Chin. Ann. Math.
35B(6), 2014, 907–940
DOI: 10.1007/s11401-014-0863-7

Chinese Annals of Mathematics, Series B

On the Rayleigh-Taylor Instability for Two Uniform Viscous Incompressible Flows*

Fei JIANG[1] Song JIANG[2] Weiwei WANG[3]

Abstract The authors study the Rayleigh-Taylor instability for two incompressible immiscible fluids with or without surface tension, evolving with a free interface in the presence of a uniform gravitational field in Eulerian coordinates. To deal with the free surface, instead of using the transformation to Lagrangian coordinates, the perturbed equations in Eulerian coordinates are transformed to an integral form and the two-fluid flow is formulated as a single-fluid flow in a fixed domain, thus offering an alternative approach to deal with the jump conditions at the free interface. First, the linearized problem around the steady state which describes a denser immiscible fluid lying above a light one with a free interface separating the two fluids, both fluids being in (unstable) equilibrium is analyzed. By a general method of studying a family of modes, the smooth (when restricted to each fluid domain) solutions to the linearized problem that grow exponentially fast in time in Sobolev spaces are constructed, thus leading to a global instability result for the linearized problem. Then, by using these pathological solutions, the global instability for the corresponding nonlinear problem in an appropriate sense is demonstrated.

Keywords Rayleigh-Taylor instability, Viscous incompressible flows, Global instability
2000 MR Subject Classification 76E17, 76D05

1 Introduction

We consider the two-phase free boundary problem for the equations of two incompressible immiscible fluids within the infinite slab $\Omega = \mathbb{R}^2 \times (-1, 1) \subset \mathbb{R}^3$ and for time $t \geq 0$. The fluids are separated by a moving free interface $\Sigma(t)$ which is given by the unknown function $\eta : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$. Hence we can define $\Sigma(t) := \{x \in \mathbb{R}^3 \mid x_3 = \eta(t, x')\}$ for each $t \geq 0$, where $x' = (x_1, x_2)^{\mathrm{T}}$, and the superscript T means matrix transposition.

The interface divides Ω into two time-dependent disjoint open subsets $\Omega_\pm(t)$, so that $\Omega = \Omega_+(t) \cup \Omega_-(t) \cup \Sigma(t)$ and $\Sigma(t) = \overline{\Omega}_+(t) \cap \overline{\Omega}_-(t)$. The motion of the fluids is driven by the constant gravitational field along e_3, i.e., the x_3 direction, and $G = (0, 0, -g)^{\mathrm{T}}$ with $g > 0$.

Manuscript received October 3, 2012. Revised February 18, 2014.
[1]Corresponding author. College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350108, China. E-mail: jiangfei0591@163.com
[2]Institute of Applied Physics and Computational Mathematics, Beijing 100088, China.
E-mail: jiang@iapcm.ac.cn
[3]College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350108, China.
E-mail: wei.wei.84@163.com
*This work was supported by the National Natural Science Foundation of China (Nos. 11101044, 11271051, 11229101, 11301083, 11371065, 11471134), the Fujian Provincial Natural Science Foundation of China (No. 2014J01011), the National Basic Research Program (No. 2011CB309705) and the Beijing Center for Mathematics and Information Interdisciplinary Sciences.

The two fluids are described by their velocity and pressure functions, which are given, for each $t \geq 0$, by

$$(u_\pm, \overline{p}_\pm)(t, \cdot) : \Omega_\pm(t) \to (\mathbb{R}^3, \mathbb{R}^+),$$

respectively. We assume that at a given time $t \geq 0$, these functions have well-defined traces onto $\Sigma(t)$.

The fluids under consideration are incompressible and viscous. Hence, for $t > 0$, the fluids satisfy the following equations of motions:

$$\begin{cases} \partial_t \eta = u_3 - u_1 \partial_1 \eta - u_2 \partial_2 \eta, & \text{on } \Sigma(t), \\ \partial_t(\varrho_\pm u_\pm) + \mathrm{div}(\varrho_\pm u_\pm \otimes u_\pm) + \mathrm{div} S_\pm = -g\varrho_\pm e_3, & \text{in } \Omega_\pm(t), \\ \mathrm{div} u_\pm = 0, & \text{in } \Omega_\pm(t), \end{cases} \tag{1.1}$$

where the first equation of (1.1) describes the motion of the free interface (see [7, 17]), $\partial_i := \partial_{x_i}$, and the positive constants $\varrho_\pm$ denote the densities of the respective fluids, and we define the stress tensor by

$$S_\pm = -\mu_\pm(\nabla u_\pm + \nabla u_\pm^{\mathrm{T}}) + \overline{p}_\pm I,$$

with $\mu_\pm$ and $\overline{p}_\pm$ being the viscosity coefficient and the pressure of the respective fluids, and I the 3×3 identity matrix.

For two viscous fluids meeting at a free boundary with surface tension, from the physical point of view, the velocity is continuous across the interface, and the jump in the normal stress is proportional to the mean curvature of the surface multiplied by the normal to the surface (see [2, 24]). Thus, we impose the jump conditions at the free interface as follows:

$$[u]|_{\Sigma(t)} = 0, \tag{1.2}$$

$$[S\nu]|_{\Sigma(t)} = \kappa H \nu, \tag{1.3}$$

where the interfacial jump is defined by

$$[f]|_{\Sigma(t)} := f_+|_{\Sigma(t)} - f_-|_{\Sigma(t)},$$

$f|_{\Sigma(t)}$ is the trace of a quantity f on $\Sigma(t)$, and

$$\nu = \frac{(-\partial_1 \eta, -\partial_2 \eta, 1)^{\mathrm{T}}}{\sqrt{1 + (\partial_1 \eta)^2 + (\partial_2 \eta)^2}}$$

denotes the normal vector to the free surface $\Sigma(t)$. The jump condition of (1.2) implies that there is no possibility for the fluid to slip past each other along $\Sigma(t)$. Here we take H to be twice the mean curvature of the surface $\Sigma(t)$ and the surface tension to be a constant $\kappa \geq 0$. Since $\Sigma(t)$ is parameterized by $(x', \eta(t, x'))$, we may employ the standard formula for the mean curvature of a parameterized surface to write

$$H = \frac{\Delta_{x'} \eta + (\partial_1 \eta)^2 \partial_2^2 \eta + (\partial_2 \eta)^2 \partial_1^2 \eta - 2\partial_1 \eta \partial_2 \eta \partial_1 \partial_2 \eta}{(1 + (\partial_1 \eta)^2 + (\partial_2 \eta)^2)^{\frac{3}{2}}}.$$

We also enforce the condition that the normal component of the fluid velocity vanishes at the fixed boundaries, that is,

$$u_+(t,x',-1)=u_-(t,x',1)=0 \quad \text{for all } t\geq 0,\ x'\in\mathbb{R}^2.$$

To complete the statement of the problem, we have to specify initial conditions. We give the initial interface $\Sigma(0)=\Sigma_0$, which yields the open sets $\Omega_\pm(0)$ on which we specify the initial data for the velocity and height of interface

$$u_\pm(0,\cdot):\Omega_\pm(0)\to\mathbb{R}^3, \quad \eta(0,\cdot):\mathbb{R}^2\to(-1,1).$$

Thus the initial datum of the pressure can be given by $\varrho_\pm$, $\eta(0,\cdot)$ and $u_\pm(0,\cdot)$. To simplify the equations, we introduce the indicator functions $\chi_{\Omega_\pm}$, denote

$$\varrho=\varrho_+\chi_{\Omega_+}+\varrho_-\chi_{\Omega_-}, \quad u=u_+\chi_{\Omega_+}+u_-\chi_{\Omega_-}, \quad \overline{p}=\overline{p}_+\chi_{\Omega_+}+\overline{p}_-\chi_{\Omega_-},$$

and define the modified pressure by

$$p=\overline{p}+g\varrho x_3.$$

Thus, for each $t>0$, (1.1) can be rewritten as

$$\begin{cases}\partial_t\eta=u_3-u_1\partial_1\eta-u_2\partial_2\eta, & \text{on } \Sigma(t),\\ \varrho\partial_t u+\varrho(\nabla u)u+\nabla p=\mu\Delta u, & \text{in } \Omega\setminus\Sigma(t),\\ \mathrm{div}u=0, & \text{in } \Omega\setminus\Sigma(t)\end{cases} \tag{1.4}$$

and the jump condition (1.3) becomes, setting $[\varrho]=\varrho_+-\varrho_-$,

$$[(pI-\mu(\nabla u+\nabla u^{\mathrm{T}}))\nu]|_{\Sigma(t)}=(g[\varrho]\eta+\kappa H)\nu.$$

For convenience in subsequent analysis, we will use the notation

$$[\![f]\!]:=f_+|_{x_3=0}-f_-|_{x_3=0}$$

for the jump of a quantity f across the set $\{x_3=0\}$.

Now, we linearize (1.4) around a steady-state solution $\eta=0$, $u=0$ and $p=$ constant, and then the resulting linearized equations read as

$$\begin{cases}\partial_t\eta=u_3, & \text{on } \mathbb{R}^+\times\{x_3=0\},\\ \varrho\partial_t u+\nabla p=\mu\Delta u, & \text{in } \mathbb{R}^+\times(\Omega\setminus\{x_3=0\}),\\ \mathrm{div}u=0, & \text{in } \mathbb{R}^+\times(\Omega\setminus\{x_3=0\}).\end{cases} \tag{1.5}$$

The corresponding linearized jump conditions are

$$[\![u]\!]=0, \quad [\![pI-\mu(\nabla u+\nabla u^{\mathrm{T}})]\!]e_3=(g[\varrho]\eta+\kappa\Delta_{x'}\eta)e_3, \tag{1.6}$$

while the boundary conditions are

$$u(t,x',-1)=u(t,x',1)=0. \tag{1.7}$$

We consider two completely plane-parallel layers of immiscible fluid, the heavier on top of the light one and both subject to the earth's gravity. In this case, the equilibrium state is unstable to sustain small perturbations or disturbances, and this unstable disturbance will grow and lead to a release of potential energy, as the heavier fluid moves down under the (effective) gravitational field, and the lighter one is displaced upwards. This phenomena was first studied by Rayleigh [18–19] and then Taylor [20], and therefore is called the Rayleigh-Taylor instability. In the last decades, many works related to this phenomena have appeared from both physical and numerical points of view. In particular, many results concerning linearized problems have been summarized in monographs (see, e.g., [2, 22]). To our best knowledge, however, there are only a few results of mathematical analysis on nonlinear problems in the literature, due to the fact that in general, the passage from a linearized instability to a dynamical nonlinear instability for a conservative nonlinear partial differential system is rather difficult. In 1987, Ebin [4] proved the ill-posedness of the equations of motion for a perfect fluid with free boundary. Then, he adapted the approach of [4] to obtain the ill-posedness of both Rayleigh-Taylor and Helmholtz problems for two-dimensional incompressible, immiscible, inviscid fluids without surface tension (see [5]). In 2003, Hwang and Guo [10] showed the nonlinear Rayleigh-Taylor instability for two-dimensional, incompressible, inviscid fluids with continuous density, and their result was extended to magnetohydrodynamic (MHD) flows (see [9]) recently. Unfortunately, the approaches in both [5] and [10] could not be applied to the viscous flow case, since the viscosity can bring some technical difficulties to the study of the nonlinear Rayleigh-Taylor instability. We should mention that Jiang et al. [11] showed the nonlinear RT instability of $\|u_3\|_{L^2(\mathbb{R}^3)}$ for the Cauchy problem of the nonhomogeneous incompressible viscous fluid with continuous density in the sense of Lipschitz structure recently.

In 2011, for two-compressible immiscible fluids evolving with a free interface (the density is discontinuous across the free interface), Guo and Tice made use of flow maps (Lagrangian coordinates) to transfer the free boundary into a fixed boundary and established a variational framework for nonlinear instability in [6], where with the help of the method of Fourier synthesis, they constructed solutions that grow arbitrarily quickly in time in the Sobolev space, leading to the ill-posedness of the perturbed problem in Lagrangian coordinates. It should be noted that they also investigated the stabilizing effect of viscosity and surface tension to the linear Rayleigh-Taylor instability (see [8]). However, the nonlinear instability for compressible flows still remains open.

In this paper, we will study the nonlinear Rayleigh-Taylor instability for two uniform viscous incompressible flows with surface tension and a free interface, across which the density is discontinuous. We will prove that in Eulerian coordinates, the corresponding linearized system is globally unstable in Sobolev spaces, and moreover, the original nonlinear problem with or without surface tension is globally unstable in an appropriate sense. For this purpose, we assume that $\kappa \geq 0$ and that the upper fluid is heavier than the lower fluid, i.e.,

$$\varrho_+ > \varrho_- \Leftrightarrow [\varrho] > 0.$$

We mention that the analogue of the Rayleigh-Taylor instability arises when the fluids are electrically conducting and a magnetic field is present, and the growth of the instability will be influenced by the magnetic field due to the generated electromagnetic induction and the Lorentz force (see [3, 9, 12–14, 23]). Some authors have extended the partial results concerning the Rayleigh-Taylor instability of superposed flows to the case of MHD flows by overcoming additional difficulties induced by the presence of the magnetic field.

This paper is organized as follows. In Section 2, we state our results on the linearized system (1.5) and the nonlinear system (1.4), i.e., Theorems 2.1–2.2. In Section 3, we construct the growing solutions to the linearized equations, while in Section 4, we analyze the linear problem and prove the uniqueness and Theorem 2.1. In Section 5, we prove the global instability of order k of the nonlinear problem, i.e., Theorem 2.2.

2 Main Results

Before stating the main results, we introduce the notation that will be used throughout the paper. For a function $f \in L^2(\Omega)$, we define the horizontal Fourier transform via

$$\widehat{f}(\xi, x_3) = \int_{\mathbb{R}^2} f(x', x_3) \mathrm{e}^{-\mathrm{i}x'\cdot\xi} \mathrm{d}x',$$

where $x', \xi \in \mathbb{R}^2$, and "$\cdot$" denotes the scalar product. By the Fubini and Parseval theorems, we have

$$\int_\Omega |f(x)|^2 \mathrm{d}x = \frac{1}{4\pi^2} \int_\Omega |\widehat{f}(\xi, x_3)|^2 \mathrm{d}\xi \mathrm{d}x_3.$$

We now define a function space suitable for our analysis of two disjoint fluids. For a function f defined on Ω, we write f_+ for the restriction to $\Omega_+ = \mathbb{R}^2 \times (0, 1)$ and f_- for the restriction to $\Omega_- = \mathbb{R}^2 \times (-1, 0)$. For $s \in \mathbb{R}$, we define the piecewise Sobolev space of order s by

$$H^s(\Omega_\pm) = \{f \mid f_+ \in H^s(\Omega_+),\ f_- \in H^s(\Omega_-)\} \tag{2.1}$$

endowed with the norm

$$\|f\|^2_{H^s(\Omega_\pm)} = \|f\|^2_{H^s(\Omega_+)} + \|f\|^2_{H^s(\Omega_-)}.$$

In a way similar to (2.1), for a function f defined on $(0, \infty) \times \Omega$, for which an interface divides Ω into two time-dependent disjoint open subsets $\Omega_\pm(t)$, so that $\Omega = \Omega_+(t) \cup \Omega_-(t) \cup \Sigma(t)$ and $\Sigma(t) = \overline{\Omega}_+(t) \cap \overline{\Omega}_-(t)$, we denote

$$H^s(\Omega_\pm(t)) = \{f(t) \mid f_+(t) \in H^s(\Omega_+(t)),\ f_-(t) \in H^s(\Omega_-(t))\} \tag{2.2}$$

for each $t \in [0, \infty)$.

In addition, for $k \in \mathbb{N}$, we can take the norms to be given by

$$\begin{aligned} \|f\|^2_{H^k(\Omega_\pm)} :&= \sum_{j=0}^k \int_{\mathbb{R}^2 \times I_\pm} (1 + |\xi|^2)^{k-j} |\partial^j_{x_3} \widehat{f}_\pm(\xi, x_3)|^2 \mathrm{d}\xi \mathrm{d}x_3 \\ &= \sum_{j=0}^k \int_{\mathbb{R}^2} (1 + |\xi|^2)^{k-j} \|\partial^j_{x_3} \widehat{f}_\pm(\xi, \cdot)\|^2_{L^2(I_\pm)} \mathrm{d}\xi, \end{aligned}$$

where $I_-=(-1,0)$ and $I_+=(0,1)$. The main difference between the piecewise Sobolev space $H^s(\Omega)$ and the usual Sobolev space lies in that we do not require functions in the piecewise Sobolev space to have weak derivatives across the set $\{x_3=0\}$. If $f:=(f_1,\cdots,f_n)^{\mathrm{T}}\in(H^s(\Omega_\pm))^n$, to shorten notation, we define

$$\|f\|^2_{H^s(\Omega_\pm)}=\sum_{i=1}^n\|f_i\|^2_{H^s(\Omega_\pm)}.$$

Now, we are in a position to state our first result, i.e., the result of global instability for the linearized problem (1.5).

Theorem 2.1 *The linearized problem* (1.5) *with the corresponding jump and boundary conditions is globally unstable in the sense of Hadamard in $H^k(\Omega)$ for every k. More precisely, there exists a constant $C_1>0$, and for any $k,j\in\mathbb{N}$ with $j\geq k$ and for any $\alpha>0$, there exists a constant $C_{j,k}$ depending on j and k, and a sequence of solutions $\{(\eta_n,u_n,p_n)\}_{n=1}^\infty$ to* (1.5) *satisfying the corresponding jump and boundary conditions* (1.6)–(1.7)*, so that*

$$\|\eta_n(0)\|_{H^j(\mathbb{R}^2)}+\|u_n(0)\|_{H^j(\Omega_\pm)}+\|p_n(0)\|_{H^j(\Omega_\pm)}\leq\frac{1}{n},\tag{2.3}$$

but

$$\|u_n(t)\|_{H^k(\Omega_\pm)}\geq\alpha\quad\textit{for all } t\geq t_n:=C_{j,k}+C_1\ln(\alpha n).\tag{2.4}$$

Moreover,

$$\|u_n(t)\|_{H^k(\Omega_\pm)}\to\infty\quad\textit{as } t\to\infty.\tag{2.5}$$

Theorem 2.1 shows globally discontinuous dependence of solutions upon initial data. The proof of Theorem 2.1 is inspired by [8] under necessary modifications and its basic idea is the following. First, we notice that the linearized equations have coefficients that depend only on the vertical variable $x_3\in(-1,1)$. This allows us to seek "normal mode" solutions by taking the horizontal Fourier transform of the equations and assuming that the solutions grow exponentially in time by the factor $\mathrm{e}^{\lambda(|\xi|)t}$, where $\xi\in\mathbb{R}^2$ is the horizontal spatial frequency and $\lambda(|\xi|)>0$. This reduces the equations to a system of ordinary differential equations with $\lambda(|\xi|)>0$ for each ξ. Then, solving the ODE system by the modified variational method, we show that $\lambda(|\xi|)>0$ is a continuous function on $(0,|\xi|_c)$, and the normal modes with spatial frequency grow in time, providing a mechanism for the Rayleigh-Taylor global instability, where $|\xi|_c=\sqrt{\frac{g[\varrho]}{\kappa}}$ if $\kappa>0$, otherwise $|\xi|_c=\infty$. Indeed, we can restrict ξ in some annulus domain, such that $\lambda(|\xi|)$ has a uniformly lower bound, and then we form a Fourier synthesis of the normal mode solutions constructed for each spatial frequency ξ to give solutions to the linearized incompressible equations that grow in time, when measured in $H^k(\Omega)$ for any $k\geq0$. Finally, we exploit the property of the boundary trace theorem to show a uniqueness result of the linearized problem (i.e., Theorem 4.1), with the help of which we obtain the global instability of the corresponding nonlinear problem (i.e., Theorem 2.2). In spite of the uniqueness, the linearized problem is globally unstable in $H^k(\Omega)$ for any k in the sense of Hadamard.

With the linear global instability established, we can show the global instability of the corresponding nonlinear problem in some sense. Recalling that the steady state solution to (1.4) is given by $\eta = 0$, $u = 0$, $p =$ constant, we now rewrite the nonlinear equations (1.4) in the form of perturbation around the steady state. Let

$$\eta = 0 + \eta, \quad u = 0 + u, \quad p = \text{constant} + \sigma.$$

Then, the system (1.4) can be rewritten for (η, u, σ) as

$$\begin{cases} \partial_t \eta = u_3 - u_1 \partial_1 \eta - u_2 \partial_2 \eta, \\ \varrho \partial_t u + \varrho(\nabla u)u + \nabla \sigma = \mu \Delta u, \\ \mathrm{div} u = 0. \end{cases} \tag{2.6}$$

The jump conditions across the interface are

$$[u]|_{\Sigma(t)} = 0, \tag{2.7}$$

$$[(\sigma I - \mu(\nabla u + \nabla u^{\mathrm{T}}))\nu]|_{\Sigma(t)} = (g[\varrho]\eta + \kappa H)\nu, \tag{2.8}$$

where

$$\Sigma(t) := \{x \in \mathbb{R}^3 \mid x_3 = \eta(t, x')\} \subset \Omega \quad \text{ for each } t \geq 0.$$

Finally, we require the boundary condition

$$u_-(t, x', -1) = u_+(t, x', 1) = 0. \tag{2.9}$$

We collectively refer to the evolution, jump, and boundary equations (2.6)–(2.9) as "the perturbed problem".

Definition 2.1 *We say that the perturbed problem has global stability of order k for some $k \geq 3$ if there exist $\delta, C_2 > 0$ and a function $F : [0, \delta) \to \mathbb{R}^+$ satisfying $F(z) \leq C_2 z$ for $z \in [0, \delta)$, so that the following holds: For any $T > 0$, η_0, u_0 satisfying*

$$\|\eta_0\|_{H^k(\mathbb{R}^2)} + \|u_0\|_{H^k(\Omega_\pm(0))} < \delta,$$

there exist $\eta(t) \in H^2(\mathbb{R}^2) \cap C^{0,1}_{\mathrm{loc}}(\mathbb{R}^2)$, $u(t) \in (H^3(\Omega_\pm(t)) \cap C^0(\overline{\Omega}) \cap H^1_0(\Omega))^3$ and $\sigma \in H^1(\Omega_\pm(t))$ for any $t \in [0, T]$, so that

(1) $(\eta, u)(0) = (\eta_0, u_0)$,

(2) *η, u, σ solve the perturbed problem* (2.6)–(2.9),

(3) $\eta \in C^0([0, T], L^2_{\mathrm{loc}}(\mathbb{R}^2))$ *and* $u \in C^0([0, T], (L^2(\Omega))^3)$,

(4) *it holds that*

$$\sup_{0 \leq t \leq T} (\|u\|_{H^3(\Omega_\pm(t))} + \|\eta\|_{H^2(\mathbb{R}^2)} + \|\sigma\|_{H^1(\Omega_\pm(t))}) \leq F(\|\eta_0\|_{H^k(\mathbb{R}^2)} + \|u_0\|_{H^k(\Omega_\pm(0))}).$$

The condition for global stability of order k is quite general and is a reasonable choice for any global stability theory. The important feature of global stability of order k is that $k \geq 3$ is arbitrary. If the initial data are extremely smooth (k very large), the failure of property $EE(k)$ means that it is impossible to control even the norm of $\sup_{0\le t\le T}(\|u\|_{H^3(\Omega_\pm(t))}+\|\eta\|_{H^2(\mathbb{R}^2)}+\|\sigma\|_{H^1(\Omega_\pm(t))})$ for all $T>0$. Theorem 2.1 shows that the velocity u results in the linear instability. However, it is still an open problem to show the nonlinear instability of u due to technical difficulties. In this paper, we can show that the property of global stability of order k can not hold for any $k\ge 3$, i.e., the following Theorem 2.2, which will be proved in Section 5.

Theorem 2.2 *The perturbed problem does not have the property of global stability of order k for any $k\ge 3$.*

The basic idea in the proof of Theorem 2.2 is to show, by utilizing the Lipschitz structure of F, that the global stability of order k would give rise to certain estimates of solutions to the linearized equations (1.5) that can not hold in general because of Theorem 2.1. We will adapt and modify the arguments in [6] to prove Theorem 2.2. Compared with the perturbed problem in [6, Theorem 5.2] where the Lagrangian coordinates were used, our problem here is coupled to a free interface, rather than the fixed interface $\{x_3=0\}$. As is well-known, the motion of the free surface $\Sigma(t)$ and the domains $\Omega_\pm(t)$ present several mathematical difficulties, so the authors [6] switched the perturbed problem in Eulerian coordinates to a perturbed problem in Lagrangian coordinates, in which the free interface is switched to the fixed interface $\{x_3=0\}$, while the domains of the upper and lower fluids stay fixed in time as $\Omega_+=\mathbb{R}^2\times(0,1)$ and $\Omega_-=\mathbb{R}^2\times(-1,0)$, respectively. Thus, the convergence for the jump conditions of the rescaled functions can be easily dealt with at the fixed interface in the proof of [6, Theorem 5.2]. To circumvent such difficulties without the aid of the transform of Lagrangian coordinates, in a way similar to [15], we transform the perturbed equation in the second line of (2.6) to the integral form. Indeed, multiplying the second equation of (2.6) by ϕ, integrating by parts over $(0,t_0)\times\Omega$, and using (2.7)–(2.9) together with the formula of surface integral, we obtain

$$\begin{aligned}&\int_0^{t_0}\int_\Omega(\varrho\partial_t u\cdot\phi+\varrho(\nabla u)u\cdot\phi)\mathrm{d}x\mathrm{d}t+\int_0^{t_0}\int_\Omega(\mu(\nabla u+\nabla u^{\mathrm{T}})-\sigma I):\nabla\phi\mathrm{d}x\mathrm{d}t\\&=\int_0^{t_0}\int_{\mathbb{R}^2}(g[\varrho]\eta+\kappa H)\phi(t,x',\eta(t,x'))\cdot(-\partial_1\eta,-\partial_2\eta,1)\mathrm{d}x'\mathrm{d}t,\end{aligned}$$

where $\phi:=(\phi_1,\phi_2,\phi_3)\in(\mathcal{D}'((0,T)\times\Omega))^3$, and

$$(\mu(\nabla u+\nabla u^{\mathrm{T}})-\sigma I):\nabla\phi=\sum_{1\le i,j\le 3}(\mu(\partial_j u_i+\partial_i u_j)-\sigma\delta_{ij})\partial_j\phi_i.$$

In this manner, we have transformed the two-fluid flow into a single-fluid flow in a fixed domain, which offers an alternative approach to deal with the jump condition (2.8) at the free interface $\Sigma(t)$, instead of using the method of Lagrangian coordinates in [6, 8]. Consequently, we can avoid the proof of convergence for the jump conditions of the rescaled viscous stress-tensor at the free boundary. This transform will play an important role in the proof of Theorem 2.2 in Section 5. Moreover, this idea is also applied to the proof of the uniqueness of solutions to the linearized equations (1.5) in Section 4.

We mention that Guo and Tice [8] recently proved the linear global instability for compressible viscous fluids in Lagrangian coordinates, while in the current paper the nonlinear global instability for incompressible viscous fluids in the sense of Definition 2.1 is established in Eulerian coodinates. Prüess and Simonett [17] developed another coordinate transformation to transform the free boundary problem (1.1) in $\Omega = \mathbb{R}^3$ to a fixed boundary problem (1.1), and then proved the nonlinear instability in the abstract Sobolev-Slobodeckii spaces for the transformed problem. Later, Tice and Wang [21] provided an alternative proof to show the nonlinear instability in the natural energy space for the horizontally periodic setting. It should be noted that we does not use any coordinate transformation in this paper. Finally, we point out that Hwang and Guo [9] constructed an unstable solution to show mathematically the Rayleigh-Taylor instability for two-dimensional incompressible inviscid flows when the density is continuous. It still needs further study whether we can construct a concrete solution to (2.6)–(2.9) which does not have global stability of order k.

3 Construction of a Growing Solution to the Linearized Equations

3.1 Growing mode ansatz

We wish to construct a solution to the linearized equations (1.5) that has a growing H^k-norm for any k. We will construct such solutions via Fourier synthesis by first constructing a growing mode for the fixed spatial frequency.

To begin with, we make a growing mode ansatz, i.e., let us assume that

$$\eta(t,x') = \widetilde{\eta}(x')\mathrm{e}^{\lambda t}, \quad u(t,x) = v(x)\mathrm{e}^{\lambda t}, \quad p(t,x) = q(x)\mathrm{e}^{\lambda t} \quad \text{for some } \lambda > 0.$$

Substituting this ansatz into (1.5), and eliminating $\widetilde{\eta}$ by using the first equation, we arrive at the time-invariant system for $v = (v_1, v_2, v_3)$ and q:

$$\begin{cases} \lambda\varrho v + \nabla q = \mu\Delta v, \\ \operatorname{div} v = 0 \end{cases} \tag{3.1}$$

with the corresponding jump conditions

$$[\![v]\!] = 0, \quad [\![qI - \mu\nabla(v + v^{\mathrm{T}})]\!]e_3 = \lambda^{-1}(g[\varrho]v_3 + \kappa\Delta_{x'}v_3)e_3 \tag{3.2}$$

and boundary conditions

$$v(t,x',-1) = v(t,x',1) = 0. \tag{3.3}$$

3.2 Horizontal Fourier transform

We take the horizontal Fourier transform of (v_1, v_2, v_3) in (3.1), which we denote with either $\widehat{\cdot}$ or $\mathcal{F}$, and fix a spatial frequency $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Define the new unknowns

$$\varphi(x_3) = \mathrm{i}\widehat{v}_1(\xi, x_3), \quad \theta(x_3) = \mathrm{i}\widehat{v}_2(\xi, x_3), \quad \psi(x_3) = \widehat{v}_3(\xi, x_3), \quad \pi(x_3) = \widehat{q}(\xi, x_3),$$

so that

$$\mathcal{F}(\mathrm{div} w) = \xi_1\varphi + \xi_2\theta + \psi',$$

where $' = \frac{\mathrm{d}}{\mathrm{d}x_3}$. Then, for φ, θ, ψ and $\lambda = \lambda(\xi)$, we can deduce from (3.1)–(3.3) that

$$\begin{cases} \lambda\varrho\varphi - \xi_1\pi + \mu(|\xi|^2\varphi - \varphi'') = 0, \\ \lambda\varrho\theta - \xi_2\pi + \mu(|\xi|^2\theta - \theta'') = 0, \\ \lambda\varrho\psi + \pi' + \mu(|\xi|^2\psi - \psi'') = 0, \\ \xi_1\varphi + \xi_2\theta + \psi' = 0 \end{cases} \tag{3.4}$$

along with the jump conditions

$$\begin{cases} [\![\varphi]\!] = [\![\theta]\!] = [\![\psi]\!] = 0, \\ [\![\mu(\xi_1\psi - \varphi')]\!] = [\![\mu(\xi_2\psi - \theta')]\!] = 0, \\ [\![-2\mu\lambda\psi' + \lambda\pi]\!] = (g[\varrho] - \kappa|\xi|^2)\psi \end{cases} \tag{3.5}$$

and the boundary conditions

$$\varphi(-1) = \varphi(1) = \theta(-1) = \theta(1) = \psi(-1) = \psi(1) = 0. \tag{3.6}$$

Eliminating π from the third equation in (3.4), we obtain the following ODE for ψ:

$$-\lambda\rho(|\xi|^2\psi - \psi'') = \mu(|\xi|^4\psi - 2|\xi|^2\psi'' + \psi'''') \tag{3.7}$$

along with the jump conditions

$$[\![\psi]\!] = [\![\psi']\!] = 0, \tag{3.8}$$

$$[\![\mu(|\xi|^2\psi + \psi'')]\!] = 0, \tag{3.9}$$

$$[\![\mu\lambda(\psi''' - 3|\xi|^2\psi')]\!] = [\![\lambda^2\varrho\psi']\!] + (g[\rho] - \kappa|\xi|^2)|\xi|^2\psi \tag{3.10}$$

and the boundary conditions

$$\psi(-1) = \psi(1) = \psi'(-1) = \psi'(1) = 0. \tag{3.11}$$

3.3 Construction of a solution to the fourth order ODE

In a way similar to [8, 23], we can apply the variational methods to construct solutions to (3.7)–(3.11). The idea of the proof can be found in the pioneering paper due to Guo and Tice [8], which was later adapted by Wang [23]. For the reader's convenience, we sketch the outline of the construction.

First, fix a non-zero vector $\xi \in \mathbb{R}^2$ and $s > 0$. From (3.7)–(3.11), we get a family of the modified problems

$$-\lambda^2\rho(|\xi|^2\psi - \psi'') = s\mu(|\xi|^4\psi - 2|\xi|^2\psi'' + \psi'''') \tag{3.12}$$

along with the jump conditions

$$[\![\psi]\!] = [\![\psi']\!] = 0, \tag{3.13}$$

$$[\![s\mu(|\xi|^2\psi + \psi'')]\!] = 0, \tag{3.14}$$

$$[\![s\mu\lambda(\psi''' - 3|\xi|^2\psi')]\!] = [\![\lambda^2\varrho\psi']\!] + (g[\rho] - \kappa|\xi|^2)|\xi|^2\psi \tag{3.15}$$

and the boundary conditions

$$\psi(-1) = \psi(1) = \psi'(-1) = \psi'(1) = 0. \tag{3.16}$$

We define the energy functional of (3.12) by

$$E(\psi) = \frac{1}{2}\int_{-1}^{1} s\mu(4|\xi|^2|\psi'|^2 + ||\xi|^2\psi + \psi''|^2)\mathrm{d}x_3 - \frac{1}{2}|\xi|^2(g[\rho] - \kappa|\xi|^2)|\psi(0)|^2 \tag{3.17}$$

with the associated admissible set

$$\mathcal{A} = \Big\{\psi \in H_0^2(-1,1) \;\Big|\; \int_{-1}^{1} \rho(|\xi|^2|\psi|^2 + |\psi'|^2)\mathrm{d}x_3 = 2\Big\}, \tag{3.18}$$

where $H_0^2(-1,1)$ is the subset of $H^2(-1,1)$ satisfying (3.16). Thus we can find a $-\lambda^2$ by minimizing

$$-\lambda^2(|\xi|, s) = \alpha(|\xi|, s) := \inf_{\psi\in\mathcal{A}} E(\psi). \tag{3.19}$$

In fact, we can show that a minimizer of (3.19) exists, and that the minimizer satisfies Euler-Lagrange equations equivalent to (3.12)–(3.16).

Proposition 3.1 *For any fixed $\xi \neq 0$ and $s > 0$, E achieves its infinimum on $\mathcal{A}$. In addition, let ψ be a minimizer and $-\lambda^2 := E(\psi)$, and then the pair (ψ, λ^2) satisfies* (3.12) *along with the jump and boundary conditions* (3.13)–(3.16)*. Moreover, ψ is smooth when restricted to* $(-1,0)$ *or* $(0,1)$.

Proof We can follow the same proof procedure as in [8, Propostition 3.2] (or [23, Proposition 3.1]) to show Proposition 3.1. Hence, we omit the details of the proof here.

Next, we want to prove that there is a fixed point, such that $\lambda = s$. To this end, we shall give some properties of $\alpha(s)$ as a function of $s > 0$.

Proposition 3.2 *$\alpha(s) \in C^{0,1}_{\mathrm{loc}}(0,\infty)$ is strictly increasing. Moreover,*

(1) *for any $a, b \in (0, |\xi|_c)$ with $a < b$, there exist constants $c_1, c_2 > 0$ depending on $\varrho_\pm$, $\mu_\pm$, g, a and b, such that*

$$\alpha(s) \le -c_1 + sc_2 \quad \textit{for all } |\xi| \in [a,b], \tag{3.20}$$

where

$$|\xi|_c := \begin{cases} \sqrt{\frac{g[\varrho]}{\kappa}}, & \textit{if } \kappa > 0, \\ +\infty, & \textit{if } \kappa = 0; \end{cases}$$

(2) *there exist constants $c_3 > 0$ depending on $\varrho_\pm$ and g, $c_4 > 0$ depending additionally on $\mu_\pm$ and $|\xi|$, such that*

$$\alpha(s) \ge -c_3|\xi| + sc_4.$$

Proof We refer to [8, Propostition 3.6] (or [23, Lemma 3.5]) for the proof.

Given $\xi \in \mathbb{R}^2$ with $|\xi| \in (0, |\xi|_c)$, by virtue of (3.20), there exists an $s_0 > 0$ depending on the quantities $\varrho_\pm$, $\mu_\pm$, g, $|\xi|$, so that for $s \le s_0$, it holds that $\alpha(s) < 0$. Hence, we can define the open set

$$\mathcal{S} = \alpha^{-1}(-\infty, 0) \subset (0,\infty).$$

Note that $\mathcal{S}$ is non-empty and this allows us to define $\lambda(s) = \sqrt{-\alpha(s)}$ for $s \in \mathcal{S}$. Therefore, as a result of Proposition 3.1, we have the following existence result for the modified problem (3.12)–(3.16).

Proposition 3.3 *For each* $\xi \in \mathbb{R}^2$ *with* $|\xi| \in (0, |\xi|_c)$ *and each* $s \in \mathcal{S}$*, there exists a solution* $\psi = \psi(|\xi|, x_3)$ *with* $\lambda = \lambda(|\xi|, s) > 0$ *to the problem* (3.12) *with the jump and boundary conditions* (3.13)–(3.16). *Moreover,* ψ *is smooth when restricted to* $(-1, 0)$ *or* $(0, 1)$ *with* $\psi(|\xi|, 0) \neq 0$.

Finally, we can use Proposition 3.2 to make a fixed-point argument to find $s \in \mathcal{S}$ such that $s = \lambda(|\xi|, s)$ to construct solutions to the original problem (3.7)–(3.11).

Proposition 3.4 *Let* $\xi \in \mathbb{R}^2$ *with* $|\xi| \in (0, |\xi|_c)$*, and then there exists a unique* $s \in \mathcal{S}$ *so that* $\lambda(|\xi|, s) = \sqrt{-\alpha(s)} > 0$ *and* $s = \lambda(|\xi|, s)$.

Proof We refer to [8, Theorem 3.8] (or [23, Lemma 3.7]) for a proof.

Consequently, in view of Propositions 3.3–3.4, we conclude the following existence result concerning the problem of (3.7)–(3.11).

Theorem 3.1 *For each* $\xi \in \mathbb{R}^2$ *with* $|\xi| \in (0, |\xi|_c)$*, there exist* $\psi = \psi(|\xi|, x_3)$ *and* $\lambda(|\xi|) > 0$ *satisfying* (3.7)–(3.11). *Moreover,* ψ *is smooth when restricted to* $(-1, 0)$ *or* $(0, 1)$ *with* $\psi(|\xi|, 0) \neq 0$.

Next, we show some properties of the solutions established in Theorem 3.1 in terms of $\lambda(|\xi|)$. The first property is given in the following proposition which shows that λ is a bounded continuous function of $|\xi|$.

Proposition 3.5 *The function* $\lambda : (0, |\xi|_c) \to (0, \infty)$ *is continuous and satisfies*

$$\sup_{0<|\xi|<\infty} \lambda(|\xi|) \leq \frac{g[\varrho]}{4\mu_-}. \tag{3.21}$$

Moreover,

$$\lim_{|\xi|\to 0} \lambda(|\xi|) = 0, \tag{3.22}$$

and if $\kappa > 0$*,then also*

$$\lim_{|\xi|\to |\xi|_c} \lambda(|\xi|) = 0. \tag{3.23}$$

Proof The continuity and the limits (3.22)–(3.23) follow from the same arguments as in [8, Proposition 3.9] with the help of (3.7)–(3.21) and Ehrling-Nirenberg-Gagliardo interpolation inequality. To complete the proof, it suffices to show (3.21). For each $|\xi| \in (0, |\xi|_c)$, there exists a function $\psi_{|\xi|} \in \mathcal{A}$ satisfying (3.7)–(3.11), so that $-\lambda^2(|\xi|) = E(\psi_{|\xi|})$. By (3.17), we find that

$$-\lambda^2(|\xi|) = \frac{\lambda(|\xi|)}{2} \int_{-1}^{1} \mu(4|\xi|^2|\psi'_{|\xi|}|^2 + ||\xi|^2\psi_{|\xi|} + \psi''_{|\xi|}|^2)\mathrm{d}x_3 - \frac{1}{2}|\xi|^2(g[\rho] - \kappa|\xi|^2)|\psi_{|\xi|}(0)|^2,$$

which yields

$$2\mu_-|\xi|^2\lambda(|\xi|) \int_{-1}^{1} |\psi'_{|\xi|}|^2\mathrm{d}x_3 \leq \frac{1}{2}|\xi|^2 g[\rho]|\psi_{|\xi|}(0)|^2. \tag{3.24}$$

Using the Hölder inequality, we can bound

$$|\psi_{|\xi|}(0)|^2 = \Big|\int_0^1 \psi'_{|\xi|}\mathrm{d}x_3\Big|^2 \leq \int_0^1 (\psi'_{|\xi|})^2\mathrm{d}x_3. \tag{3.25}$$

Substituting (3.25) into (3.24) gives then

$$|\xi|^2\Big(2\mu_-\lambda(|\xi|) - \frac{1}{2}g[\varrho]\Big)\int_{-1}^1 |\psi'_{|\xi|}|^2\mathrm{d}x_3 \leq 0. \tag{3.26}$$

Consequently, (3.26) implies (3.21), since $\|\psi'_{|\xi|}\|_{L^2(-1,1)} > 0$.

3.4 Construction of a solution to the system (3.4)–(3.6)

A solution to (3.7)–(3.11) gives rise to a solution to the system (3.4)–(3.6) for the growing mode velocity v, as well.

Theorem 3.2 *For each $\xi \in \mathbb{R}^2$ with $|\xi| \in (0,|\xi|_c)$, there exists a solution $(\widetilde{\varphi},\widetilde{\theta},\widetilde{\psi},\widetilde{\pi}) = (\widetilde{\varphi}(\xi,x_3),\widetilde{\theta}(\xi,x_3),\widetilde{\psi}(\xi,x_3),\widetilde{\pi}(\xi,x_3))$ with $\lambda = \lambda(|\xi|) > 0$ to (3.4)–(3.6), and the solution is smooth when restricted to $(-1,0)$ or $(0,1)$. Moreover,*

$$\|\widetilde{\varphi}\|^2_{L^2(-1,1)} + \|\widetilde{\theta}\|^2_{L^2(-1,1)} + \|\widetilde{\psi}\|^2_{L^2(-1,1)} = 1, \tag{3.27}$$

$$\|\widetilde{\psi}'\|_{L^2(-1,1)} \leq |\xi|\sqrt{2\varrho^+\varrho_-^{-1}}. \tag{3.28}$$

Proof By Theorem 3.1, we first construct a solution $(\psi,\lambda) = (\psi(|\xi|,x_3),\lambda(|\xi|))$ satisfying (3.7)–(3.11). Moreover, $\lambda > 0$ and $\psi \in \mathcal{A}$ is smooth when restricted to $(-1,0)$ or $(0,1)$. Then, multiplying the first and second equations in (3.4) by ξ_1 and ξ_2 respectively, adding the resulting equations, and utilizing the fourth equation in (3.4), we find that π can be expressed by ψ, i.e.,

$$\pi = \pi(|\xi|,x_3) = [\mu\psi'' - (\lambda\varrho + \mu|\xi|^2)\psi']|\xi|^{-2}. \tag{3.29}$$

Notice that the first equation in (3.4) can be rewritten as

$$\varphi'' - (\lambda\varrho + \mu|\xi|^2)\frac{\varphi}{\mu} = -\xi_1\frac{\pi}{\mu} \tag{3.30}$$

with jump and boundary conditions

$$[\![\varphi]\!] = 0, \quad [\![\mu(\xi_1\psi - \varphi')]\!] = 0, \quad \varphi(-1) = \varphi(1) = 0. \tag{3.31}$$

Hence, we can easily construct a unique solution

$$\varphi = (\xi,x_3) = \begin{cases} \xi_1(c_1\mathrm{e}^{a_+x_3} + c_2\mathrm{e}^{-a_+x_3} - f_+(x_3)), & \text{on } (0,1), \\ \xi_1(c_3\mathrm{e}^{a_-x_3} + c_4\mathrm{e}^{-a_-x_3} - f_-(x_3)), & \text{on } (-1,0) \end{cases} \tag{3.32}$$

to the equation (3.30) with jump and boundary conditions (3.31), where

$$a_\pm = \sqrt{|\xi|^2 + \frac{\lambda\varrho}{\mu_\pm}},$$

$$f_\pm(x_3) = \frac{1}{2a_\pm\mu_\pm}\int_0^{x_3} \pi(\mathrm{e}^{a_\pm(x_3-y)} - \mathrm{e}^{a_\pm(y-x_3)})\mathrm{d}y$$

and

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ (\mu_+ - \mu_-)\psi(0) \\ f(1) \\ f(-1) \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 & -1 \\ \mu_+ a_+ & -\mu_+ a_+ & -\mu_- a_- & \mu_- a_- \\ \mathrm{e}^{a_+} & \mathrm{e}^{-a_+} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-a_-} & \mathrm{e}^{a_-} \end{bmatrix}^{-1}.$$

In the way similar to (3.32),

$$\theta := \theta(\xi, x_3) = \begin{cases} \xi_2(c_1 \mathrm{e}^{a_+ x_3} + c_2 \mathrm{e}^{-a_+ x_3} - f_+(x_3)), & \text{on } (0,1), \\ \xi_2(c_3 \mathrm{e}^{a_- x_3} + c_4 \mathrm{e}^{-a_- x_3} - f_-(x_3)), & \text{on } (-1,0) \end{cases}$$

is a unique solution to the second equation in (3.4) with jump and boundary conditions:

$$[\![\theta]\!] = 0, \quad [\![\mu(\xi_2\psi - \theta')]\!] = 0, \quad \theta(-1) = \theta(1) = 0.$$

Consequently, $(\varphi, \theta, \psi, \pi)$ is a solution to the system (3.4)–(3.6). Now, we define

$$\begin{aligned} (\widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi}, \widetilde{\pi}) &:= (\widetilde{\varphi}(\xi, x_3), \widetilde{\theta}(\xi, x_3), \widetilde{\psi}(\xi, x_3), \widetilde{\pi}(\xi, x_3)) \\ &:= \frac{(\varphi, \theta, \psi, \pi)}{(\|\varphi\|^2_{L^2(-1,1)} + \|\theta\|^2_{L^2(-1,1)} + \|\psi\|^2_{L^2(-1,1)})}. \end{aligned}$$

Thus, $(\widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi}, \widetilde{\pi})$ is still a solution to the system (3.4)–(3.6), and moreover, $(\widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi}, \widetilde{\pi})$ satisfies (3.27).

Finally, making use of the fourth equation in (3.4) and (3.18), we conclude that

$$\begin{aligned} \frac{1}{\varrho_+ |\xi|^2} &= \frac{1}{2\varrho_+ |\xi|^2} \int_{-1}^{1} \varrho(|\xi|^2 |\psi|^2 + |\psi'|^2) \mathrm{d}x_3 \\ &\leq \int_{-1}^{1} (|\varphi|^2 + |\theta|^2 + |\psi|^2) \mathrm{d}x_3 \\ &= \|\varphi\|^2_{L^2(-1,1)} + \|\theta\|^2_{L^2(-1,1)} + \|\psi\|^2_{L^2(-1,1)} \end{aligned}$$

and

$$\int_{-1}^{1} |\psi'|^2 \mathrm{d}x_3 \leq \frac{2}{\varrho_-}.$$

The above two inequalities imply (3.28) immediately.

Remark 3.1 For each x_3, it is easy to see that the solution $(\widetilde{\varphi}(\xi, \cdot), \widetilde{\theta}(\xi, \cdot), \widetilde{\psi}(\xi, \cdot), \widetilde{\pi}(\xi, \cdot), \lambda(|\xi|))$ constructed in Theorem 3.2 has the following properties:

(1) $\lambda(|\xi|)$, $\widetilde{\psi}(\xi, \cdot)$ and $\widetilde{\pi}(\xi, \cdot)$ are even on ξ_1 or ξ_2, when the other variable is fixed;

(2) $\widetilde{\varphi}(\xi, \cdot)$ is odd on ξ_1, but even on ξ_2, when the other variable is fixed;

(3) $\widetilde{\theta}(\xi, \cdot)$ is even on ξ_1, but odd on ξ_2, when the other variable is fixed.

To directly estimate the H^k norm of the solution ψ from (3.7), without use of (3.17) and the continuity of λ, we shall apply the following Ehrling-Nirenberg-Gagliardo interpolation inequality, the proof of which can be found in [1, Chapter 5].

Lemma 3.1 *Let Ω be a domain in $\mathbb{R}^n$ satisfying the cone condition. For each $\varepsilon_0 > 0$, there exists a constant K depending on n, m, p and ε_0, such that if $0 < \varepsilon \leq \varepsilon_0$, $0 \leq j \leq m$ and $u \in W^{m,p}(\Omega)$, then*

$$\sum_{|\alpha|=j} \int_\Omega |D^\alpha u(x)|^p \mathrm{d}x \leq K\Big(\varepsilon \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^p \mathrm{d}x + \varepsilon^{-\frac{j}{m-j}} \int_\Omega |u|^p \mathrm{d}x\Big).$$

The next lemma provides an estimate for the H^k norm of the solution $(\widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi}, \widetilde{\pi})$ with ξ varying, which will be useful in the next section when such a solution is integrated in a Fourier synthesis. To emphasize the dependence on ξ, we write these solutions as $(\widetilde{\varphi}(\xi) = \widetilde{\varphi}(\xi, x_3),\ \widetilde{\theta}(\xi) = \widetilde{\theta}(\xi, x_3),\ \widetilde{\psi}(\xi) = \widetilde{\psi}(\xi, x_3),\ \widetilde{\pi}(\xi) = \widetilde{\pi}(\xi, x_3))$.

Lemma 3.2 *Let $\xi \in \mathbb{R}^2$ with $0 < R_1 < |\xi| < |\xi|_c$, $\varphi(\xi) := \widetilde{\varphi}(\xi)$, $\theta(\xi) := \widetilde{\theta}(\xi)$, $\psi(\xi) := \widetilde{\psi}(\xi)$, $\pi(\xi) := \widetilde{\pi}(\xi)$ and $\lambda(|\xi|)$ be constructed as in Theorem 3.2, and then for any $k \geq 0$, there exist positive constants a_k, b_k and c_k depending on R_1, $\varrho_\pm$, $\mu_\pm$ and g, so that*

$$\|\varphi(\xi)\|_{H^k(-1,0)} + \|\varphi(\xi)\|_{H^k(0,1)} + \|\theta(\xi)\|_{H^k(-1,0)} + \|\theta(\xi)\|_{H^k(0,1)} \leq a_k \sum_{j=0}^{k+1} |\xi|^j, \tag{3.33}$$

$$\|\psi(\xi)\|_{H^k(-1,0)} + \|\psi(\xi)\|_{H^k(0,1)} \leq b_k \sum_{j=0}^{k} |\xi|^j, \tag{3.34}$$

$$\|\pi(\xi)\|_{H^k(-1,0)} + \|\pi(\xi)\|_{H^k(0,1)} \leq c_k \sum_{j=0}^{k+1} |\xi|^j. \tag{3.35}$$

Moreover,

$$\|\varphi\|^2_{L^2(-1,1)} + \|\theta\|^2_{L^2(-1,1)} + \|\psi\|^2_{L^2(-1,1)} = 1. \tag{3.36}$$

Proof Throughout this proof, we denote by $\widetilde{c}_1, \cdots, \widetilde{c}_9$ generic positive constants which may depend on R_1, $\varrho_\pm$, $\mu_\pm$ and g, but not on $|\xi|$.

(i) First, (3.36) follows from (3.27) immediately. We now write (3.7) as

$$\psi''''(\xi) = \big[(\lambda\varrho + 2\mu|\xi|^2)\psi''(\xi) - (\lambda\varrho|\xi|^2 + \mu|\xi|^4)\psi(\xi)\big]\frac{1}{\mu}. \tag{3.37}$$

If we make use of (3.21), $|\xi| > R_1$ and Lemma 3.1, then we see that there exists a couple $(\widetilde{c}_1, \widetilde{c}_2)$, such that

$$\begin{aligned}\|\psi''''(\xi)\|_{L^2(I_\pm)} &\leq \widetilde{c}_1[(|\xi|^2 + |\xi|^4)\|\psi(\xi)\|_{L^2(I_\pm)} + (1 + |\xi|^2)\|\psi''(\xi)\|_{L^2(I_\pm)}] \\ &\leq (\widetilde{c}_2 + 1)[(\varepsilon^{-\frac{1}{2}} + \varepsilon^{-\frac{1}{2}}|\xi|^4 + |\xi|^2 + |\xi|^4)\|\psi(\xi)\|_{L^2(I\pm)} \\ &\quad + \sqrt{\varepsilon}\|\psi''''(\xi)\|_{L^2(I_\pm)}] \quad \text{for any } \varepsilon \in (0,1),\end{aligned} \tag{3.38}$$

respectively, where $I_+ = (0,1)$ and $I_- = (-1,0)$. Choosing $\sqrt{\varepsilon} = \frac{1}{2(\widetilde{c}_2+1)}$ in (3.38) and using (3.36), we arrive at

$$\|\psi''''(\xi)\|_{L^2(I_\pm)} \leq \widetilde{c}_3(1 + |\xi|^2 + |\xi|^4) \quad \text{for some } \widetilde{c}_3 > 0,$$

whence,

$$\|\psi''''(\xi)\|_{L^2(-1,1)} \leq \widetilde{c}_4(1+|\xi|^2+|\xi|^4). \tag{3.39}$$

Writing (3.37) as

$$\psi''(\xi) = \frac{\mu\psi''''(\xi)+(\lambda\varrho|\xi|^2+\mu|\xi|^4)\psi(\xi)}{(\lambda\varrho+2\mu|\xi|^2)},$$

we utilize (3.39) and (3.36) to get

$$\|\psi''(\xi)\|_{L^2(-1,1)} \leq \widetilde{c}_5(1+|\xi|^2). \tag{3.40}$$

Differentiating (3.37) with respect to x_3, we see that

$$\psi^{(5)}(\xi) = [(\lambda\varrho+2\mu|\xi|^2)\psi'''(\xi)-(\lambda\varrho|\xi|^2+\mu|\xi|^4)\psi'(\xi)]\frac{1}{\mu}.$$

In a way similar to (3.39)–(3.40), we obtain, by Lemma 3.1, (3.21), (3.28) and (3.36), that

$$\|\psi^{(5)}(\xi)\|_{L^2(-1,1)} \leq \widetilde{c}_6(1+|\xi|^2+|\xi|^4+|\xi|^5) \tag{3.41}$$

and

$$\|\psi'''(\xi)\|_{L^2(-1,1)} \leq \widetilde{c}_7(1+|\xi|^2+|\xi|^3). \tag{3.42}$$

Summarizing the estimates (3.28), (3.36) and (3.39)–(3.42), we conclude that, for each nonnegative integer $k\in[0,5]$, there exists a constant $\widetilde{b}_k>0$ depending on R_1, $\varrho_\pm$, $\mu_\pm$ and g, such that

$$\|\psi^{(k)}(\xi)\|_{L^2(-1,1)} \leq \widetilde{b}_k\sum_{j=0}^{k}|\xi|^j. \tag{3.43}$$

Differentiating (3.37) with respect to x_3 and using (3.43), we find, by induction on k, that (3.43) holds for any $k\geq 0$. This gives (3.34).

(ii) Recalling the expression (3.29) of π and the fact that $|\xi|>R_1$, we employ (3.34) to deduce that for any $k\geq 0$,

$$\begin{aligned}\|\pi^{(k)}(\xi)\|_{L^2(-1,1)} &\leq \frac{\mu^+}{|\xi|^2}\|\psi^{(k+3)}(\xi)\|_{L^2(-1,1)}+\Big(\frac{g[\varrho]\varrho^+}{4\mu_-|\xi|^2}+\mu^+\Big)\|\psi^{(k+1)}(\xi)\|_{L^2(-1,1)}\\ &\leq \max\Big\{\frac{\mu^+\widetilde{b}_{k+3}}{R_1^2}+\frac{\mu^+\widetilde{b}_{k+3}}{R_1}+\mu^+\widetilde{b}_{k+3}+\Big(\frac{g[\varrho]\varrho^+}{4\mu_-R_1^2}+\mu^+\Big)\widetilde{b}_{k+1},\\ &\qquad \mu+\widetilde{b}_{k+3}+\Big(\frac{g[\varrho]\varrho^+}{4\mu_-R_1^2}+\mu^+\Big)\widetilde{b}_{k+1}\Big\}\sum_{j=0}^{k+1}|\xi|^j,\end{aligned}$$

which implies (3.35).

(iii) From (3.21), (3.30), (3.35)–(3.36), we get

$$\begin{aligned}\|\varphi''(\xi)\|_{L^2(-1,1)} &\leq \frac{|\xi|}{\mu_-}\|\pi(\xi)\|_{L^2(-1,1)}+\Big(\frac{g[\varrho]\varrho^+}{4\mu_-^2}+|\xi|^2\Big)\|\varphi(\xi)\|_{L^2(-1,1)}\\ &\leq \widetilde{c}_8(1+|\xi|+|\xi|^2).\end{aligned} \tag{3.44}$$

Applying (3.44), (3.36) and Lemma 3.1, we obtain

$$\|\varphi'(\xi)\|_{L^2(-1,0)} + \|\varphi'(\xi)\|_{L^2(0,1)} \leq \widetilde{c}_9(1 + |\xi| + |\xi|^2). \tag{3.45}$$

Combining (3.36) with (3.44)–(3.45), we conclude that, for each nonnegative integer $k \in [0, 2]$, there exists a constant $\widetilde{a}_k > 0$ depending on R_1, $\varrho_\pm$, $\mu_\pm$ and g, so that

$$\|\varphi^{(k)}(\xi)\|_{L^2(-1,1)} \leq \widetilde{a}_k \sum_{j=0}^{k+1} |\xi|^j. \tag{3.46}$$

Thus, by virtue of (3.30), (3.46) and induction on k, (3.46) holds for any $k \geq 0$. Following the same procedure as used in estimating φ, we infer that for each $k \geq 0$,

$$\|\theta^{(k)}(\xi)\|_{L^2(-1,1)} \leq \widetilde{d}_k \sum_{j=0}^{k+1} |\xi|^j \tag{3.47}$$

for some constant $\widetilde{d}_k$ depending on R_1, $\varrho_\pm$, $\mu_\pm$ and g. Adding (3.47) to (3.46), we arrive at

$$\|\varphi^{(k)}(\xi)\|_{L^2(-1,1)} + \|\theta^{(k)}(\xi)\|_{L^2(-1,1)} \leq (\widetilde{a}_k + \widetilde{d}_k) \sum_{j=0}^{k+1} |\xi|^j \quad \text{for any } k \geq 0,$$

which yields (3.33). This completes the proof.

3.5 Fourier synthesis

In this section, we will use the Fourier synthesis to build growing solutions to (1.5) out of the solutions constructed in the previous section (Theorem 3.2) for the fixed spatial frequency $\xi \in \mathbb{R}^2$ with $|\xi| \in (0, |\xi|_c)$. The constructed solutions will grow in time in the piecewise Sobolev space of order k, $H^k(\Omega_\pm)$, defined by (2.1).

Theorem 3.3 *Let $0 < R_1 < R_2 < |\xi|_c$ and $f \in C_0^\infty(R_1, R_2)$ be a real-valued function. For $\xi \in \mathbb{R}^2$ with $|\xi| \in (0, |\xi|_c)$, define*

$$v(\xi, x_3) = -\mathrm{i}\varphi(\xi, x_3)e_1 - \mathrm{i}\theta(\xi, x_3)e_2 + \psi(\xi, x_3)e_3,$$

where $(\varphi, \theta, \psi, \pi)(\xi, x_3) := (\widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi}, \widetilde{\pi})(\xi, x_3)$ with $\lambda(|\xi|) > 0$ is the solution given by Theorem 3.2. Denote

$$\eta(t, x') = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(|\xi|) v_3(\xi, 0) \mathrm{e}^{\lambda(|\xi|)t} \mathrm{e}^{\mathrm{i}x'\xi} \mathrm{d}\xi, \tag{3.48}$$

$$u(t, x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \lambda(\xi) f(|\xi|) v(\xi, x_3) \mathrm{e}^{\lambda(|\xi|)t} \mathrm{e}^{\mathrm{i}x'\xi} \mathrm{d}\xi, \tag{3.49}$$

$$p(t, x) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \lambda(\xi) f(|\xi|) \pi(\xi, x_3) \mathrm{e}^{\lambda(|\xi|)t} \mathrm{e}^{\mathrm{i}x'\xi} \mathrm{d}\xi. \tag{3.50}$$

Then, (η, u, p) is a real-valued solution to the linearized problem (1.5) *along with the corresponding jump and boundary conditions. For every $k \in \mathbb{N}$, we have the estimate*

$$\begin{aligned} &\|\eta(0)\|_{H^k(\mathbb{R}^2)} + \|u(0)\|_{H^k(\Omega_\pm)} + \|p(0)\|_{H^k(\Omega_\pm)} \\ &\leq \widetilde{c}_k \left(\int_{\mathbb{R}^2} (1 + |\xi|^2)^{k+2} |f(|\xi|)|^2 \mathrm{d}\xi \right)^{\frac{1}{2}} < \infty, \end{aligned} \tag{3.51}$$

where $\widetilde{c}_k > 0$ *is a constant depending on the parameters* $\varrho_\pm$, R_1 *and* g. *Moreover, for every* $t > 0$, *we have* $\eta(t) \in H^k(\mathbb{R}^2)$, $u(t), p(t) \in H^k(\Omega_\pm)$, *and*

$$\mathrm{e}^{t\lambda_0(f)}\|\eta(0)\|_{H^k(\mathbb{R}^2)} \leq \|\eta(t)\|_{H^k(\mathbb{R}^2)} \leq \mathrm{e}^{t\Lambda}\|\eta(0)\|_{H^k(\mathbb{R}^2)}, \tag{3.52}$$

$$\mathrm{e}^{t\lambda_0(f)}\|u(0)\|_{H^k(\Omega_\pm)} \leq \|u(t)\|_{H^k(\Omega_\pm)} \leq \mathrm{e}^{t\Lambda}\|u(0)\|_{H^k(\Omega_\pm)}, \tag{3.53}$$

$$\mathrm{e}^{t\lambda_0(f)}\|p(0)\|_{H^k(\Omega_\pm)} \leq \|p(t)\|_{H^k(\Omega_\pm)} \leq \mathrm{e}^{t\Lambda}\|p(0)\|_{H^k(\Omega_\pm)}, \tag{3.54}$$

where

$$\lambda_0(f) = \inf_{|\xi| \in \mathrm{supp}(f)} \lambda(|\xi|) > 0 \tag{3.55}$$

and

$$\Lambda = \sup_{0<|\xi|<|\xi|_c} \lambda(|\xi|) < \frac{g[\varrho]}{4\mu_-}. \tag{3.56}$$

Proof By virtue of Proposition 3.5, (3.55)–(3.56) hold. For each fixed $\xi \in \mathbb{R}^2$,

$$\begin{aligned}\eta(t, x') &= f(|\xi|)v_3(\xi, 0)\mathrm{e}^{\lambda(|\xi|)t}\mathrm{e}^{ix'\xi},\\ u(t, x) &= \lambda(|\xi|)f(|\xi|)v(\xi, x_3)\mathrm{e}^{\lambda(|\xi|)t}\mathrm{e}^{ix'\xi},\\ p(t, x) &= \lambda(|\xi|)f(|\xi|)\pi(\xi, x_3)\mathrm{e}^{\lambda(|\xi|)t}\mathrm{e}^{ix'\xi}\end{aligned}$$

give a solution to (1.5). Since $f \in C_0^\infty(R_1, R_2)$, Lemma 3.2 implies that

$$\sup_{\xi \in \mathrm{supp}(f)} \|\partial_3^k v(\xi, \cdot)\|_{L^\infty(I_\pm)} < \infty \quad \text{for all } k \in \mathbb{N}.$$

Also, $\lambda(\xi) \leq \Lambda$. These bounds show that the Fourier synthesis of the solution given by (3.48)–(3.50) is also a solution to (1.5). Because f is real-valued and radial, combined with Remark 3.1, we can easily verify that the Fourier synthesis is real-valued.

The estimate (3.51) follows from Lemma 3.2 with an arbitrary $k \geq 0$ and the fact that f is compactly supported. Finally, we can use (3.55)–(3.56) and (3.48)–(3.50) to obtain the estimates (3.52)–(3.54).

4 Global Instability for the Linearized Problem

4.1 Uniqueness of linearized equations

In this subsection, we will show the uniqueness of solutions to the linearized problem, which will be applied to prove Theorem 2.2 in Section 5. For this purpose, we need a generalized formula of integrating by parts (or Gauss-Green formula). Let us first recall the boundary trace theorem (see Theorem 5.36 in [1, Chapter 5]).

Lemma 4.1 *Let* U *be a domain in* $\mathbb{R}^n$ *satisfying the uniform* C^m*-regularity condition, and assume that there exists a simple* (m, p)*-extension operator* E *for* U. *Also assume that* $mp < n$ *and* $p \leq q \leq p^* = \frac{(n-1)p}{n-mp}$. *Then, there exists a bounded linear operator*

$$\gamma^U : \; W^{m,p}(U) \to L^q(\partial U),$$

such that

$$\gamma^U(u) = u, \quad \text{on } \partial U$$

for all $u \in W^{m,p}(U) \cap C(\overline{U})$.

The function $\gamma^U(u) \in L^q(\partial U)$ is called the trace of the function of $u \in W^{1,p}(U)$ on the boundary ∂U. By the Stein extension theorem (see Theorem 5.24 in [1, Chapter 5]) and the definition of the uniform C^m-regularity condition (see Definition 4.10 in [1, Chapter 5]), it is easy to verify that Ω, Ω_+ and Ω_- have different simple (m,p)-extension operators. Keeping these facts in mind, we can start to show the following formula of integrating by parts. For convenience in the subsequent analysis, we will use the notations $\gamma_+(f) := \gamma^{\Omega_+}(f_+)$ and $\gamma_-(f) := \gamma^{\Omega_+}(f_-)$.

Lemma 4.2 *For all* $u \in H_0^1(\Omega)$ *and* $w \in H^1(\Omega_\pm)$, *we have*

$$\int_\Omega \partial_i w u \mathrm{d}x = -\int_\Omega w \partial_i u \mathrm{d}x + \int_{\mathbb{R}^2} (\gamma_+(w) - \gamma_-(w))\gamma_+(u)\alpha_i \mathrm{d}x \tag{4.1}$$

for $i = 1, 2, 3$, *where* $\alpha_1 = \alpha_2 = 0$ *and* $\alpha_3 = -1$.

Proof Temporarily suppose $\overline{u} \in C_0^1(\Omega)$, $\overline{w}_+ \in C^1(\overline{\Omega}_+)$ and $\overline{w}_- \in C^1(\overline{\Omega}_-)$. By the Gauss-Green theorem, we have

$$\int_\Omega \partial_i \overline{w}\overline{u} \mathrm{d}x = -\int_\Omega \overline{w} \partial_i \overline{u} \mathrm{d}x + \int_{\mathbb{R}^2} ((\overline{w}_+ - \overline{w}_-)\overline{u})(x', 0)\alpha_i \mathrm{d}x. \tag{4.2}$$

Using Lemma 4.1, one has

$$\begin{aligned} \|(\overline{u} - \gamma_+(u))(x', 0)\|_{L^2(\mathbb{R}^2)} &\leq \|\overline{u} - \gamma_+(u)\|_{L^2(\partial\Omega_+)} = \|\gamma_+(\overline{u} - u)\|_{L^2(\partial\Omega_+)} \\ &\leq c\|\overline{u} - u\|_{H^1(\Omega_+)} \leq c\|\overline{u} - u\|_{H_0^1(\Omega)} \end{aligned}$$

and

$$\|(\overline{w_+} - \gamma_+(w_+))(x', 0)\|_{L^2(\mathbb{R}^2)} \leq c\|\overline{w}_+ - w_+\|_{H^1(\Omega_+)}$$

for some constant $c > 0$. By the Hölder inequality, the above two estimates imply that

$$\begin{aligned} &\|(\overline{w}_+\overline{u} - \gamma_+(w)\gamma_+(u))(x', 0)\|_{L^1(\mathbb{R}^2)} \\ &\leq \|(\overline{u}(\overline{w}_+ - \gamma_+(w)))(x', 0)\|_{L^1(\mathbb{R}^2)} + \|(\gamma_+(w)(\overline{u} - \gamma_+(u)))(x', 0)\|_{L^1(\mathbb{R}^2)} \\ &\leq \|\overline{u}(x', 0)\|_{L^2(\mathbb{R}^2)}\|(\overline{w}_+ - \gamma_+(w))(x', 0)\|_{L^2(\mathbb{R}^2)} \\ &\quad + \|\gamma_+(w)(x', 0)\|_{L^2(\mathbb{R}^2)}\|(\overline{u} - \gamma_+(u))(x', 0)\|_{L^2(\mathbb{R}^2)} \\ &\leq c^2\|\overline{u}\|_{H_0^1(\Omega)}\|\overline{w}_+ - w_+\|_{H^1(\Omega_+)} + c^2\|w_+\|_{H^1(\Omega_+)}\|\overline{u} - u\|_{H_0^1(\Omega)}. \end{aligned} \tag{4.3}$$

In a way similar to (4.3), one gets

$$\begin{aligned} &\|(\overline{w}_-\overline{u} - \gamma_-(w)\gamma_+(u))(x', 0)\|_{L^1(\mathbb{R}^2)} \\ &\leq c^2(\|\overline{u}\|_{H_0^1(\Omega)}\|\overline{w}_- - w_-\|_{H^1(\Omega_-)} + \|w_-\|_{H^1(\Omega_-)}\|\overline{u} - u\|_{H_0^1(\Omega)}). \end{aligned} \tag{4.4}$$

In addition, if $\overline{u}_m \to u$ strongly in $H_0^1(\Omega)$, then there exists an $m_0 > 0$, such that

$$\|\overline{u}_m\|_{H_0^1(\Omega)} \leq \|u\|_{H_0^1(\Omega)} + 1 \quad \text{for any } m \geq m_0. \tag{4.5}$$

Note that since $C_0(\Omega)$ is dense in $H_0^1(\Omega)$ and $C_0(\overline{\Omega}_\pm)$ is dense in $H^1(\Omega_+)$ or $H^1(\Omega_-)$, thus (4.1) follows from (4.2)–(4.5), using a standard density argument.

Definition 4.1 *Given $t_0 > 0$ and the initial datum (η_0, u_0) to the linearized problem* (1.5)–(1.7)*, a triple (η, u, p) is called a strong solution of* (1.5)–(1.7)*, if*

(1) $\eta \in C^0([0,t_0], L^2_{\mathrm{loc}}(\mathbb{R}^2))$*,* $u \in C^0([0,t_0], (L^2(\Omega))^3)$*,* $\eta(0) = \eta_0$*,* $u(0) = u_0$ *and*

$$\operatorname*{ess\,sup}_{0<t<t_0} (\|u(t)\|_{H^3(\Omega_\pm)} + \|\eta(t)\|_{H^2(\mathbb{R}^2)} + \|u(t)\|_{H_0^1(\Omega)} + \|p(t)\|_{H^1(\Omega_\pm)}) < \infty; \tag{4.6}$$

(2) *the equations*

$$\varrho\partial_t u + \nabla p = \mu\Delta u, \tag{4.7}$$

$$\operatorname{div} u = 0 \tag{4.8}$$

hold a.e. in $(0,t_0) \times (\Omega \setminus \{x_3 = 0\})$*;*

(3) *for a.e.* $t \in (0,t_0)$*,*

$$\partial_t \eta = u_3, \tag{4.9}$$

$$\begin{aligned} &((\gamma_+(p)I - \mu_+(\nabla u_+ + \nabla u_+^{\mathrm{T}})) - (\gamma_-(p)I - \mu_-(\nabla u_- + \nabla u_-^{\mathrm{T}}))) \cdot e_3 \\ &= (g[\varrho]\eta + \kappa\Delta_{x'}\eta)e_3 \end{aligned} \tag{4.10}$$

hold a.e. in $\mathbb{R}^2 \times \{x_3 = 0\}$*, where* u_3 *is the third component of* u*.*

Remark 4.1 Since $u(t) \in H_0^1(\Omega) \cap H^3(\Omega_\pm)$ for each $t \geq 0$, we can make use of the embedding theorem and (4.8) to obtain

$$u(t) \in C^0(\overline{\Omega}), \quad u_+(t) \in C^1(\overline{\Omega}_+), \quad u_-(t) \in C^1(\overline{\Omega}_-) \tag{4.11}$$

and

$$u(t) \equiv 0, \qquad \text{on } \partial\Omega, \tag{4.12}$$

$$\nabla_{x'} u_+ \equiv \nabla_{x'} u_-, \quad \text{on } \mathbb{R}^2, \tag{4.13}$$

$$\operatorname{div} u(t) \equiv 0, \qquad \text{in } \overline{\Omega} \ \text{ for a.e. } t \geq 0. \tag{4.14}$$

Thus, in view of (4.13), we define for the sake of simplicity that

$$\nabla_{x'} u := \nabla_{x'} u_+ = \nabla_{x'} u_-, \quad \text{on } \mathbb{R}^2 \times \{0\}. \tag{4.15}$$

Moreover, by virtue of Lemma 4.1, there exists a constant c such that

$$\|u(t,x',0)\|_{H^1(\mathbb{R}^2)} \leq c\|u(t)\|_{H^2(\Omega_\pm)} \quad \text{for a.e. } t \geq 0. \tag{4.16}$$

Remark 4.2 It is easy to verify that any (η, u, p), which is a solution established in Theorem 3.3, is a strong solution to the linearized system (1.5)–(1.7).

Theorem 4.1 *Assume that (η_1, v, p_1) and (η_2, w, p_2) are two strong solutions to* (1.5)–(1.7)*, with $v(0) = w(0) = u_0$ and $\eta_1(0) = \eta_2(0) = \eta_0$. Then, $(\eta_1, v, p_1) = (\eta_2, w, p_2 + c)$ for some constant c.*

Proof Let $(\eta, u, p) = (\eta_1 - \eta_2, v - w, p_1 - p_2)$. Recalling Definition 4.1, (η, u, p) is still a strong solution to the linearized system (1.5)–(1.7) with zero initial data, i.e., $\eta(0) = 0$ and $u(0) = 0$.

Multiplying (4.7) by u, integrating over $(0, \tau) \times \Omega$ for any $\tau \in (0, t_0)$ and using (4.14), we find that

$$\int_0^\tau \int_\Omega \varrho \partial_t u \cdot u \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \mathrm{div}(pI - \mu(\nabla u + \nabla u^{\mathrm{T}})) \cdot u \mathrm{d}x\mathrm{d}t = 0. \tag{4.17}$$

Thanks to Lemma 4.1, (4.11)–(4.14) and the regularity of p, we obtain

$$\begin{aligned}
&\int_0^\tau \int_\Omega \mathrm{div}(pI - \mu(\nabla u + \nabla u^{\mathrm{T}})) \cdot u \mathrm{d}x\mathrm{d}t \\
=& \int_0^\tau \int_{\mathbb{R}^2} ((\gamma_-(p)I - \mu_-(\nabla u_- + \nabla u_-^{\mathrm{T}})) - (\gamma_+(p)I - \mu_+(\nabla u_+ + \nabla u_+^{\mathrm{T}})))e_3 \cdot u \mathrm{d}x'\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \mu \nabla u : (\nabla u + \nabla u^{\mathrm{T}}) \mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{4.18}$$

Notice that in view of (4.6), $p(t) \in (H^1(\Omega_\pm))^3$ and $u(t) \in (H^2(\Omega_\pm))^3$ for a.e. $t > 0$. Thus, (4.7) implies that

$$\partial_t u \in (L^2((0, t_0) \times \Omega))^3,$$

which, together with $u \in L^\infty(0, t_0; (H^1(\Omega))^3) \cap C^0([0, t_0], (L^2(\Omega))^3)$, yields

$$\int_0^\tau \int_\Omega \varrho \partial_t u \cdot u \mathrm{d}x\mathrm{d}t = \frac{1}{2} \int_\Omega \varrho u^2(\tau) \mathrm{d}x - \frac{1}{2} \int_\Omega \varrho u^2(0) \mathrm{d}x. \tag{4.19}$$

In view of (4.10), (4.17)–(4.19) and $u(0) = 0$, we find that

$$\begin{aligned}
&\frac{1}{2} \int_\Omega \varrho u^2(\tau) \mathrm{d}x + \int_0^\tau \int_\Omega \mu \nabla u : (\nabla u + \nabla u^{\mathrm{T}}) \mathrm{d}x\mathrm{d}t \\
=& \int_0^\tau \int_{\mathbb{R}^2} (g[\varrho]\eta + \kappa \Delta_{x'} \eta) u_3 \mathrm{d}x'\mathrm{d}t.
\end{aligned} \tag{4.20}$$

Since $\eta \in C^0([0, t_0], L^2_{\mathrm{loc}}(\mathbb{R}^2))$ and $\eta(0) = 0$, the equation (4.9) implies that

$$\eta(t, x') = \int_0^t u_3(s, x', 0) \mathrm{d}s \quad \text{for any } t \geq 0. \tag{4.21}$$

Using (4.15)–(4.16), (4.21) and the regularity of η, we infer that

$$\begin{aligned}
\int_0^\tau \int_{\mathbb{R}^2} \Delta_{x'} \eta u_3 \mathrm{d}x'\mathrm{d}t &= -\sum_{i=1}^2 \int_0^\tau \int_{\mathbb{R}^2} \partial_i \eta \partial_i u_3 \mathrm{d}x'\mathrm{d}t \\
&= -\sum_{i=1}^2 \int_0^\tau \int_{\mathbb{R}^2} \int_0^t \partial_i u_3(s, x', 0) \mathrm{d}s \partial_i u_3(t, x', 0) \mathrm{d}x'\mathrm{d}t,
\end{aligned} \tag{4.22}$$

where the formula of integration by parts can be shown in the same manner as in Lemma 4.1.

Consequently, inserting (4.21)–(4.22) into (4.20), we arrive at

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}\varrho u^2(\tau)\mathrm{d}x\mathrm{d}t+\int_0^{\tau}\int_{\Omega}\mu\nabla u:(\nabla u+\nabla u^{\mathrm{T}})\mathrm{d}x\mathrm{d}t\\&=g[\varrho]\int_0^{\tau}\int_{\mathbb{R}^2}\int_0^t u_3(s,x',0)\mathrm{d}s u_3(t,x',0)\mathrm{d}x'\mathrm{d}t\\&\quad-\kappa\sum_{i=1}^{2}\int_0^{\tau}\int_{\mathbb{R}^2}\int_0^t\partial_i u_3(s,x',0)\mathrm{d}s\partial_i u_3(t,x',0)\mathrm{d}x'\mathrm{d}t.\end{aligned}\tag{4.23}$$

With the help of the regularity of $\partial_i u_3$, the property of absolutely continuous functions and the Fubini theorem, we conclude that

$$\begin{aligned}&\int_0^{\tau}\int_{\mathbb{R}^2}\int_0^t\partial_i u_3(s,x',0)\mathrm{d}s\partial_i u_3(t,x',0)\mathrm{d}x'\mathrm{d}t\\&=\int_{\mathbb{R}^2}\int_0^{\tau}\int_0^t\partial_i u_3(s,x',0)\mathrm{d}s\partial_i u_3(t,x',0)\mathrm{d}t\mathrm{d}x'\\&=\int_{\mathbb{R}^2}\int_0^{\tau}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\int_0^t\partial_i u_3(s,x',0)\mathrm{d}s\Big]^2\mathrm{d}t\mathrm{d}x'\\&=\int_{\mathbb{R}^2}\Big[\int_0^{\tau}\partial_i u_3(t,x',0)\mathrm{d}t\Big]^2\mathrm{d}x'\geq 0.\end{aligned}\tag{4.24}$$

On the other hand, applying the Cauchy-Schwarz inequality, we get

$$2\sum_{i=1}^{3}\int_{\Omega}\mu(\partial_i u_i)^2\mathrm{d}x\mathrm{d}t\leq\int_{\Omega}\mu\nabla u:(\nabla u+\nabla u^{\mathrm{T}})\mathrm{d}x\mathrm{d}t.\tag{4.25}$$

Hence, by (4.23)–(4.25), we have

$$\int_{\Omega}\varrho u^2(\tau)\mathrm{d}x+4\sum_{i=1}^{3}\int_{\Omega}\mu(\partial_i u_i)^2\mathrm{d}x\mathrm{d}t\leq 2g[\varrho]\int_0^{\tau}\int_{\mathbb{R}^2}\int_0^t u_3(s,x',0)\mathrm{d}s u_3(t,x',0)\mathrm{d}x'\mathrm{d}t.\tag{4.26}$$

In a way similar to (4.24), the right-hand side of (4.26) can be bounded as follows:

$$\begin{aligned}&2\int_0^{\tau}\int_{\mathbb{R}^2}\int_0^t u_3(s,x',0)u_3(t,x',0)\mathrm{d}x'\mathrm{d}s\mathrm{d}t\\&=\int_{\mathbb{R}^2}\Big(\int_0^{\tau}u_3(t,x',0)\mathrm{d}t\Big)^2\mathrm{d}x'\\&\leq\tau\int_0^{\tau}\int_{\mathbb{R}^2}|u_3(t,x',0)|^2\mathrm{d}x'\mathrm{d}t\\&=2\tau\int_0^{\tau}\int_{\mathbb{R}^2}\int_1^0 u_3(t,x',x_3)\partial_3 u_3(t,x',x_3)\mathrm{d}x_3\mathrm{d}x'\mathrm{d}t\\&\leq\tau\int_0^{\tau}\int_{\mathbb{R}^2}\Big(\frac{\mu}{2\tau g[\varrho]}\int_0^1|\partial_3 u_3(t)|^2\mathrm{d}x_3+\frac{2\tau g[\varrho]}{\mu}\int_0^1|u_3(t)|^2\mathrm{d}x_3\Big)\mathrm{d}x'\mathrm{d}t\\&\leq\frac{1}{2g[\varrho]}\int_0^{\tau}\|\sqrt{\mu}\partial_3 u_3(t)\|^2_{L^2(\Omega)}\mathrm{d}t+\frac{2\tau^2 g[\varrho]}{\mu_-}\int_0^{\tau}\|u(t)\|^2_{L^2(\Omega)}\mathrm{d}t.\end{aligned}\tag{4.27}$$

Substituting (4.27) into (4.26), we deduce

$$\|\sqrt{\varrho}u(\tau)\|^2_{L^2(\Omega)}+3\mu_-\int_0^{\tau}\|\nabla u(t)\|^2_{L^2(\Omega)}\mathrm{d}t\leq 4\tau^2 g^2[\varrho]^2\mu_-^{-1}\int_0^{\tau}\|u(t)\|^2_{L^2(\Omega)}\mathrm{d}t,$$

which results in

$$\|u(\tau)\|^2_{L^2(\Omega)} \le \frac{4t_0^2 g^2[\varrho]^2}{\mu_-\varrho_-}\int_0^\tau \|u(t)\|^2_{L^2(\Omega)}\mathrm{d}t. \tag{4.28}$$

Moreover, if we apply the Gronwall inequality to (4.28), we obtain

$$\|u(\tau)\|^2_{L^2(\Omega)} = 0 \quad \text{for any } \tau \in [0, t_0],$$

which implies $u = 0$, i.e., $v = w$. This, combined with (4.7) and (4.9), proves Theorem 4.1.

Remark 4.3 In addition, employing arguments similar to those used for (4.1), the regularity of u stated in Remark 4.1 and the fact that $\operatorname{div} u = 0$, we can show

$$\int_0^\tau \int_\Omega \mu\nabla u : \nabla u^{\mathrm{T}}\mathrm{d}x\mathrm{d}t \equiv 0.$$

4.2 Proof of Theorem 2.1

We define

$$\beta_1 = \begin{cases} \frac{|\xi|_c}{3}, & \text{if } \kappa > 0,\\ 1, & \text{if } \kappa = 0 \end{cases} \quad \text{and} \quad \beta_2 = \begin{cases} \frac{2|\xi|_c}{3}, & \text{if } \kappa > 0,\\ 2, & \text{if } \kappa = 0. \end{cases}$$

Fix $j \ge k \ge 0$, $\alpha > 0$ and let $\widetilde{c}_j$ be the constants from Theorem 3.3. For each $n \in \mathbb{N}$, let t_n be sufficiently large, so that

$$\frac{\exp(2t_n\lambda_0)}{(1+\beta_2^2)^{j-k+2}} = \alpha^2 n^2 \widetilde{c}_j^2, \tag{4.29}$$

i.e.,

$$t_n = \frac{\ln(\widetilde{c}_j(1+\beta_2^2)^{\frac{j-k+2}{2}})}{\lambda_0} + \frac{\ln(\alpha n)}{\lambda_0} := C_{jk} + C_1\ln(\alpha n),$$

where

$$\lambda_0 = \inf_{\xi \in B(0,\beta_2)\backslash B(0,\beta_1)} \lambda(|\xi|) > 0.$$

Choose $f_n \in C_0^\infty(\mathbb{R}^2)$, such that $\operatorname{supp}(f_n) \subset B(0,\beta_2)\backslash B(0,\beta_1)$, f_n is real-valued and radial, and

$$\int_{\mathbb{R}^2} (1+|\xi|^2)^{j+2}|f_n(|\xi|)|^2\mathrm{d}\xi = \frac{1}{\widetilde{c}_j^2 n^2}. \tag{4.30}$$

Now, we can apply Theorem 3.3 with $f = f_n$, $R_1 = \beta_1$ and $R_2 = \beta_2$ to find that $(\eta_n(t), u_n(t), p_n(t)) \in H^j(\Omega)$ solves the problem (1.5)–(1.7). It follows thus from (3.51) and (4.30) that (2.3) holds for all n.

Recalling $\operatorname{supp}(f_n) \subset B(\beta_1, \beta_2)$ and $\lambda(|\xi|) \ge \lambda_0$, we have, after a straightforward calculation and using (3.36) and (4.29), that

$$\begin{aligned} \|u_n(t)\|^2_{H^k} &\ge \int_{\mathbb{R}^2} (1+|\xi|^2)^k \mathrm{e}^{2t\lambda(|\xi|)}|f_n(|\xi|)|^2\|v(\xi,x_3)\|^2_{L^2(-1,1)}\mathrm{d}\xi \\ &\ge \frac{\exp(2t\lambda_0)}{(1+\beta_2^2)^{j-k+2}}\int_{\mathbb{R}^2}(1+|\xi|^2)^{j+2}|f_n(|\xi|)|^2\|v(\xi,x_3)\|^2_{L^2(-1,1)}\mathrm{d}\xi \\ &= \alpha^2 n^2\widetilde{c}_j^2 \int_{\mathbb{R}^2}(1+|\xi|^2)^{j+2}|f_n(|\xi|)|^2\mathrm{d}\xi \\ &= \alpha^2 \quad \text{for any } t \ge t_n, \end{aligned}$$

which implies (2.4)–(2.5). This completes the proof of Theorem 2.1

5 Proof of Theorem 2.2

In this section, we will argue by contradiction to show Theorem 2.2. Therefore, we suppose that the perturbed problem has the global stability of order k for some $k \geq 3$.

5.1 Regularity under the assumption of the global stability

Let $\delta, C_2 > 0$ and $F : [0, \delta] \to \mathbb{R}^+$ be the constants and the function provided by the global stability of order k, respectively. Letting $n \in \mathbb{N}$ and applying Theorem 2.1 with this n, $t_n = \frac{T_0}{2}$, $k \geq 3$, and $\alpha = 2$, we let $(\widetilde{\eta}, \widetilde{u}, \widetilde{\sigma})$ solve (1.5), such that

$$\|\widetilde{\eta}(0)\|_{H^k(\mathbb{R}^2)} + \|\widetilde{u}(0)\|_{H^k(\Omega_\pm)} + \|\widetilde{\sigma}(0)\|_{H^k(\Omega_\pm)} < n^{-1}, \tag{5.1}$$

but

$$\|\widetilde{u}(t)\|_{H^3(\Omega_\pm)} \geq 2 \quad \text{for } t \geq \frac{T_0}{2}. \tag{5.2}$$

By the Sobolev embedding theorem, $\widetilde{\eta}(0) \equiv \widetilde{\eta}(0, x') \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^2)$. Employing the Stein extension theorem, there exist two linear operators E_+ and E_- which map $H^k(\Omega_+)$ and $H^k(\Omega_-)$ to $H^k(\mathbb{R}^3)$, respectively, such that

$$E_+(\widetilde{u}_+(0)) = \widetilde{u}_+(0), \quad \text{a.e. in } \Omega^+, \quad E_-(\widetilde{u}_-(0)) = \widetilde{u}_-(0), \quad \text{a.e. in } \Omega^-$$

and

$$\|E_+(\widetilde{u}_+(0))\|_{H^k(\mathbb{R}^3)} \leq c(k)\|\widetilde{u}_+(0)\|_{H^k(\Omega_+)}, \quad \|E_-(\widetilde{u}_-(0))\|_{H^k(\mathbb{R}^3)} \leq c(k)\|\widetilde{u}_-(0)\|_{H^k(\Omega_-)}$$

for some constant $c(k)$ depending on k. Keeping in mind that $\|\widetilde{\eta}(0)\|_{H^k(\mathbb{R}^2)} < n^{-1}$, we can apply the embedding theorem to see that there exists a sufficiently small constant $\varepsilon_0 > 0$, such that $\|\varepsilon\widetilde{\eta}(0)\|_{L^\infty(\mathbb{R}^2)} < 1$ for any $\varepsilon \in (0, \varepsilon_0)$. Thus, we define

$$\begin{aligned}
\overline{\eta}_0^\varepsilon &:= \varepsilon\widetilde{\eta}(0), \\
\Sigma^\varepsilon(0) &:= \{(x', x_3) \in \mathbb{R}^3 \mid x_3 = \overline{\eta}_0^\varepsilon(x')\}, \\
\Omega_+^\varepsilon(0) &:= \{(x', x_3) \in \mathbb{R}^3 \mid \overline{\eta}_0^\varepsilon(x') < x_3 < 1\}, \\
\Omega_-^\varepsilon(0) &:= \{(x', x_3) \in \mathbb{R}^3 \mid -1 < x_3 < \overline{\eta}_0^\varepsilon(x')\}, \\
v^\varepsilon(0) &:= \begin{cases} E_+(\widetilde{u}_+(0)), & \text{if } x \in \Omega_+^\varepsilon(0), \\ E_-(\widetilde{u}_-(0)), & \text{if } x \in \Omega_-^\varepsilon(0), \end{cases} \\
\overline{u}_0^\varepsilon &:= \varepsilon v^\varepsilon(0).
\end{aligned}$$

Now, we fix $n \in \mathbb{N}$ so that $n > \max\{1, C_2\}\max\{1, c(k)\}$. By construction, $[\overline{u}_0^\varepsilon]|_{\Sigma^\varepsilon(0)} = 0$. Moreover, for $\varepsilon < \widetilde{\varepsilon}_0 := \min\{\varepsilon_0, \delta n(\max\{1, c(k)\})^{-1}\}$, we have

$$\begin{aligned}
&\|\overline{\eta}_0^\varepsilon\|_{H^k(\mathbb{R}^2)} + \|\overline{u}_0^\varepsilon\|_{H^k(\Omega_\pm^\varepsilon(0))} \\
&= \varepsilon\Big(\|\widetilde{\eta}(0)\|_{H^k(\mathbb{R}^2)} + \sqrt{\|E_+(\widetilde{u}_+(0))\|^2_{H^k(\Omega_+^\varepsilon(0))} + \|E_-(\widetilde{u}_-(0))\|^2_{H^k(\Omega_-^\varepsilon(0))}}\Big) \\
&< \varepsilon\Big(\|\widetilde{\eta}(0)\|_{H^k(\mathbb{R}^2)} + c(k)\sqrt{\|\widetilde{u}_+(0)\|^2_{H^k(\Omega_+)} + \|\widetilde{u}_-(0)\|^2_{H^k(\Omega_-)}}\Big) \\
&\leq \varepsilon\max\{1, c(k)\}(\|\widetilde{\eta}(0)\|_{H^k(\mathbb{R}^2)} + \|\widetilde{u}(0)\|_{H^k(\Omega_\pm)}) < \delta.
\end{aligned}$$

Thus, according to the global stability of order k, there exist $\eta^\varepsilon, u^\varepsilon, \sigma^\varepsilon$ in the function class described in Definition 2.1, which solve the perturbed problem

$$\begin{cases} \partial_t \eta^\varepsilon = u_3^\varepsilon - u_1^\varepsilon \partial_1 \eta^\varepsilon - u_2^\varepsilon \partial_2 \eta^\varepsilon, \\ \varrho \partial_t u^\varepsilon + \varrho (\nabla u^\varepsilon) u^\varepsilon + \nabla \sigma^\varepsilon = \mu \Delta u^\varepsilon, \\ \mathrm{div} u^\varepsilon = 0 \end{cases} \tag{5.3}$$

with the jump condition

$$[u^\varepsilon]|_{\Sigma^\varepsilon(t)} = 0, \tag{5.4}$$

$$[(\sigma^\varepsilon I - \mu(\nabla u^\varepsilon + \nabla (u^\varepsilon)^T))\nu^\varepsilon]|_{\Sigma^\varepsilon(t)} = (g[\varrho]\eta^\varepsilon + \kappa H^\varepsilon)\nu^\varepsilon, \tag{5.5}$$

where

$$\Sigma^\varepsilon(t) := \{x \in \mathbb{R}^3 \mid x_3 = \eta^\varepsilon(t, x')\},$$

and the initial data satisfying $\|\overline{\eta}_0^\varepsilon\|_{H^k(\mathbb{R}^2)} + \|\overline{u}_0^\varepsilon\|_{H^k(\Omega_\pm^\varepsilon(0))} < \delta$. Furthermore, the solution satisfies

$$\begin{aligned} &\sup_{0 \le t < T} (\|u^\varepsilon\|_{H^3(\Omega_\pm^\varepsilon(t))} + \|\eta^\varepsilon\|_{H^2(\mathbb{R}^2)} + \|\sigma^\varepsilon\|_{H^1(\Omega_\pm^\varepsilon(t))}) \\ &\le F(\|\overline{\eta}_0^\varepsilon\|_{H^k(\mathbb{R}^2)} + \|\overline{u}_0^\varepsilon\|_{H^k(\Omega_\pm^\varepsilon(0))}) \\ &\le \varepsilon C_2 \max\{1, c(k)\}(\|\widetilde{\eta}(0)\|_{H^k(\mathbb{R}^2)} + \|\widetilde{u}(0)\|_{H^k(\Omega_\pm)}) < \varepsilon \quad \text{for any } T > 0. \end{aligned} \tag{5.6}$$

Now, define the rescaled functions $\overline{\eta}^\varepsilon = \frac{\eta^\varepsilon}{\varepsilon}$, $\overline{u}^\varepsilon = \frac{u^\varepsilon}{\varepsilon}$, $\overline{\sigma}^\varepsilon = \frac{\sigma^\varepsilon}{\varepsilon}$. If we rescale (5.6), then we see that

$$\sup_{0 \le t < T} (\|\overline{u}^\varepsilon(t)\|_{H^3(\Omega_\pm^\varepsilon(t))} + \|\overline{\eta}^\varepsilon(t)\|_{H^2(\mathbb{R}^2)} + \|\overline{\sigma}^\varepsilon(t)\|_{H^1(\Omega_\pm^\varepsilon(t))}) \le 1 \quad \text{for any } T > 0, \tag{5.7}$$

where

$$\begin{aligned} &\Omega_+^\varepsilon(t) = \big\{(x', x_3) \mid x' \in \mathbb{R}^2,\ \eta^\varepsilon(t, x') < x_3 < 1\big\}, \\ &\Omega_-^\varepsilon(t) = \big\{(x', x_3) \mid x' \in \mathbb{R}^2,\ -1 < x_3 < \eta^\varepsilon(t, x')\big\}, \quad t > 0. \end{aligned}$$

Moreover, recalling the definition of $H^3(\Omega_\pm^\varepsilon(t))$ in (2.2), and using (5.4) and the regularity of $\overline{u}^\varepsilon(t)$ in (5.7), we can easily verify that

$$\sup_{0 \le t < T} \|\overline{u}^\varepsilon(t)\|_{H_0^1(\Omega)} \le 1 \quad \text{for any } T > 0. \tag{5.8}$$

The second equation in (5.3), together with (5.7), yields

$$\begin{aligned} &\sup_{0 \le t < T} \|\partial_t \overline{u}^\varepsilon\|_{L^2(\Omega)} \\ &\le \sqrt{5}(\varepsilon \|\overline{u}^\varepsilon\|_{L^2(\Omega_\pm^\varepsilon(t))} \|\nabla \overline{u}^\varepsilon\|_{L^2(\Omega_\pm^\varepsilon(t))} + \varrho_-^{-1} \|\nabla \overline{\sigma}^\varepsilon\|_{L^2(\Omega_\pm^\varepsilon(t))} + \mu_+ \varrho_-^{-1} \|\Delta \overline{u}^\varepsilon\|_{L^2(\Omega_\pm^\varepsilon(t))}) \\ &\le \sqrt{5}\Big[\delta n + \frac{1 + \mu_+}{\varrho_-}\Big] \quad \text{for any } T > 0. \end{aligned} \tag{5.9}$$

Now, letting $j \in \mathbb{Z}^+$, and employing the first equation of (5.3), (5.7)–(5.8), we infer that for each square domain

$$\mathcal{R}_j := \{x' \in \mathbb{R}^2 \mid |x_1|, |x_2| < j\},$$

there exists a constant $c_1(j)$ depending on j, such that

$$\sup_{0 \le t < T} \|\partial_t \overline{\eta}^\varepsilon\|_{L^2(\mathcal{R}_j)} \le c_1(j) \quad \text{for any } T > 0. \tag{5.10}$$

5.2 Taking limits in the rescaled function sequences

First, making use of (5.7)–(5.10), an abstract version of the Arzela-Ascoli theorem (see [16, Theorem 1.70]), and the Aubin-Lions Theorem (see [16, Theorem 1.71]), we can extract a subsequence $(\overline{\eta}_m, \overline{u}_m) := (\overline{\eta}^{\varepsilon_m}, \overline{u}^{\varepsilon_m})$, with $\{\varepsilon_m\} \subset \{\varepsilon \mid 0 < \varepsilon < \widetilde{\varepsilon}_0\}$, such that, for any $p \ge 1$ and $j \in \mathbb{Z}^+$,

$$\begin{aligned}
\overline{u}_m &\to \overline{u} \ \text{ weakly star in } L^\infty(0, T_0; H_0^1(\Omega)), && (5.11)\\
\overline{u}_m &\to \overline{u} \ \text{ strongly in } C^0([0, T_0], L^2(\Omega)), && (5.12)\\
\partial_t \overline{u}_m &\to \partial_t \overline{u} \ \text{ weakly star in } L^\infty(0, T_0; L^2(\Omega)), && (5.13)\\
\overline{\eta}_m &\to \overline{\eta} \ \text{ weakly star in } L^\infty(0, T_0; H^2(\mathbb{R}^2)), && (5.14)\\
\overline{\eta}_m &\to \overline{\eta} \ \text{ strongly in } C^0([0, T_0], L^2(\mathcal{R}_j)) \cap L^p(0, T_0; H^1(\mathcal{R}_j)), && (5.15)\\
\partial_t \overline{\eta}_m &\to \partial_t \overline{\eta} \ \text{ weakly star in } L^\infty(0, T_0; L^2(\mathcal{R}_j)), &&\\
\overline{\sigma}_m &\to \overline{\sigma} \ \text{ weakly star in } L^\infty(0, T_0; L^2(\Omega)). && (5.16)
\end{aligned}$$

Hereafter, for simplicity we denote f^{ε_m} by f_m, where f represents $\overline{u}$, $\overline{\eta}$, $\overline{\sigma}$, Σ, or η, and so on.

Denoting

$$\Omega_\pm^j = \Big\{(x', x_3) \in \Omega_\pm \ \Big|\ \frac{1}{j} < |x_3| < 1\Big\},$$

and using the regularity of η^ε in (5.6) and the Sobolev embedding theorem, we find that for any positive integer j, there exists an $\varepsilon_{m_j} > 0$ depending on j, such that, for any $\varepsilon_m \in (0, \varepsilon_{m_j})$ and $t \ge 0$, we have

$$\Omega_\pm^j \subset \big\{(x', x_3) \in \Omega_\pm \mid |\eta_m(t, x')| < |x_3| < 1\big\} \subset \Omega. \tag{5.17}$$

Therefore, for any j, making use of (5.7), (5.9), (5.17) and the Lions-Aubin Lemma, we can show by induction that there exists $\{m_i^j\} \subset \{m_i^{j-1}\} \subset \{m\}$, such that

$$\begin{aligned}
&\varepsilon_{m_i^j} < \varepsilon_{m_j} \ \text{ for any } i > 0,\\
&\overline{u}_{m_i^j} \to \overline{u} \ \text{ weakly star in } L^\infty(0, T_0; H^3(\Omega_\pm^j)),\\
&\overline{u}_{m_i^j} \to \overline{u} \ \text{ strongly in } L^p(0, T_0; H^2(\Omega_\pm^j)) \quad \text{for any } p \ge 1,\\
&\overline{\sigma}_{m_i^j} \rightharpoonup \overline{\sigma} \ \text{ weakly star in } L^\infty(0, T_0; H^1(\Omega_\pm^j)) \quad \text{as } i \to \infty,
\end{aligned}$$

where we have defined $\{m_i^0\} = \{m\}$.

From the lower semicontinuity, one gets

$$\sup_{0\leq t<T_0}(\|\overline{u}(t)\|_{H^3(\Omega^j_\pm)}+\|\overline{\eta}(t)\|_{H^2(\mathbb{R}^2)}+\|\overline{\sigma}(t)\|_{H^1(\Omega^j_\pm)})\leq 1.$$

Choosing $m'_i=m^i_i$, one has, for any $j\in\mathbb{N}$ and $i>j$, that

$$\begin{aligned}&\overline{u}_{m'_i}\to\overline{u}\ \text{ weakly star in } L^\infty(0,T_0;H^3(\Omega^j_\pm)),\\ &\overline{u}_{m'_i}\to\overline{u}\ \text{ strongly in } L^p(0,T_0;H^2(\Omega^j_\pm))\quad\text{ for any } p\geq 1, &(5.18)\\ &\overline{\sigma}_{m'_i}\rightharpoonup\overline{\sigma}\ \text{ weakly star in } L^\infty(0,T_0;H^1(\Omega^j_\pm))\quad\text{ as } i\to\infty. &(5.19)\end{aligned}$$

Moreover, by (5.13) and (5.18)–(5.19), we find that

$$\begin{cases}\varrho\partial_t\overline{u}+\nabla\overline{\sigma}=\mu\Delta\overline{u},\\ \mathrm{div}\overline{u}=0\end{cases}\tag{5.20}$$

hold a.e. in $(0,T_0)\times(\Omega\setminus\{x_3=0\})$, and

$$\sup_{0\leq t<T_0}\big(\|\overline{u}(t)\|_{H^3(\Omega_\pm)}+\|\overline{\eta}(t)\|_{H^2(\mathbb{R}^2)}+\|\overline{\sigma}(t)\|_{H^1(\Omega_\pm)}\big)\leq 1.\tag{5.21}$$

In addition, by construction, we have

$$\begin{aligned}&\overline{\eta}^\varepsilon(0)=\widetilde{\eta}(0):=\widetilde{\eta}(0,x'),\quad\text{ in }\mathbb{R}^2, &(5.22)\\ &\overline{u}_{m'_i}(0)=\widetilde{u}(0),\quad\text{ in }\Omega^j_\pm\ \text{ for any } i>j.\end{aligned}$$

The estimates (5.20) and (5.22), combined with (5.12) and (5.15), imply

$$\overline{\eta}(0)=\widetilde{\eta}(0)\quad\text{and}\quad\overline{u}(0)=\widetilde{u}(0).\tag{5.23}$$

5.3 Convergence of the interface equation

Replacing ε by ε_m, we rewrite the first equation in (5.3) as

$$\partial_t\overline{\eta}_m=(\overline{v}_{m,3}-\varepsilon_m\overline{v}_{m,1}\partial_1\overline{\eta}_m-\varepsilon_m\overline{v}_{m,2}\partial_2\overline{\eta}_m),\tag{5.24}$$

where

$$\overline{v}_m:=\overline{v}_m(t,x')=\overline{u}_m(t,x',\eta_m(t,x')),\quad t\in(0,T_0),$$

and $\overline{v}_{m,1}$, $\overline{v}_{m,2}$ denote the first and second components of the vector function $\overline{v}_m$, respectively.

Multiplying (5.24) with $\varphi\in\mathcal{D}(\mathbb{R}^2)$ and integrating over $\mathbb{R}^2$, then we arrive at

$$\int_{\mathbb{R}^2}\partial_t\overline{\eta}_m\varphi\mathrm{d}x'=\int_{\mathbb{R}^2}(\overline{v}_{m,3}-\varepsilon_m\overline{v}_{m,1}\partial_1\overline{\eta}_m-\varepsilon_m\overline{v}_{m,2}\partial_2\overline{\eta}_m)\varphi\mathrm{d}x'.\tag{5.25}$$

Recalling $\overline{\eta}_m\in C^0([0,T_0],L^2(\mathcal{R}_j))$, we use (5.22) and (5.25) to deduce that

$$\begin{aligned}\int_{\mathbb{R}^2}\overline{\eta}_m(t)\varphi\mathrm{d}x'=\int_0^t\Big[&\int_{\mathbb{R}^2}(\overline{v}_{m,3}-\varepsilon_m\overline{v}_{m,1}\partial_1\overline{\eta}_m-\varepsilon_m\overline{v}_{m,2}\partial_2\overline{\eta}_m)\varphi\mathrm{d}x'\\ &+\int_{\mathbb{R}^2}\widetilde{\eta}(0,x')\varphi\mathrm{d}x'\Big]\mathrm{d}s.\end{aligned}\tag{5.26}$$

Next, we analyze the convergence of each integral in (5.26).

First, there exists a square domain $\mathcal{R}_{j_1}$ satisfying

$$\mathrm{supp}\ \varphi \subset \mathcal{R}_{j_1} \subset \mathbb{R}^2.$$

Due to (5.15), we get

$$\lim_{m\to\infty}\int_{\mathbb{R}^2}\overline{\eta}_m\varphi \mathrm{d}x' = \lim_{m\to\infty}\int_{\mathcal{R}_{j_1}}\overline{\eta}_m\varphi \mathrm{d}x' = \int_{\mathcal{R}_{j_1}}\overline{\eta}\varphi \mathrm{d}x' = \int_{\mathbb{R}^2}\overline{\eta}\varphi \mathrm{d}x'. \tag{5.27}$$

Then, from (5.7), we get

$$\sup_{0\le t<T}\|\overline{\eta}_m(t)\|_{H^2(\mathbb{R}^2)} \le 1 \quad \text{ for any } T>0. \tag{5.28}$$

Noticing that $\overline{u}_{m,i}(t)\in C^0(\overline{\Omega})\cap H_0^1(\Omega)$ for $t>0$, we utilize the Hölder inequality, (5.8) and (5.28) to obtain

$$\begin{aligned}
&\left|\int_0^t\int_{\mathbb{R}^2}\overline{v}_{m,i}\partial_i\overline{\eta}_m\varphi \mathrm{d}x'\mathrm{d}s\right| \\
&\le \sqrt{2}\|\varphi\|_\infty\int_0^t\Big(\int_{\mathbb{R}^2}\int_{\eta_m(t,x')}^1|\partial_3\overline{u}_{m,i}|^2\mathrm{d}x_3\mathrm{d}x'\Big)^{\frac12}\Big(\int_{\mathbb{R}^2}|\partial_i\overline{\eta}_m|^2\mathrm{d}x'\Big)^{\frac12}\mathrm{d}s \\
&\le \sqrt{2}\|\varphi\|_\infty\|\partial_3\overline{u}_{m,i}\|_{L^1((0,T_0),L^2(\Omega))}\sup_{0\le t<\infty}\|\partial_i\overline{\eta}_m(t)\|_{L^2(\mathbb{R}^2)} < \sqrt{2}\|\varphi\|_\infty T_0,
\end{aligned} \tag{5.29}$$

where $i=1,2$, and

$$\|\varphi\|_\infty := \sup_{x'\in\mathbb{R}^2}|\varphi(x')|>0.$$

Hence, from (5.29) it follows that

$$\lim_{m\to\infty}\int_0^t\int_{\mathbb{R}^2}(\varepsilon_m\overline{v}_{m,1}\partial_1\overline{\eta}_m+\varepsilon_m\overline{v}_{m,2}\partial_2\overline{\eta}_m)\varphi \mathrm{d}x'\mathrm{d}s=0. \tag{5.30}$$

Finally, keeping in mind that $\{m_i'\}\subset\{m\}$ and $\partial_3\overline{u}_3\in L^\infty(0,T_0;L^2(\Omega))$, we use (5.8), (5.18) and the absolute continuity of integrals to deduce that for any $\delta>0$, there exists a $j_2>j_1>0$ depending on j_1, T_0 and $\|\varphi\|_\infty$, such that, for any $i>j_2$,

$$\begin{aligned}
&\int_0^{T_0}\int_{\mathcal{R}_{j_1}}\int_{-j_2^{-1}}^{j_2^{-1}}|\partial_3\overline{u}_3|\,\mathrm{d}x_3\mathrm{d}x'\mathrm{d}s<\frac{\delta}{3\|\varphi\|_\infty}, \\
&\int_0^{T_0}\int_{\mathcal{R}_{j_1}}\int_{-j_2^{-1}}^{j_2^{-1}}\left|\partial_3\overline{u}_{m_i',3}-\partial_3\overline{u}_3\right|\mathrm{d}x_3\mathrm{d}x'\mathrm{d}s<\frac{\delta}{3\|\varphi\|_\infty}, \\
&\int_0^{T_0}\int_{\mathcal{R}_{j_1}}\int_{j_2^{-1}}^{1}\left|\partial_3\overline{u}_{m_i',3}-\partial_3\overline{u}_3\right|\mathrm{d}x_3\mathrm{d}x'\mathrm{d}s<\frac{\delta}{3\|\varphi\|_\infty},
\end{aligned}$$

which, recalling $\|\eta_{m_i'}(t,x')\|_{L^\infty(\mathbb{R}^2)} < j_2^{-1}$ by the construction of $\{m_i'\}$ and (5.17), imply

$$\begin{aligned}
&\Big|\int_0^t\int_{\mathbb{R}^2}\big(\overline{v}_{m_i',3}-\overline{u}_3(s,x',0)\big)\varphi\mathrm{d}x'\mathrm{d}s\Big| \\
=&\Big|\int_0^t\int_{\mathbb{R}^2}\Big[\int_{\eta_{m_i'}(t,x')}^1(\partial_3\overline{u}_{m_i',3}-\partial_3\overline{u}_3)\mathrm{d}x_3+\int_{\eta_{m_i'}(t,x')}^0\partial_3\overline{u}_3\mathrm{d}x_3\Big]\varphi\mathrm{d}x'\mathrm{d}s\Big| \\
\leq&\|\varphi\|_\infty\Big[\int_0^{T_0}\int_{\mathcal{R}_{j_1}}\int_{\eta_{m_i'}(t,x')}^1\big|\partial_3\overline{u}_{m_i',3}-\partial_3\overline{u}_3\big|\,\mathrm{d}x_3\mathrm{d}x'\mathrm{d}s+\int_0^{T_0}\int_{\mathcal{R}_{j_1}}\Big|\int_{\eta_{m_i'}(t,x')}^0\partial_3\overline{u}_3\mathrm{d}x_3\Big|\mathrm{d}x'\mathrm{d}s\Big] \\
\leq&\|\varphi\|_\infty\Big(\int_0^{T_0}\int_{\mathcal{R}_{j_1}}\int_{j_2^{-1}}^1\big|\partial_3\overline{u}_{m_i',3}-\partial_3\overline{u}_3\big|\,\mathrm{d}x_3\mathrm{d}x'\mathrm{d}s \\
&+\int_0^{T_0}\int_{\mathcal{R}_{j_1}}\int_{-j_2^{-1}}^{j_2^{-1}}\big|\partial_3\overline{u}_{m_i',3}-\partial_3\overline{u}_3\big|\,\mathrm{d}x_3\mathrm{d}x'\mathrm{d}s+\int_0^{T_0}\int_{\mathcal{R}_{j_1}}\int_{-j_2^{-1}}^{j_2^{-1}}|\partial_3\overline{u}_3|\,\mathrm{d}x_3\mathrm{d}x'\mathrm{d}s\Big)<\delta,
\end{aligned}$$

whence

$$\lim_{i\to\infty}\int_0^t\int_{\mathbb{R}^2}\overline{v}_{m_i',3}\varphi\mathrm{d}x'\mathrm{d}s=\int_0^t\int_{\mathbb{R}^2}\overline{u}_3(t,x',0)\varphi\mathrm{d}x'\mathrm{d}s. \tag{5.31}$$

Consequently, letting $i\to\infty$, then $m_i'\to\infty$ in (5.26) (i.e., $\varepsilon_{m_i'}\to 0$ with $\varepsilon_{m_i'}$ in place of ε_m), we conclude, with the help of (5.27) and (5.30)–(5.31), that

$$\int_{\mathbb{R}^2}\overline{\eta}\varphi\mathrm{d}x'=\int_0^t\int_{\mathbb{R}^2}\overline{u}_3(s,x',0)\varphi\mathrm{d}x'\mathrm{d}s+\int_{\mathbb{R}^2}\widetilde{\eta}(0,x')\varphi\mathrm{d}x',$$

which implies that, for a.e. $t\in(0,T_0)$, $\partial_t\overline{\eta}=\overline{u}_3$ a.e. in $\mathbb{R}^2$.

5.4 Convergence of the momentum equations

Multiplying the second equation in (5.3) by $\phi=(\phi_1,\phi_2,\phi_3)\in(\mathcal{D}((0,T_0)\times\Omega))^3$ with ε_m in place of ε, integrating over $(0,T_0)\times\Omega$, and using the jump conditions (5.5), we deduce

$$\begin{aligned}
&\int_0^{T_0}\int_\Omega(\varrho\partial_t\overline{u}_m\cdot\phi+\varepsilon_m\varrho(\nabla\overline{u}_m)\overline{u}_m\cdot\phi)\mathrm{d}x\mathrm{d}t+\int_0^{T_0}\int_\Omega(\mu(\nabla\overline{u}_m+\nabla\overline{u}_m^{\mathrm{T}})-\overline{\sigma}_mI):\nabla\phi\mathrm{d}x\mathrm{d}t \\
=&\,g[\varrho]\int_0^{T_0}\int_{\Sigma_m(t)}\overline{\eta}_m\varphi_m\cdot\nu_m\mathrm{d}S\mathrm{d}t+\kappa\int_0^{T_0}\int_{\Sigma_m(t)}\overline{H}_m\varphi_m\cdot\nu_m\mathrm{d}S\mathrm{d}t,
\end{aligned}\tag{5.32}$$

where

$$\Sigma_m(t):=\{(x',x_3)\in\mathbb{R}^3\mid x_3=\eta_m(t,x')\}\quad\text{for each }t>0, \tag{5.33}$$

$$\varphi_m:=\varphi_m(t,x')=\phi(t,x',\eta_m(t,x')), \tag{5.34}$$

$$\nu_m=\frac{(-\partial_1\eta_m,-\partial_2\eta_m,1)^{\mathrm{T}}}{\sqrt{|\partial_1\eta_m|^2+|\partial_2\eta_m|^2+1}}, \tag{5.35}$$

$$\overline{H}_m=\frac{\Delta_{x'}\overline{\eta}_m+(\partial_1\eta_m)^2\partial_2^2\overline{\eta}_m+(\partial_2\eta_m)^2\partial_1^2\overline{\eta}_m-2\partial_1\overline{\eta}_m\partial_2\eta_m\partial_1\partial_2\eta_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{\frac{3}{2}}}. \tag{5.36}$$

By virtue of (5.11)–(5.13) and (5.16),

$$\begin{aligned}&\lim_{m\to\infty}\Big[\int_0^{T_0}\int_\Omega(\varrho\partial_t\overline{u}_m\cdot\phi+\varepsilon_m\varrho(\nabla\overline{u}_m)\overline{u}_m\cdot\phi)\mathrm{d}x\mathrm{d}t\\&+\int_0^{T_0}\int_\Omega(\mu(\nabla\overline{u}_m+\nabla\overline{u}_m^{\mathrm{T}})-\overline{\sigma}_mI):\nabla\phi\mathrm{d}x\mathrm{d}t\Big]\\=&\int_0^{T_0}\int_\Omega\varrho\partial_t\overline{u}\cdot\phi+\int_0^{T_0}\int_\Omega(\mu(\nabla\overline{u}+\nabla\overline{u}^{\mathrm{T}})-\overline{\sigma}I):\nabla\phi\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{5.37}$$

Next, we analyze the convergence of the terms on the right-hand side of (5.32).

(i) Recalling $\mathrm{d}S=\sqrt{|\partial_1\eta_m|^2+|\partial_2\eta_m|^2+1}\,\mathrm{d}x'$, we use the formula of surface integral and (5.33)–(5.35) to infer that

$$\int_0^{T_0}\int_{\Sigma_m(t)}\overline{\eta}_m\varphi_m\cdot\nu_m\mathrm{d}S\mathrm{d}t=\int_0^{T_0}\int_{\mathbb{R}^2}\overline{\eta}_m(\varphi_{m,3}-\varphi_{m,1}\partial_1\eta_m-\varphi_{m,2}\partial_2\eta_m)\mathrm{d}x'\mathrm{d}t.$$

Now we define

$$\|\phi\|_\infty:=\sup_{(t,x)\in(0,T_0)\times\Omega}|\phi(t,x)|,$$

thus, from the Hölder inequality and (5.28), it follows that

$$\begin{aligned}\int_0^{T_0}\int_{\mathbb{R}^2}\overline{\eta}_m\varphi_{m,i}\partial_i\eta_m\mathrm{d}x'\mathrm{d}t&=\varepsilon_m\int_0^{T_0}\int_{\mathbb{R}^2}\overline{\eta}_m\varphi_{m,i}\partial_i\overline{\eta}_m\mathrm{d}x'\mathrm{d}t\\&\le\varepsilon_mT_0\|\phi\|_\infty\sup_{0\le t\le T_0}(\|\overline{\eta}_m\|_{L^2(\mathbb{R}^2)}\|\partial_i\overline{\eta}_m\|_{L^2(\mathbb{R}^2)})\\&\le\varepsilon_mT_0\|\phi\|_\infty,\quad i=1,2,\end{aligned}$$

whence

$$\lim_{m\to\infty}\int_0^{T_0}\int_{\mathbb{R}^2}\overline{\eta}_m(\varphi_{m,1}\partial_1\eta_m+\varphi_{m,2}\partial_2\eta_m)\mathrm{d}x'\mathrm{d}t=0.\tag{5.38}$$

Noticing that

$$\begin{aligned}&\int_0^{T_0}\int_{\mathbb{R}^2}|\overline{\eta}_m(\varphi_{m,3}-\phi_3(t,x',0))|\mathrm{d}x'\mathrm{d}t\\\le&\sup_{t\in(0,T_0)}\|\overline{\eta}_m\|_{L^2(\mathbb{R}^2)}\int_0^{T_0}\Big[\int_{\mathbb{R}^2}\Big|\int_0^{\eta_m(t,x')}\partial_3\phi_3\mathrm{d}x_3\Big|^2\mathrm{d}x'\Big]^{\frac12}\mathrm{d}t\\\le&T_0\sup_{t\in(0,T_0)}\|\overline{\eta}_m\|_{L^2(\mathbb{R}^2)}\|\partial_3\phi_3\|_\infty\sup_{t\in(0,T_0)}\|\eta_m\|_{L^2(\mathbb{R}^2)}\\\le&\varepsilon_mT_0\|\partial_3\phi_3\|_\infty\to0\quad\text{as }m\to\infty,\end{aligned}\tag{5.39}$$

we make use of (5.15) and (5.39) to obtain

$$\begin{aligned}&\Big|\int_0^{T_0}\int_{\mathbb{R}^2}(\overline{\eta}_m\varphi_{m,3}-\overline{\eta}\phi_3(t,x',0))\mathrm{d}x'\mathrm{d}t\Big|\\\le&\int_0^{T_0}\int_{\mathbb{R}^2}|(\overline{\eta}_m-\overline{\eta})\varphi_{m,3}|\mathrm{d}x'\mathrm{d}t+\int_0^{T_0}\int_{\mathbb{R}^2}|\overline{\eta}(\varphi_{m,3}-\phi_3(t,x',0))|\mathrm{d}x'\mathrm{d}t\\\le&\|\phi_3\|_\infty\|\overline{\eta}_m-\overline{\eta}\|_{L^1((0,T)\times\mathcal{R}_{j_3})}+\varepsilon_mT_0\|\partial_3\phi_3\|_\infty\to0\quad\text{as }m\to\infty,\end{aligned}$$

which gives

$$\lim_{m\to\infty}\int_0^{T_0}\int_{\mathbb{R}^2}\overline{\eta}_m\varphi_{m,3}\mathrm{d}x'\mathrm{d}t=\int_0^{T_0}\int_{\mathbb{R}^2}\overline{\eta}\phi_3(t,x',0)\mathrm{d}x'\mathrm{d}t. \tag{5.40}$$

Here we have assumed that $\mathcal{R}_{j_3}$ satisfies

$$\bigcup_{i=1}^{3}\bigcup_{t\in(0,T_0)}\operatorname{supp}\phi_i(t,x)\subset\mathcal{R}_{j_3}.$$

Combining (5.38) with (5.40), we arrive at

$$\lim_{m\to\infty}\int_0^{T_0}\int_{\Sigma_m(t)}\overline{\eta}_m\varphi_m\cdot\nu_m\mathrm{d}S\mathrm{d}t=\int_0^{T_0}\int_{\mathbb{R}^2}\overline{\eta}\phi_3(t,x',0)\mathrm{d}x'\mathrm{d}t. \tag{5.41}$$

(ii) For the second term on the right-hand side of (5.32), taking into account that $\eta_m=\varepsilon\overline{\eta}_m$, we employ (5.33)–(5.36) to deduce that

$$\begin{aligned}
&\int_0^{T_0}\int_{\Sigma_m(t)}\overline{H}_m\varphi_m\cdot\nu_m\mathrm{d}S\mathrm{d}t\\
=&-\sum_{i=1}^{2}\int_0^{T_0}\int_{\mathbb{R}^2}\Big[\frac{\Delta_{x'}\overline{\eta}_m+(\partial_1\eta_m)^2\partial_2^2\overline{\eta}_m+(\partial_2\eta_m)^2\partial_1^2\overline{\eta}_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{\frac{3}{2}}}\\
&-\frac{2\partial_1\eta_m\partial_2\eta_m\partial_1\partial_2\overline{\eta}_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{\frac{3}{2}}}\Big]\varphi_{m,i}\partial_i\eta_m\mathrm{d}x'\mathrm{d}t\\
&+\int_0^{T_0}\int_{\mathbb{R}^2}\frac{(\partial_1\eta_m)^2\partial_2^2\overline{\eta}_m+(\partial_2\eta_m)^2\partial_1^2\overline{\eta}_m-2\partial_1\overline{\eta}_m\partial_2\eta_m\partial_1\partial_2\eta_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{\frac{3}{2}}}\varphi_{m,3}\mathrm{d}x'\mathrm{d}t\\
&+\int_0^{T_0}\int_{\mathbb{R}^2}\frac{\varphi_{m,3}\Delta_{x'}\overline{\eta}_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{\frac{3}{2}}}\mathrm{d}x'\mathrm{d}t,
\end{aligned} \tag{5.42}$$

where the first two terms on the right-hand side can be estimated as follows, using the Sobolev imbedding theorem and (5.28), while the third term can be bounded below, following a procedure similar to that used for (5.39):

$$\begin{aligned}
&\Big|\int_0^{T_0}\int_{\mathbb{R}^2}\frac{(\partial_1\eta_m)^2\partial_2^2\overline{\eta}_m+(\partial_2\eta_m)^2\partial_1^2\overline{\eta}_m-2\partial_1\overline{\eta}_m\partial_2\eta_m\partial_1\partial_2\eta_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{\frac{3}{2}}}\varphi_{m,3}\mathrm{d}x'\mathrm{d}t\Big|\\
\le&\|\phi_3\|_\infty\int_0^{T_0}\int_{\mathcal{R}_{j_3}}\big|(\partial_1\eta_m)^2\partial_2^2\overline{\eta}_m+(\partial_2\eta_m)^2\partial_1^2\overline{\eta}_m-2\partial_1\overline{\eta}_m\partial_2\eta_m\partial_1\partial_2\eta_m\big|\mathrm{d}x'\mathrm{d}t\\
=&\varepsilon_m^2\|\phi_3\|_\infty\int_0^{T_0}\int_{\mathcal{R}_{j_3}}\big|(\partial_1\overline{\eta}_m)^2\partial_2^2\overline{\eta}_m+(\partial_2\overline{\eta}_m)^2\partial_1^2\overline{\eta}_m-2\partial_1\overline{\eta}_m\partial_2\overline{\eta}_m\partial_1\partial_2\overline{\eta}_m\big|\mathrm{d}x'\mathrm{d}t\\
\le&4\varepsilon_m^2\|\phi_3\|_\infty T_0\|\nabla\overline{\eta}_m\|^2_{L^4(\mathcal{R}_{j_3})}\|\nabla^2\overline{\eta}_m\|_{L^2(\mathcal{R}_{j_3})}\\
\le&4\varepsilon_m^2\|\phi_3\|_\infty T_0c_2(j_3),
\end{aligned} \tag{5.43}$$

$$\begin{aligned}
&\Big|\int_0^{T_0}\int_{\mathbb{R}^2}\Big[\frac{\Delta_{x'}\overline{\eta}_m+(\partial_1\eta_m)^2\partial_2^2\overline{\eta}_m+(\partial_2\eta_m)^2\partial_1^2\overline{\eta}_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{3/2}}\\
&-\frac{2\partial_1\eta_m\partial_2\eta_m\partial_1\partial_2\overline{\eta}_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{3/2}}\Big]\varphi_{m,i}\partial_i\eta_m\mathrm{d}x'\mathrm{d}t\Big|\\
\le{}&\varepsilon_m\|\phi\|_\infty\int_0^{T_0}\int_{\mathcal{R}_{j_3}}|(\Delta_{x'}\overline{\eta}_m+(\partial_1\eta_m)^2\partial_2^2\overline{\eta}_m+(\partial_2\eta_m)^2\partial_1^2\overline{\eta}_m\\
&-2\partial_1\eta_m\partial_2\eta_m\partial_1\partial_2\overline{\eta}_m)\partial_i\overline{\eta}_m|\mathrm{d}x'\mathrm{d}t\\
\le{}&\varepsilon_m\|\phi\|_\infty T_0(3\varepsilon^2\|\nabla\overline{\eta}_m\|^3_{L^6(\mathcal{R}_{j_3})}\|\nabla^2\overline{\eta}_m\|_{L^2(\mathcal{R}_{j_3})}+\|\nabla\overline{\eta}_m\|_{L^2(\mathcal{R}_{j_3})}\|\nabla^2\overline{\eta}_m\|_{L^2(\mathcal{R}_{j_3})})\\
\le{}&\varepsilon_m\|\phi_3\|_\infty T_0c_3(j_3),\quad\varepsilon_m<3^{-1},\ \ i=1,2,
\end{aligned}\tag{5.44}$$

where $c_2(j_3)$ and $c_3(j_3)$ are two constants depending on j_3, and

$$\begin{aligned}
&\Big|\int_0^{T_0}\int_{\mathbb{R}^2}\frac{(\varphi_{m,3}-\phi_3(t,x',0))\Delta_{x'}\overline{\eta}_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{\frac32}}\mathrm{d}x'\mathrm{d}t\Big|\\
\le{}&T_0\sup_{t\in(0,T_0)}\|\Delta_{x'}\overline{\eta}\|_{L^2(\mathbb{R}^2)}\|\partial_3\phi_3\|_\infty\sup_{t\in(0,T_0)}\|\eta_m\|_{L^2(\mathbb{R}^2)}\le\varepsilon_mT_0\|\partial_3\phi\|_\infty\to0.
\end{aligned}\tag{5.45}$$

On the other hand, applying (5.15) and the dominated convergence theorem, we conclude that

$$\begin{aligned}
(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{-\frac32}&=(1+\varepsilon_m^2(\partial_1\overline{\eta}_m)^2+\varepsilon_m^2(\partial_2\overline{\eta}_m)^2)^{-\frac32}\\
&\to1\quad\text{strongly in }\ L^2(0,T_0;L^2(\mathcal{R}_{j_3}))
\end{aligned}\tag{5.46}$$

as $m\to\infty$, while $\varepsilon_m\to0$. Thus, from (5.46) and (5.14), we get

$$\begin{aligned}
&\lim_{m\to\infty}\int_0^{T_0}\int_{\mathbb{R}^2}\frac{\phi_3\Delta_{x'}\overline{\eta}_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{\frac32}}\mathrm{d}x'\mathrm{d}t\\
={}&\lim_{m\to\infty}\int_0^{T_0}\int_{\mathcal{R}_{j_3}}\frac{\phi_3\Delta_{x'}\overline{\eta}_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{\frac32}}\mathrm{d}x'\mathrm{d}t\\
={}&\int_0^{T_0}\int_{\mathcal{R}_{j_3}}\frac{\phi_3\Delta_{x'}\overline{\eta}}{(1+(\partial_1\eta)^2+(\partial_2\eta)^2)^{\frac32}}\mathrm{d}x'\mathrm{d}t\\
={}&\int_0^{T_0}\int_{\mathbb{R}^2}\frac{\phi_3\Delta_{x'}\overline{\eta}}{(1+(\partial_1\eta)^2+(\partial_2\eta)^2)^{\frac32}}\mathrm{d}x'\mathrm{d}t.
\end{aligned}\tag{5.47}$$

In view of (5.45) and (5.47), we find that

$$\lim_{m\to\infty}\int_0^{T_0}\int_{\mathbb{R}^2}\frac{\varphi_{m,3}\Delta_{x'}\overline{\eta}_m}{(1+(\partial_1\eta_m)^2+(\partial_2\eta_m)^2)^{\frac32}}\mathrm{d}x'\mathrm{d}t=\int_0^{T_0}\int_{\mathbb{R}^2}\Delta_{x'}\overline{\eta}\phi_3(t,x',0)\mathrm{d}x'\mathrm{d}t.\tag{5.48}$$

Combining (5.43) with (5.44) and (5.48), we conclude that

$$\lim_{m\to\infty}\int_0^{T_0}\int_{\Sigma_m(t)}\overline{H}_m\varphi_m\cdot\nu_m\mathrm{d}S\mathrm{d}t=\int_0^{T_0}\int_{\mathbb{R}^2}\Delta_{x'}\overline{\eta}\phi_3(t,x',0)\mathrm{d}x'\mathrm{d}t.\tag{5.49}$$

Consequently, it follows from (5.32), (5.37), (5.41) and (5.49) that

$$\begin{aligned}
&\int_0^{T_0}\int_\Omega\varrho\partial_t\overline{u}\cdot\phi+\int_0^{T_0}\int_\Omega(\mu(\nabla\overline{u}+\nabla\overline{u}^{\mathrm{T}})-\overline{\sigma}I):\nabla\phi\mathrm{d}x\mathrm{d}t\\
={}&g[\varrho]\int_0^{T_0}\int_{\mathbb{R}^2}\overline{\eta}\phi_3(t,x',0)\mathrm{d}x'\mathrm{d}t+\kappa\int_0^{T_0}\int_{\mathbb{R}^2}\Delta_{x'}\overline{\eta}\phi_3(t,x',0)\mathrm{d}x'\mathrm{d}t.
\end{aligned}\tag{5.50}$$

5.5 Contradiction argument

In a way similar to (4.18), we multiply the first equation in (5.20) with $\phi \in (\mathcal{D}((0,T_0)\times\Omega))^3$ and integrate over $(0,T_0)\times\Omega$ to infer that

$$\begin{aligned}&\int_0^{T_0}\int_\Omega \varrho\partial_t\overline{u}\cdot\phi+\int_0^{T_0}\int_\Omega(\mu(\nabla\overline{u}+\nabla\overline{u}^{\mathrm{T}})-\overline{\sigma}I):\nabla\phi\mathrm{d}x\mathrm{d}t\\&=\int_0^{T_0}\int_{\mathbb{R}^2}((\gamma_+(\overline{\sigma})I-\mu_+(\nabla\overline{u}_++\nabla\overline{u}_+^{\mathrm{T}}))-(\gamma_-(\overline{\sigma})I-\mu_-(\nabla\overline{u}_-+\nabla\overline{u}_-^{\mathrm{T}})))e_3\cdot\phi\mathrm{d}x'\mathrm{d}t.\end{aligned}\tag{5.51}$$

Comparing (5.51) with (5.50), we get

$$\begin{aligned}&\int_0^{T_0}\int_{\mathbb{R}^2}((\gamma_+(\overline{\sigma})I-\mu_+(\nabla\overline{u}_++\nabla\overline{u}_+^{\mathrm{T}}))-(\gamma_-(\overline{\sigma})I-\mu_-(\nabla\overline{u}_-+\nabla\overline{u}_-^{\mathrm{T}})))e_3\cdot\phi(t,x',0)\mathrm{d}x'\mathrm{d}t\\&=\int_0^{T_0}\int_{\mathbb{R}^2}(g[\varrho]\overline{\eta}+\kappa\Delta_{x'}\overline{\eta})e_3\cdot\phi(t,x',0))\mathrm{d}x'\mathrm{d}t.\end{aligned}\tag{5.52}$$

On the other hand, by Lemma 4.1, (4.11), (5.11) and (5.21), we have

$$\big(\gamma_+(\overline{\sigma})I-\mu_+(\nabla\overline{u}_++\nabla\overline{u}_+^{\mathrm{T}}))-(\gamma_-(\overline{\sigma})I-\mu_-(\nabla\overline{u}_-+\nabla\overline{u}_-^{\mathrm{T}})\big)e_3\in L^\infty(0,T_0;(L^2(\mathbb{R}^2))^3),\tag{5.53}$$

while by virtue of (5.28),

$$g[\varrho]\overline{\eta}+\kappa\Delta_{x'}\overline{\eta}\in L^\infty(0,T_0;L^2(\mathbb{R}^2)).\tag{5.54}$$

Hence, by a density argument, we get from (5.52)–(5.54) that

$$[(\gamma_+(\overline{\sigma})I-\mu_+(\nabla\overline{u}_++\nabla\overline{u}_+^{\mathrm{T}}))-(\gamma_-(\overline{\sigma})I-\mu_-(\nabla\overline{u}_-+\nabla\overline{u}_-^{\mathrm{T}}))]e_3=(g[\varrho]\overline{\eta}+\kappa\Delta_{x'}\overline{\eta})e_3$$

holds a.e. in $\mathbb{R}^2$ and for a.e. $t\in(0,T_0)$.

In view of Definition 4.1, we find that $(\overline{\eta},\overline{u},\overline{\sigma})$ is just a strong solution to the linearized problem (1.5)–(1.7). By Remark 4.2, $(\widetilde{\eta},\widetilde{u},\widetilde{\sigma})$ is also a strong solution to (1.5)–(1.7). Moreover, $\widetilde{\eta}(0)=\overline{\eta}(0)$ and $\widetilde{u}(0)=\overline{u}(0)$ (see (5.23)). Then, according to Theorem 4.1,

$$\overline{u}=\widetilde{u}\quad\text{ on }[0,T_0)\times\Omega.$$

Hence, we may chain together the inequalities (5.2) and (5.21) to get

$$2\le\sup_{\frac{T_0}{2}\le t<T_0}\|\widetilde{u}(t)\|_{H^3(\Omega_\pm)}\le\sup_{0\le t<T_0}\|\overline{u}\|_{H^3(\Omega_\pm)}\le 1,$$

which is a contradiction. Therefore, the perturbed problem does not have the global stability of order k for any $k\ge 3$. This completes the proof of Theorem 2.2.

Acknowledgement The authors are grateful to the referees for their comments which are helpful in improving the presentation of this paper.

References

[1] Adams, R. A. and John, J., Sobolev Space, Academic Press, New York, 2005.

[2] Chandrasekhar, S., Hydrodynamic and Hydromagnetic Stability, The International Series of Monographs on Physics, Clarendon Press, Oxford, 1961.

[3] Duan, R., Jiang, F. and Jiang, S., On the Rayleigh-Taylor instability for incompressible, inviscid magnetohydrodynamic flows, *SIAM J. Appl. Math.*, **71**, 2011, 1990-2013.

[4] Ebin, D., The equations of motion of a perfect fluid with free boundary are not well-posed, *Comm. Part. Diff. Eq.*, **12**, 1987, 1175–1201.

[5] Ebin, D., Ill-posedness of the Rayleigh-Taylor and Helmholtz problems for incompressible fluids, *Comm. Part. Diff. Eq.*, **13**, 1988, 1265–1295.

[6] Guo, Y. and Tice, I., Compressible, inviscid Rayleigh-Taylor instability, *Indiana Univ. Math. J.*, **60**, 2011, 677–712.

[7] Guo, Y. and Tice, I., Decay of viscous surface waves without surface tension, 2010. arXiv: 1011.5179v1

[8] Guo, Y. and Tice, I., Linear Rayleigh-Taylor instability for viscous, compressible fluids, *SIAM J. Math. Anal.*, **42**, 2011, 1688–1720.

[9] Hwang, H. J., Variational approach to nonlinear gravity-driven instability in an MHD setting, *Quart. Appl. Math.*, **66**, 2008, 303–324.

[10] Hwang, H. J. and Guo, Y., On the dynamical Rayleigh-Taylor instability, *Arch. Rational Mech. Anal.*, **167**, 2003, 235–253.

[11] Jiang, F., Jiang, S. and Ni, G. X., Nonlinear instability for nonhomogeneous incompressible viscous fluids, *Sci. China Math.* **56**, 2013, 665–686.

[12] Jiang, F., Jiang, S. and Wang, Y. J., On the Rayleigh-Taylor instability for incompressible viscous magnetohydrodynamic equations, *Commun. Part. Diff. Eq.*, **39**, 2014, 399–438.

[13] Jiang, F., Jiang, S. and Wang, W. W., Nonlinear Rayleigh-Taylor instability in nonhomogeneous incompressible viscous magnetohydrodynamic fluids, 2013. arXiv: 1304.5636

[14] Kruskal, M. and Schwarzschild, M. Some instabilities of a completely ionized plasma, *Proc. Roy. Soc.* (*London*) *A*, **233**, 1954, 348–360.

[15] Nespoli, G. and Salvi, R., On the existence of two-phase problem for incompressible flow, Advances in Fluid Dynamics, Quaderni di Mathematica, **4**, P. Maremonti (ed.), Dept. Math., Seconda Univ., Napoli, 1999, 245–268.

[16] Novotnỳ, A. and Straškraba, I., Introduction to the Mathematical Theory of Compressible Flow, Oxford University Press, Oxford, 2004.

[17] Prüess, J. and Simonett, G., On the Rayleigh–Taylor instability for the two-phase Navier–Stokes equations, *Indiana Univ. Math. J.*, **59**, 2010, 1853–1871.

[18] Rayleigh, L., Analytic solutions of the Rayleigh equations for linear density profiles, *Proc. London. Math. Soc.*, **14**, 1883, 170–177.

[19] Rayleigh, L., Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density, *Scientific Paper*, **II**, 1990, 200–207.

[20] Taylor, G. I., The stability of liquid surface when accelerated in a direction perpendicular to their planes, *Proc. Roy Soc. A*, **201**, 1950, 192–196.

[21] Tice, I. and Wang, Y. J., The viscous surface-internal wave problem: Nonlinear Rayleigh-Taylor instability, *Commun. Part. Diff. Eq.*, **37**, 2012, 1967–2028.

[22] Wang, J., Two-Dimensional Nonsteady Flows and Shock Waves (in Chinese), Science Press, Beijing, 1994.

[23] Wang, Y., Critical magnetic number in the MHD Rayleigh-Taylor instability, *J. Math. Phys.*, **53**, 2012, 073701.

[24] Wehausen, J. and Laitone, E., Surface Waves, Handbuch der Physik, Vol. 9, Part 3, Springer-Verlag, Berlin, 1960.