [a, b]\rangle$ at the same time.

If $[b, c] \notin \langle [a, b]\rangle$, then there exists $M \lessdot G$ such that $\overline{M} = \langle \overline{a}^p, \overline{b}, \overline{c}\rangle \cong C_{p^n} \times C_{p^m} \times C_{p^k}$. Thus M is nonmetacyclic. By calculation, $o([a^p, b]) = p$ and $o([b, c]) = p$. Since $\langle [a^p, b]\rangle \neq \langle [b, c]\rangle$, $|M'| > p$. It follows that G is not a $\mathcal{P}$-group. If $[a, c] \notin \langle [a, b]\rangle$ or $[a, b, a] \notin \langle [a, b]\rangle$, then, by similar argument as that of the case $[b, c] \notin \langle [a, b]\rangle$, we get that G is not a $\mathcal{P}$-group.

To sum up, there does not exist a $\mathcal{P}$-group which is not a D_1-group with $|G'| = p^3$.

Case 2 $|G'| = p^2$.

If $G' \cong C_p^2$, then, since G is a $\mathcal{P}$-group and not a D_1-group, there exists $M \lessdot G$ such that $|M'| > p$ and M is metacyclic. It follows from Lemma 2.11 that M' is cyclic. Since $M' \leq G'$, $M' = G' \cong C_p^2$. This is a contradiction. Hence $G' \cong C_{p^2}$.

Since G is not a D_1-group, $d(G) = 3$ by [5, Theorem 3.1]. Since G' is abelian,

$$|\Omega_1(G')| = |G'/\mho_1(G')| = |G'/\Phi(G')| = p.$$

Thus $\Omega_1(G') \leq Z(G)$. Let $\overline{G} = G/\Omega_1(G')$. Then $d(\overline{G}) = 3$ and $|\overline{G}'| = p$. Thus $\overline{G}$ is isomorphic to one of the groups in [13, Theorem 3.1].

If $\overline{G}$ is isomorphic to one of the groups (1)–(4) in [13, Theorem 3.1], then, by a similar argument as that of Case 1, we get that G is not a $\mathcal{P}$-group.

Assume that $\overline{G}$ is isomorphic to one of the group (5) in [13, Theorem 3.1], that is,

$$\begin{aligned}\overline{G} &\cong \langle \overline{a}, \overline{b}, \overline{c} \mid \overline{a}^{p^{n+1}} = \overline{b}^{p^m} = \overline{c}^{p^k} = 1,\ [\overline{a}, \overline{b}] = \overline{a}^{p^n},\ [\overline{c}, \overline{a}] = 1,\ [\overline{c}, \overline{b}] = 1\rangle \\ &\cong \mathrm{M}_p(n+1, m) \times C_{p^k},\end{aligned}$$

then $\Phi(\overline{G}) = \langle \overline{a}^p, \overline{b}^p, \overline{c}^p\rangle$. Thus $\overline{M} = \langle \overline{a}, \overline{b}, \overline{c}^p\rangle \lessdot \overline{G}$ and $|\overline{M}'| = p$. If $k > 1$, then $\overline{M} \cong \mathrm{M}_p(n+1, m) \times C_{p^{k-1}}$. It follows from Lemma 2.18 that G is not a $\mathcal{P}$-group. Assume $k = 1$.

Since $d(\overline{G}) = 3$, $G = \langle a, b, c\rangle$. Since $|\overline{G}'| = p$, $G_4 = 1$. Thus

$$G_3 = \langle [a, b, b]\rangle = \langle [a, b]^{p^n}\rangle \quad \text{and} \quad G' = \langle [a, b],\ [a, c],\ [b, c],\ [a, b, b]\rangle.$$

Since $G' \cong C_{p^2}$, $[a, c]$ and $[b, c]$ are contained in $\Omega_1(G')$. Hence $G' = \langle [a, b]\rangle = \langle a^{p^n}\rangle$. It follows that $\Omega_1(G') = \langle a^{p^{n+1}}\rangle$.

If $n \geq 2$, then $c(G) = 2$. If $n = 1$, then $c(G) = 3$. We discuss the two cases: $c(G) = 2$ and $c(G) = 3$.

Case 2.1 $c(G) = 2$.

It is easy to see that $o(a) = p^{n+2}$. Notice that $\Omega_1(G') = \langle a^{p^{n+1}} \rangle$. Assume $[a, b] = a^{p^n}$, $[a, c] = a^{ip^{n+1}}$ and $[b, c] = a^{jp^{n+1}}$ without loss of generality. We discuss the possible value of i and j.

(i) $i = j = 0$. Then

$$[a, b] = a^{p^n}, \quad [a, c] = 1 \quad \text{and} \quad [b, c] = 1. \tag{4.1}$$

(ii) $i \neq 0$ and $j = 0$. Let $c_1 = c^{i^{-1}}$. Then

$$[a, b] = a^{p^n}, \quad [a, c] = a^{p^{n+1}} \quad \text{and} \quad [b, c] = 1. \tag{4.2}$$

(iii) $j \neq 0$ and $i = 0$. Let $c_1 = c^{j^{-1}}$. Then

$$[a, b] = a^{p^n}, \quad [a, c] = 1 \quad \text{and} \quad [b, c] = a^{p^{n+1}}. \tag{4.3}$$

(iv) $j \neq 0$, $i \neq 0$.

If $o(a) \geq o(b)$, then, letting $a_1 = ab^t$, where $t = -ij^{-1}$, we get $[a_1, c] = 1$. This is reduced to the case (iii).

If $o(a) < o(b)$, then, letting $b_1 = ba^t$, where $t = -ji^{-1}$, we get $[b_1, c] = 1$. This is reduced to the case (ii).

We discuss the possible value of $o(b)$ and $o(c)$.

First, we claim $o(b) \geq p^2$ and $o(c) = p$. If not, then $o(b) = p$. Thus $[a, b]^p = [a, b^p] = 1$, a contradiction. If $o(c) = p^2$, then $c^p = a^{tp^{n+1}}$. Let $c_1 = ca^{-tp^n}$. Then $c_1^p = (ca^{-tp^n})^p = 1$ which satisfies the relations. Thus $o(c) = p^2$ can be reduced to the case of $o(c) = p$. Hence we only need to discuss the possible value of $o(b)$.

If $o(b) = p^m$, then $m \geq 2$.

If a, b and c are of the relation (4.1), then

$$G = \langle a, b, c \mid a^{p^{n+2}} = b^{p^m} = c^p = 1,\ [a, b] = a^{p^n},\ [a, c] = 1,\ [b, c] = 1 \rangle,$$

where $m \geq 2$ and $n \geq 2$. This is the group (1).

If a, b and c are of the relation (4.2), then

$$G = \langle a, b, c \mid a^{p^{n+2}} = b^{p^m} = c^p = 1,\ [a, b] = a^{p^n},\ [a, c] = a^{p^{n+1}},\ [b, c] = 1 \rangle,$$

where $n \geq 2$. If $m = 2$, then, letting $c_1 = cb^{-p}$, we get $[a, c_1] = 1$. This is reduced to the group (1). If $m \geq 3$, then we get the group (2).

If a, b and c are of the relation (4.3), then

$$G = \langle a, b, c \mid a^{p^{n+2}} = b^{p^m} = c^p = 1,\ [a, b] = a^{p^n},\ [a, c] = 1,\ [b, c] = a^{p^{n+1}} \rangle,$$

where $m \geq 2$ and $n \geq 2$. This is the group (3).

If $o(b) = p^{m+1}$, then $m \geq 1$. Assume $b^{p^m} = a^{sp^{n+1}}$, where $(p, s) = 1$ without loss of generality.

If a, b and c are of the relation (4.1), then, let $b_1 = ba^{-sp^{n+1-m}}$ if $n+1 \geq m$. We get $b_1^{p^m} = 1$. This is reduced to the case when $o(b) = p^m$. If $n+1 < m$, then

$$G = \langle a, b, c \mid a^{p^{n+2}} = b^{p^{m+1}} = c^p = 1,\ [a, b] = a^{p^n},\ [a, c] = 1,\ [b, c] = 1,\ b^{p^m} = a^{p^{n+1}} \rangle.$$

This is the group (4).

If a, b and c are of the relation (4.2), then, let $b_1 = ba^{-sp^{n+1-m}}$ if $n+1 > m$. We get $b_1^{p^m} = 1$. This is reduced to the case when $o(b) = p^m$. If $n+1 \leq m$, then

$$G = \langle a, b, c \mid a^{p^{n+2}} = b^{p^{m+1}} = c^p = 1,\ [a, b] = a^{p^n},\ [a, c] = a^{p^{n+1}},\ [b, c] = 1,\ b^{p^m} = a^{p^{n+1}} \rangle.$$

This is the group (5).

If a, b and c are of the relation (4.3), then, let $b_1 = ba^{-sp^{n+1-m}}$ if $n+1 \geq m$. We get $b_1^{p^m} = 1$. This is reduced to the case when $o(b) = p^m$. If $n+1 < m$, then

$$G = \langle a, b, c \mid a^{p^{n+2}} = b^{p^{m+1}} = c^p = 1,\ [a, b] = a^{p^n},\ [a, c] = 1,\ [b, c] = a^{p^{n+1}},\ b^{p^m} = a^{p^{n+1}} \rangle.$$

This is the group (6).

Case 2.2 $c(G) = 3$.

In this case, $o(a) = p^3$.

First, we claim $o(b) \geq p^2$ if $p > 2$. If not, then $o(b) = p$. Thus

$$1 = [a, b^p] = [a, b]^p [a, b, b]^{\binom{p}{2}} = [a, b]^p,$$

a contradiction. Notice that $\Omega_1(G') = \langle a^{p^2} \rangle$. Without loss of generality, we assume

$$[a, b] = a^p, \quad [a, c] = a^{ip^2} \quad \text{and} \quad [b, c] = a^{jp^2} \quad \text{if } p > 2.$$

If $p = 2$, then $[a, b] = a^2$ or a^6. Without loss of generality, we assume

$$[a, c] = a^{i2^2} \quad \text{and} \quad [b, c] = a^{j2^2}.$$

We discuss the possible value of i and j.

(i) $i = j = 0$. Then

$$[a, b] = a^p, \quad [a, c] = 1, \quad [b, c] = 1 \tag{4.4}$$

and

$$[a, b] = a^6, \quad [a, c] = 1, \quad [b, c] = 1 \quad \text{for } p = 2. \tag{4.5}$$

(ii) $i \neq 0$ and $j = 0$.

If $p > 2$, then let $c^{j^{-1}}$ replace c. We get

$$[a, b] = a^p, \quad [a, c] = a^{p^2} \quad \text{and} \quad [b, c] = 1. \tag{4.6}$$

If $p = 2$ and $[a, b] = a^6$, then let $b_1 = bc$ if $o(b) \geq o(c)$. We get $[a, b_1] = a^2$. This is reduced to the relation (4.6). Assume $o(b) < o(c)$. Then we have

$$[a, b] = a^6, \quad [a, c] = a^4 \quad \text{and} \quad [b, c] = 1. \tag{4.7}$$

(iii) $j \neq 0$ and $i = 0$.

If $p > 2$, then let $c^{j^{-1}}$ replace c. We get

$$[a, b] = a^p, \quad [a, c] = 1 \quad \text{and} \quad [b, c] = a^{p^2}. \tag{4.8}$$

If $p = 2$ and $[a, b] = a^6$, then let $a_1 = ac$ if $o(c) = p$. We get $[a_1, b] = a_1^2$. This is reduced to the relation (4.6). If $o(c) = p^2$, then

$$[a, b] = a^6, \quad [a, c] = 1 \quad \text{and} \quad [b, c] = a^4. \tag{4.9}$$

(iv) $j \neq 0$ and $i \neq 0$.

If $o(a) \geq o(b)$, then let $a_1 = ab^t$, where $t = -ij^{-1}$. We get $[a_1, c] = 1$. This is reduced to (iii).

If $o(a) < o(b)$, then let $b_1 = ba^t$, where $t = -ji^{-1}$. We get $[b_1, c] = 1$. This is reduced to (ii).

We discuss the possible value of $o(b)$ and $o(c)$.

Case 2.2.1 $o(b) = p^m$ and $o(c) = p$.

In this case, $m \geq 2$ if $p > 2$ and $m \geq 1$ if $p = 2$.

If a, b and c are of the relation (4.4), then

$$G = \langle a, b, c \mid a^{p^3} = b^{p^m} = c^p = 1,\ [a, b] = a^p,\ [a, c] = 1,\ [b, c] = 1 \rangle.$$

This is the group (7).

If a, b and c are of the relation (4.5), then

$$G = \langle a, b, c \mid a^8 = b^{2^m} = c^2 = 1,\ [a, b] = a^6,\ [a, c] = 1,\ [b, c] = 1 \rangle.$$

This is the group (8).

If a, b and c are of the relation (4.6), then, let $c_1 = cb^{-p}$ if $p > 2$ and $m = 2$. We get $o(c_1) = o(c)$ and $[a, c_1] = 1$. This is reduced to the group (7). If $p > 2$ and $m \geq 3$, then

$$G = \langle a, b, c \mid a^{p^3} = b^{p^m} = c^p = 1,\ [a, b] = a^p,\ [a, c] = a^{p^2},\ [b, c] = 1 \rangle.$$

This is the group (9).

If a, b and c are of the relation (4.7), then

$$G = \langle a, b, c \mid a^{p^3} = b^{p^m} = c^p = 1,\ [a, b] = a^p,\ [a, c] = 1,\ [b, c] = a^{p^2} \rangle.$$

This is the group (10).

Case 2.2.2 $o(b) = p^{m+1}$ and $o(c) = p$.

In this case, $m \geq 1$. Without loss of generality, we assume $b^{p^m} = a^{sp^2}$, where $(p, s) = 1$.

If a, b and c are of the relation (4.4), then let $b_1 = ba^{-sp^{2-m}}$ if $p > 2$ and $m \leq 2$. We get $b_1^{p^m} = 1$. This is reduced to the group (7). If $p = 2$ and $m = 1$, then let $b_1 = ba^{-1}$. We get $b_1^{p^m} = 1$. This is reduced to the group (7). If $p > 2$ and $m \geq 3$ or $p = 2$ and $m \geq 2$, then

$$G = \langle a, b, c \mid a^{p^3} = b^{p^{m+1}} = c^p = 1,\ [a, b] = a^p,\ [a, c] = 1,\ [b, c] = 1,\ b^{p^m} = a^{p^2} \rangle.$$

This is the group (11).

If a, b and c are of the relation (4.5), then let $a_1 = ab^{2^{m-1}}$ if $m \geq 2$. We get $o(a_1) = o(a)$. This is reduced to the group (11). Thus $m = 1$ and

$$G = \langle a, b, c \mid a^8 = b^4 = c^2 = 1,\ [a, b] = a^6,\ [a, c] = 1,\ [b, c] = 1,\ b^2 = a^4 \rangle.$$

This is the group (12).

If a, b and c are of the relation (4.6), then let $b_1 = ba^{-sp}$ if $p > 2$ and $m = 1$. We get $b_1^p = 1$. This is reduced to the group (10). If $p > 2$ and $m = 2$, then let $b_1 = a$ and $a_1 = b$. By symmetry, the relation keeps invariant. Let $b_1 = ba^{-s}$. Then $b_1^{p^m} = 1$. This is reduced to the group (11). Thus, if $p > 2$ and $m \geq 3$ or $p = 2$ and $m \geq 2$, then

$$G = \langle a, b, c \mid a^{p^3} = b^{p^{m+1}} = c^p = 1,\ [a, b] = a^p,\ [a, c] = a^{p^2},\ [b, c] = 1,\ b^{p^m} = a^{p^2} \rangle.$$

This is the group (13).

If a, b and c are of the relation (4.8), then let $b_1 = ba^{-sp^{2-m}}$ if $p > 2$ and $m \leq 2$. We get $b_1^{p^m} = 1$. This is reduced to the group (10). If $p = 2$ and $m = 1$, then let $b_1 = ba^{-s}$. We get $b_1^p = 1$. This is reduced to the group (10). If $p = 2$ and $m \geq 1$ or $p > 2$ and $m \geq 2$, then let $c_1 = ca^{sp}b^{-p^{m-1}}$. We get $c_1^p = 1$, $[a, c_1] = [b, c_1] = 1$. This is reduced to the group (11).

In the case of $o(b) = p^m$ and $o(c) = p^2$ or $o(b) = p^{m+1}$ and $o(c) = p^2$, by a similar argument as that of Cases 2.2.1 and 2.2.2, we get the groups (7)–(13). There is no new group to occur.

Those groups listed in Theorem 4.1 are pairwise non-isomorphic. To prove this is a tedious but not trivial work, so the details are omitted.

Finally, we prove that the groups (1)–(13) in Theorem 4.1 satisfy the hypothesis by taking the group (1) for example. Let G be the group (1), that is,

$$G = \langle a, b, c \mid a^{p^{n+2}} = b^{p^m} = c^p = 1,\ [a, b] = a^{p^n},\ [a, c] = 1,\ [b, c] = 1 \rangle.$$

By calculation, we get $\Phi(G) = \langle a^p, b^p \rangle$, and

$$M_1 = \langle b, c, a^p \rangle, \quad M_2 = \langle ab^i, a^p, b^p, c \rangle \quad \text{and} \quad M_3 = \langle ac^j, a^p, b^p, bc^t \rangle$$

are all maximal subgroups of G, where $0 \leq i, j, t < p$. It is easy to see that M_1 and M_2 contain a subgroup which is isomorphic to C_p^3. This means that G has at least two distinct nonmetacyclic maximal subgroups. On the other hand, $|M_1'| = |M_2'| = p$. This means that M_1 and M_2 satisfy the hypothesis.

For the maximal subgroup M_3, if $p = 2$, then $\Phi(M_3) = \langle a^2, b^2 \rangle$, and $H_1 = \langle bc^t, a^2 \rangle$ and $H_2 = \langle ac^j, (bc^t)^m, a^2, b^2 \rangle$ are all maximal subgroups of M_3, where $0 \leq t, j, m < 2$. It is easy to prove that $d(H_1) = d(H_2) = 2$. It follows from Lemma 2.8 that M_3 is metacyclic. If $p > 2$, then $M_3' \leq G' \leq \mho_1(M_3)$. Thus $\omega(M_3) = d(M_3) = 2$. It follows from Lemma 2.7 that M_3 is metacyclic. On the other hand, $d(G) = 3$ and $|G'| = p^2$. It follows from [5, Theorem 3.1] that G is not a D_1-group. So the group (1) satisfies the hypothesis.

Theorem 4.2 *If G is a D_1-group, then G is a $\mathcal{P}$-group satisfying Assumption* 4.1 *if and only if G is isomorphic to one of the following non-isomorphic groups*:

(1) $C_{p^n} \times C_{p^m} \times C_p$, *where* $n \geq m > 1$;

(2) $\langle a, b, c \mid a^{p^n} = b^p = c^{p^{k+1}} = 1,\ [a, b] = c^{p^k},\ [c, a] = [c, b] = 1 \rangle \cong \mathrm{M}_p(n, 1, 1) * C_{p^{k+1}}$, *where* $n > 1$;

(3) $\langle a,b,c \mid a^{p^n}=b^p=c^{p^k}=1,\ [a,b]=a^{p^{n-1}},\ [c,a]=[c,b]=1\rangle \cong \mathrm{M}_p(n,1)\times C_{p^k}$, *where* $n>1, k>1$;

(4) $\langle a,b,c \mid a^{p^n}=b^{p^m}=c^p=1,\ [a,b]=a^{p^{n-1}},\ [c,a]=[c,b]=1\rangle \cong \mathrm{M}_p(n,m)\times C_p$, *where* $n>1, m>1$;

(5) $\langle a,b,c \mid a^p=b^{p^{m+1}}=c^{p^{m+1}}=1,\ [b,c]=1,\ [c,a]=c^{p^m},\ [a,b]=b^{-p^m}\rangle$, *where* $p>2$;

(6) $\langle a,b,c \mid a^p=b^{p^{m+1}}=c^{p^{m+1}}=1,\ [b,c]=1,\ [c,a]=b^{p^m}c^{p^m},\ [a,b]=b^{-p^m}\rangle$, *where* $p>2$;

(7) $\langle a,b,c \mid a^p=b^{p^{m+1}}=c^{p^{m+1}}=1,\ [b,c]=1,\ [c,a]=b^{p^m}c^{tp^m},\ [a,b]=b^{-tp^m}c^{\nu p^m}\rangle$, *where* $p>2$, $\nu=1$ *or a fixed square non-residue modulo* p. $t^2\neq-\nu$, *and* $t\in\{0,1,\cdots,\frac{p-1}{2}\}$;

(8) $\langle a,b,c \mid a^2=b^{2^{m+1}}=c^{2^{m+1}}=1,\ [b,c]=1,\ [c,a]=b^{2^m},\ [a,b]=c^{2^m}\rangle$;

(9) $\langle a,b,c \mid a^2=b^{2^{m+1}}=c^{2^{m+1}}=1,\ [b,c]=1,\ [c,a]=c^{2^m},\ [a,b]=b^{2^m}\rangle$;

(10) $\langle a,b,c \mid a^2=b^{2^{m+1}}=c^{2^{m+1}}=1,\ [b,c]=1,\ [c,a]=b^{2^m},\ [a,b]=b^{2^m}c^{2^m}\rangle$;

(11) $\langle a,b,c \mid a^p=b^{p^{m+1}}=c^{p^{n+1}}=1,\ [b,c]=1,\ [a,b]=b^{p^m},\ [c,a]=c^{tp^n}\rangle$, *where* $m>n$ *and* $1\le t\le p-1$;

(12) $\langle a,b,c \mid a^p=b^{p^{m+1}}=c^{p^{n+1}}=1,\ [b,c]=1,\ [a,b]=c^{\nu p^n},\ [c,a]=b^{p^m}\rangle$, *where* $m>n$ *and* $\nu=1$ *or a fixed square non-residue modulo* p;

(13) $\langle a,b,c \mid a^{p^{l+1}}=b^p=c^{p^{n+1}}=1,\ [b,c]=1,\ [c,a]=c^{p^n},\ [a,b]=a^{p^l}\rangle$;

(14) $\langle a,b,c \mid a^{p^{l+1}}=b^{p^{m+1}}=c^p=1,\ [b,c]=1,\ [c,a]=b^{p^m},\ [a,b]=a^{p^l}\rangle$;

(15) $\langle a,b,c \mid b^4=c^4=1,\ a^2=b^2,\ [a,b]=c^2,\ [a,c]=b^2,\ [b,c]=1\rangle$.

Proof Since G has one metacyclic maximal subgroup, $d(G)\le 3$. By Lemma 2.19, $|G'|\le p^3$. Since G is a D_1-group, G' is one of the following possible cases by [5, Theorem 3.1]:

(i) $|G'|\le p$;

(ii) $d(G)=2$, $|G'|=p^2$;

(iii) $d(G)=2$, $c(G)=3$, $G'\cong C_p^3$, where $p>2$;

(iv) $d(G)=3$, $c(G)=2$, $G'\cong C_p^3$ or $G'\cong C_p^2$.

Case 1 $|G'|=1$.

In this case, $d(G)=3$ and $|G|\ge p^5$. Thus $G\cong C_{p^n}\times C_{p^m}\times C_{p^k}$, where $n\ge m\ge k$. We claim $k=1$. If not, then $M_1\cong C_{p^n}\times C_{p^{m-1}}\times C_{p^k}$, $M_2\cong C_{p^{n-1}}\times C_{p^m}\times C_{p^k}$ and $M_3\cong C_{p^n}\times C_{p^m}\times C_{p^{k-1}}$ are all maximal subgroups of G, and M_1, M_2 and M_3 are nonmetacyclic, a contradiction. Thus we get the group (1). Conversely, the group (1) satisfies the theorem's hypothesis.

Case 2 $|G'|=p$.

If $d(G)=2$, then G is minimal nonabelian by Lemma 2.1. Since G is nonmetacyclic, $G\cong \mathrm{M}_p(n,m,1)$ by Lemma 2.2. If $m=1$, then G is the group (4) in Theorem 3.1. Obviously, G is not the required group by the theorem's hypothesis. If $m>1$, then $\Phi(G)=\langle a^p,b^p,c\rangle$, and $d(\Phi(G))=3$. It follows from Lemma 2.11 that $\Phi(G)$ is nonmetacyclic. This contradicts Lemma 2.15.

Assume $d(G)=3$. Then G is isomorphic to one of the groups in [13, Theorem 3.1].

If G is isomorphic to the group (1) or (2) in [13, Theorem 3.1], then G is the group (3) or (4) in Theorem 3.2. So G is not the required group by the theorem's hypothesis.

If G is isomorphic to the group (3) in [13, Theorem 3.1], that is, $G\cong \mathrm{M}_p(n,m,1)\times C_{p^k}$, then $\Phi(G)=\langle a^p,b^p,c^p,d\rangle$ if $m>1$. Notice that $d(\Phi(G))\ge 3$. Then it follows from Lemma 2.11 that $\Phi(G)$ is nonmetacyclic. This contradicts Lemma 2.15. Thus $m=1$.

If $k \neq 1$, then, by the same argument as that of the case when $m > 1$, a contradiction occurs. If $k = 1$, then, it is easy to prove that all maximal subgroups of G are nonmetacyclic. This contradicts the theorem's hypothesis.

If G is isomorphic to the group (4) in [13, Theorem 3.1], that is, $G \cong \mathrm{M}_p(n,m,1) * C_{p^{k+1}}$, then, by the same argument as that of the case when $m > 1$ above, a contradiction occurs. Thus $m = 1$. If $n = 1$, then G is the group (2) in Theorem 3.1. So G is not the required group. If $n > 1$, then $\Phi(G) = \langle a^p, c^p, d\rangle$, and

$$M_1 = \langle b, c, a^p\rangle, \quad M_2 = \langle ac^i, bc^j, c^p\rangle \quad \text{and} \quad M_3 = \langle ab^t, c\rangle$$

are all maximal subgroups of G, where $0 \le i, j, t < p$. It is easy to verify that M_1 and M_2 are nonmetacyclic, and M_3 is metacyclic. On the other hand, since $|G'| = p$, $|M_1'| \le p$ and $|M_2'| \le p$. Thus we get the group (2).

If G is isomorphic to the group (5) in [13, Theorem 3.1], that is, $G \cong \mathrm{M}_p(n+1,m) \times C_{p^k}$, then, if $m = k = 1$, then G is the group (3) in Theorem 3.1. So G is not the required group.

If $m = 1$ and $k \neq 1$, then $\Phi(G) = \langle a^p, c^p, d\rangle$, and

$$M_1 = \langle b, c, a^p\rangle, \quad M_2 = \langle a, b, c^p\rangle \quad \text{and} \quad M_3 = \langle a, c\rangle$$

are all maximal subgroups of G. It is easy to verify that M_1 and M_2 are nonmetacyclic, and M_3 is metacyclic. On the other hand, since $|G'| = p$, $|M_1'| \le p$ and $|M_2'| \le p$. Thus we get the group (3).

If $m \neq 1$ and $k = 1$, then $\Phi(G) = \langle a^p, b^p, d\rangle$, and

$$M_1 = \langle b, c, a^p\rangle, \quad M_2 = \langle ab^i, b^p, c\rangle \quad \text{and} \quad M_3 = \langle ac^j, bc^t\rangle$$

are all maximal subgroups of G, where $0 \le i, j, t < p$. It is easy to verify that M_1 and M_2 are nonmetacyclic, and M_3 is metacyclic. On the other hand, since $|G'| = p$, $|M_1'| \le p$ and $|M_2'| \le p$. Thus we get the group (4).

$Z(G) = \langle a^p, c\rangle \cong C_{p^{n-1}} \times C_{p^k}$ for the group (3), and $Z(G) = \langle a^p, b^p, c\rangle \cong C_{p^{n-1}} \times C_{p^{m-1}} \times C_p$ for the group (4), so the group (3) is not isomorphic to the group (4).

Case 3 $d(G) = 2,\ |G'| = p^2$.

Let $N \le G' \cap Z(G)$ and $|N| = p$. Then $d(G/N) = 2$ and $|(G/N)'| = p$. It follows from Lemma 2.1 that G/N is minimal nonabelian.

If $G' \cong C_p^2$, then $G/N \cong \mathrm{M}_p(m,n,1)$ by [14, Lemma 8(2) and Lemma 9]. Thus G is isomorphic to one of the groups in [14, Theorem 11]. If $G' \cong C_{p^2}$, then G is isomorphic to one of the groups in [14, Theorems 10 and 12] (notice that there is a typographical error in [14], where Theorem 12 is printed to Theorem 11). By checking the list of groups in [14, Theorems 10–12], we get that G is metacyclic in [14, Theorem 10]. This is not the required group. Those groups in [14, Theorems 11 and 12] do not satisfy the theorem's hypothesis. The details are omitted. So this case does not occur.

Case 4 $d(G) = 2,\ c(G) = 3,\ G' \cong C_p^3,\ p > 2$.

Since $G' \cong C_p^3$, G' is nonmetacyclic. It follows from Lemma 2.11 that $\Phi(G)$ is nonmetacyclic. This contradicts Lemma 2.15. So this case does not occur.

Case 5 $d(G) = 3,\ c(G) = 2,\ G' \cong C_p^3$ or C_p^2.

If $G' \cong C_p^3$, then, by the same argument as that of Case 4, G is not the required group. Assume $G' \cong C_p^2$. By Lemma 2.16, $\Phi(G) \le Z(G)$. Thus G is the group in [13, Theorem 4.7]. It is enough to check that those groups satisfy the theorem's hypothesis.

If G is one of the groups (A1)–(A3) and (A7)–(A8) in [13, Theorem 4.7], then $\Phi(G) = \langle a^p, b^p, c^p\rangle$ when $l > 1$. In this case, $d(\Phi(G)) = 3$. Thus $\Phi(G)$ is nonmetacyclic. This contradicts Lemma 2.15. So $l = 1$. It is easy to get that $\Phi(G) = \langle b^p, c^p\rangle$, and

$$M_1 = \langle a, c, b^p\rangle, \quad M_2 = \langle a, b, c^p\rangle \quad \text{and} \quad M_3 = \langle b, c\rangle$$

are all maximal subgroups of G. It is easy to check that M_1 and M_2 are nonmetacyclic, and M_3 is metacyclic. On the other hand, since $G' \cong C_p^2$, we get $|M_1'| \le p$ and $|M_2'| \le p$. Thus we get the groups (5)–(7) and (11)–(12), respectively.

By the same argument as that of the above paragraph, we get the following conclusions:

If G is one of the groups (A4)–(A6) in [13, Theorem 4.7], then $l = 1$, and we get the groups (8)–(10), respectively.

If G is the group (A9) in [13, Theorem 4.7], then $m = 1$, and we get the group (13).

If G is the group (A10) in [13, Theorem 4.7], then $n = 1$, and we get the group (14).

If G is one of the groups (A12) or (D) in [13, Theorem 4.7], then $n = 1$, and we get the group (15).

If G is one of the groups (A11), (B1)–(B4) and (C) in [13, Theorem 4.7], then there is no group to be the required.

5 $\mathcal{P}$-Groups All of Whose Maximal Subgroups are Nonmetacyclic

For the convenience of classifying $\mathcal{P}$-groups, here we give the classification of D_1-groups. It is not difficult to do this by [5, Theorem 3.1].

Theorem 5.1 *Assume that G is a finite p-group. Then G is a D_1-group if and only if G is isomorphic to one of the following pairwise non-isomorphic groups*:

(I) *metacyclic groups*

(1) *an abelian group with* $d(G) \le 2$;

(2) $\mathrm{M}_p(n, m)$;

(3) *one of the groups in* [14, *Theorem* 10];

(II) *nonmetacyclic groups*

(4) *an abelian group with* $d(G) > 2$;

(5) $(A_1 * A_2 * A_3 \cdots * A_s)Z(G)$, *where* $A_1, A_2, \cdots, A_s$ *are minimal nonabelian groups. In particular, if* $s = 1$ *and* $d(G) = 2$, *then* $G \cong \mathrm{M}_p(n, m, 1)$, *where* $n \ge m \ge 2$;

(6) *one of the groups in* [14, *Theorems* 11 *and* 12];

(7) *one of the groups in* [16, *Theorem* 4.9] *with* $p > 2$;

(8) *one of the groups in* [13, *Theorem* 4.8];

(9) *one of the groups in* [17, *Theorem* 3.1].

Proof If G is metacyclic, then $d(G) \le 2$ and G' is cyclic. It follows from [5, Theorem 3.1] that $|G'| \le p$ or $G' \cong C_{p^2}$. If $G' = 1$, then we get the group (1). If $|G'| = p$, then $G \cong \mathrm{M}_p(n, m)$ by Lemmas 2.1–2.2. We get the group (2). If $G' \cong C_{p^2}$, then there exists $N < G'$ such that $N \trianglelefteq G$ and $|N| = p$. Let $\overline{G} = G/N$. Then $\overline{G}$ is minimal nonabelian. Since G is metacyclic, $\overline{G}$ is metacyclic. Hence $\overline{G} \cong \mathrm{M}_p(n,m)$ by Lemma 2.2. Such groups with $G' \cong C_{p^2}$ and $\overline{G} \cong \mathrm{M}_p(n,m)$ are classified in [14], and G is one of the groups in [14, Theorem 10]. By a simple verification, we get the group (3).

Assume that G is nonmetacyclic. Since G is a D_1-group, G' is one of the following possible cases by [5, Theorem 3.1]:

(i) $|G'| \leq p$;

(ii) $d(G) = 2$, $|G'| = p^2$;

(iii) $d(G) = 2$, $c(G) = 3$, $G' \cong C_p^3$, where $p > 2$;

(iv) $d(G) = 3$, $c(G) = 2$, $G' \cong C_p^3$ or $G' \cong C_p^2$.

If $G' = 1$, then G is abelian with $d(G) > 2$. This is the group (4).

If $|G'| = p$, then such groups are characterized in [18] and $G = (A_1 * A_2 * A_3 \cdots * A_s)Z(G)$, where $A_1, A_2, \cdots, A_s$ are minimal nonabelian groups. This is the group (5).

If $d(G) = 2$ and $|G'| = p^2$, then there exists $N < G'$ such that $N \trianglelefteq G$ and $|N| = p$. Let $\overline{G} = G/N$. Then $\overline{G}$ is minimal nonabelian. Since G is nonmetacyclic, $\overline{G}$ is nonmetacyclic by Lemma 2.10. It follows by Lemma 2.2 that $G/N \cong \mathrm{M}_p(n,m,1)$.

If $G' \cong C_p^2$, then G is isomorphic to one of the groups in [14, Theorem 11]. If $G' \cong C_{p^2}$, then G is isomorphic to one of the groups in [14, Theorem 12]. By a simple verification, we get the group (6).

If $d(G) = 2,\ c(G) = 3$ and $G' \cong C_p^3$, where $p > 2$, then, by Lemmas 2.9 and 2.17, G is isomorphic to one of the groups in [16, Theorem 4.9] with $p > 2$. This is the group (7).

If $d(G) = 3,\ c(G) = 2$ and $G' \cong C_p^2$, then, by Lemma 2.16, G is isomorphic to one of the groups in [13, Theorem 4.8]. This is the group (8).

If $d(G) = 3,\ c(G) = 2$ and $G' \cong C_p^3$, then, by Lemma 2.16, G is isomorphic to one of the groups in [17, Theorem 3.1]. This is the group (9).

Conversely, the groups in the theorem are D_1-groups. The details are omitted.

Theorem 5.2 *If all maximal subgroups of a finite p-group G are nonmetacyclic, then G is a $\mathcal{P}$-group if and only if G is isomorphic to one of the following non-isomorphic groups*:

(1) *an abelian group with* $d(G) > 3$;

(2) $(A_1 * A_2 * A_3 \cdots * A_s)Z(G)$, *where* $A_1, A_2, \cdots, A_s$ *are minimal nonabelian groups. In particular, if* $s = 1$ *and* $d(G) = 2$, *then* $G \cong \mathrm{M}_p(n,m,1)$, *where* $n \geq m \geq 2$. *In addition, the following groups are excepted: The groups* (2)–(4) *and* (6) *in Theorem* 3.2, *the group* (4) *in Lemma* 2.3, *the groups* (2)–(4) *in Theorem* 4.1 *and the groups* (2)–(4) *in Theorem* 3.1;

(3) *one of the groups in* [14, *Theorems* 11 *and* 12], *where the following groups are excepted: The group* (1) *with* $p > 2$, *groups* (7) *and* (8) *with* $p = 2$ *and* $m \leq 2$, *and groups* (2)–(4) *in* [14, *Theorem* 11]; *the groups* (5)–(6) *with* $p = 2$ *and* $m = 2$ *and group* (11) *in* [14, *Theorem* 12];

(4) *one of the groups in* [16, *Theorem* 4.9] *with* $p > 2$;

(5) *one of the groups in* [13, *Theorem* 4.8], *except for the groups* (A_1)–(A_8) *with* $l = 1$, (A_9) *with* $m = 1$, (A_{10}) *with* $n = 1$, (A_{12}) *and* (D) *in* [13, *Theorem* 4.8];

(6) *one of the groups in* [17, *Theorem* 3.1].

Proof Since G is a $\mathcal{P}$-group, G is a nonmetacyclic D_1-group by the hypothesis. Thus G is one of the groups of type II in Theorem 5.1. Since G has no metacyclic maximal subgroup, we get the groups in the theorem by checking those groups in Theorems 3.1–3.2, 4.1–4.2 and Lemma 2.3. The details are omitted.

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