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Lagrangian Mean Curvature Flow in Pseudo-Euclidean Space

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Abstract The author establishes the long-time existence and convergence results of the mean curvature flow of entire Lagrangian graphs in the pseudo-Euclidean space, which is related to the logarithmic Monge-Ampère flow.

Keywords Indefinite metric, Self-expanding solution, Interior Schauder estimates, Logarithmic Monge-Ampère flow
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1 Introduction

The mean curvature flow in higher codimension was studied extensively in the last few years (cf. [7, 8, 11, 16, 19, 20, 22, 23]). In this paper, we consider the Lagrangian mean curvature flow in the pseudo-Euclidean space.

Let $\mathbb{R}^{2n}_n$ be the $2n$-dimensional pseudo-Euclidean space with index n. The indefinite flat metric on $\mathbb{R}^{2n}_n$ (cf. [24]) is defined by

$$\mathrm{d}s^2 = \frac{1}{2}\sum_{i=1}^{n} \mathrm{d}x^i \mathrm{d}y^i.$$

The logarithmic Monge-Ampère flow (cf. [21]) can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{n}\ln\det D^2 u = 0, & t > 0,\ x \in \mathbb{R}^n, \\ u = u_0(x), & t = 0,\ x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

By Proposition 2.1, there exists a family of diffeomorphisms

$$r_t : \mathbb{R}^n \to \mathbb{R}^n,$$

such that

$$F(x,t) = (r_t, Du(r_t, t)) \subset \mathbb{R}^{2n}_n$$

is a solution to the mean curvature flow in the pseudo-Euclidean space

$$\begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}t} = \overrightarrow{H}, \\ F(x,0) = F_0(x). \end{cases} \tag{1.2}$$

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Here $\overrightarrow{H}$ is the mean curvature vector of the space-like submanifold $F(x,t) \subset \mathbb{R}^{2n}_n$ and

$$F_0(x) = (x, Du_0(x)).$$

Definition 1.1 *Assume that* $u_0(x) \in C^2(\mathbb{R}^n)$. *We call* $u_0(x)$ *satisfying*
(i) (Condition A) *if*

$$u_0(x) = \frac{u_0(Rx)}{R^2}, \quad \forall\, R > 0;$$

(ii) (Condition B) *if*

$$\Lambda I \geq D^2 u_0(x) \geq \lambda I, \quad x \in \mathbb{R}^n,$$

where Λ, λ *are positive constants and* I *is the identity matrix.*

We now state the main theorems of this paper.

Theorem 1.1 *Let* $u_0 : \mathbb{R}^n \to \mathbb{R}$ *be a* C^2 *function satisfying condition* B. *Then there exists a unique strictly convex solution of* (1.1) *such that*

$$u(x,t) \in C^\infty(\mathbb{R}^n \times (0,+\infty)) \cap C(\mathbb{R}^n \times [0,+\infty)), \tag{1.3}$$

where $u(\,\cdot\,,t)$ *satisfies condition* B. *More generally, for* $l = \{3,4,5,\cdots\}$ *and* $\epsilon_0 > 0$, *there holds*

$$\sup_{x\in\mathbb{R}^n} |D^l u(x,t)| \leq C, \quad \forall\, t \in (\epsilon_0, +\infty), \tag{1.4}$$

where C *depends only on* $n, \lambda, \Lambda, \frac{1}{\epsilon_0}$.

The existence results are based in a prior estimates on u. P. L. Lions and M. Musiela [15] introduced a class of fully nonlinear parabolic equations where the convexity properties of the solutions are preserved. So we are able to derive a positive lower bound and an upper bound for the eigenvalues of D^2u. By the Krylov-Safonov Theorem, we obtain the C^α norm of D^2u. But it seems difficult to get the bound of D^3u only using interior Schauder estimates without the assumption of $\sup\limits_{x\in\mathbb{R}^n} |Du_0| < +\infty$. To overcome the difficulty, we will use the blow-up argument to prove

$$\sup_{\substack{x\in\mathbb{R}^n \\ t\geq\epsilon_0}} |D^3 u| < +\infty,$$

and further establish (1.4) by interior Schauder estimates. Here we do not need the gradient bound of u_0.

Consider the following Monge-Ampère type equation:

$$\det D^2 u = \exp\Big\{n\Big(u - \frac{1}{2}\sum_{i=1}^n x_i \frac{\partial u}{\partial x_i}\Big)\Big\}. \tag{1.5}$$

According to the definition in [9], we can show that an entire solution to (1.5) is a self-expanding solution to Lagrangian mean curvature flow in the pseudo-Euclidean space.

The following theorem shows that we can obtain the self-expanding solutions by the logarithmic Monge-Ampère flow.

Theorem 1.2 *Let* $u_0 : \mathbb{R}^n \to \mathbb{R}$ *be a* C^2 *function which satisfies condition* B. *Assume that*

$$\lim_{\tau\to+\infty} \tau^{-2} u_0(\tau x) = U_0(x) \tag{1.6}$$

for some $U_0(x) \in C^2(\mathbb{R}^n)$. *Let* $u(x,t)$ *and* $U(x,t)$ *be two solutions to* (1.1) *with initial values* $u_0(x)$ *and* $U_0(x)$ *respectively. Then*

$$\lim_{t\to+\infty} t^{-1}u(\sqrt{t}x,t) = U(x,1). \tag{1.7}$$

Here the convergence is uniform and smooth in any compact subset of $\mathbb{R}^n$, *and* $U(x,1)$ *is a smooth self-expanding solution of* (1.2).

To describe the asymptotic behavior of Lagrangian mean curvature flow (1.2), we will prove the following theorem.

Theorem 1.3 *Suppose that* u_0 *is a smooth function which satisfies condition* B *and*

$$\sup_{x\in\mathbb{R}^n} |Du_0(x)|^2 < +\infty.$$

Then the evolution equation of the mean curvature flow (1.2) *has a long-time smooth solution and the graph* $(x, Du(x,t))$ *converges to a plane in* $\mathbb{R}^{2n}_n$ *as* t *goes to infinity. If additionally* $|Du_0(x)| \to 0$ *as* $|x| \to \infty$, *then the graph* $(x, Du(x,t))$ *converges smoothly on any compact sets to the coordinate plane* $(x,0)$ *in* $\mathbb{R}^{2n}_n$.

This paper is organized as follows. In Section 2, we show that the mean curvature flow (1.2) is equivalent to logarithmic Monge-Ampère flow and then Theorem 1.1 is proved. By Theorem 1.1, we can present the proof of Theorem 1.2. In Section 3, we obtain the convergence results by the decay estimates of the logarithmic Monge-Ampère flow.

2 Logarithmic Monge-Ampère Flow

Throughout the following Einstein's convention of summation over repeated indices will be adopted.

Let $(x^1,\cdots,x^n;y^1,\cdots,y^n)$ be null coordinates in $\mathbb{R}^{2n}_n$. Then the indefinite metric (cf. [24]) is defined by

$$\mathrm{d}s^2 = \frac{1}{2}\mathrm{d}x^i\mathrm{d}y^i. \tag{2.1}$$

Suppose that u is a smooth convex function. The graph M of ∇u can be written as

$$\Big(x^1,\cdots,x^n;\frac{\partial u}{\partial x^1},\cdots,\frac{\partial u}{\partial x^n}\Big).$$

Then the induced Riemannian metric on M is defined by

$$\mathrm{d}s^2 = \frac{\partial^2 u}{\partial x^i\partial x^j}\mathrm{d}x^i\mathrm{d}x^j.$$

Choose a tangent frame field $\{e_1,\cdots,e_n\}$ along M, where

$$e_i = \frac{\partial}{\partial x^i} + \frac{\partial^2 u}{\partial x^i\partial x^j}\frac{\partial}{\partial y^j}.$$

We use $\langle\cdot\,,\cdot\rangle$ to denote the inner product induced from (2.1). Then

$$\langle e_i,\ e_j\rangle = \frac{\partial^2 u}{\partial x^i\partial x^j}.$$

Let $\{\eta_1, \cdots, \eta_n\}$ be the normal frame field of M in $\mathbb{R}^{2n}_n$ defined by

$$\eta_i = \frac{\partial}{\partial x^i} - \frac{\partial^2 u}{\partial x^i \partial x^j} \frac{\partial}{\partial y^j}$$

with

$$\langle \eta_i, \ \eta_j \rangle = -\frac{\partial^2 u}{\partial x^i \partial x^j}.$$

The mean curvature vector of M is given by

$$\overrightarrow{H} = -\frac{1}{2ng} \frac{\partial g}{\partial x^l} g^{lk} \eta_k,$$

where $g = \det D^2 u$.

Suppose that $u(x,t)$ is a strictly convex smooth function in $\mathbb{R}^n$, and

$$F(x(t),t) = \Big(x^1, \cdots, x^n; \frac{\partial u}{\partial x^1}, \cdots, \frac{\partial u}{\partial x^n}\Big)$$

satisfies (1.2). Then

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = -\frac{1}{2ng} \frac{\partial g}{\partial x^l} g^{li}, \quad \frac{\mathrm{d}u_j}{\mathrm{d}t} = \frac{1}{2ng} \frac{\partial g}{\partial x^l} g^{lk} \frac{\partial^2 u}{\partial x^k \partial x^j}, \quad i,j = 1,2,\cdots,n,$$

where $u_j = \frac{\partial u}{\partial x^j}$, $[g_{ij}] = D^2 u$, $[g^{ij}] = [g_{ij}]^{-1}$. However,

$$\frac{\mathrm{d}u_j}{\mathrm{d}t} = \frac{\partial u_j}{\partial t} + \frac{\partial u_j}{\partial x^k} \frac{\mathrm{d}x^k}{\mathrm{d}t}, \quad j = 1,2,\cdots,n.$$

So

$$\begin{aligned} \frac{\partial u_j}{\partial t} &= \frac{1}{2ng} \frac{\partial g}{\partial x^l} g^{lk} \frac{\partial^2 u}{\partial x^k \partial x^j} + \frac{1}{2ng} \frac{\partial g}{\partial x^l} g^{lk} \frac{\partial^2 u}{\partial x^k \partial x^j} \\ &= \frac{1}{ng} \frac{\partial g}{\partial x^l} g^{lk} g_{kj} = \frac{1}{n} \frac{\partial}{\partial x^j} \ln g, \quad j = 1,2,\cdots,n. \end{aligned}$$

Then $u(x,t)$ satisfies (1.1).

Conversely, if $u(x,t)$ is a strictly convex smooth function in $\mathbb{R}^n$, then we define in the obvious way

$$\widetilde{F}(x,t) = \Big(x^1, \cdots, x^n; \frac{\partial u}{\partial x^1}, \cdots, \frac{\partial u}{\partial x^n}\Big).$$

Let $r : \mathbb{R}^n \times (0,T) \to \mathbb{R}^n$ be the solution of the following systems of ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}x^i}{\mathrm{d}t} = -\dfrac{1}{2ng} \dfrac{\partial g}{\partial x^l} g^{li}, & i = 1,2,\cdots,n, \\ x^i(0) = x^i, & i = 1,2,\cdots,n. \end{cases}$$

Then r_t is a family of diffeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ and $F(x,t) = \widetilde{F}(r(x,t),t)$ is a solution to (1.2).

In summary, by the regularity theory of parabolic equations, we have the following results.

Proposition 2.1 *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex C^2 function. Then* (1.1) *admits a strictly convex smooth solution on $\mathbb{R}^n \times (0,T)$ with initial value $u(x,0) = u_0(x)$ if and only* *if* (1.2) *admits a smooth solution* $F(x,t)$ *on* $\mathbb{R}^n \times (0,T)$ *with strictly convex potential and with initial condition* $F(x,0) = (x, \nabla u_0(x))$. *In particular, there exists a family of diffeomorphisms* $r(x,t) : \mathbb{R}^n \to \mathbb{R}^n$ *for* $t \in (0,T)$ *such that* $F(x,t) = (r(x,t), \nabla u(r(x,t),t))$ *solves* (1.2) *on* $\mathbb{R}^n \times (0,T)$.

A solution $F(\,\cdot\,,t)$ to (1.2) is called self-expanding if it has the form

$$M_t = \sqrt{t}\, M_1 \quad \text{for all } t > 0, \tag{2.2}$$

where $M_t = F(\,\cdot\,,t)$.

Assume that $F(x,t)$ is a self-expanding solution to (1.2). Following Proposition 2.1 , $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} - \frac{1}{n}\ln\det D^2 u = 0, \quad t > 0,\ x \in \mathbb{R}^n. \tag{2.3}$$

Hence,

$$D\Big(u(x,t) - tu\Big(\frac{x}{\sqrt{t}}, 1\Big)\Big) = 0,$$

i.e.,

$$u(x,t) = tu\Big(\frac{x}{\sqrt{t}}, 1\Big), \quad t > 0. \tag{2.4}$$

Thus combining (2.3), (2.4) and letting $t = 1$, we can verify that $u(x,1)$ satisfies (1.5).

We want to use the method of continuity and finite approximation to prove the solvability of (1.1).

Definition 2.1 *Given any* $T > 0$, $R > 0$, $1 > \alpha > 1$ *and set*

$$B_R = \{x \mid |x| < R,\ x \in \mathbb{R}^n\}, \quad B_{R,T} = \{x \mid |x| < R, x \in \mathbb{R}^n\} \times (0,T),$$
$$PB_{R,T} = B_R \times \{t = 0\} \cup \partial B_R \times (0,T).$$

Let $\tau \in [0,1]$. *We say* $u \in C^{5+\alpha, \frac{5+\alpha}{2}}(B_R \times (0,T)) \cap C(\overline{B}_R \times [0,T))$ *is a solution to* $(\star_\tau)$ *if* u *satisfies*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\tau}{n}\ln\det D^2 u - (1-\tau)\triangle u = 0, & (x,t) \in B_{R,T}, \\ u = u_0(x), & (x,t) \in PB_{R,T}. \end{cases} \tag{2.5}$$

Clearly, there exists a unique solution $u(x,t)$ which satisfies (2.5) with $\tau = 0$. Let

$$\mathrm{I} = \{\tau \in [0,1] : (\star_\tau) \text{ has a solution}\}.$$

The long time existence of the flow (1.2) holds if I is both closed and open and $R = +\infty$. To prove that the classical solutions to (1.1) must be strictly convex, we need the following lemma which is proved by P. L. Lions, M. Musiela (cf. [15, Theorem 3.1]).

Lemma 2.1 *Let* $u : B_{R,T} \to \mathbb{R}$ *be a classical solution of a fully nonlinear equation of the form*

$$\frac{\partial u}{\partial t} = F(D^2 u),$$

where F is a C^2 function defined on the cone Γ of definite symmetric matrices, which is monotone increasing (that is, $F(A) \leq F(A+B)$ whenever B is a positive definite matrix), and the function

$$F^*(A) = -F(A^{-1})$$

is concave on Γ_+ of positive definite symmetric matrices. If $D^2u \geq 0$ for $(x,t) \in PB_{R,T}$, then $D^2u \geq 0$ for $(x,t) \in B_{R,T}$.

By making use of Lemma 2.1, we obtain the following result.

Corollary 2.1 *Suppose that $u : B_{R,T} \to \mathbb{R}$ is a classical solution of a fully nonlinear equation of the form*

$$\frac{\partial u}{\partial t} = F(D^2u),$$

where F satisfies the conditions in Lemma 2.1 and F is concave on the cone Γ_+. If $\lambda I \leq D^2u \leq \Lambda I$ for $(x,t) \in PB_{R,T}$, then $\lambda I \leq D^2u \leq \Lambda I$ for $(x,t) \in B_{R,T}$.

Proof Step 1 We will show that $D^2u \geq \lambda I$ for $(x,t) \in B_{R,T}$. In fact, B. Andrews proved the conclusions (cf. Theorem 3.3 in [1]). Here we present another proof.

Set $\overline{u} = u - \frac{\lambda}{2}|x|^2$. Then $\overline{u}$ satisfies

$$\frac{\partial \overline{u}}{\partial t} = F(D^2\overline{u} + \lambda I)$$

with $D^2\overline{u} \geq 0$ for $(x,t) \in PB_{R,T}$. Define

$$\begin{aligned}
&\overline{F}(D^2\overline{u}) = F(D^2\overline{u} + \lambda I),\\
&\overline{F}^*(A) = -F(A^{-1} + \lambda I),\\
&\overline{F}^*(\lambda_1, \lambda_2, \cdots, \lambda_n) = -F(\lambda_1^{-1} + \lambda, \lambda_2^{-1} + \lambda, \cdots, \lambda_n^{-1} + \lambda),\\
&\Sigma = \{\lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_n > 0\}.
\end{aligned}$$

It follows from [2] that $\overline{F}^*(A)$ is concave on Γ_+ if and only if $\overline{F}^*(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is concave on Σ. Note that for all $\xi \in \mathbb{R}^n$,

$$\frac{\partial^2 \overline{F}^*}{\partial \lambda_i \partial \lambda_j} \xi_i \xi_j = -F_{ij} \overline{\xi}_i \overline{\xi}_j - 2F_i \lambda_i \overline{\xi}_i^2,$$

where $\overline{\xi}_i = \frac{\xi_i}{\lambda_i^2}$. Since $F^*(A) = -F(A^{-1})$ is concave on Γ_+, we have

$$-F_{ij} \overline{\xi}_i \overline{\xi}_j |_{\lambda=0} - 2F_i \lambda_i \overline{\xi}_i^2 |_{\lambda=0} \leq 0.$$

Replace λ_i by $\frac{\lambda_i}{1+\lambda\lambda_i}$. So

$$-F_{ij} \overline{\xi}_i \overline{\xi}_j \leq 2F_i \frac{\lambda_i}{1+\lambda\lambda_i} \overline{\xi}_i^2.$$

Clearly,

$$\frac{\partial^2 \overline{F}^*}{\partial \lambda_i \partial \lambda_j} \xi_i \xi_j = -F_{ij} \overline{\xi}_i \overline{\xi}_j - 2F_i \lambda_i \overline{\xi}_i^2 \leq 2F_i \frac{\lambda_i}{1+\lambda\lambda_i} \overline{\xi}_i^2 - 2F_i \lambda_i \overline{\xi}_i^2 \leq 0.$$

Therefore $D^2\overline{u} \geq 0$ for $(x,t) \in B_{R,T}$ by Lemma 2.1.

Step 2 We prove that $D^2u \leq \Lambda I$ for $(x,t) \in B_{R,T}$.

Introduce the Legendre transformation of u

$$\tau = t, \quad y^i = \frac{\partial u}{\partial x^i}, \quad i = 1, 2, \cdots, n, \quad u^*(y^1, \cdots, y^n) := \sum_{i=1}^n x^i \frac{\partial u}{\partial x^i} - u(x).$$

In terms of $\tau, y^1, \cdots, y^n, u^*(y^1, \cdots, y^n, \tau)$, one can easily check that

$$\frac{\partial u^*}{\partial \tau} = -\frac{\partial u}{\partial t}, \quad \frac{\partial^2 u^*}{\partial y^i \partial y^j} = \Big[\frac{\partial^2 u}{\partial x^i \partial x^j}\Big]^{-1}.$$

Then u^* is a solution of the form

$$\frac{\partial u^*}{\partial \tau} = F^*(D^2 u^*).$$

Since $F^{**} = F$ is concave on the cone Γ_+, using the conclusions of Step 1, we arrive at $D^2 u^* \geq \frac{1}{\Lambda} I$ for $(x,t) \in B_{R,T}$ and this yields our desired result.

Given $x_0 \in \mathbb{R}^n$, $\kappa > 0$, define

$$Q_{R,x_0} = \{x \,|\, |x - x_0| \leq R\} \times [\kappa, \kappa + R), \quad Q_{\frac{R}{2},x_0} = \Big\{x \,\Big|\, |x - x_0| \leq \frac{R}{2}\Big\} \times \Big[\kappa + \frac{R}{4}, \kappa + \frac{R}{2}\Big),$$
$$Q_{\frac{R}{3},x_0} = \Big\{x \,\Big|\, |x - x_0| \leq \frac{R}{3}\Big\} \times \Big[\kappa + \frac{R}{3}, \kappa + \frac{5R}{12}\Big), \quad B_{R,x_0} = \{|x - x_0| \leq R\}.$$

The following two lemmas which will be mentioned below may be used repeatedly (cf. [14]).

Lemma 2.2 (cf. [3, Lemma 14.6]) *Let $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ be a classical solution of a fully nonlinear equation of the form*

$$\begin{cases} \dfrac{\partial u}{\partial t} - F(D^2 u) = 0, & t > 0, \ x \in \mathbb{R}^n, \\ u = u_0(x), & t = 0, \ x \in \mathbb{R}^n, \end{cases}$$

where F is a C^2 concave function defined on the cone Γ of definite symmetric matrices, which is monotone increasing with

$$\lambda I \leq \frac{\partial F}{\partial r_{ij}} \leq \Lambda I.$$

Then there exists $0 < \alpha < 1$ such that

$$[D^2 u]_{C^{\alpha, \frac{\alpha}{2}}(\overline{Q}_{\frac{1}{2},x_0})} \leq C |D^2 u|_{C^0(\overline{Q}_{1,x_0})},$$

where α, C are positive constants depending only on $n, \lambda, \Lambda, \frac{1}{\kappa}$.

Lemma 2.3 (cf. [3, Theorem 4.9]) *Let $v : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ be a classical solution of a linear parabolic equation of the form*

$$\begin{cases} \dfrac{\partial v}{\partial t} - a^{ij} v_{ij} = 0, & t > 0, \ x \in \mathbb{R}^n, \\ v = v_0(x), & t = 0, \ x \in \mathbb{R}^n, \end{cases}$$

where there exists a positive constant C such that

$$\lambda I \leq a^{ij} \leq \Lambda I, \quad [a^{ij}]_{C^\alpha(\overline{Q}_{\frac{R}{2},x_0})} \leq C.$$

Then there holds

$$|Dv|_{C^0(\overline{Q}_{\frac{R}{3},x_0})}+|D^2v|_{C^0(\overline{Q}_{\frac{R}{3},x_0})}+[D^2v]_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_{\frac{R}{3},x_0})}\le C_3|v|_{C^0(\overline{Q}_{R,x_0})},$$

where C_3 is a positive constant depending only on n,λ,Λ and $C,R,\frac{1}{\kappa}$.

According to problem (2.5), we have the following lemma.

Lemma 2.4 I *is closed.*

Proof Suppose that u is a solution of $(\star_\tau)$. For $A\in\Gamma_+$, set

$$F(A)=\frac{\tau}{n}\ln\det A+(1-\tau)\mathrm{Tr}\,A.$$

Let $\lambda_1,\lambda_2,\cdots,\lambda_n$ be the eigenvalues of A. Define

$$f(\lambda_1,\lambda_2,\cdots,\lambda_n)=F(A)=\frac{\tau}{n}\ln\lambda_1\lambda_2\cdots\lambda_n+(1-\tau)(\lambda_1+\lambda_2+\cdots+\lambda_n),$$
$$f^*(\lambda_1,\lambda_2,\cdots,\lambda_n)=F^*(A)=\frac{\tau}{n}\ln\lambda_1\lambda_2\cdots\lambda_n-(1-\tau)\Big(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\cdots+\frac{1}{\lambda_n}\Big).$$

One can verify that D^2f,D^2f^* are negative in a cone $\Sigma=\{\lambda_1>0,\lambda_2>0,\cdots,\lambda_n>0\}$. By [2], we deduce that F,F^* are smooth concave functions defined on the cone Γ_+, which are monotone increasing.

It follows from Corollary 2.1 that if $u_0(x)$ satisfies condition B then $u(x,t)$ does so. For $T>s>0$, $R>\epsilon>0$, define

$$B_{R-\epsilon,T}=B_{R-\epsilon}\times(0,T),\quad B_{R-\epsilon}(T,s)=B_{R-\epsilon}\times(s,T).$$

Furthermore, combining Lemma 2.2 with Lemma 2.3, we have

$$\|u\|_{C^{2,1}(\overline{B}_{R-\epsilon,T})}\le C_1,\quad \|u\|_{C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{B}_{R-\epsilon}(T,s))}\le C_2,\tag{2.6}$$

where $0<\alpha<1$, C_1 is a positive constant depending only on u_0, R, T and C_2 relies on $u_0,\lambda,\Lambda,R,T,\frac{1}{\epsilon},\frac{1}{s}$. By (2.6), a diagonal sequence argument and the regularity theory of parabolic equations imply that I is closed.

To prove that I is open we need the following lemma (cf. [10, Theorem 17.6]).

Lemma 2.5 *Let $\mathcal{B}_1,\mathcal{B}_2$ and $\mathbf{X}$ be Banach spaces and G be a mapping from an open subset of $\mathcal{B}_1\times\mathbf{X}$ into $\mathcal{B}_2$. Let $(\widetilde{u},\widetilde{\tau})$ be a point in $\mathcal{B}_1\times\mathbf{X}$ satisfying that*

(i) *$G[\widetilde{u},\widetilde{\tau}]=0$,*

(ii) *G is continuously differentiable at $(\widetilde{u},\widetilde{\tau})$,*

(iii) *the partial Fréchet derivative $L=G^1_{(\widetilde{u},\widetilde{\tau})}$ is invertible.*

Then there exists a neighbourhood $\mathcal{N}$ of $\widetilde{\tau}$ in $\mathbf{X}$, such that the equation $G[u,\tau]=0$ is solvable for each $\tau\in\mathcal{N}$ with solution $u=u_\tau\in\mathcal{B}_1$.

By the implicit function theorem, we have the next lemma.

Lemma 2.6 I *is open.*

Proof Define the Banach spaces

$$\mathbf{X}=\mathbb{R},$$
$$\mathcal{B}_1=C^{2+\alpha,\frac{2+\alpha}{2}}(\overline{B}_{R,T}),$$
$$\mathcal{B}_2=C^{\alpha,\frac{\alpha}{2}}(\overline{B}_{R,T})\times C^{2+\alpha,\frac{2+\alpha}{2}}(PB_{R,T})$$

and a differentiable map from $\mathcal{B}_1 \times \mathbf{X}$ into $\mathcal{B}_2$,

$$G : (u, \tau) \to \Big[\frac{\partial u}{\partial t} - \frac{\tau}{n} \ln \det D^2 u - (1-\tau)\triangle u, u - u_0\Big].$$

We take an open set of $\mathcal{B}_1 \times \mathbf{X}$:

$$\Theta = \Big\{ u \,\Big|\, \frac{\lambda}{2} I < D^2 u(x,t) < \frac{3\Lambda}{2} I,\ u \in \mathcal{B}_1 \Big\} \times (0,1).$$

Suppose that $(u,\tau) \in \Theta$. Then the partial Fréchet derivative $L = G^1_{(u,\tau)}$ is invertible if and only if the following Cauchy-Dirichlet problem is solvable:

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\tau}{n} u^{ij} \dfrac{\partial^2 w}{\partial x^i \partial x^j} - (1-\tau)\triangle w = f, & (x,t) \in B_{R,T}, \\ w = g, & (x,t) \in PB_{R,T}, \end{cases}$$

where $(f,g) \in \mathcal{B}_2$. Using the theory of the linear parabolic equations (cf. [14, Chapter V, Theorem 5.6]) we can do it.

Thereby applying Lemma 2.5 and using approximate methods, we deduce that I is open.

Lemma 2.7 *Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function which satisfies condition* B*. Then there exists a unique strictly convex solution of* (1.1) *such that $u(x,t)$ satisfies condition* B *and* (1.3)*.*

Proof For $N \in \mathbf{Z}^+$, $T > 0$, consider the Cauchy-Dirichlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{n} \ln \det D^2 u = 0, & (x,t) \in B_{N,T}, \\ u = u_0, & (x,t) \in PB_{N,T}. \end{cases} \tag{2.7}$$

By Lemmas 2.4 and 2.6, there exists a unique strictly convex solution of (2.7). We denote it by $u_N(x,t)$. Corollary 2.1 tells us that $u_N(x,t)$ satisfies condition B. For $Q_{R,x_0} \subset\subset B_{N,T}$, by Lemmas 2.2 and 2.3, there exists a positive constant C independent of N such that

$$[D^2 u_N]_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_{\frac{R}{3},x_0})} \le C.$$

By condition B, there exists a positive constant $\widetilde{C}$ independent of N and $\frac{1}{\kappa}$ such that

$$|u_N|_{C^0(\overline{Q}_{\frac{R}{3},x_0})} + |Du_N|_{C^0(\overline{Q}_{\frac{R}{3},x_0})} + |D^2 u_N|_{C^0(\overline{Q}_{\frac{R}{3},x_0})} \le \widetilde{C}.$$

A diagonal sequence argument and the regularity theory of parabolic equations imply that we obtain the desired results.

Proof of Theorem 1.1 Using Lemma 2.7, there exists a unique strictly convex solution to (1.1) satisfying (1.3) and condition B.

By Lemma 2.2, we get

$$[D^2 u]_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_{\frac{1}{2},x_0})} \le C, \tag{2.8}$$

where C is a positive constant depending only on n, λ, Λ and $\frac{1}{\kappa}$.

We will derive higher order estimates (1.4) via the blow up argument for $l = 3$. To do so, by [1], we employ a parabolic scaling now. The remaining proof is routine. Define

$$\begin{aligned} &y = \mu(x - x_0), \quad s = \mu^2(t - t_0), \\ &u_\mu(y,s) = \mu^2[u(x,t) - u(x_0,t_0) - Du(x_0,t_0)\cdot(x - x_0)]. \end{aligned}$$

It is easy to see that

$$D_y^2 u_\mu = D_x^2 u, \quad \frac{\partial}{\partial s} u_\mu = \frac{\partial}{\partial t} u$$

and

$$D_y^l u_\mu = \mu^{2-l} D_x^l u$$

for all nonnegative integers l. By computing, $u_\mu(y,s)$ satisfies

$$\begin{cases} \dfrac{\partial u_\mu}{\partial s} - \dfrac{1}{n} \ln\det D^2 u_\mu = 0, & s>0,\ y\in\mathbb{R}^n, \\ u_\mu = u_\mu(y,s)|_{t=t_0}, & s=0,\ y\in\mathbb{R}^n \end{cases}$$

with

$$u_\mu(0,0) = Du_\mu(0,0) = 0. \tag{2.9}$$

Suppose that $|D^3u|^2$ is not bounded on $\mathbb{R}^n \times [\epsilon_0, +\infty)$. By [12, Lemma 3.5], there would be a sequence $\{t_k\}$ $(t_k \geq \epsilon_0)$ and $R_k \to +\infty$, such that

$$2\rho_k := \sup_{x\in B_{R_k,x_0}} |D^3u(x,t_k)|^2 \to +\infty \tag{2.10}$$

and

$$\sup_{\substack{x\in B_{R_k,x_0} \\ t\leq t_k}} |D^3u(x,t)|^2 \leq 2\rho_k. \tag{2.11}$$

Then there exists x_k such that

$$|D^3u(x_k,t_k)|^2 \geq \rho_k \to +\infty, \quad \text{as } k\to+\infty. \tag{2.12}$$

Let $(y, Du_{\mu_k}(y,s))$ be a parabolic scaling of $(x, Du(x,t))$ by $\mu_k = (\rho_k)^{\frac{1}{2}}$ at (x_k,t_k) for each k. Thus $u_{\mu_k}(y,s)$ is a solution of a fully nonlinear parabolic equation

$$\frac{\partial u_{\mu_k}}{\partial s} - \frac{1}{n}\ln\det D^2 u_{\mu_k} = 0, \quad 0 < s \leq \mu_k^2 t_k,\ y\in\mathbb{R}^n. \tag{2.13}$$

Combining (2.10)–(2.12) and (2.8), we arrive at

$$\lambda I \leq D_y^2 u_{\mu_k} = D_x^2 u \leq \Lambda I, \quad (y,s)\in\mathbb{R}^n\times[0,+\infty); \tag{2.14}$$

for all $y_1, y_2 \in \mathbb{R}^n,\ y_1 = \mu_k(x_1 - x_0),\ y_2 = \mu_k(x_2 - x_0)$,

$$\frac{|D_{y_1}^2 u_{\mu_k} - D_{y_2}^2 u_{\mu_k}|}{|y_1 - y_2|^\alpha} = \mu_k^{-\alpha} \frac{|D_{x_1}^2 u - D_{x_2}^2 u|}{|x_1 - x_2|^\alpha} \leq \mu_k^{-\alpha} C \to 0,$$

and

$$\begin{aligned} &|D_y^3 u_{\mu_k}|^2 = \mu_k^{-2}|D_x^3 u|^2 \leq 2, \quad \forall\, y\in\mathbb{R}^n, \\ &|D_y^3 u_{\mu_k}(0,0)| \geq 1. \end{aligned} \tag{2.15}$$

For each i, set $w = D_{x^i} u_{\mu_k}$. From (2.13), w satisfies

$$\frac{\partial w}{\partial s} - \frac{1}{n} u_{\mu_k}^{ij} w_{ij} = 0.$$

Using (2.15) and Lemma 2.2, there exists a constant C depending only on $n, \lambda, \Lambda, \frac{1}{\epsilon_0}$, such that we derive

$$[D_y^3 u_{\mu_k}]_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_{\frac{1}{2},y_0})} \le C, \quad \forall\, y \in \mathbb{R}^n. \tag{2.16}$$

Combining (2.9) and (2.14)–(2.16) together, a diagonal sequence argument shows that u_{μ_k} converges subsequentially and uniformly on any compact subsets in $\mathbb{R}^n \times [0, +\infty)$ to a smooth function u_∞ with

$$[D_y^2 u_\infty]_{C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_{\frac{1}{2},y_0})} = 0, \quad \forall (y, s) \in \mathbb{R}^n \times [0, +\infty)$$

and

$$|D_y^3 u_\infty(0,0)| \ge 1.$$

It is a contradiction. So $\sup\limits_{\substack{x\in\mathbb{R}^n \\ t\ge\epsilon_0}} |D^3 u(x,t)| \le C$. From equation (1.1), using the interior Schauder estimates, we obtain (1.4) for $l = 3, 4, 5 \cdots$.

The following lemma shows that how the self-expanding solutions are constructed by the flow (1.1).

Lemma 2.8 *If* u_0 *satisfies conditions* A *and* B. *Then* $u(x,1)$ *is a smooth solution to* (1.5).

Proof The main idea comes from [6], which we present here for completeness.

If u_0 satisfies conditions A and B. Then by Theorem 1.1, there exists a unique smooth solution $u(x,t)$ to (1.1) for all $t > 0$ with initial value u_0. One can verify that

$$u_R(x,t) := R^{-2} u(Rx, R^2 t)$$

is a solution to (1.1) with initial value

$$u_R(x,0) := R^{-2} u_0(Rx) = u_0(x).$$

Here condition A is used. Since $u_R(x,0) = u_0$, the uniqueness results in Theorem 1.1 imply

$$u(x,t) = u_R(x,t)$$

for any $R > 0$. Therefore $u(x,t)$ satisfies (2.4), and hence $u(x,1)$ solves (1.5). In other words, $u(x,1)$ is a smooth self-expanding solution.

We present here the proof of Theorem 1.2 by the methods of [6].

Proof of Theorem 1.2 Assume that

$$U_0(x) = \lim_{R\to+\infty} R^{-2} u_0(Rx).$$

So $U(x,0)$ satisfies condition B and we obtain

$$U_0(x) = \lim_{R\to\infty} R^{-2} u_0(Rx) = \lim_{R\to\infty} R^{-2} l^{-2} u_0(Rlx) = l^{-2} U_0(lx),$$

namely, $U_0(x)$ satisfies condition A. Then by Lemma 2.8, we conclude that $U(x,1)$ is a self-expanding solution.

Define

$$u_R(x,t) := R^{-2}u(Rx, R^2t).$$

It is clear that $u_R(x,t)$ is a solution to (1.1) with initial value $u_R(x,0) = R^{-2}u_0(Rx)$ satisfying condition B.

For any sequence $R_i \to +\infty$, we consider the limitation of $u_{R_i}(x,t)$. For $t > 0$, there holds

$$D^2u_{R_i}(x,t) = D^2u(R_ix, R_i^2t).$$

Using Theorem 1.1, we have

$$\lambda I \le D^2u_{R_i}(x,t) \le \Lambda I$$

for all x and $t > 0$. Moreover, according to (1.4) in Theorem 1.1, we get

$$\sup_{x\in\mathbb{R}^n} |D^l u_{R_i}(\,\cdot\,,t)| \le C, \quad \forall\, t \ge \epsilon_0,\ l = \{3,4,5\cdots\}.$$

For any $m \ge 1,\ l \ge 0$, using (1.1), there exists a constant C such that

$$\sup_{x\in\mathbb{R}^n} \Big|\frac{\partial^m}{\partial t^m} D^l u_{R_i}\Big| \le C, \quad \forall\, t \ge \epsilon_0,\ l = \{3,4,5\cdots\}.$$

We observe that

$$u_{R_i} = R_i^{-2}u_0 \quad \text{and} \quad Du_{R_i}(0,0) = R_i^{-1}Du_0(0)$$

are both bounded. Thus $u_{R_i}(0,t)$ and $Du_{R_i}(0,t)$ are uniformly bounded with respect to i for any fixed t. By the Arzelà-Ascoli theorem, there exists a subsequence $\{R_{k_i}\}$ such that $u_{R_{k_i}}(x,t)$ converges uniformly to a solution $\widehat{U}(x,t)$ to (1.1) in any compact subsets of $\mathbb{R}^n \times (0,\infty)$, and $\widehat{U}(x,t)$ satisfies the estimates in Theorem 1.1. Since $\frac{\partial \widehat{U}}{\partial t}$ is uniformly bounded for any $t > 0$, $\widehat{U}(x,t)$ converges to some function $\widehat{U}_0(x)$ when $t \to 0$. One can verify that

$$\begin{aligned}
\widehat{U}_0(x) &= \lim_{t\to 0} \widehat{U}(x,t) \\
&= \lim_{t\to 0} \lim_{i\to+\infty} R_i^{-2}u(R_ix, R_i^2t) \\
&= \lim_{i\to+\infty} \lim_{t\to 0} R_i^{-2}u(R_ix, R_i^2t) \\
&= \lim_{i\to+\infty} R_i^{-2}u_0(R_ix) \\
&= U_0(x).
\end{aligned}$$

By the uniqueness results, the above limit is independent of the choice of the subsequence $\{R_i\}$ and

$$\widehat{U}(x,t) = U(x,t).$$

So, letting $R = \sqrt{t}$, we have $t^{-1}u(\sqrt{t}\,x,t) = u_{\sqrt{t}}(x,1)$ converging to $U(x,1)$ uniformly in compact subsets of $\mathbb{R}^n$ when $t \to +\infty$. Theorem 1.2 is established.

At the end of this section, we present the following Bernstein theorem for equation (1.5).

Proposition 2.2 *Let*

$$w = u - \frac{1}{2}\langle x, Du\rangle.$$

If u is a C^2 strictly convex solution to (1.5) *and w takes its maximum or minimum at some point $x \in \mathbb{R}^n$ with $|x| < +\infty$. Then u must be a quadratic polynomial.*

Proof It follows from Caffarelli's regularity theory of Monge-Ampère type equations and interior Schauder estimates that u is a smooth strictly convex solution. From (1.5), w satisfies

$$u^{ij}w_{ij} = \frac{1}{2}\langle x, Dw\rangle.$$

Since w takes its maximum or minimum at some point $x \in \mathbb{R}^n$ with $|x| < +\infty$. For every $R > 0$, by strong maximum principle (cf. [18]), we deduce that w must be some constant in $B_R(x)$ also in $\mathbb{R}^n$. Using the Pogorelov's theorem in [17], we show that u must be a quadratic polynomial.

3 Longtime Existence and Convergence

As in [5], we can also show that a bound on the height of the graphs is preserved along (1.1).

Lemma 3.1 *If* $u(x,t)$ *is a smooth solution to* (1.1) *and* $\sup_{x\in\mathbb{R}^n} |Du_0(x)|^2 < +\infty$. *Then*

$$\sup_{x\in\mathbb{R}^n} |Du(x,t)|^2 \leq \sup_{x\in\mathbb{R}^n} |Du_0(x)|^2. \tag{3.1}$$

Proof By (1.1), we have

$$\frac{\partial}{\partial t}|Du(x,t)|^2 - \frac{1}{n}u^{ij}(|Du(x,t)|^2)_{ij} = -\frac{2}{n}u^{pq}u_{pi}u_{qi} \leq 0.$$

Using Lemma 4.2 in [23], we obtain the desired results.

To obtain the convergence of the flow (1.2), we introduce the following decay estimates of the higher order derivatives based on Theorem 1.1 (cf. [13, Theorem 1.3]).

Proposition 3.1 *Assume that* $u(x,t)$ *is a strictly convex solution to* (1.1) *satisfying* (1.3) *and condition* B. *Then there exists a constant* C *depending only on* $n, \lambda, \Lambda, \frac{1}{\epsilon_0}$ *such that*

$$\sup_{x\in\mathbb{R}^n} |D^3u(\,\cdot\,,t)| \leq \frac{C}{t}, \quad \forall\, t \geq \epsilon_0. \tag{3.2}$$

More generally, for all $l = \{3, 4, 5, \cdots\}$ *there holds*

$$\sup_{x\in\mathbb{R}^n} |D^l u(\,\cdot\,,t)| \leq \frac{C}{t^{l-2}}, \quad \forall\, t \geq \epsilon_0. \tag{3.3}$$

Proof of Theorem 1.3 By Theorem 1.1 and Proposition 2.1, (1.2) admits a long-time smooth solution.

Using (3.3) and (3.1), a diagonal sequence argument shows that as $t \to \infty$, $Du(x,t)$ converges subsequentially and uniformly on any compact subsets of $\mathbb{R}^n$ to a smooth function Du_∞ with $|D_y^l u_\infty| = 0,\ \forall\, y \in \mathbb{R}^n$ for $l \geq 3$. So Du_∞ must be an affine linear function and $(x, Du_\infty(x))$ an affine linear subspace. It shows that the graph of the mean curvature flow (1.2) converges to a plane in $\mathbb{R}^{2n}_n$.

As the proof of Theorem 1.1 in [5], if $|Du_0(x)| \to 0$ as $|x| \to \infty$, then the graph $(x, Du(x,t))$ converges smoothly on any compact sets to the coordinate plane $(x, 0)$ in $\mathbb{R}^{2n}_n$.

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