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Carleson Type Measures for Harmonic Mixed Norm Spaces with Application to Toeplitz Operators*

Zhangjian HU[1] Xiaofen LV[2]

Abstract Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary, and let $h_{p,q}$ be the space of all harmonic functions with a finite mixed norm. The authors first obtain an equivalent norm on $h_{p,q}$, with which the definition of Carleson type measures for $h_{p,q}$ is obtained. And also, the authors obtain the boundedness of the Bergman projection on $h_{p,q}$ which turns out the dual space of $h_{p,q}$. As an application, the authors characterize the boundedness (and compactness) of Toeplitz operators T_μ on $h_{p,q}$ for those positive finite Borel measures μ.

Keywords Carleson type measure, Harmonic mixed norm space, Toeplitz operator, Bergman projection
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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary. Let $\lambda(x)$ be a defining function of Ω, that is, λ is a C^∞ real valued function and $\Omega = \{x \in \mathbb{R}^n : \lambda(x) < 0\}$ is bounded, $|\nabla\lambda(x)| \neq 0$ on the boundary $\partial\Omega$ of Ω (see [1]), where $\nabla = \big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\big)$. For $r > 0$ small enough, let $\Omega_{r,\lambda} = \{x \in \mathbb{R}^n : \lambda(x) < -r\}$. $\Omega_{r,\lambda}$ is also a C^∞ domain with the defining function $\lambda(x)+r$ and $\partial\Omega_{r,\lambda} = \{x \in \mathbb{R}^n : \lambda(x) = -r\}$. We denote by $\mathrm{d}\sigma_{r,\lambda}$ the induced surface measure on $\partial\Omega_{r,\lambda}$. Of course, there are infinitely many defining functions of Ω and any two different defining functions yield two different systems of $\{\partial\Omega_{r,\lambda}\}$ and $\{\mathrm{d}\sigma_{r,\lambda}\}$. We denote by $\mathrm{d}m$ the Lebesgue volume measure on $\mathbb{R}^n$.

Given a defining function λ, $0 < p < \infty$ and r small enough, we write

$$M_p(f,r,\lambda) = \Big\{ \int_{\partial\Omega_{r,\lambda}} |f(\zeta)|^p \mathrm{d}\sigma_{r,\lambda}(\zeta) \Big\}^{\frac{1}{p}}.$$

For $\varepsilon > 0$ small and fixed, $0 < p, q < \infty$, $\|f\|_{p,q,\lambda}$ is defined as

$$\|f\|_{p,q,\lambda} = \Big\{ \int_0^\varepsilon M_p^q(f,r,\lambda)\mathrm{d}r \Big\}^{\frac{1}{q}}.$$

Let $h(\Omega)$ be the family of all harmonic functions on Ω. The harmonic mixed norm space $h_{p,q}$ is defined to be

$$h_{p,q}(\Omega) = \{f \in h(\Omega) : \|f\|_{p,q,\lambda} < \infty\}.$$

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[1]Department of Mathematics, Huzhou Teachers College, Huzhou 313000, Zhejiang, China.
E-mail: huzj@hutc.zj.cn
[2]College of Mathematics and Science, Xiamen University, Xiamen 361005, Fujian, China.
E-mail: lvxf@hutc.zj.cn
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The mixed norm space for holomorphic functions on the unit disc (or the unit ball) has been studied in [2–5] and also by some other authors. As shown in [6], $h_{p,q}(\Omega)$ is independent of λ, and any two different defining functions λ_1 and λ_2 of Ω yield two equivalent norms $\|\cdot\|_{p,q,\lambda_1}$ and $\|\cdot\|_{p,q,\lambda_2}$. So, we will omit the subscript λ if no confusion occurs, and for example we simply write $\|f\|_{p,q}$ for $\|f\|_{p,q,\lambda}$ and Ω_r for $\Omega_{r,\lambda}$, respectively.

In what follows, we will use C to stand for positive constants whose value may change from line to line but does not depend on the functions being considered. Two quantities A and B are called equivalent (denoted by "$A \simeq B$") if there exists some C such that $C^{-1}A \le B \le CA$.

Let $r(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$. Then $r(x)$ is continuous on Ω. Write $r_0 = \max\{r(x) : x \in \Omega\}$ which is a finite positive number. For $j = 1, 2, \cdots$ and $r_j = \frac{r_0}{2^j}$, set

$$S_j = \{x \in \Omega : r_j < r(x) \le r_{j-1}\}.$$

Then $\Omega = \bigcup\limits_{j=1}^{\infty} S_j$. For fixed positive exponents p and q, we define the mixed norm space $L_{p,q}(\Omega)$ to be the set of all Lebesgue measurable functions f on Ω for which

$$\|f\|_{L_{p,q}} = \Big\{ \sum_{j=1}^{\infty} \Big(\int_{S_j} |f(x)|^p \mathrm{d}m(x) \Big)^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \Big\}^{\frac{1}{q}} < \infty.$$

It is easy to check that $L_{p,q}(\Omega)$ is a Banach space with the norm $\|\cdot\|_{L_{p,q}}$ when $1 < p, q < \infty$. In Section 2, we will prove that

$$h_{p,q}(\Omega) = L_{p,q}(\Omega) \cap h(\Omega),$$

and that $h_{p,q}(\Omega)$ is a closed subspace of $L_{p,q}(\Omega)$. Moreover, $\|f\|_{p,q} \simeq \|f\|_{L_{p,q}}$ for $f \in h(\Omega)$. If $p = q$, $L_{p,q}(\Omega)$ is just the space $L^p(\Omega)$ with the norm defined by

$$\|f\|_p = \Big(\int_{\Omega} |f(x)|^p \mathrm{d}m(x) \Big)^{\frac{1}{p}}.$$

Setting $t_1 = \max(p, q)$, $t_2 = \min(p, q)$, by the Hölder's inequality,

$$\frac{1}{C}\|f\|_{t_2} \le \|f\|_{L_{p,q}} \le C\|f\|_{t_1}.$$

And also, when $p = q$, $h_{p,q}(\Omega)$ is the harmonic Bergman space $b^p(\Omega)$ (see [7]).

Carleson measures for Hardy (and for Bergman) spaces play a very important role in the one and several complex variables analysis. It is a powerful tool to study the problems such as H^1-BMO duality, Corona problem, and many others. We know that the mixed norm space, as the generalization of Bergman space, was introduced some decades ago, but unlike the one in the Bergman space setting, there is nothing in the literature about the Carleson type measures for mixed norm spaces until now, or even on the unit disc, the simplest case. In Section 2, we will obtain an equivalent norm on $h_{p,q}$, which inspires us to introduce the Carleson type measures for $h_{p,q}$. Our definition is also suitable for holomorphic mixed norm spaces.

Let $R(x, y)$ be the reproducing kernel of the harmonic Bergman space $b^2(\Omega)$. Known from [7], $R(x, y)$ is symmetric and real-valued. And for fixed $x \in \Omega$, as a function of y, $R(x, y)$ is bounded on Ω. Hence, the integral operator P, called the Bergman projection,

$$Pf(x) = \int_{\Omega} R(x, y) f(y) \mathrm{d}m(y), \quad x \in \Omega \tag{1.1}$$

is well defined on $L^1(\Omega)$. This means that P is an integral operator that maps $L^1(\Omega)$ to $h(\Omega)$. More generally, for finite positive Borel measures μ on Ω (being simply written as $\mu \ge 0$), the integral

$$P\mu(x) = \int_{\Omega} R(x, y) \mathrm{d}\mu(y)$$

also defines a function harmonic on Ω (see [7–8] for details). The Toeplitz operator with the symbol μ is defined to be

$$T_\mu f(x) = \int_\Omega R(x,y) f(y) \mathrm{d}\mu(y) \tag{1.2}$$

for $f \in h^\infty$, where h^∞ denotes the space of all bounded functions in $h(\Omega)$. We will see in Section 2 that h^∞ is dense in $h_{p,q}$, and therefore T_μ as (1.2) is densely defined on $h_{p,q}$ for each p and q in $(1,\infty)$.

Toeplitz operators acting on holomorphic Bergman spaces were well studied (see for example [9–10]). In the hamonic Bergman space setting, the operator T_μ was also studied by many authors. Miao obtained the boundedness, compactness and Schatten classes of this operator on the harmonic Bergman spaces b^p in the unit ball for $p > 1$ in [11]. Choe, Lee and Na discussed this operator from $b^p(\Omega)$ to $b^q(\Omega)$ for $1 < p, q < \infty$ in [12–13]. Recently, Choe et al considered positive Toeplitz operators of Schatten class and Schatten-Herz class in [14–15], respectively. Meanwhile, Choe et al discussed the same problems in the setting of a half-space in [8, 16–17].

In Section 3, we obtain the boundedness of the Bergman projection on $h_{p,q}$, with which we formulate the dual space of $h_{p,q}$. In Section 4, we will apply the results of Sections 2–3 to characterize the boundedness and the compactness of Toeplitz operators T_μ on the harmonic mixed norm space for those $\mu \geq 0$. Our results will extend those mentioned above.

2 Carleson Type Measures of $h_{p,q}(\Omega)$

Carleson measures play a very important role in the Hardy and Bergman space theory (see [12, 18] and the references therein). Because the mixed norm space $h_{p,q}(\Omega)$ is an extension of Bergman spaces, we believe that the "Carleson measure" should be well defined for $h_{p,q}(\Omega)$, and also, this measure should provide a powerful tool for the study on $h_{p,q}(\Omega)$. Before giving our definition, we will exhibit an equivalent norm on the mixed norm space as follows.

Theorem 2.1 *Given $1 < p, q < \infty$, then for $f \in h(\Omega)$ there holds*

$$\|f\|_{p,q}^q \simeq \sum_{j=1}^{\infty} \Big[\int_{S_j} |f(x)|^p \mathrm{d}m(x) \Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)}.$$

Proof Applying Theorems 2–3 in [1] on the domain Ω_r, we have some positive constant ε such that, for $f \in h(\Omega)$ and $0 < r < t \leq \varepsilon$, $M_p(f,t) \leq C_r M_p(f,r)$. A careful check of the implication in [1] shows that the constant C_r depends on the smoothness and the curvature of $\partial\Omega_r$. In the present case, Ω is a bounded C^∞ domain, and hence C_r depends actually only on $\partial\Omega$. This means that we have some constant C such that

$$M_p(f,t) \leq C M_p(f,r), \quad \text{when } 0 < r < t \leq \varepsilon. \tag{2.1}$$

Fix J to be the positive integer with $2^{-J} r_0 > \varepsilon$ and $2^{-(J+1)} r_0 \leq \varepsilon$. Set $\Omega_J = \{x \in \Omega : r(x) \geq 2^{-(J+1)} r_0\}$. Thus

$$\begin{aligned}
\|f\|_{p,q}^q &\leq \sup_{x\in\Omega_J} |f(x)|^q + \sum_{j=J+2}^{\infty} \int_{r_j}^{r_{j-1}} \Big[\int_{\partial\Omega_r} |f(\zeta)|^p d\sigma_r(\zeta) \Big]^{\frac{q}{p}} \mathrm{d}r \\
&\leq \sup_{x\in\Omega_J} |f(x)|^q + C \sum_{j=J+2}^{\infty} [M_p(f,r_j)(r_{j-1}-r_j)^{\frac{1}{q}}]^q \\
&\leq \sup_{x\in\Omega_J} |f(x)|^q + C \sum_{j=J+2}^{\infty} \Big[\int_{r_{j+1}}^{r_j} M_p^p(f,r) r^{(\frac{p}{q}-1)} \mathrm{d}r \Big]^{\frac{q}{p}}
\end{aligned}$$

$$\leq C\sum_{j=1}^{\infty}\Big[\int_{r_{j+1}}^{r_j} M_p^p(f,r)\mathrm{d}r\Big]^{\frac{q}{p}}\Big(\frac{r_0}{2^j}\Big)^{1-\frac{q}{p}}$$
$$= C\sum_{j=1}^{\infty}\Big[\int_{S_j}|f(x)|^p\mathrm{d}m(x)\Big]^{\frac{q}{p}}2^{j(\frac{q}{p}-1)}.$$

By the same argument, we get

$$\|f\|_{p,q}^q \geq \sum_{j=J+2}^{\infty}[M_p(f,r_{j-1})(r_{j-1}-r_j)^{\frac{1}{q}}]^q \geq C\sum_{j=J+2}^{\infty}\Big[\int_{S_j}|f(x)|^p\mathrm{d}m(x)\Big]^{\frac{q}{p}}2^{j(\frac{q}{p}-1)}.$$

On the other hand,

$$\sum_{j=1}^{J+1}\Big[\int_{S_j}|f(x)|^p\mathrm{d}m(x)\Big]^{\frac{q}{p}}2^{j(\frac{q}{p}-1)} \leq C\sup_{x\in\Omega_J}|f(x)|^q \leq C\|f\|_{p,q}^q.$$

Thus

$$\|f\|_{p,q}^q \simeq \sum_{j=1}^{\infty}\Big[\int_{S_j}|f(x)|^p\mathrm{d}m(x)\Big]^{\frac{q}{p}}2^{j(\frac{q}{p}-1)}.$$

The proof is completed.

Corollary 2.1 *For $1<p,q<\infty$, $h_{p,q}(\Omega)=L_{p,q}(\Omega)\cap h(\Omega)$, and $h_{p,q}(\Omega)$ is a closed subspace of $L_{p,q}(\Omega)$. Moreover, $\|f\|_{p,q}\simeq\|f\|_{L_{p,q}}$ for $f\in h(\Omega)$.*

Proof The proof is routine and omitted.

Remark 2.1 The conclusion of Theorem 2.1 is valid for all possible $0<p,q<\infty$. To prove this, instead of using the estimate (2.1) we apply the claim that there exist some positive constants $c_1<c_2$ such that for all $f\in h(\Omega)$ and $0<r<\varepsilon$,

$$M_p^q(f,r)\leq C\frac{1}{r}\int_{c_1r}^{c_2r}M_p^q(f,t)\mathrm{d}t.$$

See [6] for the detail.

In view of Theorem 2.1, we give the definition of the Carleson type measure for $h_{p,q}(\Omega)$ as follows.

Definition 2.1 *A finite positive Borel measure μ on Ω is called a Carleson type measure for $h_{p,q}$ if there exists some constant C such that for $f\in h_{p,q}$,*

$$\Big\{\sum_{j=1}^{\infty}\Big[\int_{S_j}|f(x)|^p\mathrm{d}\mu(x)\Big]^{\frac{q}{p}}2^{j(\frac{q}{p}-1)}\Big\}^{\frac{1}{q}}\leq C\|f\|_{p,q};$$

and also, μ is called a vanishing Carleson type measure for $h_{p,q}$ if

$$\lim_{k\to 0}\Big\{\sum_{j=1}^{\infty}\Big[\int_{S_j}|f_k(x)|^p\mathrm{d}\mu(x)\Big]^{\frac{q}{p}}2^{j(\frac{q}{p}-1)}\Big\}^{\frac{1}{q}}=0$$

for each bounded sequence $\{f_k\}_{k=1}^{\infty}$ in $h_{p,q}(\Omega)$ which converges to 0 uniformly on each compact subset of Ω.

When $p = q$, this definition coincides with the Carleson measures for Bergman spaces in [12].

Our second theorem is about the sharp estimate for the integral mean of the Bergman kernel $R(\cdot, x)$. To get this theorem, we need the following lemma. For this purpose we set $d(x, y) = r(x) + r(y) + |x - y|$.

Lemma 2.1 *For any $s > n - 1$, then there exists C such that for all $x \in \Omega$ and $0 < r \le \varepsilon$,*

$$\int_{\partial\Omega_r} \frac{1}{d(x,y)^s} \mathrm{d}\sigma(y) \le \frac{C}{(r(x)+r)^{s-(n-1)}}.$$

Proof For $0 < a < b$,

$$\int_{\partial\Omega_r \cap B(x,a)} \mathrm{d}\sigma(y) \le Ca^{n-1}$$

and

$$\int_{\partial\Omega_r \cap \{y: a \le |y-x| < b\}} \frac{1}{[r(x)+r+|y-x|]^s} \mathrm{d}\sigma(y) \le \frac{C}{[r(x)+r+a]^s} b^{n-1}.$$

Hence, with $d(x, y) = r(x) + r(y) + |x - y|$,

$$\begin{aligned}
&\int_{\partial\Omega_r} \frac{1}{d(x,y)^s} \mathrm{d}\sigma(y) \\
&= \int_{\partial\Omega_r \cap B(x, r(x)+r)} \frac{1}{d(x,y)^s} \mathrm{d}\sigma(y) + \sum_{j=1}^{\infty} \int_{\partial\Omega_r \cap \{y: 2^{j-1}(r(x)+r) \le |y-x| < 2^j (r(x)+r)} \frac{1}{d(x,y)^s} \mathrm{d}\sigma(y) \\
&\le \frac{C}{(r(x)+r)^s} (r(x)+r)^{n-1} + \sum_{j=1}^{\infty} \frac{C}{[2^{j-1}(r(x)+r)]^s} [2^j (r(x)+r)]^{n-1} \\
&\le \frac{C}{(r(x)+r)^{s-(n-1)}}.
\end{aligned}$$

The proof is completed.

Corollary 2.2 *For $s > -1$, $t < 1$ with $s + t > 0$, there exists some C such that*

$$\int_{\Omega} \frac{\mathrm{d}m(y)}{d(x,y)^{n+s} r(y)^t} \le \frac{C}{r(x)^{s+t}}.$$

Proof It can easily follow from Lemma 2.1 and the fact that

$$\int_{\Omega} \frac{\mathrm{d}m(y)}{d(x,y)^{n+s} r(y)^t} \simeq \int_0^{\varepsilon_0} \rho^{-t} \int_{\partial\Omega_\rho} \frac{1}{d(x,y)^{n+s}} \mathrm{d}\sigma_\rho(y) \mathrm{d}\rho.$$

The proof is completed.

Remark 2.2 Restricting $s, t \ge 0$, this is just Lemma 4.1 in [7], which is the essential estimate there.

Theorem 2.2 *For $p > \frac{n-1}{n}$, we have that*

$$M_p(R(\cdot, x), r) \le \frac{C}{(r(x)+r)^{n-\frac{n-1}{p}}},$$

and the exponent $n - \frac{n-1}{p}$ in the left-hand side is best possible.

Proof By Theorem 1.1 in [7], we have

$$|R(y,x)| \le \frac{C}{d(x,y)^n} \quad \text{and} \quad |R(x,x)| \simeq r(x)^{-n}$$

for all $x, y \in \Omega$. Then, Lemma 2.1 tells us

$$M_p^p(R(\cdot,x),r) \le C\int_{\partial\Omega_r} \frac{1}{d(y,x)^{pn}}\mathrm{d}\sigma_r(y) \le \frac{C}{(r(x)+r)^{pn-(n-1)}}.$$

For any $x \in \Omega$ and $0 < \delta < 1$, we set the Euclidean ball $E_\delta(x) = \{y \in \Omega : |y - x| < \delta r(x)\}$. Now for $y \in E_\delta(x)$, by [7] again we have

$$\begin{aligned}|R(y,x) - R(x,x)| &\le |y-x| \max\Big\{\Big|\frac{\partial R(y,x)}{\partial y}\Big| : y \in \overline{E_\delta(x)}\Big\}\\ &\le \delta r(x) \max\Big\{\frac{1}{d(x,y)^{n+1}} : y \in \overline{E_\delta(x)}\Big\}\\ &\le \frac{\delta}{r(x)^n}.\end{aligned}$$

Then, we can chose some small $\delta > 0$ fixed such that for all $y \in E_\delta(x)$,

$$|R(y,x)| \ge R(x,x) - \frac{\delta}{r(x)^n} \ge \frac{C}{r(x)^n}.$$

Therefore, for $x \in \partial\Omega_r$,

$$\begin{aligned}M_p^p(R(\cdot,x),r) &\ge \int_{\partial\Omega_r\cap E_\delta(x)} |R(y,x)|^p \mathrm{d}\sigma_r(y)\\ &\ge C\int_{\partial\Omega_r\cap E_\delta(x)} \frac{1}{r(x)^{pn}}\mathrm{d}\sigma_r(y)\\ &= \frac{C}{r(x)^{pn-(n-1)}}.\end{aligned}$$

This means that the exponent $n - \frac{n-1}{p}$ is best possible. The proof is completed.

To characterize Carleson type measures for $h_{p,q}$, we need some more lemmas. The first is about the finite fold covering of Ω, which essentially comes from [19]. Much more general setting can be seen in [20].

Lemma 2.2 *Let $\delta \in (0,1)$. Then, there exists a sequence $\{a_k\}$ in Ω satisfying the following conditions:*

(1) $\Omega = \bigcup\limits_{k=1}^{\infty} E_{\frac{\delta}{3}}(a_k)$.

(2) *There exists a positive integer M such that every point in Ω belongs to at most M of the sets $E_\delta(a_k)$.*

Afterward, $\{a_k\}$ will always refer to the sequence chosen in the lemma above. Note that $a_k \to \partial\Omega$ as $k \to \infty$.

Lemma 2.3 *For $\delta \in (0,1)$, there exists some positive integer $N = N(\delta)$ such that for $k = 1, 2, \cdots$, and $x \in S_k$,*

$$E_\delta(x) \subseteq \bigcup_{j=k-N}^{k+N} S_j,$$

where $S_j = \varnothing$ if $j \le 0$.

Proof Given $\delta \in (0,1)$, fix N so that $1-\delta > \frac{r_0}{2^N}$. Let $x \in S_k$ and $y \in E_\delta(x)$. Then for any $\zeta \in \partial\Omega$, we have

$$r(y) \le |x-y| + |x-\zeta| < \delta r(x) + |x-\zeta|.$$

Taking the infimum over all $\zeta \in \partial\Omega$, then

$$r(y) \le (\delta+1)r(x) \le (\delta+1)\frac{r_0}{2^{k-1}} < \frac{r_0}{2^{k-2}}.$$

Similarly, we get

$$r(y) \ge r(x) - |x-y| > (1-\delta)r(x) > (1-\delta)\frac{r_0}{2^k} > \frac{r_0}{2^{k+N}}.$$

The proof is completed.

Remark 2.3 It is easy to see that $N=1$ if δ is small enough.

Recall that $E_\delta(x) = \{y \in \Omega : |y-x| < \delta r(x)\}$ for $x \in \Omega$ and $0<\delta<1$. Given a positive Borel measure μ on Ω, we define the averaging function

$$\widehat{\mu}_\delta(x) = \frac{\mu(E_\delta(x))}{m(E_\delta(x))} \simeq \frac{\mu(E_\delta(x))}{r(x)^n}, \quad x \in \Omega,$$

and the Berezin transform

$$\widetilde{\mu}(x) = \int_\Omega \frac{|R(x,y)|^2}{R(x,x)} \mathrm{d}\mu(y), \quad x \in \Omega.$$

Now, we are in the position to exhibit the characterization of Carleson type measures (and vanishing Carleson type measures) for mixed norm spaces, which generalizes [12].

Theorem 2.3 *Let μ be a positive Borel measure on Ω and let $1<p,q<\infty$. Then μ is a Carleson type measure for $h_{p,q}$ if and only if $\sup\limits_{x\in\Omega} \widehat{\mu}_\delta(x) \le C$ for any (or some) fixed $\delta \in (0,1)$.*

Proof Let μ be a Carleson type measure for $h_{p,q}$. Notice that, for $1<p,q<\infty$, we have

$$1 - \frac{1}{p} + \frac{1}{np} - \frac{1}{nq} > 0.$$

For fixed $x \in \Omega$, set

$$f_x(y) = \frac{R(y,x)}{R(x,x)^{1-\frac{1}{p}+\frac{1}{np}-\frac{1}{nq}}}, \quad y \in \Omega. \tag{2.2}$$

Then $f_x \in h(\Omega)$. So Theorem 2.2 implies

$$M_p(f_x,t) \le \frac{Cr(x)^{n-\frac{n}{p}+\frac{1}{p}-\frac{1}{q}}}{(r(x)+t)^{n-\frac{n}{p}+\frac{1}{p}}}$$

and

$$\|f_x\|_{p,q}^q \le C\int_0^\varepsilon \frac{r(x)^{qn-\frac{qn}{p}+\frac{q}{p}-1}}{(r(x)+t)^{nq-\frac{nq}{p}+\frac{q}{p}}} \mathrm{d}t \le C,$$

where C is independent of x. There exists some small $\delta_0 > 0$ such that $|R(x,y)| \simeq \frac{1}{r(x)^n}$ whenever $x \in \Omega$ and $y \in E_{\delta_0}(x)$. We may assume that $x \in S_k$, and thus Lemma 2.3 yields

$$\begin{aligned}
[\widehat{\mu}_{\delta_0}(x)]^{\frac{q}{p}} &\simeq \Big[\int_{E_{\delta_0}(x)} |f_x(y)|^p r(y)^{\frac{p}{q}-1} \mathrm{d}\mu(y)\Big]^{\frac{q}{p}} \\
&\le C \sum_{j=k-N}^{k+N} \Big[\int_{S_j} |f_x(y)|^p \mathrm{d}\mu(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
&\le C \sum_{j=1}^{\infty} \Big[\int_{S_j} |f_x(y)|^p \mathrm{d}\mu(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
&\le C \|f_x\|_{p,q}^q \le C.
\end{aligned}$$

Then Lemma 3.2 in [12] implies that $\sup\limits_{x\in\Omega} \widehat{\mu}_\delta(x) \le C$ for any fixed $\delta \in (0,1)$.

Conversely, we suppose $\sup\limits_{x\in\Omega} \widehat{\mu}_\delta(x) \le C$ for some δ. Without loss of generality, we may assume that δ is small enough so that $N = 1$ in Lemma 2.3. For $j = 1, 2, \cdots$, set

$$K_j = \{k : E_{\frac{\delta}{3}}(a_k) \cap S_j \neq \varnothing\} \quad \text{and} \quad \Gamma_j = \bigcup_{k\in K_j} E_{\frac{\delta}{3}}(a_k).$$

For any $f \in h_{p,q}$, by the subharmonicity of $|f|^p$, we know

$$\begin{aligned}
\max_{x\in \overline{E}_{\frac{\delta}{3}}(a_k)} |f(x)|^p m(E_{\frac{\delta}{3}}(a_k)) &\le C \max_{x\in \overline{E}_{\frac{\delta}{3}}(a_k)} \int_{E_{\frac{\delta}{3}}(x)} |f(y)|^p \mathrm{d}m(y) \\
&\le C \int_{E_\delta(a_k)} |f(y)|^p \mathrm{d}m(y).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\sum_{j=1}^{\infty} \Big[\int_{S_j} |f(y)|^p \mathrm{d}\mu(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
\le& \sum_{j=1}^{\infty} \Big[\int_{\Gamma_j} |f(y)|^p \mathrm{d}\mu(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
\le& \sum_{j=1}^{\infty} \Big[\sum_{K_j} \max_{y\in \overline{E}_{\frac{\delta}{3}}(a_k)} |f(y)|^p \mu(E_{\frac{\delta}{3}}(a_k))\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
\le& C \sum_{j=1}^{\infty} \Big[\sum_{K_j} \max_{y\in \overline{E}_{\frac{\delta}{3}}(a_k)} |f(y)|^p m(E_{\frac{\delta}{3}}(a_k))\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
\le& C \sum_{j=1}^{\infty} \Big[\sum_{K_j} \int_{E_\delta(a_k)} |f(y)|^p \mathrm{d}m(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
\le& C \sum_{j=2}^{\infty} \Big[\int_{\{y\in\Omega: r_{j+2} < r(y) \le r_{j-2}\}} |f(y)|^p \mathrm{d}m(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
\le& C \sum_{j=1}^{\infty} \Big[\int_{S_j} |f(y)|^p \mathrm{d}m(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \simeq \|f\|_{p,q}^q.
\end{aligned}$$

It turns out that μ is a Carleson type measure for $h_{p,q}$. The proof is completed.

Theorem 2.4 *Let μ be a positive Borel measure on Ω and $1 < p, q < \infty$. Then μ is a vanishing Carleson measure for $h_{p,q}$ if and only if*

$$\lim_{x\to\partial\Omega} \widehat{\mu}_\delta(x) = 0 \tag{2.3}$$

for any (or some) fixed $\delta \in (0,1)$.

Proof For $x \in \Omega$, set the test function f_x as (2.2), and then $\|f_x\|_{h_{p,q}} \le C$. And also, since $\frac{1}{R(x,x)} \simeq r(x)^n$, it is easy to check that f_x converges uniformly to 0 on any compact subset of Ω as $x \to \partial\Omega$. If μ is a vanishing Carleson type measure for $h_{p,q}$, similarly to the proof in Theorem 2.3, there exists $\delta_0 > 0$ such that

$$\Big[\widehat{\mu}_{\delta_0}(x)\Big]^{\frac{q}{p}} \le C \sum_{j=1}^{\infty} \Big[\int_{S_j} |f_x(y)|^p \mathrm{d}\mu(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \to 0, \quad x \to \partial\Omega.$$

This, together with Lemma 3.2 in [12], gives (2.3) for any $\delta \in (0,1)$.

Conversely, suppose $\lim\limits_{x\to\partial\Omega} \widehat{\mu}_\delta(x) = 0$ for some $\delta \in (0,1)$. Then, (2.3) holds for any fixed δ small enough. And for any $\epsilon > 0$, we have some integer K such that

$$\widehat{\mu}_r(a_k) < \epsilon \quad \text{as } k > K.$$

Noting that $\lim\limits_{j\to\infty} \min\{k : k \in K_j\} = \infty$, we have some J such that

$$\min\{k : k \in K_j\} > K \quad \text{whenever} \quad j > J.$$

Now for any bounded sequence $\{f_l\}$ in $h_{p,q}$, which convergs to 0 uniformly on each compact subset of Ω as $l \to \infty$, we have

$$\begin{aligned}
&\sum_{j=J+1}^{\infty} \Big[\int_{S_j} |f_l(y)|^p \mathrm{d}\mu(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
&\le \sum_{j=J+1}^{\infty} \Big[\sum_{K_j} \max_{y\in\overline{E}_{\frac{\delta}{3}}(a_k)} |f_l(y)|^p \mu(E_{\frac{\delta}{3}}(a_k))\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
&\le \epsilon^q \sum_{j=J+1}^{\infty} \Big[\sum_{K_j} \max_{y\in\overline{E}_{\frac{\delta}{3}}(a_k)} |f_l(y)|^p m(E_{\frac{\delta}{3}}(a_k))\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
&\le C\epsilon^q \sum_{j=J+1}^{\infty} \Big[\int_{S_j} |f_l(y)|^p \mathrm{d}m(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \\
&\le C\|f_l\|_{p,q}^q \epsilon^q \le C_1\epsilon^q,
\end{aligned}$$

where C_1 is independent of ϵ. On the other hand, for fixed J, $\bigcup\limits_{j=1}^{J} \overline{S_j}$ is a compact subset of Ω, f_l converges to 0 uniformly on it, and thus

$$\lim_{l\to\infty} \sum_{j=1}^{J} \Big[\int_{S_j} |f_l(y)|^p \mathrm{d}\mu(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} = 0.$$

Therefore,

$$\lim_{l\to\infty} \sum_{j=1}^{\infty} \Big[\int_{S_j} |f_l(y)|^p \mathrm{d}\mu(y)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} = 0.$$

By our definition, μ is a vanishing Carleson type measure for $h_{p,q}$. The proof is completed.

3 Projection and Duality

In this section, we will see that the Bergman projection P is bounded on the harmonic mixed norm space $h_{p,q}$ for $1 < p, q < \infty$, with which we will obtain the duality of $h_{p,q}$.

Recall that, for fixed positive exponents p and q, the mixed norm space $L_{p,q}(\Omega)$ consists of all Lebesgue measurable functions f on Ω for which

$$\|f\|_{L_{p,q}} = \Big\{ \sum_{j=1}^{\infty} \Big(\int_{S_j} |f(x)|^p \mathrm{d}m(x) \Big)^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)} \Big\}^{\frac{1}{q}} < \infty.$$

Theorem 3.1 *Suppose* $1 < p, q < \infty$. *Then P is a bounded operator from $L_{p,q}$ to $h_{p,q}$, where P is defined as* (1.1). *Moreover,* $P(L_{p,q}) = h_{p,q}$.

Proof For $1 < p, q < \infty$, then there exists $\epsilon > 0$ such that $p > \frac{p}{q} + \epsilon$. First, we prove that the operator P is bounded on

$$L^p_\alpha = \Big\{ f \text{ measurable on } \Omega : \|f\|_{L^p_\alpha} = \Big(\int_\Omega |f(x)|^p r(x)^\alpha \mathrm{d}m(x) \Big)^{\frac{1}{p}} < \infty \Big\}$$

for $\alpha = \frac{p}{q} - 1 + \epsilon$ (or $\alpha = \frac{p}{q} - 1 - \epsilon$). Fix some $s \in \big(\frac{1}{q} - \frac{1}{p} + \frac{\epsilon}{p}, \min\big(\frac{1}{p'}, \frac{1}{q} + \frac{\epsilon}{p}\big)\big)$ and set $h(x) = r(x)^{-s}$. Then by Theorem 1.1 and Lemma 4.1 in [7], we obtain

$$\int_\Omega |R(x,y)| r(y)^{-\left(\frac{p}{q}-1+\varepsilon\right)} h(y)^{p'} r(y)^{\frac{p}{q}-1+\varepsilon} \mathrm{d}m(y) \le C h(x)^{p'}, \quad x \in \Omega$$

and

$$\int_\Omega |R(x,y)| r(y)^{-\left(\frac{p}{q}-1+\varepsilon\right)} h(x)^{p} r(x)^{\frac{p}{q}-1+\varepsilon} \mathrm{d}m(x) \le C h(y)^{p}, \quad y \in \Omega.$$

Hence, P is bounded in $L^p_{\frac{p}{q}-1+\epsilon}$ by the Schur's test (see [18]). We can check that P is also bounded on $L^p_{\frac{p}{q}-1-\epsilon}$ by choosing the test function $h(x) = r(x)^{-s}$ with

$$s \in \Big(\frac{1}{q} - \frac{1}{p} - \frac{\epsilon}{p}, \min\Big(\frac{1}{p'}, \frac{1}{q} - \frac{\epsilon}{p}\Big)\Big).$$

It follows that

$$\int_\Omega |Pf(x)|^p r(x)^{\frac{p}{q}-1\pm\epsilon} \mathrm{d}m(x) \le C \int_\Omega |f(x)|^p r(x)^{\frac{p}{q}-1\pm\epsilon} \mathrm{d}m(x)$$

for all $f \in L^p_{\frac{p}{q}-1\pm\epsilon}$. In particular, if $\operatorname{supp}(f) \subset S_j$, then

$$\int_{S_k} |Pf(x)|^p r(x)^{\frac{p}{q}-1} \mathrm{d}m(x) \le C 2^{\pm\epsilon(k-j)} \int_\Omega |f(x)|^p r(x)^{\frac{p}{q}-1} \mathrm{d}m(x)$$

for all k. Let $f \in L_{p,q}$ and write $f = \sum\limits_j f\chi_{S_j}$. Assume first that the sum consists of a finite number of terms. We get

$$\begin{aligned} \|(Pf)\chi_{S_k}\|_{L^p_{\frac{p}{q}-1}} &\le \sum_j \|(P(f\chi_{S_j}))\chi_{S_k}\|_{L^p_{\frac{p}{q}-1}} \le C \sum_j 2^{\frac{\pm\epsilon(k-j)}{p}} \|f\chi_{S_j}\|_{L^p_{\frac{p}{q}-1}} \\ &\simeq C_1 \sum_{j \le k} 2^{\frac{-\epsilon(k-j)}{p}} \|f\chi_{S_j}\|_{L^p_{\frac{p}{q}-1}} + C_2 \sum_{j > k} 2^{\frac{\epsilon(k-j)}{p}} \|f\chi_{S_j}\|_{L^p_{\frac{p}{q}-1}}. \end{aligned}$$

Now for two sequences $X = \{x(j)\}_{j=-\infty}^{+\infty}$ and $Y = \{y(j)\}_{j=-\infty}^{+\infty}$ as functions on all integers, their convolution $X * Y$ is defined as the sequence $Z = \{z(j)\}_{j=-\infty}^{+\infty}$ with

$$z(j) = \sum_{k=-\infty}^{+\infty} x(k)y(k-j).$$

Set $x_j = 2^{-\frac{\epsilon|j|}{p}}$ and

$$y_j = \begin{cases} \|f\chi_{S_j}\|_{L^p_{\frac{p}{q}-1}}, & j \geq 1, \\ 0, & j < 1. \end{cases}$$

Then

$$\|(Pf)\chi_{S_k}\|_{L^p_{\frac{p}{q}-1}} \leq CX * Y(k), \quad k \in \mathbf{Z}.$$

Note that

$$\|(Pf)\chi_{S_k}\|_{L^p_{\frac{p}{q}-1}} \simeq 2^{-k(\frac{1}{q}-\frac{1}{p})}\|(Pf)\chi_{S_k}\|_{L^p}$$

and

$$\|f\chi_{S_j}\|_{L^p_{\frac{p}{q}-1}} \simeq 2^{-j(\frac{1}{q}-\frac{1}{p})}\|f\chi_{S_j}\|_{L^p}.$$

By Young's inequality (see [21, p. 53]), we have

$$\|Pf\|_{L_{p,q}} \simeq \left\|\left\{\|(Pf)\chi_{S_k}\|_{L^p_{\frac{p}{q}-1}}\right\}_k\right\|_{l^q} \leq C\|X\|_{l^1}\left\|\left\{\|f\chi_{S_j}\|_{L^p_{\frac{p}{q}-1}}\right\}_j\right\|_{l^q} \leq C\|f\|_{L_{p,q}}.$$

From this and the fact that the subspace consisting of all functions $f \in L_{p,q}$ with compact support is dense in $L_{p,q}$, we see that P is bounded from $L_{p,q}$ to $h_{p,q}$. For any $f \in h_{p,q}(\Omega)$, by $h_{p,q}(\Omega) \subset b^t(\Omega)$ with $t = \min\{p,q\} > 1$ and [7], we have $Pf = f$, which means $P(L_{p,q}) = h_{p,q}$. The proof is completed.

Corollary 3.1 *For $1 < p, q < \infty$, $h^\infty \cap C(\overline{\Omega})$ is dense in $h_{p,q}$.*

Proof Let $1 < p, q < \infty$ and fix $f \in h_{p,q}$. For $\varepsilon > 0$, set $f_\varepsilon = P(f\chi_\varepsilon) \in h_{p,q}$ with χ_ε the characteristic function of Ω_ε. Then $f_\varepsilon \in h^\infty \cap C(\overline{\Omega})$. Since P is bounded from $L_{p,q}$ to $h_{p,q}$, we have

$$\|f - f_\varepsilon\|_{p,q} = \|P(f - f_\varepsilon)\|_{p,q} \leq C\|f - f_\varepsilon\|_{L_{p,q}} \to 0$$

as $\varepsilon \to 0$. Thus, $h^\infty \cap C(\overline{\Omega})$ is dense in $h_{p,q}$. The proof is completed.

To obtain the duality for $h_{p,q}$, we need the following duality for $L_{p,q}$.

Lemma 3.1 *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then $(L_{p,q})^* = L_{p',q'}$ under the pairing*

$$\langle f, g\rangle = \int_\Omega f(x)\overline{g(x)}\mathrm{d}m(x). \tag{3.1}$$

More precisely, $T \in (L_{p,q})^*$ if and only if there exists a unique function $g \in L_{p',q'}$, such that for any $f \in L_{p,q}$, $Tf = \langle f, g\rangle$ and $\|T\| = \|g\|_{L_{p',q'}}$.

Proof Let $f \in L_{p,q}$ and $g \in L_{p',q'}$. By using the Hölder's inequality twice, we know that $|\langle f, g\rangle|$ is not more than

$$\begin{aligned}
&\sum_{j=1}^{\infty}\int_{S_j}|f(x)\overline{g(x)}|\mathrm{d}m(x)\\
&\le \sum_{j=1}^{\infty}\Big[\int_{S_j}|f(x)|^p r(x)^{p\left(\frac{1}{q}-\frac{1}{p}\right)}\mathrm{d}m(x)\Big]^{\frac{1}{p}}\Big[\int_{S_j}|g(x)|^{p'} r(x)^{p'\left(\frac{1}{p}-\frac{1}{q}\right)}\mathrm{d}m(x)\Big]^{\frac{1}{p'}}\\
&\le \Big\{\sum_{j=1}^{\infty}\Big[\int_{S_j}|f(x)|^p r(x)^{\frac{p}{q}-1}\mathrm{d}m(x)\Big]^{\frac{q}{p}}\Big\}^{\frac{1}{q}}\cdot\Big\{\sum_{j=1}^{\infty}\Big[\int_{S_j}|g(x)|^{p'} r(x)^{p'\left(\frac{1}{p}-\frac{1}{q}\right)}\mathrm{d}m(x)\Big]^{\frac{q'}{p'}}\Big\}^{\frac{1}{q'}}\\
&\simeq \Big\{\sum_{j=1}^{\infty}\Big[\int_{S_j}|f(x)|^p\mathrm{d}m(x)\Big]^{\frac{q}{p}}2^{j\left(\frac{q}{p}-1\right)}\Big\}^{\frac{1}{q}}\cdot\Big\{\sum_{j=1}^{\infty}\Big[\int_{S_j}|g(x)|^{p'}\mathrm{d}m(x)\Big]^{\frac{q'}{p'}}2^{j\left(\frac{q'}{p'}-1\right)}\Big\}^{\frac{1}{q'}}\\
&\le \|f\|_{L_{p,q}}\|g\|_{L_{p',q'}}.
\end{aligned} \tag{3.2}$$

Hence, for $g \in L_{p',q'}$, $T(\cdot) = \langle\cdot, g\rangle$ defines a bounded linear functional $T \in (L_{p,q})^*$ with

$$\|T\| \le \|g\|_{L_{p',q'}}. \tag{3.3}$$

Conversely, suppose $T \in (L_{p,q})^*$. By Theorem 6.16 in [22], we know, $\forall j$, $T|_{L^p(S_j)} \in (L^p(S_j))^*$. This means that there exist $g_j \in L^{p'}(S_j)$ such that for any $f \in L^p(S_j)$,

$$T(f\chi_{S_j}) = \int_{S_j} f(x)\overline{g_j(x)}\mathrm{d}m(x), \quad j = 1, 2, \cdots.$$

Set $F_K = \sum\limits_{j=1}^{K}\chi_{S_j}\alpha_j|g_j|^{p'-1}\mathrm{sgn}\, g_j$, where

$$\alpha_j = \begin{cases} 2^{j(\frac{q'}{p'}-1)}\Big[\Big(\displaystyle\int_{S_j}|g_j(x)|^{p'}\mathrm{d}m(x)\Big)^{\frac{q'}{p'}-\frac{q}{p}}\Big]^{\frac{1}{q}}, & g_j \ne 0 \text{ on } S_j,\\ 0, & g_j = 0 \text{ on } S_j,\end{cases}$$

$$\mathrm{sgn}\, g_j(x) = \begin{cases} \dfrac{g_j(x)}{|g_j(x)|}, & g_j(x) \ne 0,\\ 0, & g_j(x) = 0.\end{cases}$$

Then

$$\begin{aligned}
\|F_K\|^q_{L_{p,q}} &= \sum_{j=1}^{K}\alpha_j^q\Big(\int_{S_j}|g_j(x)|^{p(p'-1)}\mathrm{d}m(x)\Big)^{\frac{q}{p}}2^{j(\frac{q}{p}-1)}\\
&= \sum_{j=1}^{K}\Big(\int_{S_j}|g_j(x)|^{p'}\mathrm{d}m(x)\Big)^{\frac{q'}{p'}}2^{j(\frac{q'}{p'}-1)} < \infty.
\end{aligned} \tag{3.4}$$

By $T \in (L_{p,q})^*$ and the linearity, we obtain

$$\begin{aligned}
\|T\|\cdot\|F_K\|_{L_{p,q}} &\ge |T(F_K)|\\
&= \Big|\sum_{j=1}^{K}\alpha_j T(\chi_{S_j}|g_j|^{p'-1}\mathrm{sgn} g_j)\Big|
\end{aligned}$$

$$
\begin{aligned}
&= \Big| \sum_{j=1}^{K} \alpha_j \int_{S_j} |g_j(x)|^{p'-1} \mathrm{sgn} g_j(x) \overline{g_j(x)} \mathrm{d}m \Big| \\
&= \sum_{j=1}^{K} \Big(\int_{S_j} |g_j(x)|^{p'} \mathrm{d}m \Big)^{1+\frac{1}{q}\left(\frac{q'}{p'}-\frac{q}{p}\right)} 2^{j(\frac{q'}{p'}-1)} \\
&= \sum_{j=1}^{K} \Big(\int_{S_j} |g_j(x)|^{p'} \mathrm{d}m \Big)^{\frac{q'}{p'}} 2^{j(\frac{q'}{p'}-1)}. \qquad (3.5)
\end{aligned}
$$

The last equality holds because $1 + \frac{1}{q}\big(\frac{q'}{p'} - \frac{q}{p}\big) = \frac{q'}{p'}$. By (3.4)–(3.5), we have

$$
\sum_{j=1}^{K} \Big(\int_{S_j} |g_j(x)|^{p'} \mathrm{d}m(x) \Big)^{\frac{q'}{p'}} 2^{j(\frac{q'}{p'}-1)} \le \|T\| \Big[\sum_{j=1}^{K} \Big(\int_{S_j} |g_j(x)|^{p'} \mathrm{d}m(x) \Big)^{\frac{q'}{p'}} 2^{j(\frac{q'}{p'}-1)} \Big]^{\frac{1}{q}}.
$$

Define

$$
G_K(x) = \sum_{j=1}^{K} \chi_{S_j}(x) g_j(x),
$$

and then

$$
\|G_K\|_{L_{p',q'}} = \Big[\sum_{j=1}^{K} \Big(\int_{S_j} |g_j(x)|^{p'} \mathrm{d}m(x) \Big)^{\frac{q'}{p'}} 2^{j(\frac{q'}{p'}-1)} \Big]^{\frac{1}{q'}} \le \|T\|. \qquad (3.6)
$$

And for every $z \in \Omega$, $G_K(z) \to G(z)$ as $K \to \infty$. This, together with (3.6), gives

$$
G_K \to G \quad \text{in } L_{p',q'} \quad \text{and} \quad \|G\|_{L_{p',q'}} \le \|T\|. \qquad (3.7)
$$

Using the continuity of T and (3.7),

$$
\begin{aligned}
T(f) &= \lim_{K\to\infty} \sum_{j=1}^{K} T(\chi_{S_j} f) \\
&= \lim_{K\to\infty} \sum_{j=1}^{K} \int_{S_j} f(x) \overline{g_j(x)} \mathrm{d}m(x) \\
&= \lim_{K\to\infty} \int_{\Omega} f(x) \sum_{j=1}^{K} \overline{\chi_{S_j}(x) g_j(x)} \mathrm{d}m(x) \\
&= \int_{\Omega} f(x) \overline{G(x)} \mathrm{d}m(x).
\end{aligned}
$$

The uniqueness of g is clear, because if g and g' satisfy (3.1), then the integral of $g - g'$ over any measurable set E of finite measure is 0 (by taking χ_E for f), and the σ-finiteness of m yields that $g - g' = 0$ a.e. Moreover, (3.3) and (3.7) yield that $\|T\| = \|g\|_{L_{p',q'}}$. The proof is completed.

Theorem 3.2 *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $\frac{1}{q} + \frac{1}{q'} = 1$. Then $(h_{p,q})^* = h_{p',q'}$ under the pairing*

$$
\langle f, g \rangle = \int_{\Omega} f(x) \overline{g(x)} \mathrm{d}m(x).
$$

More precisely, $T \in (h_{p,q})^*$ if and only if there is a unique function $g \in h_{p',q'}$ such that for any $f \in h_{p,q}$, $Tf = \langle f, g \rangle$ and $\|T\| \simeq \|g\|_{p',q'}$.

Proof For $g \in h_{p',q'}$, (3.2) tells us that $T(\cdot) = \langle \cdot, g\rangle$ defines a bounded linear functional $T \in (h_{p,q})^*$ with $\|T\| \le \|g\|_{p',q'}$.

Conversely, suppose $T \in (h_{p,q})^*$. By Hahn-Banach theorem, $T \in (L_{p,q})^*$. Then we have some function $G \in L_{p',q'}$ such that $\|G\|_{L_{p',q'}} \le C\|T\|$ and for all $f \in h_{p,q}$,

$$T(f) = \int_\Omega f(x)\overline{G(x)}\mathrm{d}m(x).$$

Setting $g = PG$, Theorem 3.1 yields that $g \in h_{p',q'}$ and

$$T(f) = \int_\Omega f(x)\overline{G(x)}\mathrm{d}m(x) = \int_\Omega (Pf)(x)\overline{G(x)}\mathrm{d}m(x) = \int_\Omega f(x)\overline{g(x)}\mathrm{d}m(x).$$

Hence, by Lemma 3.1,

$$\|g\|_{p',q'} \le C\|G\|_{p',q'} \le C\|T\|.$$

Thus, $\|T\| \simeq \|g\|_{p',q'}$. The uniqueness of g comes from Lemma 3.1. The proof is completed.

4 The Toeplitz Operator

In this section, we will characterize the boundedness (and compactness) of Toeplitz operators T_μ on the mixed norm spaces, provided that μ is a finite positive Borel measure. Before doing this, we give the following "formal" equality which can be seen in [12].

Lemma 4.1 *Let $\mu \ge 0$. Then*

$$\langle T_\mu f, g\rangle = \int_\Omega f(y)\overline{g(y)}d\mu(y)$$

for $f, g \in h^\infty$.

Here is the main result of this section.

Theorem 4.1 *Let $\mu \ge 0$, $1 < p, q < \infty$. Then the following statements are equivalent:*
(1) *T_μ is bounded on $h_{p,q}$.*
(2) *μ is a Carleson type measure for $h_{p,q}$.*
(3) *$\widetilde{\mu}$ is bounded on Ω.*
(4) *$\widehat{\mu}_\delta$ is bounded on Ω for any (or some) $\delta \in (0,1)$.*
(5) *The sequence $\{\widehat{\mu}_r(a_k)\}$ is bounded for any (or some) $r \in (0,1)$ and $\{a_k\}$ as in Lemma* 2.2.

Proof Combining [12, Theorem 3.5], [13, Proposition 2.3] and Theorem 2.3, we need only to prove the implications (1)⇒(3) and (2)⇒(1).

(1)⇒(3) Suppose that T_μ is bounded on $h_{p,q}$. Let f_x be the test function as (2.2). Then $f_x \in h(\Omega)$ and $\|f_x\|_{p,q} \le C$. Hence, $T_\mu f_x \in h_{p,q}$. Assuming that $x \in S_k$, by subharmonicity, we know

$$\begin{aligned}
|T_\mu f_x(x)|^q &\le \frac{C}{r(x)^{\frac{nq}{p}+1-\frac{q}{p}}}\Big[\int_{E_{\frac{\delta}{3}}(x)} |T_\mu f_x(y)|^p r(y)^{\frac{p}{q}-1}\mathrm{d}m(y)\Big]^{\frac{q}{p}} \\
&\le \frac{C}{r(x)^{\frac{nq}{p}+1-\frac{q}{p}}}\Big\{\sum_{j=k-N}^{k+N}\Big[\int_{S_j}|T_\mu f_x(y)|^p\mathrm{d}m(y)\Big]^{\frac{q}{p}}2^{j(\frac{q}{p}-1)}\Big\} \\
&\le \frac{C}{r(x)^{\frac{nq}{p}+1-\frac{q}{p}}}\Big\{\sum_{j=1}^{\infty}\Big[\int_{S_j}|T_\mu f_x(y)|^p\mathrm{d}m(y)\Big]^{\frac{q}{p}}2^{j(\frac{q}{p}-1)}\Big\}
\end{aligned}$$

$$\begin{aligned}&\simeq \frac{C}{r(x)^{\frac{nq}{p}+1-\frac{q}{p}}}\|T_\mu f_x\|_{p,q}^q\\ &\le \frac{C}{r(x)^{\frac{nq}{p}+1-\frac{q}{p}}}\|f_x\|_{p,q}^q.\end{aligned}$$

Therefore, by the definitions of $\widetilde{\mu}$ and T_μ, we have

$$\widetilde{\mu}(x) \simeq r(x)^{\frac{n}{p}+\frac{1}{q}-\frac{1}{p}}|T_\mu f_x(x)| \le C\|f_x\|_{p,q} \le C.$$

This gives the statement (3).

(2)⇒(1) Suppose that μ is a Carleson type measure for $h_{p,q}$. For any $f, g \in h^\infty$, by Lemma 4.1 and the proof in (3.2), we have

$$\begin{aligned}|\langle T_\mu f, g\rangle| &\le \sum_{j=1}^\infty \int_{S_j} |f(x)g(x)|\mathrm{d}\mu(x)\\ &\le \Big\{\sum_{j=1}^\infty \Big[\int_{S_j} |f(x)|^p r(x)^{\frac{p}{q}-1}\mathrm{d}\mu(x)\Big]^{\frac{q}{p}}\Big\}^{\frac{1}{q}} \cdot \Big\{\sum_{j=1}^\infty \Big[\int_{S_j} |g(x)|^{p'} r(x)^{p'\left(\frac{1}{p}-\frac{1}{q}\right)} d\mu(x)\Big]^{\frac{q'}{p'}}\Big\}^{\frac{1}{q'}}\\ &\simeq \Big\{\sum_{j=1}^\infty \Big[\int_{S_j} |f(x)|^p \mathrm{d}\mu(x)\Big]^{\frac{q}{p}} 2^{j\left(\frac{q}{p}-1\right)}\Big\}^{\frac{1}{q}} \cdot \Big\{\sum_{j=1}^\infty \Big[\int_{S_j} |g(x)|^{p'} \mathrm{d}\mu(x)\Big]^{\frac{q'}{p'}} 2^{j\left(\frac{q'}{p'}-1\right)}\Big\}^{\frac{1}{q'}}\\ &\le C\|f\|_{p,q}\|g\|_{p',q'}, \qquad (4.1)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. The duality argument shows that T_μ is bounded on $h_{p,q}$, because h^∞ is dense in $h_{p,q}$ for all $1 < p, q < \infty$. The proof is completed.

Theorem 4.2 *Let $\mu \ge 0$, $1 < p, q < \infty$. Then the following statements are equivalent:*
(1) *T_μ is compact on $h_{p,q}$.*
(2) *μ is a vanishing Carleson type measure for $h_{p,q}$.*
(3) *$\widetilde{\mu}(x) \to 0$ as $x \to \partial\Omega$.*
(4) *$\widehat{\mu}_\delta(x) \to 0$ as $x \to \partial\Omega$ for any (or some) $\delta \in (0,1)$.*
(5) *$\widehat{\mu}_r(a_k) \to 0$ as $k \to \infty$ for any (or some) $r \in (0,1)$ and $\{a_k\}$ as in Lemma* 2.2.

Proof Similarly to Theorem 4.1, we need only to prove (1)⇒(3) and (2)⇒(1) as well.

(1)⇒(3) Taking f_x as (2.2) once again, then $\{f_x\} \subseteq h_{p,q}$ is bounded and converges to 0 uniformly on each compact subset of Ω as $x \to \partial\Omega$. Suppose that T_μ is compact on $h_{p,q}$. By statement (1),

$$\lim_{x\to\partial\Omega} \|T_\mu f_x\|_{p,q} = 0.$$

As in the proof of Theorem 4.1, we have

$$[\widetilde{\mu}(x)]^q \le Cr(x)^{\frac{nq}{p}+1-\frac{q}{p}}|T_\mu f_x(x)|^q \le C\|T_\mu f_x\|_{p,q}^q \to 0$$

as $x \to \partial\Omega$.

(2)⇒(1) Let $\{f_l\}$ be a sequence of functions such that $f_l \to 0$ weakly in $h_{p,q}$ as $l \to \infty$. Since μ is a vanishing Carleson type measure for $h_{p,q}$, then

$$\lim_{l\to\infty} \Big\{\sum_{j=1}^\infty \Big[\int_{S_j} |f_l(x)|^p \mathrm{d}\mu(x)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)}\Big\}^{\frac{1}{q}} = 0.$$

By (4.1) and a duality argument, we get

$$\|T_\mu f_l\|_{p,q} \le C\Big\{\sum_{j=1}^{\infty}\Big[\int_{S_j}|f_l(x)|^p \mathrm{d}\mu(x)\Big]^{\frac{q}{p}} 2^{j(\frac{q}{p}-1)}\Big\}^{\frac{1}{q}} \to 0$$

as $l \to \infty$. Therefore, $T_\mu : h_{p,q} \to h_{p,q}$ is compact. The proof is completed.

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