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On the Geometry of the $(1,1)$-Tensor Bundle with Sasaki Type Metric*

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Abstract Curvature properties are studied for the Sasaki metric on the $(1,1)$ tensor bundle of a Riemannian manifold. As an application, examples of almost para-Nordenian and para-Kähler-Nordenian B-metrics are constructed on the $(1,1)$ tensor bundle by looking at the Sasaki metric. Also, with respect to the para-Nordenian B-structure, paraholomorphic conditions for the complete lifts of vector fields are analyzed.

Keywords Tensor bundle, Sasaki metric, Vertical and horizontal lift, B-manifold
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1 Introduction

Let M be an n-dimensional differentiable manifold of class C^∞, $T_1^1(M)$ its tensor bundle of type (1,1). We denote by $\Im_s^r(M)$ the set of all tensor fields of type (r,s) on M. Similarly, we denote by $\Im_s^r(T_1^1(M))$ the corresponding set on $T_1^1(M)$.

Fiber bundles play an important role in just about every aspect of modern geometry and topology. Prime examples of fiber bundles are tensor bundles of different types (p,q) over differentiable manifolds. The tangent bundle $T(M)$ and cotangent bundle $T^*(M)$ are the special cases of a more general tensor bundle $T_q^p(M)$ of type (p,q) over M. The geometry of tangent bundles goes back to the fundamental paper [22] of Sasaki published in 1958. He used a given Riemannian metric g on a differentiable manifold M to construct a metric $\widetilde{g}$ on the tangent bundle $T(M)$ of M. Today this metric is called the Sasaki metric. The Levi-Civita connection $\widetilde{\nabla}$ of the Sasaki metric on $T(M)$ and its Riemann curvature tensor $\widetilde{R}$ are calculated by Kowalski in [9] (see also [8]). Afterwards, Aso [1], Musso and Tricerri [16] investigated interesting relations between the geometric properties of the base manifold (M,g) and its tangent bundle $(T(M),\widetilde{g})$ with the Sasaki metric. The Sasaki metric on the cotangent bundle was studied by several authors, including Mok [14], Salimov and Agca [21]. In [3, 11, 17, 18], the Sasaki metric was studied on the tensor bundles of different types (p,q) over differentiable manifolds. In [13], Mok studied the Sasaki metric on frame bundles and calculated its curvature tensor (for details, see [4]).

The work is organized as follows.

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In Section 2, some introductory materials concerning the tensor bundle $T_1^1(M)$ over the differentiable manifold M are collected, like some particular types of vector fields on the tensor bundle $T_1^1(M)$ and Lie bracket operation of vertical and horizontal vector fields.

In Section 3, the adapted frame which allows the tensor calculus to be efficiently done is inserted in the tensor bundle $T_1^1(M)$. The Sasaki metric on the tensor bundle $T_1^1(M)$ is introduced and the components of the Levi-Civita connection of the Sasaki metric with respect to the adapted frame are calculated.

In Section 4, we calculate the Riemann curvature of the Levi-Civita connection of the Sasaki metric with respect to the adapted frame, and compare the geometries of the manifold M and its tensor bundle $T_1^1(M)$ with the Sasaki metric.

In Section 5, the similar problems in Section 4 are investigated for the metric connection of the Sasaki metric on the tensor bundle $T_1^1(M)$.

Section 6 deals with the para-Nordenian property of the Sasaki metric.

Throughout this paper, we always suppose that all manifolds, functions and tensor fields are differentiable and of class C^∞.

2 Priliminieries

Let M be a differentiable manifold of class C^∞ and finite dimension n. Then the set $T_1^1(M) = \bigcup_{P\in M} T_1^1(P)$ is, by definition, the tensor bundle of type $(1,1)$ over M, where $\bigcup$ denotes the disjoint union of the tensor spaces $T_1^1(P)$ for all $P \in M$. For any point $\widetilde{P}$ of $T_1^1(M)$, the surjective correspondence $\widetilde{P} \to P$ determines the natural projection $\pi : T_1^1(M) \to M$. The projection π defines the natural differentiable manifold structure of $T_1^1(M)$, that is, $T_1^1(M)$ is a C^∞-manifold of dimension $n+n^2$. A local coordinate neighborhood $\{(U; x^j,\ j = 1,\cdots,n)\}$ in M induces on $T_1^1(M)$ a local coordinate neighborhood $\{\pi^{-1}(U); x^j, x^{\overline{j}} = t_j^i,\ j = 1,\cdots,n\}$, $\overline{j} := n + j$ $(\overline{j} = n+1,\cdots,n+n^2)$, where $x^{\overline{j}} = t_j^i$ are the components of the $(1,1)$ tensor field t in each $(1,1)$ tensor space $T_1^1(P)$, $P \in U$ with respect to the natural base.

We denote by $\Im_s^r(M)$ the module over $F(M)$ of all C^∞ tensor fields of type (r,s) on M, where $F(M)$ is the ring of real-valued C^∞ functions on M. If $\alpha \in \Im_1^1(M)$, it is regarded, by contraction, as a function on $T_1^1(M)$, which we denote by $\imath\alpha$. If α has the local expression $\alpha = \alpha_i^j \partial_j \otimes \mathrm{d}x^i$ in a coordinate neighborhood $U(x^j) \subset M$, then $\imath\alpha = \alpha(t)$ has the local expression $\imath\alpha = \alpha_i^j t_j^i$ with respect to the coordinates $(x^j, x^{\overline{j}})$ in $\pi^{-1}(U)$.

Suppose that $A \in \Im_1^1(M)$. Then there is a unique vector field ${}^V\!A \in \Im_0^1(T_1^1(M))$, such that for $\alpha \in \Im_1^1(M)$ (see [12])

$$ {}^V\!A(\imath\alpha) = \alpha(A) \circ \pi = {}^V(\alpha(A)), \tag{2.1} $$

where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in F(M)$. We note that the vertical lift ${}^Vf = f \circ \pi$ of the arbitrary function $f \in F(M)$ is constant along each fibre $\pi^{-1}(P)$. If ${}^V\!A = {}^V\!A^k \partial_k + {}^V\!A^{\overline{k}} \partial_{\overline{k}}$, from (2.1) we have

$$ {}^V\!A^k t_j^i \partial_k \alpha_i^j + {}^V\!A^{\overline{k}} \alpha_h^k = \alpha_h^k A_k^h. $$

Since α_h^k and $\partial_k \alpha_i^j$ can take any preassigned values at each point, we have from the above equation

$$^{V}A^{k}t_{j}^{i}=0,\quad {}^{V}A^{\overline{k}}=A_{k}^{h}.$$

Hence

$$^{V}A^{k}=0$$

at all points of $T_1^1(M)$ except possibly those at which all the components $x^{\overline{j}}=t_j^i$ are zero, that is, at points of the base space. Thus we see that the components $^{V}A^{k}$ are zero at a point such that $x^{\overline{j}}\neq 0$, that is, in $T_1^1(M)\to M$. But the vector field ^{V}A is continuous at every point of $T_1^1(M)$. So, we have $^{V}A^{k}=0$ at all points of $T_1^1(M)$. Consequently, the vertical lift ^{V}A of A to $T_1^1(M)$ has the components

$$^{V}A=\begin{pmatrix} ^{V}A^{j} \\ ^{V}A^{\overline{j}} \end{pmatrix}=\begin{pmatrix} 0 \\ A_j^i \end{pmatrix} \tag{2.2}$$

with respect to the coordinates $(x^j,x^{\overline{j}})$ in $T_1^1(M)$ (see [2]).

Let L_V be the Lie derivation with respect to $V\in\Im_0^1(M)$. We define the complete lift $^{C}V=\overline{L}_V$ of V to $T_1^1(M)$ by

$$^{C}V(\imath\alpha)=\imath(L_V\alpha) \tag{2.3}$$

for $\alpha\in\Im_1^1(M)$ (see [12]). If $^{C}V={}^{C}V^{k}\partial_k+{}^{C}V^{\overline{k}}\partial_{\overline{k}}$, from (2.3), we have

$$^{C}V^{k}t_j^i\partial_k\alpha_i^j+{}^{C}V^{\overline{k}}\alpha_h^k=t_j^i(V^k\partial_k\alpha_i^j-(\partial_kV^j)\alpha_i^k+(\partial_iV^k)\alpha_k^j). \tag{2.4}$$

Discussing in the same way as in the case of the vertical lift, from (2.4), we see that the complete lift ^{C}V has the components

$$^{C}V=\begin{pmatrix} ^{C}V^{j} \\ ^{C}V^{\overline{j}} \end{pmatrix}=\begin{pmatrix} V^{j} \\ t_j^m(\partial_mV^i)-t_m^i(\partial_jV^m) \end{pmatrix} \tag{2.5}$$

with respect to the coordinates $(x^j,x^{\overline{j}})$ in $T_1^1(M)$ (see [2, 15]).

Let ∇ be a symmetric affine connection on M. We define the horizontal lift $^{H}\nabla=\widetilde{\nabla}_V\in\Im_0^1(T_1^1(M))$ of $V\in\Im_0^1(M)$ to $T_1^1(M)$ by

$$^{H}V(\imath\alpha)=\imath(\nabla_V\alpha),\quad \alpha\in\Im_1^1(M)$$

(see [12]). The horizontal lift ^{H}V of $V\in\Im_0^1(M)$ to $T_1^1(M)$ has the components

$$^{H}V=\begin{pmatrix} ^{H}V^{j} \\ ^{H}V^{\overline{j}} \end{pmatrix}=\begin{pmatrix} V^{j} \\ V^s(\Gamma_{sj}^m t_m^i-\Gamma_{sm}^i t_j^m) \end{pmatrix} \tag{2.6}$$

with respect to the coordinates $(x^j,x^{\overline{j}})$ in $T_1^1(M)$, where Γ_{ij}^k are the local components of ∇ on M (see [2, 15, 23]).

Let $\varphi\in\Im_1^1(M)$, which are locally represented by $\varphi=\varphi_j^i\frac{\partial}{\partial x^i}\otimes \mathrm{d}x^j$. The vector fields $\gamma\varphi$ and $\widetilde{\gamma}\varphi$ on $T_1^1(M)$ are defined by

$$\begin{cases}\gamma\varphi=(t_j^m\varphi_m^i)\dfrac{\partial}{\partial x^{\overline{j}}},\\ \widetilde{\gamma}\varphi=(t_m^i\varphi_{j_\mu}^m)\dfrac{\partial}{\partial x^{\overline{j}}}\end{cases}$$

with respect to the coordinates $(x^j, x^{\overline{j}})$ in $T_1^1(M)$. From (2.2) we easily see that the vector fields $\gamma\varphi$ and $\widetilde{\gamma}\varphi$ determine respectively global vector fields on $T_1^1(M)$ (see [2]).

Explicit expressions for the Lie bracket [,] of the tensor bundle $T_1^1(M)$ are given in [2, 12] (for tangent and cotangent bundles, see [5, 25]). The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{cases} [{}^V A, {}^V B] = 0, \\ [{}^H X, {}^V A] = {}^V(\nabla_X A), \\ [{}^H X, {}^H Y] = {}^H[X,Y] + (\widetilde{\gamma} - \gamma)R(X,Y), \end{cases} \tag{2.7}$$

where R denotes the curvature tensor field of the connection ∇, and $\widetilde{\gamma} - \gamma : \varphi \to \Im_0^1(T_1^1(M))$ is the operator defined by

$$(\widetilde{\gamma} - \gamma)\varphi = \begin{pmatrix} 0 \\ t_m^i \varphi_j^m - t_j^m \varphi_m^i \end{pmatrix}$$

for any $\varphi \in \Im_1^1(M)$.

3 Sasaki Metric on $T_1^1(M)$

In each local chart $U(x^h)$ of M, we put

$$X_{(j)} = \frac{\partial}{\partial x^j} = \delta_j^h \frac{\partial}{\partial x^h} \in \Im_0^1(M),$$
$$A^{(\overline{j})} = (A^{(\overline{j})}) = \partial_i \otimes \mathrm{d}x^j = \delta_i^k \delta_h^j \partial_k \otimes \mathrm{d}x^h \in \Im_1^1(M), \quad \overline{j} = n+1, \cdots, n+n^2.$$

From (2.2) and (2.6), we have

$${}^H X_{(j)} = \delta_j^h \partial_h + (-\Gamma_{js}^k t_h^s + \Gamma_{jh}^s t_s^k)\partial_{\overline{h}}, \tag{3.1}$$
$${}^V A^{(\overline{j})} = \delta_i^k \delta_h^j \partial_{\overline{h}} \tag{3.2}$$

with respect to the natural frame $\left\{\frac{\partial}{\partial x^H}\right\} = \left\{\frac{\partial}{\partial x^h}, \frac{\partial}{\partial x^{\overline{h}}}\right\}$ in $T_1^1(M)$, where $x^{\overline{h}} = t_h^k$ and δ_i^j is the Kronecker's. These $n + n^2$ vector fields are linearly independent and generate, respectively, the horizontal distribution of ∇ and the vertical distribution of $T_1^1(M)$. We call the set $\left\{{}^H X_{(j)}, {}^V A^{(\overline{j})}\right\}$ the frame adapted to the affine connection ∇ on $\pi^{-1}(U) \subset T_1^1(M)$. Putting

$$e_{(j)} = {}^H X_{(j)}, \quad e_{(\overline{j})} = {}^V A^{(\overline{j})},$$

we write the adapted frame as $\{e_\beta\} = \{e_{(j)}, e_{(\overline{j})}\}$. The indices $\alpha, \beta, \gamma, \cdots$ run over the range $\{1, \cdots, n, n+1, \cdots, n+n^2\}$ and indicate the indices with respect to the adapted frame $\{e_\beta\}$.

Using (3.1) and (3.2), we have

$${}^H X = \begin{pmatrix} X^j \delta_j^h \\ -X^j(\Gamma_{js}^k t_h^s - \Gamma_{jh}^s t_s^k) \end{pmatrix} = X^j \begin{pmatrix} \delta_j^h \\ -(\Gamma_{js}^k t_h^s - \Gamma_{jh}^s t_s^k) \end{pmatrix} = X^j e_{(j)},$$
$${}^V A = \begin{pmatrix} 0 \\ A_h^k \end{pmatrix} = \begin{pmatrix} 0 \\ \delta_i^k \delta_h^j A_j^i \end{pmatrix} = A_j^i \begin{pmatrix} 0 \\ \delta_i^k \delta_h^j \end{pmatrix} = A_j^i e_{(\overline{j})},$$

i.e., the lifts ${}^{H}X$ and ${}^{V}A$ have, respectively, the components

$$ {}^{H}X = ({}^{H}X^{\beta}) = \begin{pmatrix} {}^{H}X^{j} \\ {}^{H}X^{\overline{j}} \end{pmatrix} = \begin{pmatrix} X^{j} \\ 0 \end{pmatrix}, \tag{3.3} $$

$$ {}^{V}A = ({}^{V}A^{\beta}) = \begin{pmatrix} {}^{V}A^{j} \\ {}^{V}A^{\overline{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A^{i}_{j} \end{pmatrix} \tag{3.4} $$

with respect to the adapted frame $\{e_\beta\}$, X^j and A^i_j are the local components of X and A on M, respectively (see [11]). For tensor bundles of type (p,q), see [18].

For each $P \in M$, the extension of scalar product g (denoted by G) is defined on the tensor space $\pi^{-1}(P) = T^1_1(P)$ by

$$ G(A,B) = g_{it}g^{jl}A^i_jB^t_l $$

for all $A, B \in \Im^1_1(P)$. The Sasaki metric ${}^{S}g$ (or diagonal lift of g) is defined on $T^1_1(M)$ by the following three equations

$$ {}^{S}g({}^{V}A, {}^{V}B) = {}^{V}(G(A,B)), \tag{3.5} $$

$$ {}^{S}g({}^{V}A, {}^{H}Y) = 0, \tag{3.6} $$

$$ {}^{S}g({}^{H}X, {}^{H}Y) = {}^{V}(g(X,Y)) \tag{3.7} $$

for any $X, Y \in \Im^1_0(M)$ and $A, B \in \Im^1_1(M)$. Since any tensor field of type $(0,2)$ on $T^1_1(M)$ is completely determined by its action on vector fields of type ${}^{H}X$ and ${}^{V}A$ (see [25, p. 280]), it follows that ${}^{S}g$ is completely determined by the equations (3.5)–(3.7).

From (3.5)–(3.7), we see that the Sasaki metric ${}^{S}g$ has the components

$$ {}^{S}g_{\beta\gamma} = \begin{pmatrix} {}^{S}g_{jl} & {}^{S}g_{j\overline{l}} \\ {}^{S}g_{\overline{j}l} & {}^{S}g_{\overline{j}\overline{l}} \end{pmatrix} = \begin{pmatrix} g_{jl} & 0 \\ 0 & g_{it}g^{jl} \end{pmatrix}, \quad x^{\overline{l}} = t^t_l, \tag{3.8} $$

$$ {}^{S}g^{\beta\gamma} = \begin{pmatrix} {}^{S}g^{jl} & {}^{S}g^{j\overline{l}} \\ {}^{S}g^{\overline{j}l} & {}^{S}g^{\overline{j}\overline{l}} \end{pmatrix} = \begin{pmatrix} g^{jl} & 0 \\ 0 & g^{it}g_{lj} \end{pmatrix}, \quad x^{\overline{j}} = t^i_j \tag{3.9} $$

with respect to the adapted frame $\{e_\beta\}$, where g_{ij} and g^{ij} are the local covariant and contravariant components of g on M (see [11]). For tensor bundles of type (p,q), see [18].

We now consider local 1-forms ω^α in $\pi^{-1}(U)$ defined by

$$ \omega^\alpha = \widetilde{A}^{\alpha}{}_{B}\,\mathrm{d}x^B, $$

where

$$ A^{-1} = (\widetilde{A}^{\alpha}{}_{B}) = \begin{pmatrix} \widetilde{A}^{h}{}_{j} & \widetilde{A}^{h}{}_{\overline{j}} \\ \widetilde{A}^{\overline{h}}{}_{j} & \widetilde{A}^{\overline{h}}{}_{\overline{j}} \end{pmatrix} = \begin{pmatrix} \delta^h_j & 0 \\ \Gamma^k_{ls}t^s_h - \Gamma^s_{lh}t^k_s & \delta^k_i\delta^j_h \end{pmatrix}. \tag{3.10} $$

The matrix (3.10) is the inverse of the matrix

$$ A = (A_\beta{}^{A}) = \begin{pmatrix} A_j{}^{h} & A_{\overline{j}}{}^{h} \\ A_j{}^{\overline{h}} & A_{\overline{j}}{}^{\overline{h}} \end{pmatrix} = \begin{pmatrix} \delta^h_j & 0 \\ -\Gamma^k_{ls}t^s_h + \Gamma^s_{lh}t^k_s & \delta^k_i\delta^j_h \end{pmatrix} \tag{3.11} $$

of the transformation $e_\beta = A_\beta{}^{A}\,\partial_A$. We easily see that the set $\{\omega^\alpha\}$ is the coframe dual to the adapted frame $\{e_\beta\}$, i.e., $\omega^\alpha(e_\beta) = \widetilde{A}^{\alpha}{}_{B}A_\beta{}^{B} = \delta^\alpha_\beta$.

For various types of indices, we have

$$\begin{cases} e_j = A_j{}^A\, \partial_A = \partial_j + (-\Gamma^k_{js} t^s_h + \Gamma^s_{jh} t^k_s)\partial_{\overline{h}}, \\ e_{\overline{j}} = A_{\overline{j}}{}^A\, \partial_A = \partial_{\overline{j}}, \end{cases} \tag{3.12}$$

$$\begin{cases} \omega^j = \widetilde{A}^j{}_B\, \mathrm{d}x^B = \mathrm{d}x^j, \\ \omega^{\overline{j}} = \widetilde{A}^{\overline{j}}{}_B\, \mathrm{d}x^B = \delta t^i_j, \end{cases} \tag{3.13}$$

where $\delta t^i_j = \mathrm{d}t^i_j + (\Gamma^i_{km} t^m_j - \Gamma^m_{kj} t^i_m)\mathrm{d}x^k$.

Since the adapted frame field $\{e_\beta\}$ is non-holonomic, we put

$$[e_\alpha, e_\beta] = \Omega_{\alpha\beta}{}^\gamma\, e_\gamma$$

from which we have $\Omega_{\gamma\beta}{}^\alpha = (e_\gamma\, A_\beta{}^A - e_\beta\, A_\gamma{}^A)\widetilde{A}^\alpha{}_A$.

According to (3.10), (3.11) and (3.12), the components of non-holonomic object $\Omega_{\gamma\beta}{}^\alpha$ are given by

$$\begin{cases} \Omega_{l\overline{j}}{}^{\overline{r}} = -\Omega_{\overline{j}l}{}^{\overline{r}} = \Gamma^v_{ls}\delta^j_r - \Gamma^j_{lr}\delta^v_i, \\ \Omega_{lj}{}^{\overline{r}} = R_{ljr}{}^s\, t^v_s - R_{ljs}{}^v\, t^s_r \end{cases} \tag{3.14}$$

with all the others being zero, where $R_{ijk}{}^h$ are the local components of the curvature tensor R of the metric g on M.

Let ${}^S\nabla$ be the Levi-Civita connection of the Sasaki metric ${}^Sg$. Putting ${}^S\nabla_{e_\alpha} e_\beta = {}^S\Gamma^\gamma_{\alpha\beta} e_\gamma$, from the equation ${}^S\nabla_{\widetilde{X}}\widetilde{Y} - {}^S\nabla_{\widetilde{Y}}\widetilde{X} = [\widetilde{X}, \widetilde{Y}],\ \forall \widetilde{X}, \widetilde{Y} \in \Im^1_0(T^1_1(M))$, we have

$${}^S\Gamma^\alpha_{\gamma\beta} - {}^S\Gamma^\alpha_{\beta\gamma} = \Omega_{\gamma\beta}{}^\alpha \tag{3.15}$$

with respect to the adapted frame $\{e_\beta\}$, where ${}^S\Gamma^\alpha_{\gamma\beta}$ are the components of the Levi-Civita connection ${}^S\nabla$.

The equation $({}^S\nabla_{\widetilde{X}}{}^Sg)(\widetilde{Y}, \widetilde{Z}) = 0,\ \forall \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im^1_0(T^1_1(M))$, has the form

$$e_\delta\, {}^Sg_{\gamma\beta} - {}^S\Gamma^\varepsilon_{\delta\gamma}\, {}^Sg_{\varepsilon\beta} - {}^S\Gamma^\varepsilon_{\delta\beta}\, {}^Sg_{\gamma\varepsilon} = 0 \tag{3.16}$$

with respect to the adapted frame $\{e_\beta\}$. Thus, we have from (3.15) and (3.16)

$${}^S\Gamma^\alpha_{\gamma\beta} = \frac{1}{2}{}^Sg^{\alpha\varepsilon}(e_\alpha\, {}^Sg_{\varepsilon\beta} + e_\beta\, {}^Sg_{\gamma\varepsilon} - e_\varepsilon\, {}^Sg_{\gamma\beta}) + \frac{1}{2}(\Omega_{\gamma\beta}{}^\alpha + \Omega^\alpha{}_{\gamma\beta} + \Omega^\alpha{}_{\beta\gamma}), \tag{3.17}$$

where $\Omega^\alpha{}_{\gamma\beta} = {}^Sg^{\alpha\varepsilon}\, {}^Sg_{\delta\beta}\, \Omega_{\varepsilon\gamma}{}^\delta$.

Taking account of (3.9), (3.14) and (3.17), for various types of indices, we find

$$\begin{aligned} &{}^S\Gamma^r_{lj} = \Gamma^r_{lj}, \quad {}^S\Gamma^{\overline{r}}_{lj} = \frac{1}{2}R_{ljr}{}^s\, t^v_s - \frac{1}{2}R_{ljs}{}^v\, t^s_r, \\ &{}^S\Gamma^r_{\overline{l}\overline{j}} = 0, \quad {}^S\Gamma^{\overline{r}}_{\overline{l}\overline{j}} = 0, \quad {}^S\Gamma^{\overline{r}}_{\overline{l}j} = 0, \\ &{}^S\Gamma^r_{\overline{l}j} = \frac{1}{2}g_{ta}R^{s\,l}_{\bullet\bullet\,j}{}^r\, t^a_s - \frac{1}{2}g^{lb}R_{tsj}{}^r\, t^s_b, \\ &{}^S\Gamma^{\overline{r}}_{l\overline{j}} = \Gamma^v_{li}\delta^j_r - \Gamma^j_{lr}\delta^v_i, \\ &{}^S\Gamma^r_{l\overline{j}} = \frac{1}{2}g_{ia}R^{s\,j}_{\bullet\bullet\,l}{}^r\, t^a_s - \frac{1}{2}g^{jb}R_{isl}{}^r\, t^s_b \end{aligned} \tag{3.18}$$

with respect to the adapted frame $\{e_\beta\}$ (see [11]). For tensor bundles of type (p,q), see [18].

Let $\widetilde{X},\widetilde{Y}\in\Im^1_0(T^1_1(M))$ and $\widetilde{X}=\widetilde{X}^\alpha e_\alpha,\ \widetilde{Y}=\widetilde{Y}^\beta e_\beta$. Then the covariant derivative ${}^S\nabla_{\widetilde{Y}}\widetilde{X}$ along $\widetilde{Y}$ has the components

$$ {}^S\nabla_{\widetilde{Y}}\widetilde{X}^\alpha=\widetilde{Y}^\gamma e_\gamma(\widetilde{X}^\alpha)+{}^S\Gamma^\alpha_{\gamma\beta}\widetilde{X}^\beta\widetilde{Y}^\gamma \tag{3.19}$$

with respect to the adapted frame $\{e_\beta\}$.

Using (3.3), (3.4), (3.18) and (3.19), we have the following theorem.

Theorem 3.1 *Let M be a Riemannian manifold with the metric g, and $T^1_1(M)$ be its $(1,1)$ tensor bundle equipped with the Sasaki metric ${}^Sg$. Then the corresponding Levi-Civita connection satisfies the following relations:*

(i) ${}^S\nabla_{{}^HX}{}^HY={}^H(\nabla_XY)+\dfrac{1}{2}(\widetilde{\gamma}-\gamma)R(X,Y)$,

(ii) ${}^S\nabla_{{}^VA}{}^HY=\dfrac{1}{2}{}^H(g^{bl}\,R(t_b,A_l)Y+g_{at}(t^a\,(g^{-1}\circ R(\,\cdot\,,Y)\widetilde{A}^{\,t})))$,

(iii) ${}^S\nabla_{{}^HX}{}^VB={}^V(\nabla_XB)+\dfrac{1}{2}{}^H(g^{bj}\,R(t_b,B_j)X+g_{ai}\,(t^a(g^{-1}\circ R(\,\cdot\,,X)\widetilde{B}^{\,i})))$,

(iv) ${}^S\nabla_{{}^VA}{}^VB=0$

for all $X,Y\in\Im^1_0(M)$ and $A,B\in\Im^1_1(M)$, where $A_l=(A_l{}^i)$, $\widetilde{A}^t=(g^{bl}A_l{}^{\ t})=(A^{b\,t}_{\bullet})$, $t_l=(t_l{}^a)$, $t^a=(t_b{}^a)$, $R(\ \ ,X)Y\in\Im^1_1(M)$ and $g^{-1}\circ R(\ \ ,X)Y\in\Im^1_0(M)$.

4 Curvature Tensor of ${}^S\nabla$

The curvature tensor R of the connection ∇ is obtained from the well-known formula

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z$$

for all $X,\,Y\in\Im^1_0(M)$. With respect to the adapted frame $\{e_\beta\}$, we write ${}^S\nabla_{e_\alpha}e_\beta={}^S\Gamma^\gamma_{\alpha\beta}e_\gamma$, where ${}^S\Gamma^\gamma_{\alpha\beta}$ denotes the Levi-Civita connection constructed by ${}^Sg$. Then the curvature tensor ${}^SR$ has the components

$${}^SR_{\alpha\beta\gamma}{}^\sigma=\widetilde{e}_\alpha{}^S\Gamma^\sigma_{\beta\gamma}-\widetilde{e}_\beta{}^S\Gamma^\sigma_{\alpha\gamma}+{}^S\Gamma^\sigma_{\alpha\varepsilon}{}^S\Gamma^\varepsilon_{\beta\gamma}-{}^S\Gamma^\sigma_{\beta\varepsilon}{}^S\Gamma^\varepsilon_{\alpha\gamma}-\Omega_{\alpha\beta}{}^\varepsilon\,{}^S\Gamma^\sigma_{\varepsilon\gamma}.$$

By using (3.14) and (3.18), we find that the components of the curvature tensor ${}^SR$ of ${}^S\nabla$ are as follows:

$$\begin{aligned}
&{}^SR_{\overline{m}\overline{l}\overline{j}}{}^r=0,\\
&{}^SR_{\overline{m}\overline{l}\overline{j}}{}^{\overline{r}}=0,\\
&{}^SR_{\overline{m}\overline{l}j}{}^r=g_{tn}R^{m\,l}_{\bullet\,\bullet\,j}{}^r-g^{lm}R_{tnj}{}^r+\frac{1}{4}(g_{na}R^{s\,m}_{\bullet\,\bullet\,h}{}^r\,g_{tb}R^{p\,l}_{\bullet\,\bullet j}{}^h-g_{ta}R^{s\,l}_{\bullet\bullet h}{}^r\,g_{nb}R^{p\,m}_{\bullet\,\bullet\,j}{}^h)t^a_st^b_p\\
&\qquad+\frac{1}{4}(g_{ta}R^{s\,l}_{\bullet\bullet h}{}^r\,g^{mb}R_{npj}{}^h-g_{na}R^{s\,m}_{\bullet\,\bullet\,h}{}^r\,g^{lb}R_{tpj}{}^h)t^a_st^p_b\\
&\qquad+\frac{1}{4}(g^{lb}R_{tph}{}^rg_{na}R^{s\,m}_{\bullet\,\bullet j}{}^h-g^{mb}R_{nph}{}^rg_{ta}R^{s\,l}_{\bullet\,\bullet j}{}^h)t^p_bt^a_s\\
&\qquad+\frac{1}{4}(g^{ma}R_{nsh}{}^r\,g^{lb}R_{tsj}{}^h-g^{la}R_{tsh}{}^r\,g^{mb}R_{npj}{}^h)t^s_at^p_b,
\end{aligned}$$

$$
\begin{aligned}
{}^{S}R_{\overline{m}\overline{l}j}{}^{\overline{r}} &= 0,\\
{}^{S}R_{m\overline{l}\overline{j}}{}^{\overline{r}} &= 0,\\
{}^{S}R_{m\overline{l}\,\overline{j}}{}^{r} &= -\frac{1}{2}g_{it}R^{l\,j}_{\bullet\bullet\,m}{}^{r} + \frac{1}{2}g^{jl}R_{itm}{}^{r} - \frac{1}{4}(g_{ta}R^{s\,l}_{\bullet\bullet\,h}{}^{r}\,g_{ib}R^{p\,l}_{\bullet\bullet\,m}{}^{h})t^{a}_{s}t^{b}_{p}\\
&\quad + \frac{1}{4}(g_{ta}R^{s\,l}_{\bullet\bullet\,h}{}^{r}\,g^{jb}R_{ipm}{}^{h})t^{a}_{s}t^{p}_{b} + \frac{1}{4}(g^{lb}R_{tph}{}^{r}\,g_{ia}R^{s\,j}_{\bullet\bullet\,m}{}^{h})t^{p}_{b}t^{a}_{s}\\
&\quad - \frac{1}{4}(g^{la}R_{tsh}{}^{r}g^{jb}R_{ipm}{}^{h})t^{s}_{a}t^{p}_{b},\\
{}^{S}R_{m\,\overline{l}\,j}{}^{r} &= \frac{1}{2}g_{ta}(\nabla_{m}R^{s\,l}_{\bullet\bullet\,j}{}^{r})t^{a}_{s} - \frac{1}{2}g^{lb}(\nabla_{m}R_{tsj}{}^{r})t^{s}_{b},\\
{}^{S}R_{m\,\overline{l}\,j}{}^{\overline{r}} &= -\frac{1}{2}R_{mjr}{}^{l}\,\delta^{v}_{t} + \frac{1}{2}R_{mjt}{}^{v}\,\delta^{l}_{r} + \frac{1}{4}(R_{mhr}{}^{s}\,g_{va}R^{p\,l}_{\bullet\bullet\,j}{}^{h})t^{v}_{s}t^{a}_{p} \qquad (4.1)\\
&\quad - \frac{1}{4}(R_{mhr}{}^{s}\,g^{lb}R_{tpj}{}^{h})t^{v}_{s}t^{p}_{b} - \frac{1}{4}(R_{mhp}{}^{v}\,g_{ta}R^{s\,l}_{\bullet\bullet\,j}{}^{h})t^{p}_{r}t^{a}_{s}\\
&\quad + \frac{1}{4}(R_{mhs}{}^{v}\,g^{lb}R_{tpj}{}^{h})t^{s}_{r}t^{p}_{b},\\
{}^{S}R_{ml\,\overline{j}}{}^{r} &= \frac{1}{2}(g_{ia}\nabla_{m}R^{s\,j}_{\bullet\bullet\,l}{}^{r} - g_{ia}\nabla_{l}R^{s\,j}_{\bullet\bullet\,m}{}^{r})t^{a}_{s} + \frac{1}{2}(g^{jb}\nabla_{l}R_{ism}{}^{r} - g^{jb}\nabla_{m}R_{isl}{}^{r})t^{s}_{b},\\
{}^{S}R_{ml\,\overline{j}}{}^{\overline{r}} &= R_{mli}{}^{v}\,\delta^{j}_{r} - R_{mlr}{}^{j}\,\delta^{v}_{i} + \frac{1}{4}(R_{mhr}{}^{s}\,g_{ia}R^{p\,j}_{\bullet\bullet\,l}{}^{h} - R_{lhr}{}^{s}\,g_{ia}R^{p\,j}_{\bullet\bullet\,m}{}^{h})t^{v}_{s}t^{a}_{p}\\
&\quad + \frac{1}{4}(R_{lhr}{}^{s}\,g^{jb}R_{lpm}{}^{h} - R_{mhr}{}^{s}\,g^{jb}R_{ipl}{}^{h})t^{v}_{s}t^{p}_{b}\\
&\quad + \frac{1}{4}(R_{lhp}{}^{v}\,g_{ia}R^{s\,j}_{\bullet\bullet\,m}{}^{h} - R_{mhp}{}^{v}\,g_{ia}R^{s\,j}_{\bullet\bullet\,l}{}^{h})t^{p}_{r}t^{a}_{s}\\
&\quad + \frac{1}{4}(R_{mhs}{}^{v}\,g^{jb}R_{ipl}{}^{h} - R_{lhs}{}^{v}\,g^{jb}R_{ipm}{}^{h})t^{s}_{r}t^{p}_{b},\\
{}^{S}R_{mlj}{}^{\overline{r}} &= \frac{1}{2}(\nabla_{m}R_{ljr}{}^{s} - \nabla_{l}R_{mjr}{}^{s})t^{v}_{s} + \frac{1}{2}(\nabla_{l}R_{mjs}{}^{v} - \nabla_{m}R_{ljs}{}^{v})t^{s}_{r},\\
{}^{S}R_{mlj}{}^{r} &= R_{mlj}{}^{r} + \frac{1}{4}(g_{ka}R^{s\,h}_{\bullet\bullet\,m}{}^{r}R_{ljh}{}^{p} - g_{ka}R^{s\,h}_{\bullet\bullet\,l}{}^{r}R_{mjh}{}^{p} - 2g_{ka}R^{s\,h}_{\bullet\bullet\,j}{}^{r}R_{mlh}{}^{p})t^{a}_{s}t^{k}_{p}\\
&\quad + \frac{1}{4}(g_{ka}R^{s\,h}_{\bullet\bullet\,l}{}^{r}R_{mjp}{}^{k} - g_{ka}R^{s\,h}_{\bullet\bullet\,m}{}^{r}R_{ljp}{}^{k} + 2g_{ka}R^{s\,h}_{\bullet\bullet\,j}{}^{r}R_{mlp}{}^{k})t^{a}_{s}t^{p}_{h}\\
&\quad + \frac{1}{4}(g^{hb}R_{kpl}{}^{r}R_{mjh}{}^{s} - g^{hb}R_{kpm}{}^{r}R_{ljh}{}^{s} + 2g^{hb}R_{kpj}{}^{r}R_{mlh}{}^{s})t^{p}_{b}t^{k}_{s}\\
&\quad + \frac{1}{4}(g^{hb}R_{ksm}{}^{r}R_{ljp}{}^{k} - g^{hb}R_{ksl}{}^{r}R_{mjp}{}^{k} - 2g^{hb}R_{ksj}{}^{r}R_{mlp}{}^{k})t^{s}_{b}t^{p}_{h}
\end{aligned}
$$

with respect to the adapted frame $\{e_{\beta}\}$. Thus we have the following result.

Theorem 4.1 *Let M be a Riemannian manifold with the metric g, and $T^{1}_{1}(M)$ be its $(1,1)$ tensor bundle with the Sasaki metric ${}^{S}g$. Then $T^{1}_{1}(M)$ is flat if and only if M is flat.*

Proof It is a direct consequence of (4.1) that $R = 0$ implies ${}^{S}R = 0$. If we assume that ${}^{S}R = 0$, then from the last equation at the point $(x^{i}, t^{j}_{i}) = (x^{i}, 0) \in T^{1}_{1}(M)$, we get

$$
\begin{aligned}
({}^{S}R_{mlj}{}^{r})_{(x^{i},0)} = \Big[R_{mlj}{}^{r} &+ \frac{1}{4}(g_{ka}R^{s\,h}_{\bullet\bullet\,m}{}^{r}R_{ljh}{}^{p} - g_{ka}R^{s\,h}_{\bullet\bullet\,l}{}^{r}R_{mjh}{}^{p} - 2g_{ka}R^{s\,h}_{\bullet\bullet\,j}{}^{r}R_{mlh}{}^{p})t^{a}_{s}t^{k}_{p}\\
&+ \frac{1}{4}(g_{ka}R^{s\,h}_{\bullet\bullet\,l}{}^{r}R_{mjp}{}^{k} - g_{ka}R^{s\,h}_{\bullet\bullet\,m}{}^{r}R_{ljp}{}^{k} + 2g_{ka}R^{s\,h}_{\bullet\bullet\,j}{}^{r}R_{mlp}{}^{k})t^{a}_{s}t^{p}_{h}\\
&+ \frac{1}{4}(g^{hb}R_{kpl}{}^{r}R_{mjh}{}^{s} - g^{hb}R_{kpm}{}^{r}R_{ljh}{}^{s} + 2g^{hb}R_{kpj}{}^{r}R_{mlh}{}^{s})t^{p}_{b}t^{k}_{s}
\end{aligned}
$$

$$+\frac{1}{4}(g^{hb}R_{ksm}{}^{r}R_{ljp}{}^{k}-g^{hb}R_{ksl}{}^{r}R_{mjp}{}^{k}-2g^{hb}R_{ksj}{}^{r}R_{mlp}{}^{k})t_{b}^{s}t_{h}^{p}\Big]_{(x^{i},0)}$$
$$=R_{mlj}{}^{r}(x^{i})=0.$$

We now turn our attention to the Ricci tensor and scalar curvature of the Sasaki metric ${}^{S}g$. Let ${}^{S}R_{\alpha\beta}={}^{S}R_{\sigma\alpha\beta}{}^{\sigma}$ and ${}^{S}r={}^{S}g^{\alpha\beta}\,{}^{S}R_{\alpha\beta}$ denote the Ricci tensor and scalar curvature of the Sasaki metric ${}^{S}g$, respectively. From (4.1), the components of the Ricci tensor ${}^{S}R_{\alpha\beta}$ are characterized by

$$\begin{aligned}
{}^{S}R_{\bar{l}\,\bar{j}}&=-\frac{1}{4}(g_{ta}R^{s\,l}_{\bullet\bullet h}{}^{r}\;g_{ib}R^{p\,j}_{\bullet\,\bullet r}{}^{h})t_{s}^{a}t_{p}^{b}-\frac{1}{4}(g^{lb}R_{tsh}{}^{r}\;g^{ja}R_{ipr}{}^{h})t_{b}^{s}t_{a}^{p}\\
&\quad+\frac{1}{4}(g^{lb}R_{tph}{}^{r}g_{ia}R^{s\,j}_{\bullet\,\bullet r}{}^{h})t_{b}^{p}t_{s}^{a}+\frac{1}{4}(g_{ta}R^{s\,l}_{\bullet\,\bullet h}{}^{r}\;g^{jb}R_{ipr}{}^{h})t_{s}^{a}t_{b}^{p},\\
{}^{S}R_{\bar{l}\,j}&=\frac{1}{2}g_{ta}(\nabla_{r}R^{s\,l}_{\bullet\,\bullet j}{}^{r})t_{s}^{a}-\frac{1}{2}g^{lb}(\nabla_{r}R_{tsj}{}^{r})t_{b}^{s},\\
{}^{S}R_{l\,\bar{j}}&=\frac{1}{2}g_{ia}(\nabla_{r}R^{s\,j}_{\bullet\,\bullet l}{}^{r})t_{s}^{a}-\frac{1}{2}g^{jb}(\nabla_{r}R_{isl}{}^{r})t_{b}^{s},\\
{}^{S}R_{lj}&=R_{lj}-\frac{1}{4}(g_{ka}R^{s\,h}_{\bullet\,\bullet l}{}^{r}\;R_{rjh}{}^{p})t_{s}^{a}t_{p}^{k}-\frac{1}{2}(g_{ka}R^{s\,h}_{\bullet\,\bullet j}{}^{r}\;R_{rlh}{}^{p})t_{s}^{a}t_{p}^{k}\\
&\quad-\frac{1}{4}(R_{lhr}{}^{s}\;g_{va}R^{p\,r}_{\bullet\,\bullet j}{}^{h})t_{s}^{v}t_{p}^{a}-\frac{1}{4}(g^{hb}R_{ksl}{}^{r}\;R_{rjp}{}^{k})t_{b}^{s}t_{h}^{p}\\
&\quad-\frac{1}{2}(g^{hb}R_{ksj}{}^{r}\;R_{rlp}{}^{k})t_{b}^{s}t_{h}^{p}-\frac{1}{4}(R_{lhs}{}^{v}\;g^{rb}R_{vpj}{}^{h})t_{r}^{s}t_{b}^{p}\\
&\quad+\frac{1}{2}(g_{ka}R^{s\,h}_{\bullet\,\bullet j}{}^{r}\;R_{rlp}{}^{k})t_{s}^{a}t_{h}^{p}+\frac{1}{2}(g^{hb}R_{kpj}{}^{r}\;R_{rlh}{}^{s})t_{b}^{p}t_{s}^{k}
\end{aligned}\tag{4.2}$$

with respect to the adapted frame $\{e_{\beta}\}$. From (3.9) and (4.2), the scalar curvature of the Sasaki metric ${}^{S}g$ is given by

$$\begin{aligned}
{}^{S}r&=r-\frac{1}{4}g^{ab}g^{hk}g^{lj}g^{ti}R_{slhv}R_{pjkr}t_{a}^{s}t_{b}^{p}\\
&\quad-\frac{1}{4}g_{cd}g^{lj}g^{hk}g^{rv}R_{rlh}{}^{s}\;R_{vjk}{}^{p}\;t_{s}^{c}t_{p}^{d}+\frac{1}{2}R_{cpr}{}^{h}R^{\;r\;b\;s}_{h\bullet\;\bullet}\;t_{s}^{c}t_{b}^{p}.
\end{aligned}$$

Thus, we have the next theorem.

Theorem 4.2 *Let M be a Riemannian manifold with the metric g, and $T_1^1(M)$ be its $(1,1)$ tensor bundle equipped with the Sasaki metric ${}^{S}g$. Let r be the scalar curvature of g, and ${}^{S}r$ be the scalar curvature of ${}^{S}g$. Then the following equation holds:*

$${}^{S}r=r-\frac{1}{4}\|tR\|^{2}-\frac{1}{4}\|R_{t}\|^{2}+\frac{1}{2}T,$$

where $\|tR\|^{2}=g^{ab}g^{hk}g^{lj}g^{vr}R_{slhv}R_{pjkr}t_{a}^{s}t_{b}^{p}$, $\|R_{t}\|^{2}=g_{cd}g^{lj}g^{hk}g^{rv}R_{rlh}{}^{s}R_{vjk}{}^{p}\;t_{s}^{c}t_{p}^{d}$ and $T=R_{cpr}{}^{h}R^{\;r\;b\;s}_{h\bullet\;\bullet}\;t_{s}^{c}t_{b}^{p}$.

Let now (M,g), $n>2$, be a Riemannian manifold of constant curvature κ, i.e.,

$$R_{kmj}{}^{s}=\kappa\,(\delta_{k}^{s}g_{mj}-\delta_{m}^{s}g_{kj})$$

and

$$r=n\,(n-1)\,\kappa,$$

where δ_k^s is the Kronecker's. Then, from Theorem 4.2, we have

$$\begin{aligned}
{}^S r &= r - \frac{1}{4} g^{ab} g^{hk} g^{vr} g_{lj} R_{hvs}{}^{l} R_{krp}{}^{j} t_a^s t_b^p \\
&\quad - \frac{1}{4} g_{cd} g^{lj} g^{hk} g^{rv} R_{rlh}{}^{s}\, R_{vjk}{}^{p}\; t_s^c t_p^d + \frac{1}{2} g^{re} g^{bz} R_{cpr}{}^{h} R_{hez}{}^{s}\; t_s^c t_b^p \\
&= r - \frac{1}{4} g^{ab} g^{hk} g^{vr} g_{lj} (\kappa(\delta_h^j g_{vs} - \delta_v^j g_{hs}) \kappa(\delta_k^l g_{rp} - \delta_r^l g_{kp})) t_a^s t_b^p \\
&\quad - \frac{1}{4} g_{cd} g^{lj} g^{hk} g^{rv} (\kappa(\delta_r^s g_{lh} - \delta_l^s g_{rh}) \kappa(\delta_v^p g_{jk} - \delta_j^p g_{vk}))\; t_s^c t_p^d \\
&\quad + \frac{1}{2} g^{re} g^{bz} (\kappa(\delta_c^h g_{pr} - \delta_p^h g_{cr}) \kappa(\delta_h^s g_{ez} - \delta_e^s g_{hz}))\; t_s^c t_b^p \\
&= n(n-1)\kappa - \frac{1}{4}\kappa^2 n g^{ab} g_{sp} t_a^s t_b^p + \frac{1}{4}\kappa^2 g^{ab} g_{sp} t_a^s t_b^p + \frac{1}{4}\kappa^2 g^{ab} g_{sp} t_a^s t_b^p \\
&\quad - \frac{1}{4}\kappa^2 n g^{ab} g_{sp} t_a^s t_b^p - \frac{1}{4}\kappa^2 n g_{cd} g^{rv} t_r^c t_v^d + \frac{1}{4}\kappa^2 g_{cd} g^{rj} t_r^c t_j^d \\
&\quad + \frac{1}{4}\kappa^2 g_{cd} g^{lv} t_l^c t_v^d - \frac{1}{4}\kappa^2 n g_{cd} g^{lj} t_l^c t_j^d + \frac{1}{2}\kappa^2 \delta_c^s \delta_p^b t_s^c t_b^p - \frac{1}{2}\kappa^2 \delta_c^b \delta_p^s t_s^c t_b^p \\
&\quad - \frac{1}{2}\kappa^2 \delta_c^b \delta_p^s t_s^c t_b^p + \frac{1}{2}\kappa^2 \delta_c^s \delta_p^b t_s^c t_b^p \\
&= n(n-1)\kappa - \frac{1}{2}\kappa^2 \|t\|^2 (n-1) - \frac{1}{2}\kappa^2 \|t\|^2 (n-1) + \kappa^2 t_c^c t_p^p - \kappa^2 t_p^c t_c^p \\
&= (n-1)\kappa(n - \|t\|^2 \kappa) + \kappa^2 ((\operatorname{tr} t)^2 - (\operatorname{tr} t^2)).
\end{aligned}$$

Thus, we have the theorem below.

Theorem 4.3 *Let (M, g), $n > 2$, be a Riemannian manifold of constant curvature κ. Then the scalar curvature ${}^S r$ of $(T_1^1(M), {}^S g)$ is*

$$ {}^S r = (n-1)\kappa(n - \|t\|^2 \kappa) + \kappa^2 ((\text{tr } t)^2 - (\text{tr } t^2)), $$

where $\|t\|^2 = g^{kl} g_{ij} t_k^i t_l^j$.

It is known that for a local ortonormal frame a sectional curvature on $(T_1^1(M), {}^S g)$ is given by

$$ {}^S \kappa(\Delta_2) = -{}^S R_{kmij} U^k V^m U^i V^j, \tag{4.3} $$

where $\Delta_2 = (U, V)$ denotes the plane spanned by (U, V).

Let now $\{X_i\}$, $i = 1, \cdots, n$, be a local ortonormal frame and $\|A^{\bar{i}}\|_G^2 = G(A^{\bar{i}}, A^{\bar{i}}) = 1$, $G(A^{\bar{i}}, A^{\bar{j}}) = 0$, $\bar{i} \neq \bar{j}$ for $A^{\bar{i}} \in \Im_1^1(M)$, $\bar{i} = n+1, \cdots, n^2$. Then from (3.5)–(3.7) we see that $\{{}^H X_1, \cdots, {}^H X_n, {}^V A^1, \cdots, {}^V A^{n^2}\}$ is a local ortonormal frame on $T_1^1(M)$. Let ${}^S\kappa({}^H X, {}^H Y)$, ${}^S\kappa({}^H X, {}^V A)$ and ${}^S\kappa({}^V A, {}^V B)$ denote the sectional curvature of the plane spanned by $({}^H X, {}^H Y)$, $({}^H X, {}^V A)$ and $({}^V A, {}^V B)$ on $(T_1^1(M), {}^S g)$, respectively. Then, using (3.3), (3.4), (3.8) and (4.1), we have from (4.3) that

(i) ${}^S\kappa({}^V A, {}^V B) = 0$,

(ii) $$ {}^S\kappa({}^H X, {}^V B) = \frac{1}{4} g^{ez} R_{emr}{}^{s}\, R_{zjd}{}^{p}\; g_{ca} g_{th} t_s^c t_p^a X^m B_\bullet^{d\;t} X^j B_\bullet^{r\;h} + \frac{1}{4} g_{ve} R_{tsm}{}^{v}\, R_{hpj}{}^{e} g^{cb} t_c^s t_b^p g^{lk} X^m B_l^t X^j B_k^h $$

$$+\frac{1}{4}g^{kr}g_{hv}g^{lb}R_{mer}{}^{s}\ R_{tpj}{}^{e}\ t_s^v t_b^p X^m B_l^t X^j B_k^h$$
$$+\frac{1}{4}g^{kr}g_{hv}g_{ta}R_{mep}{}^{v}\ R^{s\ l}_{\bullet\bullet\ j}{}^{e}\ t_r^p t_s^a X^m B_l^t X^j B_k^h,$$

(iii) $${}^S\kappa({}^HX,{}^HY) = -R_{mljk}X^mY^lX^jY^k$$
$$-\frac{3}{4}g^{zh}R_{mkz}{}^{s}\ R_{ljh}{}^{p}g_{ab}t_s^a t_p^b X^mY^kX^lY^j$$
$$-\frac{3}{4}g_{ve}R_{mks}{}^{v}\ R_{jlp}{}^{e}g^{ab}t_a^s t_b^p X^mY^lX^jY^k$$
$$-\frac{3}{4}g_{fa}g^{zh}R_{mkz}{}^{s}\ R_{ljp}{}^{f}t_s^a t_h^p X^mY^lX^jY^k$$
$$-\frac{3}{4}g^{hb}g_{kz}R_{mfp}{}^{z}R_{ljh}{}^{s}t_b^p t_s^k X^mY^lX^jY^f.$$

From (i)–(iii), we have the result as follows.

Theorem 4.4 *Let (M,g) be a Riemannian manifold, and $T_1^1(M)$ be its $(1,1)$ tensor bundle with the Sasaki metric ${}^Sg$. If $(T_1^1(M),{}^Sg)$ is a Riemannian manifold of constant sectional curvature ${}^S\kappa$, then ${}^S\kappa=0$.*

5 Scalar Curvature of the Metric Connection with Respect to the Sasaki Metric ${}^Sg$

In Section 2, we give the Sasaki metric ${}^Sg$ on the tensor bundle $T_1^1(M)$ and consider the Levi-Civita connection ${}^S\nabla$ of ${}^Sg$. This is the unique connection which satisfies ${}^S\nabla\ {}^Sg=0$, and has no torsion. But there exists another connection which satisfies $\widetilde{\nabla}{}^Sg=0$, and has non-trivial torsion tensor. We call this connection the metric connection of ${}^Sg$.

The horizontal lift ${}^H\nabla$ of any connection ∇ on the tensor bundle $T_1^1(M)$ is defined by

$$\begin{cases}{}^H\nabla_{{}^VA}{}^VB=0, \quad {}^H\nabla_{{}^VA}{}^HY=0,\\ {}^H\nabla_{{}^HX}{}^VB={}^V(\nabla_XB), \quad {}^H\nabla_{{}^HX}{}^HY={}^H(\nabla_XY)\end{cases} \tag{5.1}$$

for all vector fields $X,Y\in\Im_0^1(M)$ and $A,B\in\Im_1^1(M)$ (see [6, 11, 12]). For tensor bundles of type (p,q), see [18].

We put ${}^H\nabla_\alpha={}^H\nabla_{\widetilde{e}_{(\alpha)}}$. Then taking account of ${}^H\nabla_\alpha\widetilde{e}_{(\beta)}={}^H\Gamma_{\alpha\beta}^\gamma\widetilde{e}_{(\gamma)}$ and writing ${}^H\Gamma_{\alpha\beta}^\gamma$ for the different indices, from (5.1) it follows that the horizontal lift ${}^H\nabla$ of ∇ has the components

$$\begin{cases}{}^H\Gamma_{\overline{l}\,\overline{j}}^r={}^H\Gamma_{\overline{l}\,\overline{j}}^{\overline{r}}={}^H\Gamma_{\overline{l}\,j}^r={}^H\Gamma_{\overline{l}\,j}^{\overline{r}}={}^H\Gamma_{lj}^{\overline{r}}={}^H\Gamma_{l\,\overline{j}}^r=0,\\ {}^H\Gamma_{lj}^r=\Gamma_{lj}^r, \quad {}^H\Gamma_{l\,\overline{j}}^{\overline{r}}=\Gamma_{li}^v\ \delta_r^j-\Gamma_{lr}^j\ \delta_i^v.\end{cases} \tag{5.2}$$

Denote by $\widetilde{T}$ the torsion tensor of ${}^H\nabla$. Then $\widetilde{T}$ is the skew-symmetric tensor field of type $(1,2)$ on $T_1^1(M)$ determined by

$$\begin{aligned}\widetilde{T}({}^VA,{}^VB)&={}^H\nabla_{{}^VA}{}^VB-{}^H\nabla_{{}^VB}{}^VA-[{}^VA,{}^VB]=0,\\ \widetilde{T}({}^VA,{}^HY)&=-\widetilde{T}({}^HY,{}^VA)={}^H\nabla_{{}^VA}{}^HY-{}^H\nabla_{{}^HY}{}^VA-[{}^VA,{}^HY]\\ &=-{}^V(\nabla_YA)+{}^V(\nabla_YA)=0,\\ \widetilde{T}({}^HX,{}^HY)&={}^H\nabla_{{}^HX}{}^HY-{}^H\nabla_{{}^HY}{}^HX-[{}^HX,{}^HY]\\ &={}^H(\nabla_XY)-{}^H(\nabla_YX)-{}^H[X,Y]+(\widetilde{\gamma}-\gamma)R(X,Y)\end{aligned}$$

$$= {}^H(\nabla_X Y - \nabla_Y X - [X,Y]) + (\widetilde{\gamma} - \gamma)R(X,Y)$$
$$= (\widetilde{\gamma} - \gamma)R(X,Y),$$

where R is the curvature tensor of ∇ and

$$(\widetilde{\gamma} - \gamma)R(X,Y) = \sum_i (t^i_m R_{klj}{}^m X^k Y^l - t^m_j R_{mkl}{}^i X^k Y^l)\frac{\partial}{\partial x^{\overline{j}}}.$$

Thus, the connection ${}^H\nabla$ has non-trivial torsion even for the Levi-Civita connection ∇_g determined by g, unless g is locally flat.

A straightforward computation, using (3.5)–(3.7) and (5.1), leads to the following set of formulas:

$$\begin{aligned}
&({}^H\nabla_{{}^VC}{}^Sg)({}^VA, {}^VB) = 0, \quad ({}^H\nabla_{{}^HZ}{}^Sg)({}^VA, {}^VB) = 0,\\
&({}^H\nabla_{{}^VC}{}^Sg)({}^VA, {}^HY) = 0, \quad ({}^H\nabla_{{}^HZ}{}^Sg)({}^VA, {}^HY) = 0,\\
&({}^H\nabla_{{}^VC}{}^Sg)({}^HX, {}^VB) = 0, \quad ({}^H\nabla_{{}^HZ}{}^Sg)({}^HX, {}^VB) = 0,\\
&({}^H\nabla_{{}^VC}{}^Sg)({}^HX, {}^HY) = 0, \quad ({}^H\nabla_{{}^HZ}{}^Sg)({}^HX, {}^HY) = 0
\end{aligned}$$

for any $A, B, C \in \Im^1_1(M)$ and $X, Y, Z \in \Im^1_0(M)$, i.e., the horizontal lift ${}^H\nabla$ of ∇_g is the metric connection with respect to the Sasaki metric ${}^Sg$.

Let now ${}^HR$ be the curvature tensor field of ${}^H\nabla$. The curvature tensor ${}^HR$ of the metric connection ${}^H\nabla$ of ${}^Sg$ has the components

$${}^HR_{\delta\gamma\beta}{}^\alpha = 2(\widetilde{e}_{[\delta}\,{}^H\Gamma^\alpha_{\gamma]\beta} + {}^H\Gamma^\alpha_{[\delta|\varepsilon|}\,{}^H\Gamma^\varepsilon_{\gamma]\beta}) - \Omega_{\delta\gamma}{}^\varepsilon\,{}^H\Gamma^\alpha_{\varepsilon\beta}$$

with respect to the adapted frame $\{e_\beta\}$. Using (3.12), (3.14), (5.2) and computing the components of the curvature tensor ${}^HR$ of the metric connection ${}^H\nabla$, we obtain

$$\begin{aligned}
&{}^HR_{\overline{m}\,\overline{l}\,\overline{j}}{}^{\overline{r}} = 0, \quad {}^HR_{\overline{m}\,\overline{l}\,\overline{j}}{}^{r} = 0, \quad {}^HR_{\overline{m}\,\overline{l}\,j}{}^{r} = 0, \quad {}^HR_{\overline{m}\,\overline{l}\,j}{}^{\overline{r}} = 0,\\
&{}^HR_{m\,\overline{l}\,j}{}^{r} = 0, \quad {}^HR_{m\,\overline{l}\,j}{}^{\overline{r}} = 0, \quad {}^HR_{ml\,\overline{j}}{}^{r} = 0, \quad {}^HR_{ml\,\overline{j}}{}^{\overline{r}} = R_{mli}{}^v\,\delta^j_r + R_{lmr}{}^j\,\delta^v_i,\\
&{}^HR_{m\overline{l}\,\overline{j}}{}^{r} = 0, \quad {}^HR_{m\overline{l}\,\overline{j}}{}^{\overline{r}} = 0, \quad {}^HR_{mlj}{}^{r} = R_{mlj}{}^{r}, \quad {}^HR_{mlj}{}^{\overline{r}} = 0.
\end{aligned} \tag{5.3}$$

The contracted curvature tensor field (Ricci tensor field) ${}^HR_{\gamma\beta} = {}^HR_{\alpha\gamma\beta}{}^\alpha$ of the metric connection ${}^H\nabla$ has the components

$${}^HR_{\overline{l}\,\overline{j}} = 0, \quad {}^HR_{l\overline{j}} = 0, \quad {}^HR_{\overline{l}\,j} = 0, \quad {}^HR_{lj} = R_{lj},$$

where R_{kj} is the Ricci tensor field of ∇_g on M.

For the scalar curvature of ${}^H\nabla$ with respect to the Sasaki metric ${}^Sg$, we have

$${}^Hr = {}^Sg^{\gamma\beta}\,{}^HR_{\gamma\beta} = g^{kj}R_{kj} = r$$

by means of (5.3) and ${}^Sg^{\overline{k}j} = {}^Sg^{k\overline{j}} = 0$.

Thus we have the following theorem.

Theorem 5.1 *Let (M,g) be a Riemannian manifold, and the tensor bundle $T^1_1(M)$ be equipped with the Sasaki metric ${}^Sg$. Then the tensor bundle $T^1_1(M)$ with the metric connection ${}^H\nabla$ has vanishing scalar curvature ${}^Hr$ with respect to the Sasaki metric ${}^Sg$ if and only if the scalar curvature r of ∇_g on M is zero.*

6 Para-Nordenian Structures on $(T_1^1(M), {}^S g)$

An almost paracomplex manifold is an almost product manifold (M,φ), $\varphi^2 = I$, such that the two eigenbundles T^+M and T^-M associated with the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even. Considering the paracomplex structure φ, we obtain the set $\{I,\varphi\}$ on M, which forms a base of an isomorphic representation of the algebra of order 2, which is called the algebra of paracomplex (or double) numbers and is denoted by $R(j)$, $j^2 = 1$.

A tensor field $\omega \in \Im_q^0(M_{2n})$ is said to be pure with respect to the paracomplex structure φ, if

$$\omega(\varphi X_1, X_2, \cdots, X_q) = \omega(X_1, \varphi X_2, \cdots, X_q) = \cdots = \omega(X_1, X_2, \cdots, \varphi X_q)$$

for any $X_1, X_2, \cdots, X_q \in \Im_0^1(M_{2n})$.

We consider the operator ϕ_φ associated with φ and applied to the pure tensor field ω by [24]

$$\begin{aligned}(\phi_\varphi \omega)(Y, X_1, \cdots, X_q) &= (\varphi Y)(\omega(X_1, \cdots, X_q)) - Y(\omega(\varphi X_1, X_2, \cdots, X_q)) \\ &\quad + \omega((L_{X_1}\varphi)Y, X_2, \cdots, X_q) + \cdots + \omega(X_1, X_2, \cdots, (L_{X_q}\varphi)Y).\end{aligned}$$

If $\phi_\varphi \omega = 0$, then ω is said to be almost paraholomorphic with respect to the paracomplex algebra $R(j)$ (see [10, 19]).

A Riemannian manifold (M_{2n}, g) with an almost paracomplex structure φ, is said to be almost para-Nordenian, if the Riemannian metric g is pure with respect to φ. It is well-known that, the almost para-Nordenian B-manifold is para-Kähler ($\nabla_g \varphi = 0$) if and only if g is paraholomorphic ($\phi_\varphi g = 0$) (see [19, 20]).

Let $\left(T_1^1(M), {}^S g\right)$ be the $(1,1)$ tensor bundle with the Sasaki metric ${}^S g$. From the equations (3.5)–(3.7), we easily see that the horizontal distribution H, induced by ∇_g and determined by the horizontal lifts, is orthogonal to the fibres of $T_1^1(M)$.

Let now $E \in \Im_0^1(M)$ be a nowhere zero vector field on M. For any $X \in \Im_0^1(M)$ and $\widetilde{E} = g \circ E \in \Im_1^0(M)$, we define the vertical lift ${}^V(X \otimes \widetilde{E})$ of X with respect to E. The map $X \to {}^V(X \otimes \widetilde{E})$ is a monomorphism of $\Im_0^1(M) \to \Im_0^1(T_1^1(M))$. Hence an $n-$dimensional C^∞ vertical distribution V^E is defined on $T_1^1(M)$. Let $V^\perp$ be the distribution on $T_1^1(M)$ which is orthogonal to H and V^E. Then H, V^E and $V^\perp$ are mutually orthogonal distributions with respect to the Sasaki metric ${}^S g$. We define a tensor field F of type (1,1) on $T_1^1(M)$ by

$$\begin{cases} F^H X = {}^V(X \otimes \widetilde{E}), \\ F^V(X \otimes \widetilde{E}) = {}^H X, \\ F({}^V A) = {}^V A \end{cases} \tag{6.1}$$

for any $X \in \Im_0^1(M)$ and $A \in \Im_1^1(M)$, where $\widetilde{E} = g \circ E \in \Im_1^0(M)$. The restrictions of F to $H + V^E$ and $V^\perp$ are endomorphisms, and hence F is a tensor field of type $(1,1)$ on $T_1^1(M)$. It is easy to see that $F^2 = \mathrm{I}$. In fact, we have by virtue of (6.1)

$$\begin{aligned}&F^2({}^H X) = F(F^H X) = F({}^V(X \otimes \widetilde{E})) = {}^H X, \\ &F^2({}^V(X \otimes \widetilde{E})) = F(F^V(X \otimes \widetilde{E})) = F({}^H X) = {}^V(X \otimes \widetilde{E}), \\ &F^2({}^V A) = F(F^V A) = F({}^V A) = {}^V A\end{aligned}$$

for any $X \in \Im_0^1(M)$ and $A \in \Im_1^1(M)$, which implies $F^2 = I$.

Theorem 6.1 *The triple $\left(T_1^1(M), {}^S g, F\right)$ is an almost para-Nordenian B-manifold if and only if $g(E,E) = 1$.*

Proof We put

$$A(\widetilde{X},\widetilde{Y}) = {}^S g(F\widetilde{X},\widetilde{Y}) - {}^S g(\widetilde{X},F\widetilde{Y})$$

for any $\widetilde{X},\widetilde{Y} \in \Im_0^1(T_1^1(M))$. From (3.5)–(3.7) and (6.1), we have

$$\begin{aligned}
A({}^H X, {}^H Y) &= {}^S g(F\,{}^H X, {}^H Y) - {}^S g({}^H X, F\,{}^H Y)\\
&= {}^S g({}^V (X\otimes \widetilde{E}), {}^H Y) - {}^S g({}^H X, {}^V (Y\otimes \widetilde{E})) = 0,\\
A({}^V (X\otimes \widetilde{E}), {}^V (Y\otimes \widetilde{E})) &= {}^S g(F\,{}^V (X\otimes \widetilde{E}), {}^V (Y\otimes \widetilde{E})) - {}^S g({}^V (X\otimes \widetilde{E}), F\,{}^V (Y\otimes \widetilde{E}))\\
&= {}^S g({}^H X, {}^V (Y\otimes \widetilde{E})) - {}^S g({}^V (X\otimes \widetilde{E}), {}^H Y) = 0,\\
A({}^V (X\otimes \widetilde{E}), {}^H Y) &= {}^S g(F\,{}^V (X\otimes \widetilde{E}), {}^H Y) - {}^S g({}^V (X\otimes \widetilde{E}), F\,{}^H Y)\\
&= {}^S g({}^H X, {}^H Y) - {}^S g({}^V (X\otimes \widetilde{E}), {}^V (Y\otimes \widetilde{E}))\\
&= {}^V (g(X,Y)) - {}^V (g(X,Y) g^{-1}(\widetilde{E},\widetilde{E}))\\
&= {}^V (g(X,Y) - g(X,Y) g(E,E)) = 0,\\
A({}^V A, {}^V B) &= {}^S g(F\,{}^V A, {}^V B) - {}^S g({}^V A, F\,{}^V B)\\
&= {}^S g({}^V A, {}^V B) - {}^S g({}^V A, {}^V B) = (G(A,B)) - (G(A,B)) = 0,\\
A({}^V A, {}^V (X\otimes \widetilde{E})) &= {}^S g(F\,{}^V A, {}^V (X\otimes \widetilde{E})) - {}^S g({}^V A, F\,{}^V (X\otimes \widetilde{E}))\\
&= {}^S g({}^V A, {}^V (X\otimes \widetilde{E})) - {}^S g({}^V A, {}^H X) = {}^V (G(A, X\otimes \widetilde{E})) = 0,\\
A({}^V A, {}^H Y) &= {}^S g(F\,{}^V A, {}^H Y) - {}^S g({}^V A, F\,{}^H Y)\\
&= {}^S g({}^V A, {}^H Y) - {}^S g({}^V A, {}^V (Y\otimes \widetilde{E})) = -{}^V (G(A, Y\otimes \widetilde{E})) = 0,
\end{aligned}$$

i.e., ${}^S g$ is pure with respect to F. Thus Theorem 6.1 is proved.

We now consider the covariant derivative of F. Taking Theorem 3.1(i)–(iv) and (6.1) into account, we obtain

$$\begin{aligned}
&({}^S\nabla_{{}^H X} F)({}^H Y)\\
&= {}^S\nabla_{{}^H X}(F\,{}^H Y) - F({}^S\nabla_{{}^H X}{}^H Y)\\
&= {}^S\nabla_{{}^H X}{}^V (Y\otimes \widetilde{E}) - F\Big({}^H(\nabla_X Y) + \frac{1}{2}(\widetilde{\gamma} - \gamma) R(X,Y)\Big)\\
&= {}^V (\nabla_X (Y\otimes \widetilde{E}))\\
&\quad + \frac{1}{2}{}^H (g^{bj}\, R(t_b, (Y\otimes \widetilde{E})_j) X + g_{ai}\,(t^a (g^{-1}\circ R(\,\cdot\,,X)\,(Y\tilde{\otimes}\widetilde{E})\ \ ^i)))\\
&\quad - {}^V ((\nabla_X Y)\otimes \widetilde{E}) - \frac{1}{2}(\widetilde{\gamma} - \gamma) R(X,Y)\\
&= {}^V (Y\otimes \nabla_X \widetilde{E}) - \frac{1}{2}(\widetilde{\gamma} - \gamma) R(X,Y)\\
&\quad + \frac{1}{2}{}^H (g^{bj}\, R(t_b, (Y\otimes \widetilde{E})_j) X + g_{ai}\,(t^a (g^{-1}\circ R(\,\cdot\,,X)\,(Y\tilde{\otimes}\widetilde{E})\ ^i)))\\
&= {}^V (Y\otimes \nabla_X (g\circ E) - \frac{1}{2}(\widetilde{\gamma} - \gamma) R(X,Y)
\end{aligned}$$

$$+\frac{1}{2}{}^H(g^{bj}\,R(t_b,(Y\otimes\widetilde{E})_j)X+g_{ai}\,(t^a(g^{-1}\circ R(\,\cdot\,,X)\,(Y\,\widetilde{\otimes}\,\widetilde{E})\ ^i)))$$

$$={}^V(Y\otimes[(\nabla_X g)\circ E+g\circ\nabla_X E])-\frac{1}{2}(\widetilde{\gamma}-\gamma)R(X,Y)$$

$$+\frac{1}{2}{}^H(g^{bj}\,R(t_b,(Y\otimes\widetilde{E})_j)X+g_{ai}\,(t^a(g^{-1}\circ R(\,\cdot\,,X)\,(Y\,\widetilde{\otimes}\,\widetilde{E})\ ^i)))$$

$$={}^V(Y\otimes[g\circ\nabla_X E])-\frac{1}{2}(\widetilde{\gamma}-\gamma)R(X,Y)$$

$$+\frac{1}{2}{}^H(g^{bj}\,R(t_b,(Y\otimes\widetilde{E})_j)X+g_{ai}\,(t^a(g^{-1}\circ R(\,\cdot\,,X)\,(Y\,\widetilde{\otimes}\,\widetilde{E})\ ^i))), \tag{6.2}$$

$$({}^S\nabla_{{}^HX}F)({}^VB)$$

$$={}^S\nabla_{{}^HX}(F\,{}^VB)-F({}^S\nabla_{{}^HX}{}^VB)$$

$$={}^S\nabla_{{}^HX}{}^VB-F({}^S\nabla_{{}^HX}{}^VB)$$

$$={}^V(\nabla_X B)+\frac{1}{2}{}^H(g^{bj}\,R(t_b,B_j)X+g_{ai}\,(t^a(g^{-1}\circ R(\,\cdot\,,X)\widetilde{B}^{\,i})))$$

$$-F({}^V(\nabla_X B)+\frac{1}{2}{}^H(g^{bj}\,R(t_b,B_j)X+g_{ai}\,(t^a(g^{-1}\circ R(\,\cdot\,,X)\widetilde{B}^{\,i})))$$

$$=\frac{1}{2}{}^H(g^{bj}\,R(t_b,B_j)X+g_{ai}\,(t^a(g^{-1}\circ R(\,\cdot\,,X)\widetilde{B}^{\,i})))$$

$$-\frac{1}{2}{}^V([g^{bj}\,R(t_b,B_j)X+g_{ai}\,(t^a(g^{-1}\circ R(\,\cdot\,,X)\widetilde{B}^{\,i}))]\otimes\widetilde{E}), \tag{6.3}$$

$$({}^S\nabla_{{}^HX}F)\,{}^V(Y\otimes\widetilde{E})$$

$$={}^S\nabla_{{}^HX}(F\,{}^V(Y\otimes\widetilde{E}))-F({}^S\nabla_{{}^HX}{}^V(Y\otimes\widetilde{E}))$$

$$={}^S\nabla_{{}^HX}{}^HY-F({}^S\nabla_{{}^HX}{}^V(Y\otimes\widetilde{E}))$$

$$={}^H(\nabla_X Y)+\frac{1}{2}(\widetilde{\gamma}-\gamma)R(X,Y)-F\,{}^V(\nabla_X(Y\otimes\widetilde{E}))$$

$$-\frac{1}{2}F\,{}^H(g^{bj}\,R(t_b,(Y\otimes\widetilde{E})_j)X+g_{ai}\,(t^a(g^{-1}\circ R(\,\cdot\,,X)\,(Y\,\widetilde{\otimes}\,\widetilde{E})\ ^i)))$$

$$=-{}^V(Y\otimes(g\circ\nabla_X E))+\frac{1}{2}(\widetilde{\gamma}-\gamma)R(X,Y)$$

$$-\frac{1}{2}{}^V([g^{bj}\,R(t_b,(Y\otimes\widetilde{E})_j)X+g_{ai}\,(t^a(g^{-1}\circ R(\,\cdot\,,X)\,(Y\,\widetilde{\otimes}\,\widetilde{E})\ ^i))]\otimes\widetilde{E}), \tag{6.4}$$

$$({}^S\nabla_{{}^VA}F)({}^VB)$$

$$={}^S\nabla_{{}^VA}(F\,{}^VB)-F({}^S\nabla_{{}^VA}{}^VB)$$

$$={}^S\nabla_{{}^VA}{}^VB-F({}^S\nabla_{{}^VA}{}^VB)=0, \tag{6.5}$$

$$({}^S\nabla_{{}^VA}F)({}^HY)$$

$$={}^S\nabla_{{}^VA}(F\,{}^HY)-F({}^S\nabla_{{}^VA}{}^HY)$$

$$={}^S\nabla_{{}^VA}{}^V(Y\otimes\widetilde{E})-\frac{1}{2}F\,{}^H(g^{bl}\,R(t_b,A_l)Y+g_{at}(t^a\,(g^{-1}\circ R(\,\cdot\,,Y)\widetilde{A}^{\,t}))$$

$$=-\frac{1}{2}{}^V([g^{bl}\,R(t_b,A_l)Y+g_{at}(t^a\,(g^{-1}\circ R(\,\cdot\,,Y)\widetilde{A}^{\,t}))]\otimes\widetilde{E}), \tag{6.6}$$

$$({}^S\nabla_{{}^VA}F)\,{}^V(Y\otimes\widetilde{E})$$

$$={}^S\nabla_{{}^VA}(F\,{}^V(Y\otimes\widetilde{E}))-F({}^S\nabla_{{}^VA}{}^V(Y\otimes\widetilde{E}))$$

$$= {}^S\nabla_{{}^V A}{}^H Y = \frac{1}{2}{}^H(g^{bl}\, R(t_b, A_l)Y + g_{at}(t^a\,(g^{-1}\circ R(\,\cdot\,,Y)\widetilde{A}^{\,t}))), \tag{6.7}$$

$$\begin{aligned}
&({}^S\nabla_{{}^V(X\otimes\widetilde{E})}F)^V(Y\otimes\widetilde{E})\\
&= {}^S\nabla_{{}^V(X\otimes\widetilde{E})}(F^V(Y\otimes\widetilde{E})) - F({}^S\nabla_{{}^V(X\otimes\widetilde{E})}{}^V(Y\otimes\widetilde{E}))\\
&= {}^S\nabla_{{}^V(X\otimes\widetilde{E})}{}^H Y - F({}^S\nabla_{{}^V(X\otimes\widetilde{E})}{}^V(Y\otimes\widetilde{E}))\\
&= \frac{1}{2}{}^H(g^{bl}\, R(t_b,(X\otimes\widetilde{E})_l)Y + g_{at}(t^a\,(g^{-1}\circ R(\,\cdot\,,Y)\,(X\overset{\sim}{\otimes}\widetilde{E})^{\;t}))),
\end{aligned}\tag{6.8}$$

$$\begin{aligned}
&({}^S\nabla_{{}^V(X\otimes\widetilde{E})}F)({}^H Y)\\
&= {}^S\nabla_{{}^V(X\otimes\widetilde{E})}(F^H Y) - F({}^S\nabla_{{}^V(X\otimes\widetilde{E})}{}^H Y)\\
&= {}^S\nabla_{{}^V(X\otimes\widetilde{E})}{}^V(Y\otimes\widetilde{E})\\
&\quad - \frac{1}{2}F^H(g^{bl}\, R(t_b,(X\otimes\widetilde{E})_l)Y + g_{at}(t^a\,(g^{-1}\circ R(\,\cdot\,,Y)\,(X\overset{\sim}{\otimes}\widetilde{E})^{\;t})))\\
&= -\frac{1}{2}{}^V([g^{bl}\, R(t_b,(X\otimes\widetilde{E})_l)Y + g_{at}(t^a\,(g^{-1}\circ R(\,\cdot\,,Y)\,(X\overset{\sim}{\otimes}\widetilde{E})^{\;t}))]\otimes\widetilde{E}),
\end{aligned}\tag{6.9}$$

$$\begin{aligned}
&({}^S\nabla_{{}^V(X\otimes\widetilde{E})}F)({}^V B)\\
&= {}^S\nabla_{{}^V(X\otimes\widetilde{E})}(F^V B) - F({}^S\nabla_{{}^V(X\otimes\widetilde{E})}{}^V B)\\
&= {}^S\nabla_{{}^V(X\otimes\widetilde{E})}{}^V B - F({}^S\nabla_{{}^V(X\otimes\widetilde{E})}{}^V B) = 0.
\end{aligned}\tag{6.10}$$

From (6.2)–(6.10), we have the following theorem.

Theorem 6.2 *The tensor bundle $T_1^1(M)$ of a Riemannian manifold M is a para-Kählerian (paraholomorphic Nordenian) B-manifold with respect to the Sasaki metric ${}^S g$ and the almost para-Nordenian B-structure F defined by* (6.1) *if and only if $R = 0$ and $\nabla E = 0$.*

A vector field $\widetilde{Z} \in \Im_0^1(T_1^1(M))$ with respect to which an almost para-Nordenian B-structure F has a vanishing Lie derivative ($L_{\widetilde{Z}}F = 0$) is said to be almost paraholomorphic (see [10]).

It is well known that

$$\begin{cases} \left[{}^C X, {}^H Y\right] = {}^H\,[X,Y] + (\widetilde{\gamma}-\gamma)((L_X\nabla)Y),\\ \left[{}^C X, {}^V A\right] = {}^V\,(L_X A) \end{cases}\tag{6.11}$$

for any $X, Y \in \Im_0^1(M)$, where $(L_X\nabla)\,Y = \nabla_Y\nabla X + R\,(X,Y)$ and $(L_X\nabla)\,(Y,Z) = L_X\,(\nabla_Y X) - \nabla_Y\,(L_X Z) - \nabla_{[X,Y]}Z$ (see [7]).

A vector field $Z \in \Im_0^1\,(M)$ is called a Killing vector field (or infinitesimal isometry) if $L_Z g = 0$, and Z is called an infinitesimal affine transformation if $L_Z\nabla_g = 0$. A Killing vector field is necessarily an infinitesimal affine transformation, i.e., we have $L_Z\nabla_g = 0$ as a consequence of $L_Z g = 0$.

We now consider the Lie derivative of F with respect to the complete lift ${}^C Z$. Taking account of (6.1) and (6.11), we obtain

$$\begin{aligned}
(L_{{}^C Z}F)^V A &= L_{{}^C Z}(F^V A) - F(L_{{}^C Z}{}^V A)\\
&= L_{{}^C Z}{}^V A - F({}^V(L_Z A)) = 0,
\end{aligned}\tag{6.12}$$

$$(L_{{}^C Z}F)^V(Y\otimes\widetilde{E}) = L_{{}^C Z}(F^V(Y\otimes\widetilde{E})) - F(L_{{}^C Z}{}^V(Y\otimes\widetilde{E}))$$

$$\begin{aligned}
&= L_{{}^C Z}{}^H Y - F^V(L_Z(Y \otimes \widetilde{E})) \\
&= L_{{}^C Z}{}^H Y - F^V((L_Z Y) \otimes \widetilde{E}) - F^V(Y \otimes (L_Z \widetilde{E})) \\
&= {}^H(L_Z Y) + (\widetilde{\gamma} - \gamma)((L_Z \nabla)Y) - {}^H(L_Z Y) - {}^V(Y \otimes (L_Z \widetilde{E})) \\
&= (\widetilde{\gamma} - \gamma)((L_Z \nabla)Y) - {}^V(Y \otimes L_Z(g \circ E)) \\
&= (\widetilde{\gamma} - \gamma)((L_Z \nabla)Y) - {}^V(Y \otimes [(L_Z g) \circ E + g \circ L_Z E]), \qquad (6.13)
\end{aligned}$$

$$\begin{aligned}
(L_{{}^C Z} F)^H Y &= L_{{}^C Z}(F^H Y) - F(L_{{}^C Z}{}^H Y) \\
&= L_{{}^C Z}{}^V(Y \otimes \widetilde{E}) - F({}^H(L_Z Y) + (\widetilde{\gamma} - \gamma)(L_Z \nabla)Y) \\
&= {}^V(L_Z(Y \otimes \widetilde{E})) - {}^V((L_Z Y) \otimes \widetilde{E}) - (\widetilde{\gamma} - \gamma)((L_Z \nabla)Y) \\
&= {}^V(Y \otimes L_Z(g \circ E)) - (\widetilde{\gamma} - \gamma)((L_Z \nabla)Y) \\
&= {}^V(Y \otimes [(L_Z g) \circ E + g \circ L_Z E) - (\widetilde{\gamma} - \gamma)((L_Z \nabla)Y). \qquad (6.14)
\end{aligned}$$

Let $(F, {}^S g)$ be the para-Nordenian B-structure on $T_1^1(M)$ and Z be a Killing vector field ($L_Z g = 0$). From the equation $g(E, E) = 1$ (see Theorem 6.1), we have $L_Z E = 0$ for any $Z \in \Im_0^1(M)$. By virtue of $L_Z E = 0$ and $L_Z \nabla = 0$, from (6.13) and (6.14), we have $L_{{}^C Z} F = 0$, i.e., ${}^C Z$ is paraholomorphic with respect to F. If we assume that $L_{{}^C Z} F = 0$ and calculate the equation (6.13) (or (6.14)) at $(x^i, 0)$, $t_i^j = 0$, we get $L_Z g = 0$. We hence have the following result.

Theorem 6.3 *Let $(T_1^1(M)$, ${}^S g$, $F)$ be an almost para-Nordenian B-manifold. An infinitesimal transformation Z of Riemannian manifold (M, g) is a Killing vector field if and only if its complete lift ${}^C Z$ to the tensor bundle $T_1^1(M)$ is an almost paraholomorphic vector field with respect to the almost para-Nordenian B-structure $(F, {}^S g)$.*

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