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On Exact Controllability of Networks of Nonlinear Elastic Strings in 3-Dimensional Space*

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Abstract This paper concerns a system of nonlinear wave equations describing the vibrations of a 3-dimensional network of elastic strings. The authors derive the equations and appropriate nodal conditions, determine equilibrium solutions, and, by using the methods of quasilinear hyperbolic systems, prove that for tree networks the natural initial, boundary value problem has classical solutions existing in neighborhoods of the "stretched" equilibrium solutions. Then the local controllability of such networks near such equilibrium configurations in a certain specified time interval is proved. Finally, it is proved that, given two different equilibrium states satisfying certain conditions, it is possible to control the network from states in a small enough neighborhood of one equilibrium to any state in a suitable neighborhood of the second equilibrium over a sufficiently large time interval.

Keywords Nonlinear strings, Network, Quasilinear system of hyperbolic equations, Controllability
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1 Introduction

We consider networks of elastic strings in $\mathbb{R}^3$, following ideas introduced and developed successively in [6, 13] (which also introduced networks of beams) and Chapter 2 of the monograph in [5] (which went on to treat much more general multi-link structures). In those publications, the strings were modeled as parametrized curves, and, while nonlinear equations for the position vectors along the curve were derived, the main emphasis was on network equations obtained by linearization about equilibrium configurations in which all the strings are stretched, with displacements which necessarily also are vectors in $\mathbb{R}^3$. The latter were studied in detail and existence theorems for the linearized system were proved along with exact controllability results in the case of tree networks controlled at the extremities. In both [13] and [5], the nonlinear equations were derived from Hamilton's principle based on a quadratic potential energy corresponding to Hooke's law with the remark that the potential function could be generalized. No results on those nonlinear systems were obtained in those references. In [15], some results were obtained on the existence of equilibria in situations where gravitational forces were included in the model. In [14], local well-posedness and controllability results were proved for a single

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nonlinear string governed by the nonlinear system. This was done by using results on quasilinear hyperbolic systems. In [4], Li and Gu proved well-posedness and exact controllability for a model of a planar tree network of strings governed by quasilinear wave equations governing scalar (necessarily transverse) displacements of the strings. In this paper, we prove analogous results in $\mathbb{R}^3$ where the displacements of the strings from "stretched" equilibrium configurations are necessary in 3-dimensional space. This covers a much wider range of realistic physical situations, and so we consider this a major step towards application in mechanics and material sciences. We stay with the concept of semi-global classical solutions developed by Li [7]. For a similar model of nonlinear strings, see [9–11].

The paper is organized as follows. In Section 2, we derive the model of the string network to be studied. In Section 3, we provide some results on the existence of "stretched equilibria". In Section 4, we show the existence of semi-global classical solutions to the network equations for tree networks near a stretched equilibrium. In Section 5, we first prove a local exact controllability result around an equilibrium for a star-like network, where all but simple nodes are under control. Here we use the arguments of [4]. Further, we prove a global-local exact controllability result, where neighborhoods of two different equilibria can be exactly controlled for such a situation. In Section 6, we comment briefly on how these controllability results can be extended to tree networks.

We end this introduction with some comments on notation. Vectors, or vector valued functions, will be indicated in boldface. For a vector $\mathbf{v}$ in a Euclidean space, we let $|\mathbf{v}|$ denote the Euclidean length. We denote the Frechét derivative of a function f with respect to a scalar or vector argument ξ by D_ξ. We shall also often write f_s for the partial derivative of f with respect to a scalar variable s and Df for the Frechét derivative of f with respect to its complete argument.

We let $C^n([0,L];W)$ denote the space of n times continuously differentiable functions $\mathbf{f}(x)$ from the interval $[0,L]$ to an open subset W of a Euclidean space with corresponding norms

$$\|\mathbf{f}\|_0 = \|\mathbf{f}\| := \sup_{x\in[0,L]} |\mathbf{f}(x)|, \quad \|\mathbf{f}\|_n := \max\{\|\mathbf{f}\|, \|\mathbf{f}_x\|, \cdots, \|D_x^n\mathbf{f}\|\}.$$

Similar notation will be used for spaces of functions of variables $(x,t) \in [0,L]\times[0,T]$ or of $t\in[0,T]$ and the associated norms.

2 Modeling of Networks of Nonlinear Elastic Strings

In this section, we describe a nonlinear model for networks of elastic strings. We suppose that there are n strings indexed by $i \in \mathcal{I} = \{1,\cdots,n\}$. We let the i-th string be parameterized by its rest arc length x with $x \in [0, L_i]$, L_i of course being the natural length of that string. The position at time t of the point corresponding to the parameter x will be denoted by the vector $\mathbf{R}^i(x,t)$. We shall let $\mathbf{R}$ denote $\{\mathbf{R}^i\}_{i\in\mathcal{I}}$. The positions of the endpoints, which we refer to as nodes, are given by functions $\mathbf{N}^j(t)$ with $j \in \mathcal{J} = \{1,\cdots,m\}$. At multiple nodes where several strings meet there is a common location $\mathbf{N}^j$. Simple nodes are those corresponding to the endpoints of only one string. We let $\mathcal{I}^j = \{i \in \mathcal{I} : \mathbf{N}^j \text{ is an end point of the } i\text{-th string}\}$, $\mathcal{J}^M$ be the subset of $\mathcal{J}$ corresponding to multiple nodes while $\mathcal{J}^S$ contains the indices of simple nodes. We assume that there are simple nodes so that $\mathcal{J}^S$ is not empty. For $j \in \mathcal{J}^S$, we have $\mathcal{I}^j = \{i_j\}$. For $i \in \mathcal{I}^j$, we let $x_{ij} := 0$ if $\mathbf{R}^i(0,t) = \mathbf{N}^j(t)$, or $x_{ij} := L_i$ if $\mathbf{R}^i(L^i,t) = \mathbf{N}^j(t)$. For purposes of integration by parts, we also introduce ϵ_{ij} to equal 1 or -1 depending on whether x_{ij} is equal to L_i or 0. Then $\epsilon_{ij}\mathbf{R}^i_x$ is the outward pointing derivative at the boundary point x_{ij} of the interval $[0, L^i]$.

Let ρ_i be the constant density of that string. Then the kinetic energy of a single string, labeled by i, at time t is given by

$$\mathcal{K}^i(\mathbf{R}^i(\,\cdot\,,t)) := \frac{1}{2}\int_0^{L_i} \rho_i |\mathbf{R}^i_t(x,t)|^2 \,\mathrm{d}x. \tag{2.1}$$

We shall assume that the potential energy of the same string is of the form

$$\mathcal{V}^i(\mathbf{R}^i(\,\cdot\,,t)) := \int_0^{L_i} [V^i(|\mathbf{R}^i_x(x,t)|) + \rho_i g \mathbf{R}(x,t)^i \cdot \mathbf{e}] \,\mathrm{d}x, \tag{2.2}$$

where

(1) $V^i(s)$ is a twice continuously differentiable, convex real valued function defined on an open subinterval $I^i = (a^i, b^i)$ of the positive real axis, with $a^i < 1 < b^i$, satisfying $V^i_{ss}(s) > 0$ and $V^i(1) = V^i_s(1) = 0$;

(2) $\mathbf{e}$ is the vertical unit vector and g is the gravitational constant.

As for the total kinetic energy and total potential energy, we define

$$\mathcal{K}(\mathbf{R}(\,\cdot\,,t)) := \sum_{i\in\mathcal{I}} \mathcal{K}^i(\mathbf{R}^i) = \sum_{i\in\mathcal{I}} \frac{1}{2}\int_0^{L_i} \rho_i |\mathbf{R}^i_t(x,t)|^2 \,\mathrm{d}x \tag{2.3}$$

and

$$\mathcal{V}(\mathbf{R}(\,\cdot\,,t)) := \sum_{i\in\mathcal{I}} \mathcal{V}^i(\mathbf{R}^i) = \sum_{i\in\mathcal{I}} \int_0^{L_i} [V^i(|\mathbf{R}^i_x(x,t)|) + \rho_i g \mathbf{R}(x,t)^i \cdot \mathbf{e}] \,\mathrm{d}x, \tag{2.4}$$

respectively.

Remark 2.1 The two terms in the potential energy correspond to potential energy due respectively to extension or compression, as measured by $|\mathbf{R}^i_x(x)|$, and to gravity. The generic hypotheses on V^i are quite broad and in the nature of minimum physically plausible assumptions. At points where $|\mathbf{R}^i_x(x)| = 1$, there is neither compression nor extension, and hence there should be no contribution to the first term in the potential energy while at points where there is compression or extension there is a positive contribution. This leads to the condition that $V^i(s)$ takes minimum value 0 at 1. Certainly, the potential energy should increase with increasing extension ($|\mathbf{R}^i_x(x)| > 1$ and increasing) or compression ($|\mathbf{R}^i_x(x)| < 1$ and decreasing). The convexity assumption seems appropriate at least over a small interval of values of $|\mathbf{R}^i_x(x)|$. In [5–6, 13], these functions took the form $V^i(s) = \frac{h^i(s-1)^2}{2}$ for $s \in I = (0, \infty)$. See also Remark 2.4.

Remark 2.2 While the function V^i may extend naturally to a bigger domain the interval I^i can be regarded as the domain for which the potential energy is physically realistic.

Definition 2.1 *The string is respectively stretched, limp or compressed at x depending on whether $|\mathbf{R}^i_x(x)| > 1,\ = 1$ or < 1. We say that the string is stretched if $|\mathbf{R}^i_x(x)| > 1$ for all $x \in [0, L^i]$ and compressed if $|\mathbf{R}^i_x(x)| < 1$ for all $x \in [0, L^i]$.*

Physically, one expects significantly different phenomena in these three situations and transitions between them are certainly complicated.

We now apply Hamilton's principle to the Lagrangian functional defined for fixed $T > 0$ by

$$\mathcal{L}(\mathbf{R}) := \int_0^{\mathrm{T}} \sum_{i\in\mathcal{I}} \int_0^{L_i} \Big[\frac{1}{2}\rho_i|\mathbf{R}_t^i(x,t)|^2 - V^i(|\mathbf{R}_x^i(x,t)|) - \rho_i g \mathbf{R}^i(x,t)\cdot\mathbf{e}\Big]\,\mathrm{d}x\mathrm{d}t. \tag{2.5}$$

This requires $\mathcal{L}$ to be stationary at $\mathbf{R}$, where the domain of $\mathcal{L}$ consists of those $\mathbf{R}$ whose component functions $\mathbf{R}^i$ are in

$$\mathcal{O}^i := \{\mathbf{R}^i \in C^2([0,L_i]\times[0,T];\mathbb{R}^3) : |\mathbf{R}_x^i(x,t)| \in I^i \text{ for } (x,t)\in[0,L_i]\times[0,T]\}, \tag{2.6}$$

which satisfy the given initial conditions

$$\mathbf{R}^i(\,\cdot\,,0) = \mathbf{R}^{0,i} \quad \text{and} \quad \mathbf{R}_t^i(\,\cdot\,,0) = \mathbf{R}^{1,i}, \tag{2.7}$$

as well as prescribed Dirichlet boundary conditions at simple nodes

$$\mathbf{R}^{i_j}(x_{i_j j},t) = \mathbf{U}^j(t) \quad \text{for } j\in\mathcal{J}^S \text{ and for } t\in[0,T], \tag{2.8}$$

and the continuity condition at multiple nodes

$$\mathbf{R}^{i_1}(x_{i_1 j},t) = \mathbf{R}^{i_2}(x_{i_2 j},t) \quad \text{for } j\in\mathcal{J}^M,\ i_1,i_2\in\mathcal{I}^j \text{ and for } t\in[0,T]. \tag{2.9}$$

Now, one can consider a perturbation $\mathbf{R}^i(x,t)+\lambda\mathbf{r}^i(x,t)$ for each $i\in\mathcal{I}$, where $\mathbf{r}^i(x,t)\in C^2([0,L_i]\times[0,T];\mathbb{R}^3)$, satisfying homogeneous initial conditions, vanishing at the simple nodes and satisfying the continuity condition at multiple nodes. Because the intervals $I^i = (a^i,b^i)$ are open, the perturbations belong to the domain of $\mathcal{L}$ for small enough values of λ.

A necessary condition for $\mathcal{L}$ to be stationary at $\mathbf{R}$ is

$$\begin{aligned} &D_\lambda\mathcal{L}(\mathbf{R}+\lambda\mathbf{r})|_{\lambda=0}\\ &= \int_0^{\mathrm{T}} \sum_{i\in\mathcal{I}} \int_0^{L_i} \Big[\rho_i\mathbf{R}_t^i(x,t)\cdot\mathbf{r}_t^i(x,t) - V_s^i(|\mathbf{R}_x^i|)\frac{\mathbf{R}_x^i}{|\mathbf{R}_x^i|}(x,t)\cdot\mathbf{r}_x^i(x,t) - \rho_i g\mathbf{r}^i\cdot\mathbf{e}\Big]\mathrm{d}x\mathrm{d}t\\ &= 0. \end{aligned}$$

First, we choose perturbations with compact support in $(0,L_i)\times(0,T)$. One can then integrate by parts and obtain in the usual way the following nonlinear partial differential equation:

$$\rho_i\mathbf{R}_{tt}^i(x,t) = \mathbf{G}^i(\mathbf{R}_x^i(x,t))_x - \rho_i g\mathbf{e} \quad \text{for each } i\in\mathcal{I} \tag{2.10}$$

with $\mathbf{G}^i : \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by

$$\mathbf{G}^i(\mathbf{v}) := V_s^i(|\mathbf{v}|)\frac{\mathbf{v}}{|\mathbf{v}|}. \tag{2.11}$$

Next, for $j\in\mathcal{J}^M$, we choose perturbations with $\mathbf{r}^i=\mathbf{0}$ for $i\notin\mathcal{I}^j$ and with support in a small neighborhood of x_{ij} for $i\in\mathcal{I}^j$. Because of the continuity condition at the multiple node, we are led to the multiple node condition

$$\sum_{i\in\mathcal{I}^j} \epsilon_{ij}\mathbf{G}^i(\mathbf{R}_x^i(x_{ij},t)) = \mathbf{0} \quad \text{for each } j\in\mathcal{J}^M. \tag{2.12}$$

We refer to this as a "balance of forces" or Kirchoff condition.

To summarize, the motion of the string network is governed by the quasilinear vector valued equations (2.10), coupled by the multiple node conditions (2.9) and (2.12), subject to the initial conditions (2.7) and the Dirichlet boundary conditions (2.8) at the simple nodes. The solutions are also constrained by the requirement that $a_i < |\mathbf{R}^i_x(x,t)| < b_i$.

Remark 2.3 There are other conditions which could be imposed instead of the Dirichlet conditions acting on all simple nodes.

(1) Some simple nodes could be left free which would lead to a Neumann condition

$$\mathbf{G}^i(\mathbf{R}^i_x(x_{ij},t)) = \mathbf{0} \quad \text{or} \quad |\mathbf{R}^i(x_{ij},t)| = 1.$$

(2) Some multiple nodes could have their positions prescribed

$$\mathbf{R}^i(x_{ij},t) = \mathbf{U}^j(t) \qquad \text{for all } i \in \mathcal{I}^j,$$

in which case the multiple node condition (2.12) falls aside.

(3) One could also impose forces at certain simple or multiple node (as is done in [5, p. 14]) which leads to a nonlinear Neumann condition

$$\sum_{i\in\mathcal{I}^j} \epsilon_{ij}\mathbf{G}^i(\mathbf{R}^i_x(x_{ij},t)) = \mathbf{F}^j(t).$$

Remark 2.4 With the choice of $V^i(s) = \frac{h^i(s-1)^2}{2}$, one obtains the equations

$$\rho_i \mathbf{R}^i_{tt}(x,t) = h^i\Big[\mathbf{R}^i_x - \frac{\mathbf{R}^i_x}{|\mathbf{R}^i_x|}\Big]_x(x,t) - \rho_i g\mathbf{e}.$$

In [5–6, 13], the emphasis was on the equations obtained by linearization about stretched equilibria. The system of linearized equations decouple into wave equations for the transversal and tangential displacements of the string corresponding to different wave velocities interacting at the multiple nodes. The requirement that the equilibria be stretched is essential to ensure that the equations are in fact wave equations.

3 Equilibrium Solutions on the Network

The equilibria $\mathbf{R}^e = \{\mathbf{R}^{e,i}\}_{i\in\mathcal{I}}$ of the network are solutions of the following stationary (time-independent) version of (2.7), (2.9)–(2.10) and (2.12):

$$\begin{cases} [\mathbf{G}^i(\mathbf{R}^i_x(x))]_x = \rho_i g\mathbf{e} & \text{for } i \in \mathcal{I}, \\ \mathbf{R}^{i_j}(x_{i_j j}) = \mathbf{U}^j & \text{for } j \in \mathcal{J}^S, \\ \mathbf{R}^{i_1}(x_{i_1 j}) = \mathbf{R}^{i_2}(x_{i_2 j}) & \text{for } j \in \mathcal{J}^M,\ i_1, i_2 \in \mathcal{I}^j, \\ \sum_{i\in\mathcal{I}^j} \epsilon_{ij}\mathbf{G}^i(\mathbf{R}^i_x(x_{ij})) = \mathbf{0} & \text{for each } j \in \mathcal{J}^M. \end{cases} \tag{3.1}$$

This is a coupled system of quasilinear elliptic equations with the additional complication of the requirement $a_i < |\mathbf{R}^i_x(x)| < b_i$. We need to pay particular attention to the stretched equilibria for which $1 < |\mathbf{R}^i_x(x,t)| < b_i$ for all $x \in [0, L_i]$.

Ideally, one would be able to characterize those boundary data $\{\mathbf{U}^1, \cdots, \mathbf{U}^n\}$ for which such stretched solutions exist. This seems hopeless at this degree of generality although partial results are possible. For the case of a single string this is done in [14, Theorem 1] which in particular guarantees that given two stretched equilibrium configurations of a single string there is a curve of such equilibria joining them.

We first show how the argument in [14] can be adapted to characterize stretched equilibria in the case of star networks with one multiple node corresponding to the parameter value 0 of the n strings.

We need to solve the main equation in (3.1) for $\mathbf{R}^i_x(x)$, and then integrate to get $\mathbf{R}^i(x)$ by taking into account the nodal conditions at the center node. One integration gives

$$\mathbf{G}^i(\mathbf{R}^i_x(x)) = \mathbf{V}^i + \rho_i g x \mathbf{e}, \quad \text{where } \mathbf{V}^i = \mathbf{G}^i(\mathbf{R}^i_x(0)). \tag{3.2}$$

In order to solve this for $\mathbf{R}^i_x(x)$, we need to solve for $\mathbf{v}$ in equations of the form

$$\mathbf{G}^i(\mathbf{v}) = m^i(|\mathbf{v}|)\frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{w}, \tag{3.3}$$

where $m^i(s) := V^i_s(s)$ is a strictly increasing function of $s \in I^i = (a^i, b^i)$. We denote the range of m^i by $J^i = (c^i, d^i)$ and its inverse function by $(m^i)^{-1}$. We note $c^i < 0 < d^i$, and let $J^i_+ := (0, d^i)$ and $J^i_- := (c^i, 0)$. We note that $0 \in J^i$ corresponds to $1 \in I^i$, and $\mathbf{v} \in I^i$ satisfies $|v| > 1$ or $|v| < 1$ depending on whether $m^i(\mathbf{v}) \in J^i_+$ or $m^i(\mathbf{v}) \in J^i_-$.

If $|\mathbf{w}| \in J^i_+$, (3.3) has a unique solution with $|\mathbf{v}| > 1$ given by

$$\mathbf{v} = (m^i)^{-1}(|\mathbf{w}|)\frac{\mathbf{w}}{|\mathbf{w}|}. \tag{3.4}$$

If $-|\mathbf{w}| \in J^i_-$, (3.3) has a unique solution with $|\mathbf{v}| < 1$ given by

$$\mathbf{v} = -(m^i)^{-1}(-|\mathbf{w}|)\frac{\mathbf{w}}{|\mathbf{w}|}. \tag{3.5}$$

If $\mathbf{w} = 0$, the solution set to (3.3) consists of all $\mathbf{v}$ with $|\mathbf{v}| = 1$.

When $|\mathbf{w}| \in J^i_+$ and $-|\mathbf{w}_-| \in J^i_-$, equation (3.3) has two solutions $\mathbf{v}_\pm$ with $|\mathbf{v}_+| > 1$ and $|\mathbf{v}_-| < 1$. If neither $|\mathbf{w}|$ nor $-|\mathbf{w}|$ lies in J^i, (3.3) has no solution.

It follows from (3.2) that $|\mathbf{R}^i_x(x)| = 1$ (the condition for the string to be limp at x) can hold at only one point and only if $\mathbf{V}^i = -\alpha\rho_i g\mathbf{e}$ with $\alpha \in [0, L_i]$. For other choices of $\mathbf{V}^i$, the right-hand side of (3.2) never vanishes and consequently $|\mathbf{R}^i_x(x)| \neq 1$ for $x \in [0, L_i]$. If we then require $\mathbf{R}^i_x(x)$ to be continuous, the corresponding equilibria are either stretched or compressed.

The nowhere compressed equilibria of strings, which can be limp at one point, arise naturally, but we need to concentrate on stretched equilibria, which are stretched at every point along the string. As we shall see later, we can rewrite the equations close to stretched equilibria as strictly hyperbolic quasilinear systems. We shall for the moment consider both situations.

We remark that in [14], it is shown that when $|\mathbf{R}^i_x(x)| > 1$ for all $x \in [0, L_i]$, the string lies along a convex curve in a vertical plane. If on the other hand, $|\mathbf{R}^i_x(x)| = 1$ for some $x \in [0, L_i]$, the string lies along a vertical line with one "limp" point. If that point corresponds to an interior point the string has a "kink" and bends back vertically on itself.

This occurs, for example, when $\mathbf{R}^i(0) = \mathbf{R}^i(L_i)$. Alternatively, if that point occurs at $x = L_i$ (corresponding to $\mathbf{V}^i = -L_i\rho_i g\mathbf{e}$), we have the situation of a "dangling string" suspended at $\mathbf{R}^i(0)$. If the limp point corresponds to $x = 0$ (corresponding to $\mathbf{V}^i = \mathbf{0}$), we again have a dangling string now suspended at $\mathbf{R}^i(L_i)$. If the limp end is located at a multiple node, it contributes no force at the node. These are all physically plausible although somewhat exceptional situations.

Equation (3.2) has a nowhere compressed stretched solution for a particular value of i if and only if

$$|\mathbf{V}^i + \rho_i g x \mathbf{e}| \in J_+^i \cup \{0\} = [0, d^i) \quad \text{for all } x \in [0, L_i],$$

which occurs when

$$\mathbf{V}^i \in \widetilde{\mathcal{S}}_+^i := \{\mathbf{v} \in \mathbb{R}^3 : |\mathbf{v}| < d_i, |\mathbf{v} + \rho_i g L_i \mathbf{e}| < d_i\}.$$

The equation (3.2) has a stretched solution if in addition $\mathbf{v} + \rho_i g x \mathbf{e} \neq \mathbf{0}$ for all $x \in [0, L_i]$, i.e., if

$$\mathbf{V}^i \in \mathcal{S}_+^i := \widetilde{\mathcal{S}}_+^i \setminus \{-\rho_i g x \mathbf{e} : x \in [0, L_i]\}.$$

To visualize these conditions note that $\{\mathbf{V}^i + \rho_i g x \mathbf{e} \mid x \in [0, L_i]\}$ is a vertical segment of length $\rho_i g L_i$ in $\mathbb{R}^3$ emanating from $\mathbf{V}^i$.

The sets $\widetilde{\mathcal{S}}_+^i$ and $\mathcal{S}_+^i$ are nonempty if and only if

$$\rho_i g L_i < 2d^i. \tag{3.6}$$

$\widetilde{\mathcal{S}}_+^i$ is an open and convex subset of $\mathbb{R}^3$, while $\mathcal{S}_+^i$ is open and connected.

From (3.4), by assuming that $\mathbf{V}^i \in \mathcal{S}_+^i$, the previous considerations lead to the following formula for stretched equilibria:

$$\mathbf{R}_x^i(x) = (m^i)^{-1}(|\mathbf{V}^i + \rho_i g x \mathbf{e}|)\frac{\mathbf{V}^i + \rho_i g x \mathbf{e}}{|\mathbf{V}^i + \rho_i g x \mathbf{e}|}$$

and

$$\mathbf{R}^i(\widehat{x}) = \mathbf{R}^i(0) + \int_0^{\widehat{x}} (m^i)^{-1}(|\mathbf{V}^i + \rho_i g x \mathbf{e}|)\frac{\mathbf{V}^i + \rho_i g x \mathbf{e}}{|\mathbf{V}^i + \rho_i g x \mathbf{e}|}\, \mathrm{d}x. \tag{3.7}$$

The discussion up to now relates only to equilibrium solutions on a single string leaving aside the question of the node conditions.

We now impose the node conditions for the star network. The continuity condition at the center node requires a common value $\mathbf{U}^0$ for all $\mathbf{R}^i(0)$, which together with the Dirichlet conditions at simple nodes leads to the following requirements on the $\mathbf{V}^i$'s:

$$\mathbf{U}^i - \mathbf{U}^0 = \int_0^{L_i} (m^i)^{-1}(|\mathbf{V}^i + \rho_i g x \mathbf{e}|)\frac{\mathbf{V}^i + \rho_i g x \mathbf{e}}{|\mathbf{V}^i + \rho_i g x \mathbf{e}|}\mathrm{d}x. \tag{3.8}$$

Moreover, the Kirchhoff condition at the center node becomes

$$\sum_{i \in \mathcal{I}} \mathbf{V}^i = \mathbf{0}.$$

For nowhere compressed solutions, we need

$$\mathbf{V}^i \in \widetilde{\mathcal{S}}_+^i \quad \text{for all } i \in \mathcal{I} \quad \text{and} \quad \sum_{i \in \mathcal{I}} \mathbf{V}^i = \mathbf{0}, \tag{3.9}$$

while for stretched solutions, we need

$$\mathbf{V}^i \in \mathcal{S}_+^i \quad \text{for all } i \in \mathcal{I} \quad \text{and} \quad \sum_{i \in \mathcal{I}} \mathbf{V}^i = \mathbf{0}. \tag{3.10}$$

We denote by $\widetilde{\mathcal{S}}_+$ and $\mathcal{S}_+$ the set of n-tuples $\{\mathbf{V}^i\}_{i\in\mathcal{I}}$ satisfying respectively the above conditions (3.9) or (3.10). Note that both sets are open subsets of the subspace

$$\Big\{\{\mathbf{V}^i\}_{i\in\mathcal{I}} \subset \prod_{i\in\mathcal{I}} \mathbb{R}^3 \equiv \mathbf{r}\mathrm{e}^{3n} : \sum_{i\in\mathcal{I}} \mathbf{V}^i = \mathbf{0}\Big\}, \tag{3.11}$$

and that $\widetilde{\mathcal{S}}_+$ is a convex set. A necessary condition for $\mathcal{S}_+$ and hence $\widetilde{\mathcal{S}}_+$ to be nonempty is that there is at least one index $i \in \mathcal{I}$, such that $\mathcal{S}^i_+$ contains vectors for which $\mathbf{V}^i \cdot \mathbf{e} > 0$, since otherwise it would not be possible to satisfy the Kirchhoff condition with $\mathbf{V}^i \neq \mathbf{0}$. This occurs for index i if and only if

$$\rho_i g L_i < d^i, \tag{3.12}$$

to be contrasted with our previous condition $\rho_i g L_i < 2d^i$ for an individual $\mathcal{S}^i_+$ to be nonempty. A sufficient condition, easily seen not to be necessary, for $\mathcal{S}_+$ to be nonempty is that (3.12) holds for all i. One shows this by firstly choosing nonzero vectors $\mathbf{V}^i$ satisfying the Kirchoff condition without regard to the length of the vectors, and by then scaling them down by a common factor α small enough to ensure that each $\alpha\mathbf{V}^i \in \mathcal{S}^i_+$.

Being convex, $\widetilde{\mathcal{S}}_+$ is arcwise connected. $\mathcal{S}_+$ is also arcwise connected, a fact which is essential for some of our results on controllability. To see this, consider $\{\mathbf{V}^i\}_{i\in\mathcal{I}}$ and $\{\mathbf{W}^i\}_{i\in\mathcal{I}}$ belonging to $\mathcal{S}_+$. Consider the curve of convex combinations

$$\{\mathbf{V}^i(\lambda)\}_{i\in\mathcal{I}} \qquad \text{with } V^i(\lambda) := (1-\lambda)\mathbf{V}^i + \lambda\mathbf{W}^i$$

parametrized by $\lambda \in [0,1]$. Its range is of course the convex hull $\mathrm{co}(\mathbf{V}^i, \mathbf{W}^i)$ of $\mathbf{V}^i$ and $\mathbf{W}^i$. In the case that for all $i \in \mathcal{I}$ this does not intersect with the vertical segment $\{-\rho_i g x \mathbf{e} : x \in [0, L_i]\} = \mathrm{co}(\mathbf{0}, -\rho_i g L_i \mathbf{e})$, the convex combinations remain in $\mathcal{S}_+$ and join $\{\mathbf{V}^i\}_{i\in\mathcal{I}}$ to $\{\mathbf{W}^i\}_{i\in\mathcal{I}}$. In the case that for some i's $\mathbf{V}^i(\lambda)$ intersect the vertical segment, we use a perturbation argument. We shall strategically choose $\{\mathbf{v}^i\}_{i\in\mathcal{I}}$, and then introduce

$$\widetilde{\mathbf{V}}^i(\lambda) = \mathbf{V}^i(\lambda) + \epsilon\mathbf{v}^i.$$

We must have

$$\sum_{i\in\mathcal{I}} \mathbf{v}^i = \mathbf{0}.$$

We also require that $\mathbf{v}^i$ are not in the span of $\mathbf{e}$ for those i for which $\mathrm{co}(\mathbf{V}^i, \mathbf{W}^i)$ intersects $\mathrm{co}(\mathbf{0}, -\rho_i g L_i \mathbf{e})$. We note that

$$\mathrm{co}(\widetilde{\mathbf{V}}^i, \widetilde{\mathbf{W}}^i) = \mathrm{co}(\mathbf{V}^i, \mathbf{W}^i) + \epsilon\mathbf{v}^i,$$

where of course $\widetilde{\mathbf{V}}^i = \mathbf{v}^i + \epsilon\mathbf{v}^i$ and $\widetilde{\mathbf{W}}^i = \mathbf{W}^i + \epsilon\mathbf{v}^i$. For ϵ sufficiently small, we then have that the curve $\{\widetilde{\mathbf{V}}^i(\lambda)\}_i \in \mathcal{I}$ remains in $\mathcal{S}_+$. Now, again requiring ϵ to be sufficiently small, $\widetilde{\mathbf{V}}^i$ and $\widetilde{\mathbf{W}}^i$ lie in convex neighborhoods within $\mathcal{S}_+$ of $\mathbf{V}^i$ and $\mathbf{W}^i$, respectively. Hence, we can concatenate linear segments obtained by convex combinations leading from $\mathbf{V}^i$ to $\widetilde{\mathbf{V}}^i$, from $\widetilde{\mathbf{V}}^i$ to $\widetilde{\mathbf{W}}^i$, and finally from $\widetilde{\mathbf{W}}^i$ to $\mathbf{W}^i$. This completes the proof of the connectedness of $\widetilde{\mathcal{S}}_+$ and $\mathcal{S}_+$.

We have therefore proved the following theorem.

Theorem 3.1 *For the star network as described above, there exists a noncompressed solution to the equilibrium system* (3.1) *subject to requirement* $1 \le |R_x^i(x)| < d_i$ *if and only if one can find* $\mathbf{U}^0 \in \mathbb{R}^3$ *and* $\{\mathbf{V}^i\}_{i\in\mathcal{I}} \in \widetilde{\mathcal{S}}_+$ *such that* (3.8) *holds. It has a stretched solution (satisfying* $1 < |\mathbf{R}_x^i(x)| < d_i$*) if* $\{\mathbf{V}^i\}_{i\in\mathcal{I}} \in \mathcal{S}_+$*.* $\widetilde{\mathcal{S}}_+$ *and* $\mathcal{S}_+$ *are both open, the former is convex and the latter is arcwise connected.*

Remark 3.1 For any $\{\mathbf{V}^i\}_{i\in\mathcal{I}} \in \mathcal{S}$ and any $\mathbf{U}^0$, one can find Dirichlet data $\mathbf{U}^1, \cdots, \mathbf{U}^n$ corresponding to a stretched equilibrium configuration of the star configuration of string. It is enough to consider $\mathbf{U}^0 = \mathbf{0}$, because other equilibria with multiple nodes at different locations can be obtained by simple translation $\mathbf{U}^i \mapsto \mathbf{U}^0 + \mathbf{U}^i$.

Remark 3.2 One can characterize the Dirichlet data $\mathbf{U}^1, \cdots, \mathbf{U}^n$ at the simple nodes, which correspond to stretched equilibria as lying in the image $\mathcal{A}_+$ of the map $\mathbf{\Phi}(\mathbf{U}^0, \mathbf{V}^1, \cdots, \mathbf{V}^n)$ defined on $\mathcal{S}_+$ by using the formula (3.8). A somewhat intricate calculation, adapting an argument given in [14], shows that one can use the open mapping theorem to conclude that $\mathcal{A}_+$ is open and then also connected. However, there is no transparent criterion to determine which Dirichlet data belongs to $\mathcal{A}_+$.

Remark 3.3 It is physically natural to require the length of the string to be short enough to satisfy conditions such as (3.3) and (3.6). A long piece of "flimsy" string could sag under its own weight to the extent that it is stretched beyond the range covered by the dynamical law, i.e., with $|\mathbf{R}_x^i(x)| \notin I^i$ for some range of x.

We can obtain a similar result for tree networks, in which there are no closed circuits. We single out one of the simple nodes, which we can suppose to be indexed by $1 \in \mathcal{J}$ and to correspond to parameter value $x = 0$. Every other node can then be joined to $\mathbf{N}^1$ by a minimal succession of strings. By a possible switch of parameter $x \mapsto L_i - x$, one can ensure that at each multiple node all the parameter values of the strings meeting that node are either 0 or L_i depending on whether the number of strings leading from $\mathbf{N}^j$ to $\mathbf{N}^1$ is even or odd. This simple remark will be the key to our proof of existence theorems later and will simplify the notation below.

To generalize Theorem 3.1 to tree networks, we introduce vectors $\mathbf{V}^{ij}$ corresponding to $j \in \mathcal{J}$ and $i \in \mathcal{I}^j$, which represent possible values of $\mathbf{G}^i(\mathbf{R}_x^i(x_{ij}))$ where $\mathbf{R}^i(x)$ satisfy the equilibrium equations. These vectors have to satisfy

$$\begin{cases} \sum_{i\in\mathcal{I}^j} \mathbf{V}^{ij} = \mathbf{0} & \text{for } j \in \mathcal{J},\ i \in \mathcal{I}^j, \\ \mathbf{V}^{ij} - \mathbf{V}^{ik} = \epsilon_{ik} g\rho_i L_i \mathbf{e} & \text{when } i \in \mathcal{I}^j \cap \mathcal{I}^k. \end{cases} \tag{3.13}$$

Obviously, the condition $i \in \mathcal{I}^j \cap \mathcal{I}^k$ identifies $\mathbf{N}^j$ and $\mathbf{N}^k$ as the nodes at the ends of the i-th string. We can now also adapt the definitions of $\widetilde{\mathcal{S}}_+$ and $\mathcal{S}_+$ in a consistent way to tree networks as follows:

$$\widetilde{\mathcal{S}}_+ := \{\{\mathbf{V}^{ij}\}_{j\in\mathcal{J},i\in\mathcal{I}^j} : (3.13) \text{ is satisfied and } |\mathbf{V}^{ij}| < d_i\}, \tag{3.14}$$

$$\mathcal{S}_+ := \{\{\mathbf{V}^{ij}\}_{j\in\mathcal{J},i\in\mathcal{I}^j} \in \widetilde{\mathcal{S}}_+ : \mathbf{0} \notin \mathrm{co}(\mathbf{V}^{ij}, \mathbf{V}^{ik}) \text{ when } i \in \mathcal{I}^j \cap \mathcal{I}^k\}. \tag{3.15}$$

Given $\{\mathbf{V}^{ij}\}_{j\in\mathcal{J},i\in\mathcal{I}^j}$, we then obtain the following formulas for equilibrium solutions:

$$\mathbf{R}_x^i(x) = \mathbf{V}^{ij} + g\rho_i x\mathbf{e} = \mathbf{V}^{ik} - g\rho_i[L_i - x]\mathbf{e},$$

when $x_{ij} = 0$ and the corresponding formula with j and k reversed when $x_{ik} = 0$. Given the location of one of the nodes, the root node N^1, one can successively obtain formulas for all of the $\mathbf{R}^i(x)$ by integration along the lines of (3.7). In particular, one is led to formulas for the Dirichlet data $\mathbf{U}^j$ corresponding to the location of simple nodes $\mathbf{N}^j$. To write these down, we suppose that $\mathbf{N}^j$ is joined to $\mathbf{N}^1$ by the minimal succession of p strings indexed successively by $\{i^1, i^2, \cdots, i^p\}$ with $i^1 = 1$ and $\mathcal{I}^j = \{i^p\}$ (or $i^p = i_j$). One gets

$$\mathbf{U}^j = \mathbf{U}^1 + \sum_{l=1}^{p} \int_0^{L_{i^l}} (m^{i^l})^{-1}(|\mathbf{V}^{i^l} + \rho_{i^l} gx\mathbf{e}|) \frac{\mathbf{V}^{i^l} + \rho_{i^l} gx\mathbf{e}}{|\mathbf{V}^{i^l} + \rho_{i^l} gx\mathbf{e}|} \, \mathrm{d}x, \tag{3.16}$$

where $\mathbf{U}^{i_l}$ is set to be equal to $\mathbf{V}^{i_l j}$ where $\mathbf{N}^j$ corresponds to parameter value 0 for the i^p-th string.

With additional arguments, entirely analogous to those for star networks, we can complete the proof of the following theorem.

Theorem 3.2 *For the tree network as described above, there exists a noncompressed solution to the equilibrium system* (3.1) *subject to requirement* $1 \le |R^i_x(x)| < d_i$ *if and only if one can find* $\{\mathbf{V}^{ij}\}_{j\in\mathcal{J}, i\in\mathcal{I}^j} \in \widetilde{\mathcal{S}}_+$ *such that* (3.16) *holds. It has a stretched solution* (*satisfying* $1 < |\mathbf{R}^i_x(x)| < d_i$) *if* $\{\mathbf{V}^i\}_{i\in\mathcal{I}} \in \mathcal{S}_+$*.* $\widetilde{\mathcal{S}}_+$ *and* $\mathcal{S}_+$ *are both open, the former is convex and the latter is arcwise connected.*

Remark 3.4 The proof of the arcwise connectedness of $\mathcal{S}_+$ can also be adapted to showing that $\mathcal{S}_+$ is dense in $\widetilde{\mathcal{S}}_+$.

Remark 3.5 In the absence of gravity, setting $g = 0$ versions of Theorems 3.1 and 3.2 continue to hold with minor simplifications and adjustments to the notation.

Under the additional strong assumptions that the potential functions $V^i(x)$ have infinite intervals (a^i, ∞) as domains, and satisfy a certain growth condition, one can in fact prove the existence of nowhere compressed equilibria for networks corresponding to any prescribed locations $\{\mathbf{U}^j\}_{j\in\mathcal{J}^S}$ of the simple nodes. In this situation, $\widetilde{\mathcal{S}}_+$ is characterized exclusively by the requirement (3.13). Moreover, since $\mathcal{S}_+$ is dense in $\widetilde{\mathcal{S}}_+$, nowhere compressed equilibria can be approximated arbitrarily closely by stretched equilibria. Existence is proved by using a variational argument, slightly adapted from [15]. The idea is to prove the existence of stationary points for the potential energy functional appearing in the derivation of our string model

$$\mathcal{V}(\mathbf{R}) := \sum_{i\in\mathcal{I}} \int_0^{L_i} [V^i(|\mathbf{R}^i_x(x)|) + \rho_i g \mathbf{R}^i(x) \cdot \mathbf{e}] \, \mathrm{d}x, \tag{3.17}$$

(recalling that $\mathbf{R} = \{\mathbf{R}^i\}_{i\in\mathcal{I}}$) defined on the Hilbert space

$$H := \Big\{ \mathbf{R} \in \prod_{i\in\mathcal{I}} W^1(0, L^i) : \mathbf{R}^i(x_{ij}) \text{ coincide for all } i \in \mathcal{I}^j, \text{ for each } j \in \mathcal{J}^M \Big\}.$$

Here $W^1(0, L)$ denotes the Sobolev space of square integrable functions with square integrable distribution derivative on $(0, L)$.

Theorem 3.3 *Suppose that the functions* $V^i(s)$ *are defined in the infinite intervals* (a^i, ∞) *and that they satisfy the growth conditions*

$$\liminf_{s\to\infty} V^i(s) > 0 \quad or \quad V^i(s) > \alpha s^2 + \beta s + \gamma \quad with\ \alpha > 0. \tag{3.18}$$

Then there exists a unique $\mathbf{R}^e \in H$ *which minimizes the function* $\mathcal{V}(\mathbf{R})$ *subject to the boundary conditions*

$$\mathbf{R}^{i_j}(x_{i_j j}) = \mathbf{U}^j \quad \textit{for } j \in \mathcal{J}^S. \tag{3.19}$$

These functions satisfy the equilibrium system (3.1)*. Each string is stretched except possibly at one point.*

Proof We note that the functional $\mathcal{V}(\mathbf{R})$ is not convex, because $V^i(|\mathbf{R}^i_x(x)|)$ is not a convex function of $\mathbf{R}^i$. We convexify the functional by introducing

$$V^i_c(s) := \begin{cases} V^i(s), & \text{when } 1 \le s, \\ 0, & \text{when } 0 \le s \le 1 \end{cases} \tag{3.20}$$

and

$$\mathcal{V}_c(\mathbf{R}) := \sum_{i \in \mathcal{I}} \int_0^{L_i} [V^i_c(|\mathbf{R}^i_x(x)|) + \rho_i g \mathbf{R}^i(x) \cdot \mathbf{e}]\, \mathrm{d}x. \tag{3.21}$$

The new functional has the following properties:

(i) $\mathcal{V}_c(\mathbf{R})$ is convex on H;

(ii) $\mathcal{V}_c(\mathbf{R}) \le \mathcal{V}(\mathbf{R})$ for all $\mathbf{R} \in H$;

(iii) $\mathcal{V}_c(\mathbf{R}) = \mathcal{V}(\mathbf{R})$, when each $|\mathbf{R}^i_x(x)| \ge 1$ for all x in $[0, L_i]$.

Properties (ii) and (iii) are obvious; (i) holds because $\mathcal{V}_c(s)$ can be written as the composition of a convex, monotone increasing function with convex function and is therefore convex. Because of these properties, one can obtain a minimizer for $\mathcal{V}(\mathbf{R})$ by obtaining a minimizer $\mathbf{R}$ for $\mathcal{V}_c(\mathbf{R})$ and showing that the minimizer necessarily satisfies $|\mathbf{R}^i(x)| \ge 1$ for each i and x in $[0, L_i]$.

To prove the existence of a minimizer for $\mathcal{V}_c$, we let

$$H_0 := \{\mathbf{R} \in H : \mathbf{R}^{i_j}(x_{i_j j}) = \mathbf{0} \ \text{ for each } j \in \mathcal{J}^S\}.$$

Since $\mathcal{J}^S \ne \emptyset$, one can show by a standard argument that the norms defined by

$$\|\mathbf{R}\|^2_{H_0} := \sum_{i \in \mathcal{I}} \int_0^{L_i} |\mathbf{R}^i_x(x)|^2 \, \mathrm{d}x \quad \text{and} \quad \|\mathbf{R}\|^2_H := \sum_{i \in \mathcal{I}} \int_0^{L_i} [|\mathbf{R}^i(x)|^2 + |\mathbf{R}^i_x(x)|^2] \, \mathrm{d}x$$

are equivalent on H_0. Letting $\widehat{\mathbf{R}}$ be any fixed element of H satisfying (3.20), we rewrite the minimization problem for $\mathcal{V}_c$ as

$$\text{to minimize } \ \mathcal{V}_c(\widehat{\mathbf{R}} + \mathbf{R}_0) \ \text{ subject to } \mathbf{R}_0 \in H_0.$$

As a function of $\mathbf{R}_0$, $\mathcal{V}_c(\widehat{\mathbf{R}} + \mathbf{R}_0)$ is convex and continuous. It is also coercive, since as a consequence of (3.14), one can easily verify that

$$\mathcal{V}_c(\widehat{\mathbf{R}} + \mathbf{R}_0) \ge A\|\mathbf{R}_0\|^2_{H_0} + B\|\mathbf{R}_0\|_{H_0} + C \qquad \text{with } A > 0.$$

It follows from a standard theorem in convex analysis (see [3, p. 35]) that $\mathcal{V}_c(\widehat{\mathbf{R}} + \mathbf{R}_0)$ has a minimizer $\underline{\mathbf{R}}_0$.

We now set $\underline{\mathbf{R}} = \widehat{\mathbf{R}} + \underline{\mathbf{R}}_0$ to obtain the minimizer of $\mathcal{V}_c(\mathbf{R})$ subject to (3.19). We show that this is also a minimizer for $\mathcal{V}(\mathbf{R})$ by proving that $|\underline{\mathbf{R}}^i_x(x)| \ge 1$ almost everywhere for each $i \in \mathcal{I}$.

Suppose that this fails to hold for some $i = k$. Then there exists a subset S of $(0, L_k)$ having positive measure along with a constant r such that $|\underline{\mathbf{R}}_x^k(x)| \le r < 1$ for $x \in S$. This leads to a contradiction, since we can modify $\underline{\mathbf{R}}$ to achieve $\mathcal{V}_c(\mathbf{R}) < \mathcal{V}_c(\underline{\mathbf{R}})$. To do this, we set $\mathbf{R}^i = \underline{\mathbf{R}}^i$ for $i \neq k$ and define

$$\mathbf{R}^k(x) := \underline{\mathbf{R}}^k(x)) - \delta \mathbf{e} \int_0^x [\chi_{S\cap[0,x_0]}(y) - \chi_{S\cap[x_0,L^k]}(y)] \mathrm{d}y,$$

where x_0 is chosen so that $S \cap [0, x_0]$ and $S \cap [x_0, L_k]$ have the same measure. $\chi_A(x)$ is the characteristic function of A and $\delta < 1 - r$. Consequently,

$$|\mathbf{R}_x^k(x)| = |\underline{\mathbf{R}}_x^k(x) - \delta[\chi_{S\cap[0,x_0]}(x) - \chi_{S\cap[x_0,L^k]}(x)]| < 1 \quad \text{for a.e. } x \in S.$$

Moreover, $\mathbf{R}^k(0) = \underline{\mathbf{R}}^k(0)$ and $\mathbf{R}^k(L_k) = \underline{\mathbf{R}}^k(L_k)$ so that $\mathbf{R}$ satisfies the conditions (3.19) and the multiple node conditions in the definition of H. Then $\mathcal{V}_c^k(|\mathbf{R}_x^k(x)|) = \mathcal{V}_c^k(|\underline{\mathbf{R}}_x^k(x)|)$, since $\mathbf{R}_x^k(x)$ differs from $\underline{\mathbf{R}}_x^k(x)$ only in S. Finally, since $\mathbf{R}^k(x) \cdot \mathbf{e} < \underline{\mathbf{R}}^k(x \cdot \mathbf{e})$ on a subinterval of $[0, L_k]$, we are led to the contradiction

$$\mathcal{V}_c(\mathbf{R}) < \mathcal{V}_c(\underline{\mathbf{R}}).$$

This completes the proof of the existence of a minimizer for $\mathcal{V}(\mathbf{R})$.

Next, we prove the uniqueness of the minimizer. We do this by showing that the set C of minimizers of $\mathcal{V}_c$ contains only one element. Since C is a nonempty convex subset, and by what was proved above, we have

$$C \subset \{\mathbf{R} \in H : |\mathbf{R}_x^i| \ge 1 \text{ a.e. for all } i \in \mathcal{I}\}. \tag{3.22}$$

Since $\mathcal{V}(\mathbf{R}) = \mathcal{V}_c(\mathbf{R})$ for $\mathbf{R} \in C$ and $\mathcal{V}_c(\mathbf{R}) \le \mathcal{V}(\mathbf{R})$ for all $\mathbf{R} \in H$, it follows that C is also the set of minimizers for $\mathcal{V}$. Let $\underline{\mathbf{R}}$ and $\underline{\widetilde{\mathbf{R}}}$ be two elements in C and $\lambda \in [0, 1]$. Then

$$\mathcal{V}_c(\lambda \underline{\mathbf{R}} + (1-\lambda)\underline{\widetilde{\mathbf{R}}}) = \lambda \mathcal{V}_c(\underline{\mathbf{R}}) + (1-\lambda)\mathcal{V}_c(\underline{\widetilde{\mathbf{R}}}).$$

Disregarding the linear terms in $\mathcal{V}_c$, one gets

$$\begin{aligned} &\sum_{i\in\mathcal{I}} \int_0^{L_i} V_c^i(|\lambda \underline{\mathbf{R}}_x^i(x) + (1-\lambda)\underline{\widetilde{\mathbf{R}}}_x^i(x)|)\, \mathrm{d}x \\ &= \sum_{i\in\mathcal{I}} \int_0^{L_i} [\lambda V_c^i(|\underline{\mathbf{R}}_x^i(x)|) + (1-\lambda) V_c^i(|\underline{\widetilde{\mathbf{R}}}_x^i(x)|)]\, \mathrm{d}x. \end{aligned} \tag{3.23}$$

Since, by convexity,

$$V_c^i(|\lambda \underline{\mathbf{R}}_x^i(x) + (1-\lambda)\underline{\widetilde{\mathbf{R}}}_x^i(x)|) \le \lambda V_c^i(|\underline{\mathbf{R}}_x^i(x)|) + (1-\lambda) V_c^i(|\underline{\widetilde{\mathbf{R}}}_x^i(x)|),$$

one can conclude from (3.23) that

$$V_c^i(|\lambda \underline{\mathbf{R}}_x^i(x) + (1-\lambda)\underline{\widetilde{\mathbf{R}}}_x^i(x)|) = \lambda V_c^i(|\underline{\mathbf{R}}_x^1(x)|) + (1-\lambda) V_c^i(|\underline{\widetilde{\mathbf{R}}}_x^i(x)|).$$

Recalling the definition of V_c^1 and using (3.22) in conjunction with the fact that $\mathbf{V}^i(s)$ is strictly convex for $s \ge 1$, one sees that necessarily

$$|\lambda \underline{\mathbf{R}}_x^i(x) + (1-\lambda)\underline{\widetilde{\mathbf{R}}}_x^i(x)| = \lambda |\underline{\mathbf{R}}_x^i(x)| + (1-\lambda)|\underline{\widetilde{\mathbf{R}}}_x^i(x)|.$$

Hence, it follows from properties of the Euclidean norm that $\widetilde{\underline{\mathbf{R}}}_x^i(x) = \underline{\mathbf{R}}_x^i(x)$. Since $\widetilde{\underline{\mathbf{R}}}$ and $\underline{\mathbf{R}}$ both satisfy (3.19) as well as the continuity conditions at multiple nodes embodied in the definition of H, we conclude that $\widetilde{\underline{\mathbf{R}}}^i(x) = \underline{\mathbf{R}}^i(x)$ and this uniqueness is proved.

To show that $\underline{\mathbf{R}}$ is in fact an equilibrium solution to the network equations (3.1) is standard starting from the Frechét derivative of $\mathcal{V}(\mathbf{R})$ at $\underline{\mathbf{R}}$ and using the variations of $\underline{\mathbf{R}}$ as test functions to be chosen in various ways.

Remark 3.6 In the case that the $V^i(s)$ are defined in bounded open intervals (a_i, b_i), the existence assertion of the previous theorem is no longer valid. However, one can adapt the argument to prove that noncompressed equilibria corresponding to given boundary data $\mathbf{U}^j$, if they exist, are unique. One can as before introduce the functional $\mathcal{V}(\mathbf{R})$ and its convexification $\mathcal{V}_c(\mathbf{R})$ on a convex subset of H. Noncompressed equilibria are then stationary points of both functionals, and hence minimize the convex functional $\mathcal{V}_c(\mathbf{R})$ subject to the simple node conditions and do the same for $\mathcal{V}(\mathbf{R})$. The uniqueness proof then goes through.

Remark 3.7 The condition (3.18) is satisfied for our canonical example $V^i(s) = \frac{1}{2}h_i(s-1)^2$. The condition could also be relaxed to $V^i(s) > \alpha s^p + \beta s + \gamma$ with $\alpha > 0$ and $p > 1$.

Remark 3.8 The above proof does not in fact require the absence of closed circuits in the network, so the theorem is valid for general networks having some simple nodes.

Remark 3.9 In the proof of Theorem 3.3, the gravitational term is essential, since this is what ensures that the minimizers correspond to nowhere compressed equilibria.

4 Existence of Solutions Near Stretched Equilibria on Tree Networks

We continue to restrict our attention to tree networks with prescribed positions for the simple nodes. As pointed out before, we can reparametrize individual strings to ensure that at each multiple node all the parameter values are either 0 or L_i depending on whether the number of strings leading to that node from the simple node $\mathbf{N}^1$ is even or odd.

We consider the equations (2.10) accompanied by initial conditions (2.7), boundary conditions (2.8) at the simple nodes as well as the multiple node conditions (2.9) and (2.12).

One can only hope to find solutions $\mathbf{R}^i(x,t)$ in $C^2([0,L_i]\times[0,T];\mathbb{R}^3)$ if

$$\begin{cases} \mathbf{U}^j(t) \in C^2([0,T];\mathbb{R}^3) & \text{for } j \in \mathcal{J}^S, \\ \mathbf{R}^{0,i}(x) \in C^2([0,L_i];\mathbb{R}^3) & \text{for } i \in \mathcal{I}, \\ \mathbf{R}^{1,i}(x) \in C^1([0,L_i];\mathbb{R}^3) & \text{for } i \in \mathcal{I}, \end{cases} \tag{4.1}$$

and if these functions satisfy the following C^2 compatibility conditions:

$$\begin{cases} \mathbf{R}^{0,i}(x_{ij}) \text{ coincide} & \text{for each } j \in \mathcal{J}^M \text{ and } i \in \mathcal{I}^j, \\ \mathbf{R}^{1,i}(x_{ij}) \text{ coincide} & \text{for each } j \in \mathcal{J}^M \text{ and } i \in \mathcal{I}^j, \\ \sum_{i\in\mathcal{I}^j} \epsilon_{ij}\mathbf{G}^i(\mathbf{R}_x^{0,i}(x_{ij})) = \mathbf{0} & \text{for each } j \in \mathcal{J}^M, \\ \sum_{i\in\mathcal{I}^j} \epsilon_{ij}\mathbf{G}_{\mathbf{v}}^i(\mathbf{R}_x^{0,i}(x_{ij}))\mathbf{R}_x^{1,i}(x_{ij}) = \mathbf{0} & \text{for each } j \in \mathcal{J}^M, \end{cases} \tag{4.2}$$

as well as

$$\begin{cases} \mathbf{U}^j(0) = \mathbf{R}^{i_j,0}(x_{ij}) & \text{for } j \in \mathcal{J}^S, \\ \mathbf{U}_t^j(0) = \mathbf{R}^{i_j,1}(x_{ij}) & \text{for } j \in \mathcal{J}^S, \\ \mathbf{U}_{tt}^j(0) = (\rho_{i_j})^{-1}[\mathbf{G}^{i_j}(\mathbf{R}_x^{i_j,0})]_x|_{x=x_{ij}} & \text{for } j \in \mathcal{J}^S, \end{cases} \tag{4.3}$$

where $\mathcal{I}^j = \{i_j\}$ for $j \in \mathcal{J}^S$. After specifying the initial data $\mathbf{R}^{0,i} \times \mathbf{R}^{1,i} \in C^2([0,L_i];\mathbb{R}^3) \times C^1([0,L_i];\mathbb{R}^3)$ for all i subject to the conditions (4.2), the control data $U^j(t)$ at simple nodes can be uniquely represented by

$$\mathbf{U}^j(t) = \mathbf{u}^j(t) + \mathbf{R}^{i_j,0}(x_{ij}) + t\mathbf{R}^{i_j,1}(x_{ij}) + \frac{t^2}{2}\,(\rho_{i_j})^{-1}[\mathbf{G}^{i_j}(\mathbf{R}_x^{i_j,0})]_x|_{x=x_{ij}}, \tag{4.4}$$

where $\mathbf{u}^j(t)$ belongs to

$$C_0^2([0,T];\mathbb{R}^3) := \{\mathbf{u}(t) \in C^2([0,T];\mathbb{R}^3) \mid \mathbf{u}(0) = \mathbf{u}_t(0) = \mathbf{u}_{tt}(0) = \mathbf{0}\}.$$

We are left with the system

$$\begin{cases} \rho_i \mathbf{R}_{tt}^i(x,t) = [\mathbf{G}^i(\mathbf{R}_x^i(x,t))]_x - \rho_i g \mathbf{e} & \text{for } i \in \mathcal{I}, \\ \mathbf{R}^i(x,0) = \mathbf{R}^{0,i}(x), \quad \mathbf{R}_t^i(x,0) = \mathbf{R}^{1,i}(x) & \text{for } i \in \mathcal{I}, \\ \mathbf{R}^{i_j}(x_{ij},t) = \mathbf{u}^j(t) + \mathbf{R}^{0,i_j}(x_{ij}) + t\mathbf{R}^{1,i}(x_{ij}) & \\ \qquad\qquad + \dfrac{t^2}{2}(\rho_{i_j})^{-1}[\mathbf{G}^{i_j}(\mathbf{R}_x^{0,i_j})]_x|_{x=x_{ij}} & \text{for } j \in \mathcal{J}^S, \\ \mathbf{R}^i(x_{ij},t) \ \text{ coincide} & \text{for each } j \in \mathcal{J}^M \text{ and } i \in \mathcal{I}^j, \\ \sum_{i\in\mathcal{I}^j} \mathbf{G}^i(\mathbf{R}_x^i(x_{ij},t)) = \mathbf{0} & \text{for } j \in \mathcal{J}^M \end{cases} \tag{4.5}$$

required to hold for $x \in [0,L_i]$ and $t \in [0,T]$, respectively.

We shall prove a local existence theorem close to a specified stretched equilibrium $\mathbf{R}^e = \{\mathbf{R}^{e,i}\}_{i\in\mathcal{I}}$. We shall in fact first show that the above system is equivalent to a coupled set of quasilinear hyperbolic systems corresponding to each of the strings, and then apply an existence theorem for such systems. As an intermediate step, we introduce perturbations away from the given equilibrium by setting

$$\mathbf{r}^i(x,t) := \mathbf{R}^i(x,t) - \mathbf{R}^{e,i}(x). \tag{4.6}$$

Noting that the $\mathbf{R}^{e,i}$ do not depend on t, the system (4.5) is equivalent to

$$\begin{cases} \rho_i \mathbf{r}_{tt}^i(x,t) = [\mathbf{G}^i(\mathbf{R}_x^{e,i}(x) + \mathbf{r}_x^i(x,t))]_x - \rho_i g \mathbf{e} & \text{for } i \in \mathcal{I}, \\ \mathbf{r}^i(x,0) = \mathbf{r}^{0,i}(x), \quad r_t^i(x,0) = \mathbf{r}^{1,i}(x) & \text{for } i \in \mathcal{I}, \\ \mathbf{r}^{i_j}(x_{i_j j},t) = \mathbf{u}^j(t) + \mathbf{r}^{0,i_j}(x_{i_j j}) + t\mathbf{r}^{1,i_j}(x_{i_j j}) & \\ \qquad\qquad + \dfrac{t^2}{2}(\rho_{i_j})^{-1}[\mathbf{G}^{i_j}(\mathbf{R}_x^{e,i_j} + \mathbf{r}_x^{0,i_j})]_x|_{x=x_{i_j j}} & \text{for } j \in \mathcal{J}^S, \\ \mathbf{r}^i(x_i,t) \ \text{ coincide} & \text{for each } j \in \mathcal{J}^M \text{ and } i \in \mathcal{I}^j, \\ \sum_{i\in\mathcal{I}^j} \mathbf{G}^i(\mathbf{R}_x^{e,i}(x_{ij}) + \mathbf{r}_x^i(x_{ij},t)) = \mathbf{0} & \text{for } j \in \mathcal{J}^M, \end{cases} \tag{4.7}$$

where we have set $\mathbf{r}^{0,i} := \mathbf{R}^{0,i} - \mathbf{R}^{e,i}$ and $\mathbf{r}^{1,i} := \mathbf{R}^{1,i}$. Note that this initial data has to satisfy the compatibility conditions

$$\begin{cases} \mathbf{r}^{0,i}(x_{ij}) \text{ coincide} & \text{for each } j \in \mathcal{J}^M \text{ and } i \in \mathcal{I}^j, \\ \mathbf{r}^{1,i}(x_{ij}) \text{ coincide} & \text{for each } j \in \mathcal{J}^M \text{ and } i \in \mathcal{I}^j, \\ \sum_{i\in\mathcal{I}^j} \epsilon_{ij}\mathbf{G}^i(\mathbf{R}_x^{e,i}(x_{ij}) + \mathbf{r}_x^{0,i}(x_{ij})) = \mathbf{0} & \text{for each } j \in \mathcal{J}^M, \\ \sum_{i\in\mathcal{I}^j} \epsilon_{ij}\mathbf{G}_{\mathbf{v}}^i(\mathbf{R}_x^{e,i}(x_{ij}) + \mathbf{r}_x^{0,i}(x_{ij}))\mathbf{r}_x^{1,i}(x_{ij}) = \mathbf{0} & \text{for each } j \in \mathcal{J}^M. \end{cases} \tag{4.8}$$

We transform the system (4.7) to an equivalent initial boundary value problem for a coupled system of first order quasilinear hyperbolic systems valued function $\mathbf{w}^i = (\mathbf{w}_1^i, \mathbf{w}_2^i) := (\mathbf{r}_x^i, \mathbf{r}_t^i)$ of x and t. It is easily seen that equations in (4.7) can be rewritten as

$$\mathbf{w}_t^i + \mathbf{f}^i(x, \mathbf{w}^i)_x = -(\mathbf{0}, g\mathbf{e})$$

with $(x,t) \in [0, L_i] \times [0, T]$, where

$$\mathbf{f}^i(x, \mathbf{w}) = -(\mathbf{w}_2^i, {\rho_i}^{-1}[\mathbf{G}^i(\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1^i(x,t))]).$$

This, in turn, can be rewritten in the form of a quasilinear hyperbolic system

$$\mathbf{w}_t^i + A^i(x, \mathbf{w}^i)\mathbf{w}_x^i = \mathbf{g}^i(x, \mathbf{w}^i), \tag{4.9}$$

with

$$A^i(x, \mathbf{w}^i) = A^i(x, \mathbf{w}_1^i) := -\begin{pmatrix} 0 & I \\ \rho_i^{-1}\mathbf{G}_{\mathbf{v}}^i(\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1^i(x,t)) & 0 \end{pmatrix} \tag{4.10}$$

and

$$\mathbf{g}^i(x, \mathbf{w}^i) = \mathbf{g}^i(x, \mathbf{w}_1^i) := (\mathbf{0}, \rho_i^{-1}\mathbf{G}_{\mathbf{v}}^i(\mathbf{R}_x^{e,i}(x) + w_1^i)\mathbf{R}_{xx}^{e,i} - g\mathbf{e}). \tag{4.11}$$

We note that

$$\begin{aligned} \mathbf{g}^i(x, \mathbf{0}) &= (\mathbf{0}, \rho_i^{-1}\mathbf{G}_{\mathbf{v}}^i(\mathbf{R}_x^{e,i}(x) = \mathbf{0})\mathbf{R}_{xx}^{e,i} - g\mathbf{e}) \\ &= (\mathbf{0}, \rho^{-1}\left[\mathbf{G}(\mathbf{R}_x^e)_x - \rho g\mathbf{e}\right]) = (\mathbf{0}, \mathbf{0}). \end{aligned} \tag{4.12}$$

Rewriting system (4.7), we get

$$\begin{cases} \mathbf{w}_t^i + A^i(x, \mathbf{w}^i)\mathbf{w}_x^i = \mathbf{g}^i(x, \mathbf{w}^i) & \text{for } i \in \mathcal{I}, \\ \mathbf{w}^i(x, 0) = \mathbf{w}^{0,i}(x) = (\mathbf{r}_x^{0,i}(x), \mathbf{r}^{1,i}(x)) & \text{for } i \in \mathcal{I}, \\ \mathbf{w}_2^{i_j}(x_{i_j j}, t) = \mathbf{v}^j(t) = \mathbf{u}_t^j(t) + \mathbf{w}_2^{0,i_j}(x_{i_j j}) & \\ \qquad + t\,\rho_{i_j}^{-1}[\mathbf{G}^{i_j}(\mathbf{R}_x^{e,i_j} + \mathbf{w}_1^{0,i_j})]_x|_{x=x_{i_j j}} & \text{for } j \in \mathcal{J}^S, \\ \mathbf{w}_2^i(x_{ij}, t) \ \text{ coincide} & \text{for each } j \in \mathcal{J}^M \text{ and } i \in \mathcal{I}^j, \\ \sum_{i\in\mathcal{I}^j} \mathbf{G}^i(\mathbf{R}_x^{e,i}(x_{ij}) + \mathbf{w}_1^i(x_{ij}, t)) = \mathbf{0} & \text{for } j \in \mathcal{J}^M. \end{cases} \tag{4.13}$$

Note that $\mathbf{w}^{0,i}(x) \in C^1([0,L]; \mathbb{R}^3) \times C^1([0,L]; \mathbb{R}^3)$. C^1-compatibility conditions between the boundary and initial data are built into the boundary data $\mathbf{v}^j(t)$, and if the initial data in the system (4.5) satisfy the C^2-compatibility conditions at multiple nodes, the initial data in system (4.13) satisfy the following C^1-conditions at multiple nodes:

$$\begin{cases} \mathbf{w}_2^{0,i}(x_{ij}) \text{ coincide} & \text{for each } j \in \mathcal{J}^M \text{ and } i \in \mathcal{I}^j, \\ \sum_{i\in\mathcal{I}^j} \epsilon_{ij}\mathbf{G}^i(\mathbf{R}_x^{e,i}(x_{ij}) + \mathbf{w}_1^{0,i}(x_{ij})) = \mathbf{0} & \text{for each } j \in \mathcal{J}^M, \\ \sum_{i\in\mathcal{I}^j} \epsilon_{ij}\mathbf{G}_{\mathbf{v}}^i(\widetilde{\mathbf{R}}_x^{0,i}(x_{ij}) + \mathbf{w}_1^{0,i}(x_{ij}))\mathbf{w}_{2x}^{0,i}(x_{ij}) = \mathbf{0} & \text{for each } j \in \mathcal{J}^M. \end{cases} \tag{4.14}$$

From the preceding considerations, it follows trivially that if $\mathbf{r} = \{\mathbf{r}^i\}_{i\in\mathcal{I}}$ is a twice continuously differentiable solution to (4.7), $\mathbf{w} = \{\mathbf{w}^i\}_{i\in\mathcal{I}}$ is continuously differentiable and satisfies (4.13). Conversely, if $\mathbf{w}$ is a continuously differentiable solution to the latter system, one can recover the solutions $\mathbf{r}$ to (4.7). One has $\partial_x\mathbf{w}_2^i = \partial_t\mathbf{w}_1^i$, implying the existence of a twice differentiable $\mathbf{r}^i \in C^2$ with $\mathbf{r}_x^i = \mathbf{w}_1^i$ and $\mathbf{r}_t^i = \mathbf{w}_2^i$. In fact, one can easily check, condition by condition and taking into account the compatibility conditions, that the functions

$$\mathbf{r}^i(x,t) = \mathbf{r}^{0,i}(x) + \int_0^t \mathbf{w}_2^i(x,s)\mathrm{d}s,$$

then satisfy all the requirements of (4.7).

Theorem 4.1 *Consider a tree network as described in the beginning of this section. Let* $\mathbf{R}^e$ *be a given stretched equilibrium. For a specified value of* $T > 0$*, there exist constants* c_0 *and* c_T*, such that if the initial data*

$$\mathbf{w}^{0,i} = (\mathbf{w}_1^{0,i}, \mathbf{w}_2^{0,i}) \in C^1([0,L_i]\times[0,T];\mathbb{R}^3)\times C^1([0,L_i]\times[0,T];\mathbb{R}^3)$$

and the boundary data

$$\mathbf{v}^j(t) \in C_0^1([0,T];\mathbb{R}^3)$$

given in (4.13) *satisfy the* C^1*-compatibility conditions* (4.14) *and satisfy*

$$\max\{\|\mathbf{w}_1^{0,i}\|_1, \|\mathbf{w}_2^{0,i}\|_1, \|\mathbf{v}^j\|_1\}_{i\in\mathcal{I},j\in\mathcal{J}^S} < c_0,$$

there exists a unique solution

$$\mathbf{w} \in \prod_{i\in\mathcal{I}} C^1([0,L_i]\times[0,T];\mathbb{R}^3)\times C^1([0,L_i]\times[0,T];\mathbb{R}^3)$$

to (4.13)*, depending continuously on the data in the sense that for each* $i \in \mathcal{I}$*,*

$$\|\mathbf{w}_1^i\|_1 + \|\mathbf{w}_2^i\|_1 \le c_T \max\{\|\mathbf{w}_1^{0,i}\|_1, \|\mathbf{w}_2^{0,i}\|_1, \|\mathbf{v}^j\|_1\}_{i\in\mathcal{I},j\in\mathcal{J}^S}.$$

Proof We shall begin with the assumption that the lengths L_i are all the same and equal to L. When this is not the case, one can perform a rescaling by introducing on each string a new parameter $y := \frac{L}{L_i}x$, so that each string is parametrized over the same interval $[0,L]$. We then set $\widetilde{\mathbf{R}}^i(y,t) := \mathbf{R}^i(\frac{L_i}{L}y)$. Then $\frac{L}{L_i}\widetilde{\mathbf{R}}_x^i(y,t) = \mathbf{R}_x^i(\frac{L_i}{L}y)$, and it is similar for other functions of x which occur in our equations. Moreover, the values $y_{ij} := \frac{L}{L_i}x_{ij}$ of course now take on the values 0 or L. The system (4.13) can then be rewritten in terms of the variable $y \in [0,L]$ for all i with some added purely notational complexity.

With this initial assumption the 6-dimensional hyperbolic systems (4.14) occurring in the system (4.13) can now be assembled into a system in $6n$ unknowns on the domain $[0, L] \times [0, T]$:

$$\mathbf{w}_t + A(x, \mathbf{w})\mathbf{w}_x = \mathbf{g}(x, \mathbf{w}) \tag{4.15}$$

by setting

$$\begin{cases} A := \operatorname{diag}(A^1, A^2, \cdots, A^n), \quad \mathbf{w} = (\mathbf{w}^1, \mathbf{w}^2, \cdots, \mathbf{w}^n), \\ \mathbf{g}(x, \mathbf{w}) := (\mathbf{g}^1(x, \mathbf{w}^1), \mathbf{g}^2(x, \mathbf{w}^2), \cdots, \mathbf{g}^n(x, \mathbf{w}^n)). \end{cases} \tag{4.16}$$

We shall see later that this is a hyperbolic system near the equilibrium. We note that

$$A(x, \mathbf{w}) = A(x, \mathbf{w}_1), \qquad \text{where } \mathbf{w}_1 = (\mathbf{w}_1^1, \mathbf{w}_1^2, \cdots, \mathbf{w}_1^n).$$

The associated initial conditions are simply

$$\mathbf{w}(x, 0) = (\mathbf{w}^{0,1}(x), \cdots, \mathbf{w}^{0,2}(x), \cdots, \mathbf{w}^{0,n}(x)) \in C^1([0, L]; \mathbf{re}^{6n}). \tag{4.17}$$

The boundary conditions at the simple nodes take the form

$$\mathbf{w}_2^{i_j}(x_{i_j j}, t) = \mathbf{u}_t^j(t) + \mathbf{w}_2^{0,i_j}(x_{i_j j}) + t\,\rho_{i_j}^{-1}[\mathbf{G}^{i_j}(\mathbf{R}_x^{e,i_j} + \mathbf{w}_1^{0,i_j})]_x|_{x = x_{i_j j}} \quad \text{for } j \in \mathcal{J}^S. \tag{4.18}$$

The multiple node conditions become

$$\begin{cases} \mathbf{w}_2^i(x_{ij}, t) \ \text{ coincide} & \text{for each } j \in \mathcal{J}^M \text{ and } i \in \mathcal{I}^j, \\ \sum\limits_{i \in \mathcal{I}^j} \mathbf{G}^i(\mathbf{R}_x^{e,i}(x_{ij}) + \mathbf{w}_1^i(x_{ij}, t)) = \mathbf{0} & \text{for } j \in \mathcal{J}^M. \end{cases} \tag{4.19}$$

These can also be considered as boundary conditions, since at the multiple node indexed by j the x_{ij} are either all 0 or all L.

We shall prove that the boundary conditions both at simple nodes and corresponding to multiple nodes are of the type allowing solutions to hyperbolic systems.

We need to analyse the structure of the system. Let $P^i = P^i(x, \mathbf{w}_1^i)$ denote the matrix of the linear transformation $\rho_i^{-1}\mathbf{G}_{\mathbf{v}}^i(\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1^i(x, t))$, so that A^i has the block matrix structure

$$A^i = \begin{pmatrix} 0 & I \\ P^i(x, \mathbf{w}_1^i) & 0 \end{pmatrix}. \tag{4.20}$$

Starting from the definition (2.11), one finds

$$\mathbf{G}_{\mathbf{v}}^i(\mathbf{V})\mathbf{v} = V_{ss}^i(|\mathbf{V}|)\frac{\mathbf{V} \cdot \mathbf{v}}{|\mathbf{V}|^2}\mathbf{V} + \frac{V_s^i(|\mathbf{V}|)}{|\mathbf{V}|}\Big[\mathbf{v} - \frac{\mathbf{V} \cdot \mathbf{v}}{|\mathbf{V}|^2}\mathbf{V}\Big].$$

The linear transformation $\mathbf{G}_{\mathbf{v}}^i(\mathbf{V})$ therefore has eigenvalues $V_{ss}^i(|\mathbf{V}|)$ and $\frac{V_s^i(|\mathbf{V}|)}{|\mathbf{V}|}$ corresponding respectively to the eigenspaces spanned by $\mathbf{V}$ and its 2-dimensional orthogonal complement. These eigenvalues are both positive if $|\mathbf{V}| > 1$. An orthonormal basis of eigenvectors of $\mathbf{G}_{\mathbf{v}}^i(\mathbf{V})$, depending smoothly on $\mathbf{V}$ in the set theoretic complement of a specified 1-dimensional subspace of $\mathbb{R}^3$, can be chosen as follows:

$$\frac{\mathbf{V}}{|\mathbf{V}|}, \quad \frac{M^i\mathbf{V}}{|M^i\mathbf{V}|}, \quad \frac{\mathbf{V}}{|\mathbf{V}|} \times \frac{M^i\mathbf{V}}{|M^i\mathbf{V}|},$$

where M^i is a fixed, invertible, skew-symmetric 3×3 matrix with the specified subspace as nullspace, and the third vector is obtained by taking the cross product of the previous vectors. We restrict our attention to small perturbations of stretched equilibria, so that $|\mathbf{R}_x^{e,i}(x)| > 1$, and by continuity, $|\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1^i(x,t)| > 1$ for sufficiently small $\mathbf{w}_1^i$. If $\mathbf{R}^{e,i}(x)$ does not correspond to a string in vertical position, so that $\mathbf{R}_x^{e,i}(x)$ is never aligned with $\mathbf{e}$, we pick M^i to have the vertical subspace spanned by $\mathbf{e}$ as nullspace. If on the other hand, the equilibrium position is vertical, we choose M^i to have a subspace spanned by a vector orthogonal to $\mathbf{e}$ as nullspace. In either case, sufficiently small perturbations $\mathbf{R}^{e,i} + \mathbf{w}_1^i$ cannot lie in the nullspace of M^i. Let $Q^i(x, \mathbf{w}_1^i)$ denote the orthogonal matrix having the vectors constructed above with $\mathbf{V} = \mathbf{R}_x^{e,i}(x) + \mathbf{w}_1^i$ as columns. Then the symmetric matrix $P^i(x, \mathbf{w}_1^i)$ can be diagonalized as follows:

$$P^i(x, \mathbf{w}_1^i) = Q^i(x, \mathbf{w}_1^i)[D^i(x, \mathbf{w}_1^i)]^2 Q^i(x, \mathbf{w}_1^i)^{\mathrm{T}}, \tag{4.21}$$

where $D^i(x, \mathbf{w}_1^i) = \mathrm{diag}(\mu_1^i(x, \mathbf{w}_1^i), \mu_2^i(x, \mathbf{w}_1^i), \mu_3^i(x, \mathbf{w}_1^i))$ is diagonal with positive diagonal entries given by

$$\begin{cases} [\mu_1^i(x, \mathbf{w}_1^i)]^2 = \rho_i^{-1} V_{ss}^i(|\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1^i|), \\ [\mu_2^i(x, \mathbf{w}_1^i)]^2 = [\mu_3^i(x, \mathbf{w}_1^i)]^2 = \dfrac{\rho_i^{-1} V_s^i(|\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1|)}{|\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1|}. \end{cases} \tag{4.22}$$

The following easily proved lemma on partitioned matrices will be applied to each of the 6×6 matrices A^i, which appear along the diagonal of the matrix $A(x, w_1)$ in (4.16).

Lemma 4.1 *Let A be a $2p \times 2p$ matrix partitioned into $p \times p$ matrices:*

$$A = \begin{pmatrix} 0 & I \\ P & 0 \end{pmatrix} \tag{4.23}$$

with P symmetric and positive definite, having eigenvalues $\mu_1^2, \cdots, \mu_p^2$ with each $\mu_i > 0$. Let D denote the $p \times p$ diagonal matrix having the μ_i's along the diagonal, and $P = QD^2Q^{\mathrm{T}}$ where Q is orthogonal. Then the matrix A has eigenvalues $\pm\mu_1, \pm\mu_2, \cdots, \pm\mu_p$ and is diagonalizable as follows:

$$A = S \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} S^{-1} \tag{4.24}$$

with

$$S = \begin{pmatrix} Q & Q \\ QD & -QD \end{pmatrix}, \quad S^{-1} = \frac{1}{2} \begin{pmatrix} Q^{\mathrm{T}} & D^{-1}Q^{\mathrm{T}} \\ Q^{\mathrm{T}} & -D^{-1}Q^{\mathrm{T}} \end{pmatrix}.$$

Applying this to each of the diagonal blocks $A^i(x, \mathbf{w}_1^i)$, we conclude that

$$A(x, \mathbf{w}^1) = S(x, \mathbf{w}^1) D(x, \mathbf{w}_1) S(x, \mathbf{w}_1)^{-1}, \tag{4.25}$$

where

$$\begin{cases} D(x, \mathbf{w}_1) = \mathrm{diag}(\widehat{D}^1(x, \mathbf{w}_1^1), \widehat{D}^2(x, \mathbf{w}_1^2), \cdots, \widehat{D}^n(x, \mathbf{w}_1^n)), \\ S(x, \mathbf{w}_1) = \mathrm{diag}(S^1(x, \mathbf{w}_1^1), S^2(x, \mathbf{w}_1^2), \cdots, S^n(x, \mathbf{w}_1^n)), \\ S(x, \mathbf{w}_1)^{-1} = \mathrm{diag}(S^1(x, \mathbf{w}_1^1)^{-1}, S^2(x, \mathbf{w}_1^2)^{-1}, \cdots, S^n(x, \mathbf{w}_1^n)^{-1}) \end{cases} \tag{4.26}$$

with

$$\begin{cases} \widehat{D}^i(x, \mathbf{w}_1^i) = \begin{pmatrix} D^i(x, \mathbf{w}_1^i) & 0 \\ 0 & -D^i(x, \mathbf{w}_1^i) \end{pmatrix}, \\ S^i(x, \mathbf{w}_1^i) = \begin{pmatrix} Q^i(x, \mathbf{w}_1^i) & Q^i(x, \mathbf{w}_1^i) \\ Q^i(x, \mathbf{w}_1^i)D^i(x, \mathbf{w}_1^i) & -Q^i(x, \mathbf{w}_1^i)D^i(x, \mathbf{w}_1^i) \end{pmatrix}, \\ S^i(x, \mathbf{w}_1^i)^{-1} = \dfrac{1}{2} \begin{pmatrix} Q^i(x, \mathbf{w}_1^i)^{\mathrm{T}} & D^i(x, \mathbf{w}_1^i)^{-1} Q^i(x, \mathbf{w}_1^i)^{\mathrm{T}} \\ Q^i(x, \mathbf{w}_1^i)^{\mathrm{T}} & -D^i(x, \mathbf{w}_1^i)^{-1} Q^i(x, \mathbf{w}_1^i)^{\mathrm{T}} \end{pmatrix}. \end{cases} \tag{4.27}$$

We note that the eigenvalues of the diagonalisable $6n \times 6n$ matrix $A(x, \mathbf{w}_1)$, namely $\pm\mu_k^i$ with $i \in \mathcal{I}$ and $k = 1, 2, 3$, are all nonzero for $\mathbf{w}$ sufficiently small, with half of them positive and half negative. Thus we have a quasilinear hyperbolic system in 1-dimensional space variable for which boundary value problems have received considerable attention. We shall rely on [16], which treats the situation where A and $\mathbf{g}$ can depend on x, t and $\mathbf{w}$. That paper draws heavily on the proofs of previous results, for example in [8] where the coefficients depend only on $\mathbf{w}$.

Premultiplying the system (4.15) by $S(x, \mathbf{w}_1)^{-1}$, we get

$$S(x, \mathbf{w}_1)^{-1}\mathbf{w}_t + S(x, \mathbf{w}_1)^{-1}A(x, \mathbf{w}_1)\mathbf{w}_x = S(x, \mathbf{w}_1)^{-1}\mathbf{g}(x, \mathbf{w}). \tag{4.28}$$

We denote the k-th row of $S(x, \mathbf{w}_1)^{-1}$ by $\mathbf{l}^k(x, \mathbf{w}_1)$. This is a left eigenvector of $A(x, \mathbf{w}_1)$ corresponding to the k-th eigenvalue $\lambda_k(x, \mathbf{w}_1)$ occurring along the diagonal of D, one gets the equations

$$\mathbf{l}^k(x, \mathbf{w}_1)\,[\mathbf{w}_t + \lambda_k(x, \mathbf{w}_1)\mathbf{w}_x] = S(x, \mathbf{w}_1)^{-1}\mathbf{g}(x, \mathbf{w}), \tag{4.29}$$

which is exactly of the form of the equations studied in [16].

To study the boundary conditions, we have to rely heavily on the block structure of our matrices. It is useful to introduce $\boldsymbol{\xi}(x, \mathbf{w}) = S(x, \mathbf{w}_1)^{-1}\mathbf{w}$. As in [14], one can use the implicit function theorem to show that this change of variables is invertible near $\mathbf{w} = \mathbf{0}$. With a minor abuse of notation, we also write $\boldsymbol{\xi}(x, t) = \boldsymbol{\xi}(x, \mathbf{w}(x, t))$. The k-th component of $\boldsymbol{\xi}$ is simply $\mathbf{l}_k(x, \mathbf{w}_1)\mathbf{w}$, an expression commonly used in the formulation of boundary conditions. We have

$$S(x, \mathbf{w}_1)^{-1}\mathbf{w}_t + D(x, \mathbf{w}_1)S(x, \mathbf{w}_1)^{-1}\mathbf{w}_x = S(x, \mathbf{w}_1)^{-1}\mathbf{g}(x, \mathbf{w}), \tag{4.30}$$

or, for each $i \in \mathcal{I}$,

$$S^i(x, \mathbf{w}_1^i)^{-1}\mathbf{w}_t^i + \widehat{D}^i(x, \mathbf{w}_1^i)S^i(x, \mathbf{w}_1^i)^{-1}\mathbf{w}_x^i = S^i(x, \mathbf{w}_1^i)^{-1}\mathbf{g}^i(x, \mathbf{w}). \tag{4.31}$$

Corresponding to the block structure, we write $\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \boldsymbol{\xi}^2, \cdots, \boldsymbol{\xi}^n)$ with $\boldsymbol{\xi}^i = S^i(x, \mathbf{w}_1^i)^{-1}\mathbf{w}^i$. Taking into account (4.27), we define the modes $\boldsymbol{\xi}_\pm = (\boldsymbol{\xi}_\pm^1, \cdots, \boldsymbol{\xi}_\pm^n)$, where

$$\begin{aligned} \boldsymbol{\xi}_+^i &:= \frac{1}{2}[Q^i(x, \mathbf{w}_1^i)^{\mathrm{T}}\mathbf{w}_1^i + M^i(x, \mathbf{w}_1^i)\mathbf{w}_2^i)], \\ \boldsymbol{\xi}_-^i &:= \frac{1}{2}[Q^i(x, \mathbf{w}_1^i)^{\mathrm{T}}\mathbf{w}_1^i - M^i(x, \mathbf{w}_1^i)\mathbf{w}_2^i)] \end{aligned} \tag{4.32}$$

with $M^i(x, \mathbf{w}_1^i) = D^i(x, \mathbf{w}_1^i)^{-1}Q^i(x, \mathbf{w}_1^i)^{\mathrm{T}}$. These modes correspond respectively to the positive and the negative eigenvalues of A^i.

To apply the results of [16], one needs to show that boundary conditions at 0 and L can respectively be rewritten in the form

$$\boldsymbol{\xi}_+(0,t) = \mathbf{G}_0(t, \boldsymbol{\xi}_-(0,t)) + \mathbf{H}_0(t) \quad \text{and} \quad \boldsymbol{\xi}_-(L,t) = \mathbf{G}_L(t, \boldsymbol{\xi}_+(L,t)) + \mathbf{H}_L(t), \tag{4.33}$$

where $\mathbf{G}_0(t,\mathbf{0}) = \mathbf{0}$ and $\mathbf{G}_L(t,\mathbf{0}) = \mathbf{0}$. These conditions determine the "incoming modes" in terms of the "outgoing modes" as well as boundary data.

We consider the situation at 0, that at L being entirely analogous. The endpoint of the i-th string at 0 either is at a simple node $\mathbf{N}^j$ with $j \in \mathcal{J}^S$ and $i = i_j$ or meets a multiple node $\mathbf{N}^j$ with $j \in \mathcal{J}^M$ and $i \in \mathcal{J}^j$.

In the former case, we have the boundary condition

$$\mathbf{w}_2^i(0,t) = \mathbf{v}^j(t) = \mathbf{u}_t^j(t) + \mathbf{w}_2^{0,i}(0) + t\,\rho_i^{-1}[\mathbf{G}^i(\mathbf{R}_x^{e,i} + \mathbf{w}_1^{0,i})]_x|_{x=0}. \tag{4.34}$$

At $x = 0$, we get from (4.32) that

$$\begin{aligned} \boldsymbol{\xi}_+^i(0,t) + \boldsymbol{\xi}_-^i(0,t) &= Q^i(0, \mathbf{w}_1^i(0,t))^{\mathrm{T}} \mathbf{w}_1^i(0,t), \\ \boldsymbol{\xi}_+^i(0,t) - \boldsymbol{\xi}_-^i(0,t) &= M^i(0, \mathbf{w}_1^i(0,t)) \mathbf{w}_2^i(0,t). \end{aligned} \tag{4.35}$$

Since

$$D_{\mathbf{w}_1}[Q^i(0,\mathbf{w}_1)^{\mathrm{T}} \mathbf{w}_1]|_{\mathbf{w}_1 = \mathbf{0}} = Q(0,\mathbf{0})^{\mathrm{T}},$$

it follows from the first of the previous identities and the implicit function theorem that, near $x = 0$, $\mathbf{w}_1^i = \boldsymbol{\phi}^i(\boldsymbol{\xi}_+^i - \boldsymbol{\xi}_-^i)$. Substituting this back into the second identity, we get at $x = 0$,

$$\boldsymbol{\Phi}^i(\boldsymbol{\xi}_+^i, \boldsymbol{\xi}_-^i, \mathbf{w}_2^i) := \boldsymbol{\xi}_+^i - \boldsymbol{\xi}_-^i - M^i(0, \boldsymbol{\phi}(\boldsymbol{\xi}_+ + \boldsymbol{\xi}_-))\mathbf{w}_2 = \mathbf{0}.$$

Since

$$\boldsymbol{\Phi}^i(\mathbf{0},\mathbf{0},\mathbf{0}) = \mathbf{0}, \quad D_{\boldsymbol{\xi}_+^i} \boldsymbol{\Phi}^i(\mathbf{0},\mathbf{0},\mathbf{0}) = I,$$

a second application of the implicit function theorem gives

$$\boldsymbol{\xi}_+^i = \mathbf{G}_0^i(\boldsymbol{\xi}_-^i, \mathbf{w}_2^i)$$

with $\mathbf{G}_0^i$ a differentiable function defined near $\mathbf{0} \times \mathbf{0}$ and vanishing at that point. Hence, in this situation, we get the boundary condition

$$\boldsymbol{\xi}_+^i(0,t) = \mathbf{G}_0^i(\boldsymbol{\xi}_-^i(0,t), \widehat{\mathbf{u}}^j(t)), \tag{4.36}$$

which can be rewritten in the form (4.33) by introducing $\mathbf{H}_0^i(t) = \mathbf{G}_0^i(0, \widehat{\mathbf{u}}^j(t))$ and then replacing $\mathbf{G}_0^i(\boldsymbol{\xi}_-^i(0,t), \widehat{\mathbf{u}}^j(t))$ by $\mathbf{G}_0^i(\boldsymbol{\xi}_-^i(0,t), \widehat{\mathbf{u}}^j(t)) - \mathbf{H}_0^i(t)$.

In the alternative case where the endpoint of the i-th string at $x = 0$ meets a multiple node $\mathbf{N}^j$, one needs to rewrite the multiple node conditions (4.19) appropriately. We proceed with the following 3 steps, making repeated use of the implicit function theorem.

Step 1 For each $i \in \mathcal{J}^j$, we show that the mapping $(\mathbf{w}_1^i, \mathbf{w}_2^i) \mapsto (\boldsymbol{\xi}_+^i, \boldsymbol{\xi}_-^i)$ given by (4.32) with $y = 0$ can be inverted near $(\mathbf{0},\mathbf{0})$ to give

$$\mathbf{w}_1^i = \boldsymbol{\phi}^i(\boldsymbol{\xi}_+^i, \boldsymbol{\xi}_-^i), \quad \mathbf{w}_2^i = \boldsymbol{\psi}^i(\boldsymbol{\xi}_+^i, \boldsymbol{\xi}_-^i). \tag{4.37}$$

Step 2 We then show that $\mathbf{w}_2^i = \boldsymbol{\psi}^i(\boldsymbol{\xi}_+^i, \boldsymbol{\xi}_-^i)$ can be solved for

$$\boldsymbol{\xi}_+^i = \boldsymbol{\xi}_+^i(\mathbf{w}_2^i, \boldsymbol{\xi}_-^i) = \boldsymbol{\xi}_+^i(\mathbf{w}_2^{i_1}, \boldsymbol{\xi}_-^i), \tag{4.38}$$

where i_j is any fixed index in $\mathcal{J}^j$, and that, once $\mathbf{w}_2^i$ has been suitably determined, all the $\boldsymbol{\xi}_+^i$ at $x = 0$ have the form required for boundary conditions at 0.

Step 3 We determine $\mathbf{w}_2^i$ by first setting

$$\mathbf{w}_1^i = \mathbf{w}_1^i(\mathbf{w}_2^{i_1}, \boldsymbol{\xi}_-^i) = \boldsymbol{\phi}^i(\boldsymbol{\xi}_+^i(\mathbf{w}_2^i, \boldsymbol{\xi}_-^i), \boldsymbol{\xi}_-^i), \tag{4.39}$$

and then substituting these into the remaining node condition to get the following condition, which we show can be used to determine $\mathbf{w}_2^{i_1}$, thus complete the argument

$$\sum_{i \in \mathcal{I}^j} \mathbf{G}^i(\mathbf{R}_x^{e,i}(0) + \mathbf{w}_1^i(\mathbf{w}_2^{i_1}(0,t), \boldsymbol{\xi}_-^i(0,t))) = \mathbf{0}. \tag{4.40}$$

At this point, it is clear that $\mathbf{w}_2^{i_1}(0,t)$ is determined as a function of all the incoming modes $\boldsymbol{\xi}_-^i(0,t)$ at the multiple node so that, substituting back into $\boldsymbol{\xi}_+^i(\mathbf{w}_2^{i_1}, \boldsymbol{\xi}_-^i)$, we get that

$$\boldsymbol{\xi}_+^i(0,t) = \mathbf{G}_0^i(\{\boldsymbol{\xi}_-^l(0,t)\}_{l \in \mathcal{I}^j}) \quad \text{for } i \in \mathcal{I}^j. \tag{4.41}$$

To accomplish Step 1, we note that the Jacobian of the mapping $(\mathbf{w}_1^i, \mathbf{w}_2^i) \mapsto (\boldsymbol{\xi}_+^i, \boldsymbol{\xi}_-^i)$ at $(\mathbf{0}, \mathbf{0})$ is

$$\frac{1}{2}\begin{pmatrix} {Q_0^i}^{\mathrm{T}} & M_0^i \\ {Q_0^i}^{\mathrm{T}} & -M_0^i \end{pmatrix},$$

where the subscript "0" indicates that the matrices are evaluated at $(\mathbf{0}, \mathbf{0})$. This matrix is invertible. Recalling that $M^i = {D^i}^{-1} Q^i$, one easily verifies that its inverse is

$$\begin{pmatrix} Q_0^i & Q_0^i \\ Q_0^i D_0^i & Q_0^i D_0^i \end{pmatrix}.$$

The inverse mapping theorem implies that one can invert the mapping to get (4.37) with

$$D_{\boldsymbol{\xi}_+^i} \boldsymbol{\phi}^i = D_{\boldsymbol{\xi}_-^i} \boldsymbol{\phi}^i = Q_0^i \quad \text{and} \quad D_{\boldsymbol{\xi}_+^i} \boldsymbol{\psi}^i = -D_{\boldsymbol{\xi}_-^i} \boldsymbol{\psi}^i = Q_0^i D_0^i \tag{4.42}$$

at $(\mathbf{0}, \mathbf{0})$.

Moving to Step 2, we consider the relation

$$\mathbf{F}(\boldsymbol{\xi}_+^i, \boldsymbol{\xi}_-^i, \mathbf{w}_2^i) := \boldsymbol{\psi}^i(\boldsymbol{\xi}_+^i, \boldsymbol{\xi}_-^i) - \mathbf{w}_2^i = \mathbf{0},$$

and note that

$$D_{\boldsymbol{\xi}_+^i} \mathbf{F} = -D_{\boldsymbol{\xi}_-^i} \mathbf{F} = Q_0^i D_0^i = {M_0^i}^{-1} \quad \text{and} \quad D_{\mathbf{w}_2^i} \mathbf{F} = I$$

at $(\mathbf{0}, \mathbf{0}, \mathbf{0})$. By the implicit function theorem, one can therefore solve for ξ_+^i in the form (4.38) with

$$D_{\boldsymbol{\xi}_-^i} \boldsymbol{\xi}_+^i = I \quad \text{and} \quad D_{\mathbf{w}_2^{i_1}} \boldsymbol{\xi}_+^i = M_0^i \tag{4.43}$$

at $(\mathbf{0})$.

Finally, we implement Step 3. We need to solve (4.38) locally for $\mathbf{w}_2^{i_1}$. Let

$$\mathbf{H}(\mathbf{w}_2^{i_1}, \{\boldsymbol{\xi}_-^i\}_{i \in \mathcal{I}^j}) := \sum_{i \in \mathcal{I}^j} \mathbf{G}^i(\mathbf{R}_x^{e,i}(0) + \mathbf{w}_1^i(\mathbf{w}_2^{i_1}, \boldsymbol{\xi}_-^i)).$$

We know that $\mathbf{H}(\mathbf{0},\{\mathbf{0}\}_{i\in\mathcal{I}^j})=\mathbf{0}$. So, to solve for $\mathbf{w}_2^{i_1}$ as a function of $\{\boldsymbol{\xi}_-^i\}_{i\in\mathcal{I}^j}$, we need to prove that $D_{\mathbf{w}_2^{i_1}}\mathbf{H}(\mathbf{0},\{\mathbf{0}\}_{i\in\mathcal{I}^j})$ is invertible. Evaluating the derivative, we get

$$\begin{aligned}D_{\mathbf{w}_2^{i_1}}\mathbf{H}(\mathbf{0},\{\mathbf{0}\}_{i\in\mathcal{I}^j}) &= \sum_{i\in\mathcal{I}^j} D_{\mathbf{v}}\mathbf{G}^i(\mathbf{R}_x^{e,i}(0)t)D_{\boldsymbol{\xi}_+^i}\boldsymbol{\phi}^i(\mathbf{0},\mathbf{0})D_{\mathbf{w}_2^{i_1}}\xi_+^i(\mathbf{0},\mathbf{0})\\ &= \sum_{i\in\mathcal{I}^j} P_0^iQ_0^iM_0^i = \sum_{i\in\mathcal{I}^j} Q_0^i{D_0^i}^2{Q_0^i}^{\mathrm{T}}Q_0^iM_0^i = \sum_{i\in\mathcal{I}^j} Q_0^iD_0^i{Q_0^i}^{\mathrm{T}}, \end{aligned} \tag{4.44}$$

which is positive definite and therefore invertible.

This completes the proof of all the details needed to apply the existence theorems presented in [16] in our context, and thus the proof of Theorem 4.1 is completed.

In view of the discussion prior to the statement and proof of Theorem 4.1, the following existence theorem for the original second order system (4.5) is also valid.

Theorem 4.2 *Consider a tree network as described in the beginning of this section. Let* $\mathbf{R}^e$ *be a given stretched equilibrium. For a specified value of* $T>0$*, there exist constants* c_0 *and* c_T *such that for initial data*

$$\mathbf{R}^{0,i}(x)\in C^2([0,L_i];\mathbb{R}^3) \quad \textit{and} \quad \mathbf{R}^{1,i}(x)\in C^1([0,L_i];\mathbb{R}^3) \quad \textit{for } i\in\mathcal{I}$$

which respect the compatibility conditions (4.2) *and boundary data*

$$\mathbf{u}^j(t)\in C_0^2([0,T];\mathbb{R}^3) \quad \textit{for } j\in\mathcal{J}^S$$

satisfying

$$\max\{\|\mathbf{R}^{0,i}-\mathbf{R}^{e,i}\|_2, \|\mathbf{R}^{1,i}\|_1, \|\mathbf{u}^j\|_2\}_{i\in\mathcal{I},j\in\mathcal{J}^S} < c_0,$$

there exists a unique twice continuously differentiable solution

$$\mathbf{R}\in\prod_{i\in\mathcal{I}} C^2([0,L_i]\times[0,T])$$

to (4.5) *depending continuously on the data in the sense that for each* $i\in\mathcal{I}$,

$$\|\mathbf{R}^i\|_2 \le c_T\max\{\|\mathbf{R}^{0,i}-\mathbf{R}^{e,i}\|_2, \|\mathbf{R}^{1,i}\|_1, \|\mathbf{u}^j\|_2\}_{i\in\mathcal{I},j\in\mathcal{J}^S}.$$

Remark 4.1 Because of this theorem, it is natural to introduce the "state space"

$$\mathbf{H}_{\mathrm{s}} = \Big\{\{(\mathbf{R}^{0i},\mathbf{R}^{1i})\}_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} C^2([0,L_i];\mathbb{R}^3)\times C^1([0,L_i];\mathbb{R}^3) : (4.2) \text{ holds}\Big\}. \tag{4.45}$$

5 Controllability on a Star-Graph

We consider a star-like network with n strings stretched from zero — the common nodal point — to the ends at $x=L_i$. We exert controls at the ends for the strings labeled $i=2,\cdots,n$, i.e., we keep fixed Dirichlet conditions for the first string and control the displacements of all of the other connected strings. We have the following regularity and C^2-compatibility conditions:

$$\begin{cases}\mathbf{U}^j(t)\in C^2([0,T];\mathbb{R}^3) & \text{for } j=2,\cdots,n,\\ \mathbf{R}^{0,i}(x)\in C^2([0,L_i];\mathbb{R}^3) & \text{for } i=1,\cdots,n,\\ \mathbf{R}^{1,i}(x)\in C^1([0,L_i];\mathbb{R}^3) & \text{for } i=1,\cdots,n,\end{cases} \tag{5.1}$$

$$\begin{cases} \mathbf{R}^{0,i}(0) \text{ coincide} & \text{for each } i = 1, \cdots, n, \\ \mathbf{R}^{1,i}(0) \text{ coincide} & \text{for each } i = 1, \cdots, n, \\ \sum\limits_{i=1,\cdots,n} \mathbf{G}^i(\mathbf{R}_x^{0,i}(0) = \mathbf{0}, \\ \sum\limits_{i=1,\cdots,n} \mathbf{G}_{\mathbf{v}}^i(\mathbf{R}_x^{0,i}(0)\mathbf{R}_x^{1,i}(0) = \mathbf{0}, \end{cases} \tag{5.2}$$

as well as

$$\begin{cases} \mathbf{U}^j(0) = \mathbf{R}^{j,0}(L_j), & j = 2, \cdots, n, \\ \mathbf{U}_t^j(0) = \mathbf{R}^{i_j,1}(L_j), & j = 2, \cdots, n, \\ \mathbf{U}_{tt}^j(0) = (\rho_j)^{-1}[\mathbf{G}^j(\mathbf{R}_x^{j,0})]_x|_{x=0}, & j = 2, \cdots, n. \end{cases} \tag{5.3}$$

Controllability of the star-like network of nonlinear elastic strings will be considered in the energy space $H_{\rm s}$ which for the star configuration becomes

$$\mathbf{H}_{\rm s} = \Big\{ \{(\mathbf{R}^{0i}, \mathbf{R}^{1i})\}_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} C^2([0,L_i];\mathbb{R}^3) \times C^1([0,L_i];\mathbb{R}^3) : (5.2) \text{ holds} \Big\}. \tag{5.4}$$

In this section, we prove the following theorem concerning the exact controllability in finite time of the motion of a star-like string network from the neighborhood of one equilibrium at time 0 to a neighborhood of another equilibrium at time T. In order to make this more precise, we recall that the eigenvalues of the matrices A^i given by (4.20) are $\pm\mu_j^i(x, \mathbf{w}_1^i),\ j = 1, 2, 3$ given by (4.22). Indeed,

$$\begin{cases} \mu_1^i(x, \mathbf{w}_1^i) = \sqrt{\rho_i^{-1} V_{ss}^i(|\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1^i|)}\,, \\ \mu_2^i(x, \mathbf{w}_1^i) = \mu_3^i(x, \mathbf{w}_1^i) = \sqrt{\dfrac{\rho_i^{-1} V_s^i(|\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1|)}{|\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1|}}\,. \end{cases} \tag{5.5}$$

In terms of the original variables, we have

$$\begin{cases} \mu_1^i(x, \mathbf{r}_x^i) = \sqrt{\rho_i^{-1} V_{ss}^i(|\mathbf{R}_x^{e,i}(x) + \mathbf{r}_x^i|)}\,, \\ \mu_2^i(x, \mathbf{r}_x^i) = \mu_3^i(x, \mathbf{r}_x^i) = \sqrt{\dfrac{\rho_i^{-1} V_s^i(|\mathbf{R}_x^{e,i}(x) + \mathbf{r}_x^i|)}{|\mathbf{R}_x^{e,i}(x) + \mathbf{r}_x^i|}}\,. \end{cases} \tag{5.6}$$

In particular, under the assumed condition of stretched equilibria, the roots (5.5) are strictly positive at $\mathbf{w}_1^i = \mathbf{0},\ i \in \mathcal{I}$. We may then define a lower bound on the traveling time for signals entering the boundary points up to their arrival at the same boundary point.

$$T > 2\,T_0 := \max_{j=1,2,3} \max_{x\in[0,L_1]} \frac{2L_1}{\mu_j^1(x,0)} + \max_{i=2,\cdots,n} \max_{j=1,2,3} \max_{x\in[0,L_i]} \frac{2L_i}{\mu_j^i(x,0)}. \tag{5.7}$$

Under our assumptions, we can guarantee the existence of an $\epsilon_0 > 0$ such that

$$T > 2\,T_1 := \max_{j=1,2,3} \max_{\substack{x\in[0,L_1] \\ \|\mathbf{r}_x^1\|<\epsilon_0}} \frac{2L_1}{\mu_j^1(x,\mathbf{r}_x^1)} + \max_{i=2,\cdots,n} \max_{j=1,2,3} \max_{\substack{x\in[0,L_i] \\ \|r_x^i\|<\epsilon_0}} \frac{2L_i}{\mu_j^i(x,\mathbf{r}_x^i)}. \tag{5.8}$$

We also define the maximal traveling time for the strings labeled $i = 2, \cdots, n$ as follows:

$$T_2 := \max_{i=2,\cdots,n} \max_{j=1,2,3} \max_{\substack{x\in[0,L_i] \\ \|r_x^i\|<\epsilon_0}} \frac{L_i}{\mu_j^i(x,\mathbf{r}_x^i)}. \tag{5.9}$$

Theorem 5.1 *Let* $\mathbf{R}^{e,i}$ *be a stretched equilibrium solution to the system* (2.7)–(2.10) *and* (2.12) *according to* (3.1). *Let* $T > 2T_0$. *Then there exist neighborhoods* $\mathcal{U}_0$ *and* $\mathcal{U}_1$ *of* $(\mathbf{R}_0^e, \mathbf{0})$ *in* $\mathbf{H}_s$, *such that given*

$$\{(\mathbf{R}^{0i}, \mathbf{R}^{1i})\} \in \mathcal{U}_0 \quad \text{and} \quad \{(\widehat{\mathbf{R}}^{0i}, \widehat{\mathbf{R}}^{1i})\} \in \mathcal{U}_1,$$

one can find controls $\mathbf{u}^j(t) \in C_0^2([0,T];\mathbb{R}^3)$, *such that the corresponding solution* $\mathbf{R}$ *to* (4.5) *satisfies*

$$\mathbf{R}^i(\cdot, T) = \widehat{\mathbf{R}}^{0i} \quad \text{and} \quad \mathbf{R}_t^i(\cdot, T) = \widehat{\mathbf{R}}^{1i}.$$

Having established this local exact controllability result, we can proceed to the following global-local exact controllability result.

Theorem 5.2 *Let the assumptions of Theorem* 5.1 *hold. Given two stretched equilibrium solutions* $\mathbf{R}_0^e(x)$ *and* $\mathbf{R}_1^e(x)$ *to the system* (2.7)–(2.10) *and* (2.12) *according to* (3.1), *there are neighborhoods* $\mathcal{U}_0$, $\mathcal{U}_1$ *of* $(\mathbf{R}_0^e, \mathbf{0})$, $(\mathbf{R}_1^e, \mathbf{0})$, *respectively, in the state space* $\mathbf{H}_s$ *such that, for* T *sufficiently large, each solution to* (4.5) *in the sense of Theorem* 5.1 *with initial conditions in* $\mathcal{U}_0$ *can be steered to any state in* $\mathcal{U}_1$ *in the given time via admissible controls.*

We shall achieve this goal in a couple of steps. We first prove the local-exact controllability result Theorem 5.1. In the second step, we use the connectedness of $\mathcal{S}_+$, and hence, of the set of equilibrium points. We can find a compact λ-parametrized path $\mathbf{R}_\lambda^e$ of equilibrium solutions connecting $\mathbf{R}_0^e$ and $\mathbf{R}_1^e$ and a corresponding path $(\mathbf{R}_\lambda^e, \mathbf{0})$ in state space. Then by using a monodromy argument, we can cover this path by sufficiently small neighborhoods, such that initial and final states in these local neighborhoods can be connected via admissible controls. In this way, we can start close to the equilibrium state $(\mathbf{R}_0^e, \mathbf{0})$ and terminate at a state close to $(\mathbf{R}_1^e, \mathbf{0})$ via finitely many intermediate states located in the neighborhoods connecting $(\mathbf{R}_0^e, \mathbf{0})$ and $(\mathbf{R}_1^e, \mathbf{0})$. Thus we achieve a global-local exact controllability result for the fully nonlinear network of vibrating strings.

Proof of Theorem 5.1 We follow the spirit of the proof of Theorem 3.1 in [4]. The principal idea in boundary exact controllability of 1-hyperbolic systems, which has been used already by Littman [12], is to solve forward problems with the given initial data, backward problems with the given final data and a Cauchy from "the left and the right". In particular, for the latter, it is convenient to interchange the spatial and time variables x and t, and then solve Cauchy problems from the left and the right, once the given boundary conditions have been extended to Cauchy-data there.

As we assume stretched equilibria, $\mathbf{G}_{\mathbf{v}}^i(\mathbf{R}_x^{e,i}(x))$ is uniformly positive definite and so is $\mathbf{G}_{\mathbf{v}}^i(\mathbf{R}_x^{e,i}(x) + \mathbf{r}_x^i(x,t))$ uniformly with respect to (x,t) for ϵ_1 sufficiently small. The state equation for $\mathbf{r}^i$ in quasilinear form can be written as

$$\rho_i \mathbf{r}_{tt}^i(x,t) = \mathbf{G}_{\mathbf{v}}(\mathbf{R}_x^{e,i} + \mathbf{r}_x^i(x,t))\mathbf{r}_{xx}^i(x,t) - \rho_i g \mathbf{e}.$$

Therefore,

$$\mathbf{r}_{xx}^i(x,t) = \rho_i[\mathbf{G}_{\mathbf{v}}(\mathbf{R}_x^{e,i}(x) + \mathbf{r}_x^i(x,t))]^{-1}\mathbf{r}_{tt}^i(x,t) + \rho_i g[\mathbf{G}_{\mathbf{v}}(\mathbf{R}_x^{e,i}(x) + \mathbf{r}_x^i(x,t))]^{-1}\mathbf{e}.$$

In terms of the variables $\mathbf{w}^i = (\mathbf{w}_1^i, \mathbf{w}_2^i) = (\mathbf{r}_x^i, \mathbf{r}_t^i)$ this reads as

$$\mathbf{w}_{1x}^i(x,t) = \rho_i[\mathbf{G}_{\mathbf{v}}(\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1^i(x,t))]^{-1}\mathbf{w}_{2t}^i(x,t) + \rho_i g[\mathbf{G}_{\mathbf{v}}(\mathbf{R}_x^{e,i}(x) + \mathbf{w}_1^i(x,t))]^{-1}\mathbf{e}.$$

Thus, if one has Cauchy-data $(\mathbf{r}^i(x,t), \mathbf{r}^i_x(x,t))$ at a boundary point $x = L_i$ or in terms of $\mathbf{w}^i$ Cauchy-data $(\mathbf{w}^i_1, \mathbf{w}^i_2)(L_i, t)$, and "boundary data" $\mathbf{r}^i(x,0), \mathbf{r}^i(x,T),\ x \in [0, L_i]$, one can solve the wave equation "from $x = L_i$ to $x = 0$". A similar statement holds for the first order system involving $\mathbf{w}^i$.

We describe the idea of the proof. There are five steps.

Step 1 In the first step, we proceed forward from $t = 0$ to $t = T_1$. We solve the initial boundary value problem with controls at $x = L_i$.

For each string $i \in \mathcal{I}$, we define the set $R^i_{\rm I} := \{(x,t) \in [0, L_i] \times [0, T_1]\}$, and for the network, we set $R_{\rm I} := \bigcup_{i=1}^{n} R^i_{\rm I}$. The first string is fixed at $x = L_1$. We may, for the sake of convenience, assume that we impose a homogenous Dirichlet condition there. This, however, specifies the corresponding compatibility conditions. We impose nonhomogeneous Dirichlet conditions at $x = L_i, i = 2, \cdots, n$, i.e., $\mathbf{r}^i(L_i, t) = \mathbf{f}^i(t), i = 2, \cdots, n$, where $\mathbf{f}^i(\cdot)$ are small in $C^2(0, T_1; \mathbb{R}^3)$. We also have sufficiently small initial data $(\mathbf{r}^i(x,0), \mathbf{r}^i_t(x,0)) = (\boldsymbol{\phi}^i(x), \boldsymbol{\psi}^i(x))$ for all strings. We apply Theorem 4.2 and obtain a unique solution on $R_{\rm I}$.

We can now take traces of $(\mathbf{r}^1(L_1,t), \mathbf{r}^1_x(L_1,t)) = (\mathbf{a}^1(t), \mathbf{a}^2(t))$ (here $\mathbf{a}^1(t) = 0$) at the boundary of the first string along $\{L_1\} \times [0, T_1]$ and for $(\mathbf{r}^i(0,t), \mathbf{r}^i_x(0,t)) = (\mathbf{b}^i_1(t), \mathbf{b}^i_2(t))$ for all strings at $\{0\} \times [0, T_1]$. It is clear that $(\mathbf{b}^i_1(t), \mathbf{b}^i_2(t))$ satisfy the nodal conditions at the common node. Moreover, all data are small in the appropriate spaces.

Step 2 We perform the same procedure, but now reversing the time and progress from the final time T to $T - T_1$. More precisely, we introduce the individual domains $R^i_{\rm II} := \{(x,t) \in [0, L_i] \times [T - T_1, T]\},\ i = 1, \cdots, n$ and the global one $R_{\rm II} = \bigcup_{i=1}^{n} R^i_{\rm II}$. By the same argument, a unique semi-global small solution $(\mathbf{r}^i_{\rm II}, \mathbf{r}^i_{{\rm II},x})$ to the network problem exists, and we can take traces $(\mathbf{r}^1_{\rm II}(L_1,t), \mathbf{r}^1_{{\rm II},x}(L_1,t)) = (\overline{\mathbf{a}}^1(t), \overline{\mathbf{a}}^2(t))$ at $\{L_1\} \times [T - T_1, T]$ for the first string (again $\overline{\mathbf{a}}^1(t) = 0$) and $(\mathbf{r}^i_{\rm II}(0,t), \mathbf{r}^i_{{\rm II},x}(0,t)) = (\overline{\mathbf{b}}^i_1(t), \overline{\mathbf{b}}^i_2(t))$ at $\{L_i\} \times [T - T_1, T]$ for the strings labeled $i = 1, \cdots, n$.

In order to prepare Step 3, we extend the Cauchy-data at $\{\{L_1\} \times [0, T_1]\} \cup \{\{L_1\} \times [T - T_1, T]\}$ in the C^2-sense to $\{L_1\} \times [0, T]$ as $(\widetilde{\mathbf{a}}^1(t), \widetilde{\mathbf{a}}^2(t))$. After that, we can use these Cauchy-data along $\{L_1\} \times [0, T]$ as "initial conditions".

Step 3 We change the order of x and t as explained at the beginning of the proof. The Cauchy-data just constructed can be taken as "initial conditions" for the first string "starting" at $x = L_1$ with "boundary conditions" at $t = 0$ and $t = T$ taken from the original initial data. Applying Theorem 4.2 to that situation, we can evaluate the solution $(\mathbf{r}^1(x,t), \mathbf{r}^1_x(x,t))$ at $\{0\} \times [0, T]$. On the set $\{(x,t) \in [0, L_1],\ 0 \le t \le T_2 + \frac{(T_1 - T_2)x}{L_1}\}$, this solution $\mathbf{r}^1$ is identical to $\mathbf{r}^1_{\rm I}$. Therefore, at $t = 0$, we have

$$\mathbf{r}^1(x,0) = \boldsymbol{\phi}^1(x), \quad \mathbf{r}^1_t(x,0) = \boldsymbol{\psi}^1(x), \quad x \in [0, L_1].$$

At $x = 0$, we have

$$\mathbf{r}^1(0,t) = \mathbf{b}^1_1(t), \quad \mathbf{r}^1_x(0,t) = \mathbf{b}^1_2(t), \quad t \in [0, T_2].$$

The analogous uniqueness argument applies for the backward solution of Step 2, such that the final conditions are

$$\mathbf{r}^1(x,T) = \boldsymbol{\Phi}^1(x), \quad \mathbf{r}^1_t(x,T) = \boldsymbol{\Psi}^1(x), \quad x \in [0, L_1],$$

while the evaluation at $x = 0$ provides the Cauchy-data

$$\mathbf{r}^1(0,t) = \overline{\mathbf{b}}_1^1(t), \quad \mathbf{r}_x^1(0,t) = \overline{\mathbf{b}}_2^1(t), \quad t \in [T - T_2, T].$$

Step 4 We now extend the Cauchy-data $(\mathbf{b}_1^1(t), \mathbf{b}_2^1(t)),\ t \in [0, T_2]$, together with $(\overline{\mathbf{b}}_2^1(t), \overline{\mathbf{b}}_2^1(t)), t \in [T - T_2, T]$, to Cauchy-data $(\widetilde{\mathbf{b}}_1^1(t), \widetilde{\mathbf{b}}_2^1(t)),\ t \in [0, T]$, such that corresponding solutions satisfy the nodal conditions.

Step 5 We now have Cauchy-data on $\{0\} \times [0, T]$, such that the nodal conditions are satisfied. Therefore, we can use these as compatible initial conditions for the strings labeled $i = 2, \cdots, n$ after interchanging x and t. Thus, on the domains $R_{\rm IV}^i := \{(x,t) \in [0, L_i] \times [0, T]\}$, we solve the initial boundary value problems with Cauchy-data

$$\mathbf{r}^i(0,t) = \widetilde{\mathbf{b}}_1^i, \quad \mathbf{r}_x^i(0,t) = \widetilde{\mathbf{b}}_2^i, \quad t \in [0, T],$$

and boundary conditions

$$\mathbf{r}^i(x,0) = \boldsymbol{\phi}^i(x), \quad \mathbf{r}^i(x,T) = \boldsymbol{\Phi}^i(x), \quad x \in [0, L_i].$$

By construction, the solutions are small in the sense described above. A similar uniqueness argument applies to the region $\{(x,t) \mid x \in [0, L_i], 0 \le t \le T_2(1 - \frac{x}{L_i})\}$ to the effect that

$$\mathbf{r}^i(x,0) = \boldsymbol{\phi}^i(x), \quad \mathbf{r}_t^i(x,0) = \boldsymbol{\psi}^i(x), \quad x \in [0, L_i].$$

The analogous argument on the "upper" domain leads to

$$\mathbf{r}^i(x,T) = \boldsymbol{\Phi}^i(x), \quad \mathbf{r}_t^i(x,T) = \boldsymbol{\Psi}^i(x), \quad x \in [0, L_i].$$

This gives the solution to the problem stated.

Proof of Theorem 5.2 We now prove Theorem 5.2 for the star-graph. As stated in the theorem, we assume two different equilibria $\mathbf{R}_0^e, \mathbf{R}_1^e$ described above. These equilibria come from different fixed boundary conditions at the simple nodes. According to Section 4, there is continuous path $\mathbf{R}_\lambda^e$ with finite length connecting the two equilibria and hence a path $(\mathbf{R}_\lambda^e, \mathbf{0})$ joining the states $(\mathbf{R}_0^e, \mathbf{0})$ and $(\mathbf{R}_1^e, \mathbf{0})$. Thus, given the smallness bounds needed in order to apply Theorem 5.1, we can cover the path with finitely many, say m, such neighborhoods, the centers of which are located at an equilibrium $(\mathbf{R}_{\frac{k}{m}}^e, \mathbf{0})$ on the path. The controllability times $T^k, k = 1, \cdots, m$ are individually calculated according to the data in these neighborhoods according to Theorem 5.1. Starting in the neighborhood of $(\mathbf{R}_0^e, \mathbf{0})$, we can reach all states $(\mathbf{R}^i(x, T^0), \mathbf{R}_t^i(x, T^0)),\ i = 1, \cdots, n$ in that neighborhood, which, in turn, has a nonempty intersection with the next neighborhood. Therefore, the state reached in the first step can be steered in time T^1 to any state $(\mathbf{R}^i(x, T^1), \mathbf{R}_t^i(x, T^1)),\ i = 1, \cdots, n$ in the second neighborhood. This argument can be now applied m-times until we arrive in the neighborhood of $(\mathbf{R}_1^e, \mathbf{0})$. The total control time can be estimated below by $\sum\limits_{k=1}^{m} T^k$.

6 Exact Controllability of Tree-Networks

Our controllability Theorems 5.1 and 5.2 for star networks have their exact counterparts for general tree networks.

Theorem 6.1 *Let* $\mathbf{R}^{e,i}$ *be a stretched equilibrium solution to the system* (2.7)–(2.10) *and* (2.12) *according to* (3.1). *Let*

$$T > 2T_0 \quad \text{with } T_0 = \max\Big\{\sum_{l=1}^{p}\frac{L_{i^l}}{\mu_0^{i^l}} : \{i^1, i^2, \cdots, i^p\} \in \mathcal{P}\Big\}, \tag{6.1}$$

where

$$\mu_0^i = \min_{j=1,2,3}\min_{x\in[0,L_1]}\mu_j^i(x,0) \tag{6.2}$$

with μ_j^i *as in* (4.22), *and where* $\mathcal{P}$ *denotes the set of index sequences* $\{i^1, i^2, \cdots, i^p\}$ *corresponding to successions of strings joining* $\mathbf{N}^1$ *to a simple node* $\mathbf{N}^j$.

Then there exist neighborhoods $\mathcal{U}_0$ *and* $\mathcal{U}_1$ *of* $(\mathbf{R}_0^e, \mathbf{0})$ *in* $\mathbf{H}_s$, *such that given*

$$\{(\mathbf{R}^{0i}, \mathbf{R}^{1i})\} \in \mathcal{U}_0 \quad \text{and} \quad \{(\widehat{\mathbf{R}}^{0i}, \widehat{\mathbf{R}}^{1i})\} \in \mathcal{U}_1,$$

one can find controls $\mathbf{u}^j(t) \in C_0^2([0,T];\mathbb{R}^3)$ *such that the corresponding solution* $\mathbf{R}$ *to* (4.5) *satisfies*

$$\mathbf{R}^i(\,\cdot\,,T) = \widehat{\mathbf{R}}^{0i} \quad \text{and} \quad \mathbf{R}_t^i(\,\cdot\,,T) = \widehat{\mathbf{R}}^{1i}.$$

Consequently we can also prove the theorem below.

Theorem 6.2 *Given two stretched equilibrium solutions* $\mathbf{R}_0^e(x)$ *and* $\mathbf{R}_1^e(x)$ *to the system* (2.7)–(2.10) *and* (2.12) *according to* (3.1), *there are neighborhoods* $\mathcal{U}_0$, $\mathcal{U}_1$ *of* $(\mathbf{R}_0^e, \mathbf{0})$, $(\mathbf{R}_1^e, \mathbf{0})$, *respectively, in the state space* $\mathbf{H}_s$, *such that, for* T *sufficiently large, each solution to* (4.5) *in the sense of Theorem* 4.2 *with initial conditions in* $\mathcal{U}_0$ *can be steered to any state in* $\mathcal{U}_1$ *in the given time via admissible controls.*

We make only a few comments concerning the proofs. Concerning the proof of Theorem 6.1, we note that exact controllability results for partial differential equations on tree-like graphs are typically proved by using a peeling technique. See for instance [5]. This technique consists in going from the leaves of the tree to the root. In other words, one starts at the simple nodes and reduces the tree by cutting off the root layer by layer until one reaches the edge with fixed Dirichlet conditions. The argument given in [4], in a sense, starts at the root, and is related to the classical idea, going back to Russell, in which the controls are taken as appropriate traces at the control-boundary point for the solutions satisfying overdetermined initial-, final-, and boundary- as well as multiple nodal conditions. In this way, one works from the given boundary conditions at the root by interchanging x and t, and produces Cauchy-data at the end of this edge which have to be compatible with the multiple-node conditions. This procedure provides Cauchy-data at the next (now multiple) node. Applying the same argument to each branch initiated at this node, we continue until we arrive at the simple nodes, where we take the controls as traces of the corresponding solutions. We, thus, obtain exact local controllability of tree-like networks of nonlinear strings around a stretched equilibrium.

Once Theorem 6.1 has been proved, the proof of Theorem 6.2 is exactly the same as for star configurations. The monodromy argument depends on the pathwise connectivity of the set of stretched equilibria which follows from Theorem 3.2 concerning the equilibria on tree networks.

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