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Trajectory Attractors for Binary Fluid Mixtures in 3D***

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(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract Two different models for the evolution of incompressible binary fluid mixtures in a three-dimensional bounded domain are considered. They consist of the 3D incompressible Navier-Stokes equations, subject to time-dependent external forces and coupled with either a convective Allen-Cahn or Cahn-Hilliard equation. Such systems can be viewed as generalizations of the Navier-Stokes equations to two-phase fluids. Using the trajectory approach, the authors prove the existence of the trajectory attractor for both systems.

Keywords Navier-Stokes equations, Allen-Cahn equations, Cahn-Hilliard equations, Two-phase fluid flows, Longtime behavior, Trajectory attractors
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1 Introduction

Modelling the behavior of binary fluid mixtures can be rather challenging (see, e.g., [57]). A possible approach is based on the so-called diffuse-interface method (see [5, 14, 53] and references therein). This method consists in introducing an order parameter, accounting for the presence of two species, whose dynamics interacts with the fluid velocity. For incompressible fluids a well-known model, known as Cahn-Hilliard fluid, consists of the classical Navier-Stokes equations suitably coupled with a convective Cahn-Hilliard equation (see [33, 34], also [6, 17, 37, 42, 48, 54, 61] and references therein). In related contexts there have also been considered models in which the Cahn-Hilliard equation is replaced by the (convective) Allen-Cahn equation (see, e.g., [9, 25, 26, 29, 63, 67]) or, in the case of liquid crystals, by the convective Ginzburg-Landau equation (see [43], also [22, 23, 44, 47] and references therein). Denoting by $\boldsymbol{u} = (u_1, u_2, u_3)$ the velocity field and by ϕ the order parameter, the Cahn-Hiliard-Navier-Stokes and the Allen-Cahn-Navier-Stokes systems can be written in a unified form. Indeed, if we assume that the density is constant and equal to one, the kinematic viscosity $\nu(\phi) > 0$ and temperature differences are

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negligible, we have

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \operatorname{div}(\nu(\phi) D\boldsymbol{u}) + \nabla p = \kappa \mu \nabla \phi + \boldsymbol{g}, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1.2}$$

$$\partial_t \phi + \boldsymbol{u} \cdot \nabla \phi + A_K(\phi)\mu = 0, \tag{1.3}$$

$$\mu = -\varepsilon \Delta \phi + \alpha f(\phi), \tag{1.4}$$

in $\Omega \times (0, +\infty)$, where Ω is a bounded domain in $\mathbb{R}^3$ with a sufficiently smooth boundary Γ, $D\boldsymbol{u} = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}}}{2}$ is the deformation tensor, $\boldsymbol{g} = \boldsymbol{g}(t)$ is an external body force and $\kappa > 0$ is a capillarity (stress) coefficient. Moreover, the operator A_K has a twofold definition according to the case $K = CH$ (Cahn-Hilliard fluid) or $K = AC$ (Allen-Cahn fluid), namely,

$$A_{CH}(\phi)\mu = -\operatorname{div}(\mathcal{M}(\phi)\nabla \mu), \quad A_{AC}(\phi)\mu = \mu,$$

where $\mathcal{M}(\phi) > 0$ is the mobility of the mixture. The so-called chemical potential μ is obtained as a variational derivative of the following free energy functional

$$\mathcal{F}(\phi) = \int_\Omega \Big(\frac{\varepsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \Big) \mathrm{d}x. \tag{1.5}$$

Here $F(r) = \int_0^r f(y)\mathrm{d}y$, $r \in \mathbb{R}$, and ε, α are two positive parameters describing the interaction between the two phases. In particular, we recall that ε is related with the thickness of the interface separating the two fluids. It is thus reasonable to assume from now on that $\varepsilon \leq \alpha$ (since, in practice, α is always of the order of $\frac{1}{\varepsilon}$). This restriction is only needed in the case $K = AC$ (see [32]). The potential F is a double-well logarithmic-type function defined on a bounded interval (see [15]). In this case, F is usually named singular (or nonsmooth) potential. However, F is often replaced by a polynomial approximation of the type, e.g., $F(r) = c_1 r^4 - c_2 r^2$, where c_1 and c_2 are given positive constants. We also note that (1.1) can be equivalently rewritten in the following form:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \operatorname{div}(\nu(\phi) D\boldsymbol{u}) + \nabla \widetilde{p} = -\kappa \operatorname{div}(\nabla \phi \otimes \nabla \phi) + \boldsymbol{g}, \tag{1.6}$$

with $\widetilde{p} = p - \kappa(\frac{\varepsilon}{2}|\nabla \phi|^2 + \alpha F(\phi))$, on account of (see, e.g., [1])

$$\kappa \mu \nabla \phi = \kappa \nabla \Big(\frac{\varepsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \Big) - \kappa \operatorname{div}(\nabla \phi \otimes \nabla \phi). \tag{1.7}$$

The stress tensor $\kappa \operatorname{div}(\nabla \phi \otimes \nabla \phi)$ (also named Korteweg force) is considered the main contribution in (1.7) modelling capillary forces due to surface tension at the interface between the two phases of the fluid. However, we prefer to using equation (1.1) instead of (1.6), since energy estimates can be obtained more conveniently (see, e.g., Theorem 3.1).

Systems like (1.1)–(1.4) in the case $K = CH$ have been investigated in a number of papers (see, for instance, [12, 13, 36, 39, 41, 46]). However, confining ourselves to the most theoretical aspects (i.e., well-posedness, regularity, asymptotic behavior), to the best of our knowledge the first results can be found in [62], where the 2D case with smooth potential and constant viscosity and mobility was analyzed on the whole $\mathbb{R}^2$. A more refined analysis for bounded domains which includes the 3D case is contained in [10] (see also [11] for the nonhomogeneous case). In that contribution both singular and smooth potentials were considered as well as concentration dependent viscosity and nonconstant degenerate mobility (i.e., $\mathcal{M}(\pm 1) = 0$). We recall that this assumption forces ϕ to take values in a given bounded interval (i.e., the domain of F). More recently, singular potentials without nonconstant mobility and viscosity depending on ϕ have been considered in [1]. In this case the analysis requires nontrivial arguments, especially to establish some regularity properties of weak solutions. Non-Newtonian Cahn-Hilliard fluids were examined in [21, 40], while the compressible case has been recently carefully studied in [3]. All these contributions are mainly concerned with existence, uniqueness and regularity issues (see also [4] for the asymptotic limit as ε goes to 0, with $\alpha = \varepsilon^{-1}$ and $\kappa = \varepsilon$). Regarding the longtime behavior, results about the stability of some stationary solutions were given in [2, 10], while a theorem about the convergence to single equilibria was proven in [1] via Łojasiewicz-Simon approach. More recently, the 2D case on bounded domains with smooth potentials and constant viscosity has been investigated in [31] within the theory of dissipative dynamical systems. In particular, existence of global and exponential attractors have been established and an estimate of the fractal dimension of the global attractor in terms of ν, ε and α has been obtained. It has also been proven the convergence to single stationary solutions along with an explicit rate estimate (see also [68] for further results). The existence of a strong global attractor $\mathcal{A}$ for the above system in 2D has also been established in [2] for a singular potential, but no further properties of $\mathcal{A}$ have been demonstrated: for instance, the finite dimensionality.

Similar results have been proven in [32] for $K = AC$ always in two-dimensional bounded domains (see [35, 60, 66] for related results on nematic liquid crystal dynamics). Existence of a weak solution for compressible Allen-Cahn-Navier-Stokes systems has been recently proven in [28].

This paper is devoted to the analysis of the global dynamics of solutions to system (1.1)–(1.4) in the 3D case and, in contrast with most of the quoted papers, here we allow the presence of a time-dependent external nongradient force (see, e.g., [8] for its role in coarsening processes). The existence of a weak global attractor in the three dimensional case for $K = CH$ with $\boldsymbol{g} = \mathbf{0}$ was firstly established in [2] for singular potentials, following the approach developed in [20]. Here we follow a different strategy which allows us to say more on the global longterm dynamics under rather general conditions (e.g., on the body force $\boldsymbol{g}$).

The choice of the notion of attractor we are looking for is indeed essential, due the lack of uniqueness. In the classical theory of dissipative systems, it is usually required that the solution operator, which maps the initial condition to the solution, be well defined and continuous in a proper phase space. This theory has been successfully applied to many nonlinear differential equation of mathematical physics (see, for instance, [55, 59, 64] and references therein). On the other hand, concerning ill-posed problems, there exist basically two approaches to handle dissipative systems without uniqueness (see, however, [24] for a method inspired by nonstandard analysis). The first one allows the solution operator to be multi-valued and, accordingly, extends the theory of global attractors to the case of semigroups of multi-valued maps (see [16, 51, 52], while for 3D incompressible Navier-Stokes, see [7, 20, 38, 56] and references therein). The second is a more geometric approach which consists in taking as phase space the so-called trajectory space and the translation semigroup acting on them. This operator is single-valued so that the usual theory of attractors can be adapted (see [18, 19], also [30] and the pioneering [58]). We intend to apply this approach which seems more effective in presence of time-dependent body forces. It is also worth mentioning that, compared to the Navier-Stokes equation, the main technical difficulty of the paper is finding suitable dissipative estimates for the Leray-Hopf solutions of (1.1)–(1.4) subject to the boundary and initial conditions detailed below (see

Proposition 3.2 and Lemma 4.2), which requires employing different arguments when $K = AC$ and $K = CH$.

System (1.1)–(1.4) is subject to given initial conditions

$$\boldsymbol{u}_{|t=\tau} = \boldsymbol{u}_0, \quad \phi_{|t=\tau} = \phi_0, \quad \text{in } \Omega, \tag{1.8}$$

for a fixed $\tau \geq 0$.

As far as boundary conditions are concerned, for the velocity field we assume no-slip boundary conditions

$$\boldsymbol{u} = \mathbf{0}, \quad \text{on } \Gamma \times (\tau, +\infty), \tag{1.9}$$

while for ϕ we choose no-flux boundary conditions, namely,

$$\partial_{\boldsymbol{n}} \phi = 0, \quad \text{on } \Gamma \times (\tau, +\infty), \tag{1.10}$$

if $K = AC$, and

$$\partial_{\boldsymbol{n}} \phi = \partial_{\boldsymbol{n}} \Delta \phi = 0, \quad \text{on } \Gamma \times (\tau, +\infty), \tag{1.11}$$

in the case $K = CH$. However, the validity of our results can also be proven when other types of boundary conditions are imposed (see, for instance, [10]).

Noting that in the case $K = CH$, we have

$$\partial_{\boldsymbol{n}} \mu = 0, \quad \text{on } \Gamma \times (\tau, +\infty).$$

So that, setting,

$$\langle \phi(t) \rangle_\Omega = \frac{1}{|\Omega|} \int_\Omega \phi(t) \mathrm{d}x, \tag{1.12}$$

where $|\Omega|$ is the Lebesgue measure of Ω, we have the mass conservation,

$$\langle \phi(t) \rangle_\Omega = \langle \phi_0 \rangle_\Omega, \quad \forall\, t \geq \tau. \tag{1.13}$$

Concerning the nonlinearity, we suppose that $f \in C^1(\mathbb{R}; \mathbb{R})$ satisfies

$$\begin{cases} \liminf\limits_{|y| \to +\infty} f'(y) > 0, \\ |f(y)| \leq c_f (1 + |y|^3), \quad \forall y \in \mathbb{R}, \end{cases} \tag{1.14}$$

where c_f is some positive constant. Note that the derivative f of the typical double-well potential F satisfies both assumptions (1.14). We also assume that both the viscosity and mobility functions $\nu,\ \mathcal{M}$ belong to $C^1(\mathbb{R}, \mathbb{R}^+)$ and satisfy

$$\nu_1 \geq \nu(s) \geq \nu_0 > 0, \quad m_1 \geq \mathcal{M}(s) \geq m_0 > 0, \quad \forall\, s \in \mathbb{R}. \tag{1.15}$$

The plan of the paper goes as follows. In Section 2, we introduce the functional setup and the class of Leray-Hopf solutions for problems (1.1)–(1.4), (1.9), (1.10) (if $K = AC$) or (1.11) (if $K = CH$) and (1.8). In Section 3, we prove the existence of weak solutions and define the corresponding trajectory space. Section 4, is devoted to establish the existence of the trajectory attractor for our problems.

2 Functional Setup and Leray-Hopf Solutions

The main goal of this section is to introduce the class of weak solutions which we use to define the trajectory dynamical systems in both the cases $K = AC$ and $K = CH$. This is a natural generalization of the well-known notion of Leray-Hopf solution (see, e.g., [59, Chapter 6]).

First we need to fix some notation. If X is a (real) Hilbert space with inner product $(\cdot\,,\cdot)_X$, the induced norm will be denoted by $|\cdot|_X$ and X^* will be the dual space. Moreover, we set $\mathbb{X} = X^3$ endowed with the product structure. Let us now introduce the classical functional spaces related to Navier-Stokes equations (see, for instance, [65])

$$\mathbb{H} = \overline{\{\boldsymbol{u} \in \mathbb{C}_0^\infty(\Omega) : \operatorname{div} \boldsymbol{u} = 0, \quad \text{in } \Omega\}}^{\mathbb{L}^2(\Omega)},$$
$$\mathbb{V} = \overline{\{\boldsymbol{u} \in \mathbb{C}_0^\infty(\Omega) : \operatorname{div} \boldsymbol{u} = 0, \quad \text{in } \Omega\}}^{\mathbb{H}^1(\Omega)}.$$

The canonical scalar product of the Hilbert space $\mathbb{H}$ is denoted by $(\cdot\,,\cdot)$ and $|\cdot|$ is the induced norm. Moreover, the scalar product and the related norm in the Hilbert space $\mathbb{V}$ are defined by

$$((\boldsymbol{u},\boldsymbol{v})) = \sum_{i=1}^{3}(\partial_{x_i}\boldsymbol{u}, \partial_{x_i}\boldsymbol{v}), \quad \|\boldsymbol{u}\| = ((\boldsymbol{u},\boldsymbol{u}))^{\frac{1}{2}}.$$

We recall that the norm in $\mathbb{V}$ is equivalent to the standard $\mathbb{H}^1(\Omega)$-norm, due to the Poincaré inequality

$$|\boldsymbol{v}| \le C_\Omega \|\boldsymbol{v}\|, \quad \forall \boldsymbol{v} \in \mathbb{V} \tag{2.1}$$

and, because of Korn's inequality, $\mathbb{V}$ can also be normed by $|D\boldsymbol{v}|$. As usual, we identify $\mathbb{H}$ with its dual so that we will have the Hilbert triplet $\mathbb{V} \hookrightarrow \mathbb{H} \equiv \mathbb{H}^* \hookrightarrow \mathbb{V}^*$. The duality pairing between $\mathbb{V}^*$ and $\mathbb{V}$ are indicated by $\langle\cdot\,,\cdot\rangle$. Then we introduce the positive (monotone) operator $A_0(\phi)$ by the formula

$$A_0(\phi) : D(A_0(\phi)) \to \mathbb{H}, \quad A_0(\phi)\boldsymbol{u} = -\mathbb{P}\operatorname{div}(\nu(\phi)D\boldsymbol{u}), \tag{2.2}$$

where $D(A_0(\phi)) = \{\boldsymbol{u} \in \mathbb{H}^2(\Omega) \cap \mathbb{V} : A_0(\phi) \in \mathbb{H}\}$ and $\mathbb{P}$ is the Leray-Helmholtz projector in $\mathbb{L}^2(\Omega)$ on $\mathbb{H}$ (and its extensions). Note that $A_0(\phi)$ is also symmetric and invertible on $\mathbb{H}$, as it can be easily seen from the following standard calculation:

$$(A_0(\phi)\boldsymbol{u},\boldsymbol{v}) = (\nu(\phi)D\boldsymbol{u}, D\boldsymbol{v}) = (\boldsymbol{u}, A_0(\phi)\boldsymbol{v}), \quad \forall\, \boldsymbol{u},\boldsymbol{v} \in D(A_0(\phi)). \tag{2.3}$$

Moreover, $|A_0(\phi)|_{\mathcal{L}(\mathbb{V};\mathbb{V}^*)} \le C$, for some positive constant C that depends only on ν_1.

To give a rigorous and unified formulation of the order parameter equations, we define first the self-adjoint positive operator on $L^2(\Omega)$

$$A_1\psi = (-\Delta + \varepsilon^{-1}\alpha\gamma)\psi, \quad \forall \psi \in D(A_1) = \{\psi \in H^2(\Omega) : \partial_{\boldsymbol{n}}\psi = 0, \text{ on } \Gamma\}, \tag{2.4}$$

where $\gamma > 0$ is such that (see (1.14))

$$\liminf_{|y|\to+\infty} f'(y) > 2\gamma.$$

Then, we set

$$A_2(\phi)\psi = -\mathrm{div}(\mathcal{M}(\phi)\nabla\psi), \quad \forall\psi \in D(A_2(\phi)) = D(A_1). \tag{2.5}$$

Note that A_2 is a nonnegative (symmetric) monotone operator on $L^2(\Omega)$, but is positive on

$$L_0^2(\Omega) = \{\psi \in L^2(\Omega) : \langle\psi\rangle_\Omega = 0\}$$

and $A_2(\phi)$ is a bounded from $H^1(\Omega) \cap L_0^2(\Omega)$ into $(H^1(\Omega) \cap L_0^2(\Omega))^*$ (see (1.15)). In addition, we need to introduce the bilinear operators B_0 and B_1 (and the related trilinear forms b_0 and b_1) and $\boldsymbol{R}_0$:

$$\begin{aligned}
(B_0(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}) &= \int_\Omega [(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}]\cdot\boldsymbol{w}\mathrm{d}x := b_0(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}),\\
(B_1(\boldsymbol{u},\phi),\psi)_{L^2} &= \int_\Omega [(\boldsymbol{u}\cdot\nabla)\phi]\psi\mathrm{d}x := b_1(\boldsymbol{u},\phi,\psi),\\
(\boldsymbol{R}_0(\xi,\phi),\boldsymbol{w}) &= \int_\Omega \xi\nabla\phi\cdot\boldsymbol{w}\mathrm{d}x.
\end{aligned}$$

As is well-known, $B_0(\boldsymbol{u},\boldsymbol{u}) \in \mathbb{V}^*$ for all $\boldsymbol{u} \in \mathbb{V}$. On the other hand, $B_1(\boldsymbol{u},\phi)$ maps $\mathbb{V}\times H^1(\Omega)$ into $(H^1(\Omega))^*$ and $\boldsymbol{R}_0$ maps $L^2(\Omega)\times H^2(\Omega)$ into $\mathbb{V}^*$. We also recall the following inequality (see e.g., [64]):

$$|(\boldsymbol{R}_0(A_1\phi,\phi),\boldsymbol{v})| = |b_1(\boldsymbol{v},\phi,A_1\phi)| \le c_0\|\boldsymbol{v}\||\phi|_{H^1}^{\frac{1}{2}}|\phi|_{D(A_1)}^{\frac{3}{2}} \tag{2.6}$$

for all $\boldsymbol{v} \in \mathbb{V}$, $\phi \in D(A_1)$, which implies

$$\|\boldsymbol{R}_0(A_1\phi,\phi)\|_{\mathbb{V}^*} \le c_0|\phi|_{H^1}^{\frac{1}{2}}|\phi|_{D(A_1)}^{\frac{3}{2}}. \tag{2.7}$$

Let us now set

$$\|\psi\|_{H^1}^2 = \begin{cases} \varepsilon|\nabla\psi|_{L^2}^2 + \langle\psi\rangle_\Omega^2, & \text{if } K = CH,\\ \varepsilon|\nabla\psi|_{L^2}^2 + \alpha\gamma|\psi|_{L^2}^2, & \text{if } K = AC. \end{cases} \tag{2.8}$$

Then we introduce

$$\mathbb{Y}_K = \begin{cases} \mathbb{H}\times H^1(\Omega), & \text{if } K = AC,\\ \{(\boldsymbol{v},\psi) \in \mathbb{H}\times H^1(\Omega) : |\langle\psi\rangle_\Omega| \le M\}, & \text{if } K = CH, \end{cases} \tag{2.9}$$

where $M \ge 0$ is given (see (1.12)–(1.13)). We endow $\mathbb{H}\times H^1(\Omega)$ with the norm (cf. (2.8))

$$\|(\boldsymbol{v},\psi)\|_K^2 = \frac{1}{\kappa}|\boldsymbol{v}|^2 + \|\phi\|_{H^1}^2,$$

and we observe that $\mathbb{Y}_{CH}$ is a complete metric space with respect to the metric induced by the $\mathbb{H}\times H^1$-norm.

We now have all the ingredients to introduce the notion of Leray-Hopf solution to our problems.

Problem $\boldsymbol{P_K}$ Let $\tau \in \mathbb{R}$ and $T > \tau$. Given $\boldsymbol{g} \in L^2([\tau,T];\mathbb{V}^*)$ and $(\boldsymbol{u}_0,\phi_0) \in \mathbb{H}\times H^1(\Omega)$, we find

$$(\boldsymbol{u},\phi) \in L^\infty([\tau,T];\mathbb{Y}_K) \cap L^2([\tau,T];\mathbb{V}\times D(A_1)), \tag{2.10}$$

such that

$$\partial_t \boldsymbol{u} \in L^{\frac{4}{3}}([\tau,T];\mathbb{V}^*), \quad \partial_t \phi \in L^2([\tau,T];(H^1(\Omega))^*) \tag{2.11}$$

and, if $K = CH$,

$$\mu \in L^2([\tau,T];H^1(\Omega)), \quad \phi \in L^2([\tau,T];H^3(\Omega)), \tag{2.12}$$

which solves

$$\partial_t \boldsymbol{u} + A_0(\phi)\boldsymbol{u} + B_0(\boldsymbol{u},\boldsymbol{u}) - \kappa \boldsymbol{R}_0(\varepsilon A_1\phi,\phi) = \boldsymbol{g}, \quad \text{in } \mathbb{V}^*, \text{ a.e. in } (\tau,+\infty), \tag{2.13}$$

$$\mu = \varepsilon A_1\phi + \alpha f_\gamma(\phi), \quad \text{a.e. in } \Omega \times (\tau,+\infty), \tag{2.14}$$

$$\partial_t \phi + A_K(\phi)\mu + B_1(\boldsymbol{u},\phi) = 0, \quad \text{in } (H^1(\Omega))^*, \text{ a.e. in } (\tau,+\infty), \tag{2.15}$$

and fulfills initial conditions (1.8) and, if $K = CH$, mass conservation (1.13). Here $A_K = \mathbb{I}$, if $K = AC$ and $A_K = A_2$, if $K = CH$.

Remark 2.1 Note that μ does no longer appear in equation (2.13). More precisely, the term $\mu\nabla\phi$ (see (1.1)) has been replaced by $A_1\phi\nabla\phi$ because $f'_\gamma(\phi)\nabla\phi = \nabla F_\gamma(\phi)$ can be incorporated into the pressure gradient term.

Remark 2.2 It follows from (2.3), the assumption on $\nu(\phi)$ and (2.11) that $A_0(\phi)\boldsymbol{u} \in L^2([\tau,T];\mathbb{V}^*)$ and (see (2.7))

$$B_0(\boldsymbol{u},\boldsymbol{u}),\ \boldsymbol{R}_0(A_1\phi,\phi) \in L^{\frac{4}{3}}([\tau,T];\mathbb{V}^*). \tag{2.16}$$

Thus, from equation (2.13), we deduce that $\partial_t \boldsymbol{u} \in L^{\frac{4}{3}}([\tau,T];\mathbb{V}^*)$. In addition, we can also easily deduce that $\boldsymbol{u} \in C([\tau,T];\mathbb{V}^*) \cap C_w([\tau,T];\mathbb{H})$. Hence, the velocity initial datum makes sense in the usual way. On the other hand, we have $B_1(\boldsymbol{u},\phi) \in L^2([\tau,T];(H^1(\Omega))^*)$. Then, it is not difficult to deduce that $\partial_t\phi \in L^2([\tau,T];(H^1(\Omega))^*)$ and, on account of (2.11), this entails $\phi \in C([\tau,T];L^2(\Omega))$. Therefore, the initial condition for ϕ makes sense as well. Summing up, we have $(\boldsymbol{u},\phi) \in C_w([\tau,T];\mathbb{Y}_K)$ so that initial conditions (1.8) hold weakly.

3 Weak Solutions and Dissipative Estimates

Here we first prove an existence theorem for P_K by means of a classical Faedo-Galerkin scheme. Let us consider the following energy functional $\mathcal{L}_K : \mathbb{Y}_K \to \mathbb{R}_+$,

$$\mathcal{L}_K(\boldsymbol{u},\phi) = \|(\boldsymbol{u},\phi)\|_K^2 + 2\alpha(F_K(\phi),1)_{L^2} + c_{F_K}, \tag{3.1}$$

where $F_{AC} = F_\gamma$ while $F_{CH} = F$. Moreover, $c_{F_K} > 0$ is sufficiently large so that

$$2\alpha(F_K(\psi),1)_{L^2} + c_{F_K} \geq 0 \tag{3.2}$$

for all $\psi \in H^1(\Omega)$. Note that such a constant exists, since F_K is bounded below, due to the first assumption of (1.14). In addition, thanks to the same assumption, we can find positive constant c_{f_γ}, c'_f, c''_f, and c_M such that, for all $y \in \mathbb{R}$,

$$|F_\gamma(y)| - c_{f_\gamma} \leq 2f_\gamma(y)y, \tag{3.3}$$

$$F_\gamma(y) - f_\gamma(y)y \leq c'_f|y|^2 + c''_f, \tag{3.4}$$

$$|F(y)| \leq 2f(y)(y-M) + c_M, \tag{3.5}$$

$$F(y) - f(y)(y-M) \leq c'_f|y-M|^2 + c_M. \tag{3.6}$$

Let us now set

$$\rho_{AC} = \min\Big\{\frac{\nu_0}{\kappa C_\Omega^2}, 1, \frac{\alpha c_{f_\gamma}}{1+\alpha c_{f_\gamma} + 2\alpha c_f'}\Big\}, \tag{3.7}$$

where C_Ω is the Poincaré constant (see (2.1)). We also set

$$\rho_{CH} = \min\Big\{\frac{\nu_0}{\kappa C_\Omega^2}, \frac{\varepsilon m_0}{\widetilde{C}_\Omega^2}, \frac{\varepsilon^2 m_0}{\widetilde{C}_\Omega^2(\varepsilon + 2\alpha c_f' \widetilde{C}_\Omega^2)}\Big\}, \tag{3.8}$$

where $\widetilde{C}_\Omega$ is the Poincaré-Wirtinger constant, that is,

$$|\psi - \langle\psi\rangle_\Omega|_{L^2} \le \widetilde{C}_\Omega |\nabla\psi|_{L^2}, \quad \forall \psi \in H^1(\Omega). \tag{3.9}$$

We now state and prove the following basic result.

Theorem 3.1 *Let assumptions* (1.14) *be satisfied. If* $\boldsymbol{g} \in L^2([\tau,T];\mathbb{V}^*)$ *and* $(\boldsymbol{u}_0,\phi_0) \in \mathbb{Y}_K$. *Then there exists a solution* $(\boldsymbol{u},\phi)$ *to* $\boldsymbol{P}_K$ *such that, if* $K = AC$, *the following inequality holds:*

$$\begin{aligned} &-\int_\tau^T \mathcal{L}_{AC}(\boldsymbol{u}(s),\phi(s))\Lambda'(s)\mathrm{d}s + \rho_{AC}\int_\tau^T \mathcal{L}_{AC}(\boldsymbol{u}(s),\phi(s))\Lambda(s)\mathrm{d}s \\ &+\int_\tau^T \Big(\frac{\nu_0}{\kappa}\|\boldsymbol{u}(s)\|^2 + 2|\mu(s)|_{L^2}^2\Big)\Lambda(s)\mathrm{d}s \\ \le &\int_\tau^T \Big(\frac{2}{\kappa}\langle \boldsymbol{g}(s),\boldsymbol{u}(s)\rangle + \Theta_{AC}\Big)\Lambda(s)\mathrm{d}s \end{aligned} \tag{3.10}$$

for any $\Lambda \in C_0^\infty((\tau,T);\mathbb{R}_+)$, *where*

$$\Theta_{AC} = \alpha|\Omega|(c_{f_\gamma} + 2c_f'') + \rho_{AC} C_{F_{AC}}.$$

Instead, if $K = CH$, *then* $(\boldsymbol{u},\phi)$ *satisfies*

$$\begin{aligned} &-\int_\tau^T \mathcal{L}_{CH}(\boldsymbol{u}(s),\phi(s))\Lambda'(s)\mathrm{d}s + \rho_{CH}\int_\tau^T \mathcal{L}_{CH}(\boldsymbol{u}(s),\phi(s))\Lambda(s)\mathrm{d}s \\ &+\int_\tau^T \Big(\frac{\nu_0}{\kappa}\|\boldsymbol{u}(s)\|^2 + m_0|\nabla\mu(s)|_{L^2}^2\Big)\Lambda(s)\mathrm{d}s \\ \le &\int_\tau^T \Big(\frac{2}{\kappa}\langle \boldsymbol{g}(s),\boldsymbol{u}(s)\rangle + \Theta_{CH}\Big)\Lambda(s)\mathrm{d}s \end{aligned} \tag{3.11}$$

for any $\Lambda \in C_0^\infty((\tau,T);\mathbb{R}_+)$, *where* $\rho = \rho_{CH}$ *and*

$$\Theta_{CH} = 2\alpha\rho c_f''|\Omega| + \alpha(\varepsilon\widetilde{C}_\Omega^{-2} - \rho)c_{M_0}|\Omega| + \rho c_{F_{CH}}.$$

Proof The existence argument does not depend on K but for some details. Let $\{\boldsymbol{w}_j\} \subset \mathbb{V}$ be a sequence which is dense and orthogonal in $\mathbb{V}$ (see, for instance, [45, Chapter 1, 6.3] or [65, Chapter 1, 2.6]), and let $\{q_j\}$ be the sequence of eigenfunctions of A_1. Then set

$$\boldsymbol{u}_m(x,t) = \sum_{i=1}^m a_{j,m}(t)\boldsymbol{w}_j(x), \quad \phi_m(x,t) = \sum_{i=1}^m b_{j,m}(t) q_j(x), \tag{3.12}$$

where $a_{j,m}$, $b_{j,m}$ are functions to be determined in $C^1([\tau,T])$ in such a way that $(\boldsymbol{u}_m,\phi_m)$ satisfy the following Cauchy problem $\boldsymbol{P}_K^m$:

$$\frac{\mathrm{d}\boldsymbol{u}_m}{\mathrm{d}t} + \nu P_{0,m}A_0(\phi_m)\boldsymbol{u}_m + P_{0,m}B_0(\boldsymbol{u}_m,\boldsymbol{u}_m) - \kappa P_{0,m}\boldsymbol{R}_0(\mu_m,\phi_m) = \boldsymbol{g}_m, \tag{3.13}$$

$$\mu_m = \varepsilon P_{1,m}A_1\phi_m + \alpha P_{1,m}f_\gamma(\phi_m), \tag{3.14}$$

$$\frac{\mathrm{d}\phi_m}{\mathrm{d}t} + P_{1,m}A_K(\phi_m)\mu_m + P_{1,m}B_1(\boldsymbol{u}_m,\phi_m) = 0, \tag{3.15}$$

$$\boldsymbol{u}_m(\tau) = \boldsymbol{u}_{0,m}, \quad \phi_m(\tau) = \phi_{0,m}. \tag{3.16}$$

Here $P_{0,m}$ (respectively, $P_{1,m}$) is the orthogonal projector from $\mathbb{H}$ (respectively, $L^2(\Omega)$) onto the linear space $\langle \boldsymbol{u}_1,\cdots,\boldsymbol{u}_m\rangle$ (respectively, $\langle \phi_1,\cdots,\phi_m\rangle$). Moreover, we have

$$\boldsymbol{g}_m(t) = P_{0,m}\boldsymbol{g}(t), \quad \boldsymbol{u}_{0,m} = P_{0,m}\boldsymbol{u}_0, \quad \phi_{0,m} = P_{1,m}\phi_0,$$

so that $\boldsymbol{g}_m \to \boldsymbol{g}$ strongly in $L^2([\tau,T];\mathbb{V}^*)$ and $(\boldsymbol{u}_{0,m},\phi_{0,m}) \to (\boldsymbol{u}_0,\phi_0)$ strongly in $\mathbb{Y}_K$. Note that, in equation (3.13), the (approximated) chemical potential μ_m appears again since it allows us to simplify the computations (see Remark 2.1). On the other hand, due to the incompressibility and boundary conditions (1.11), we always have

$$(\boldsymbol{R}_0(\mu_m,\phi_m),\boldsymbol{v}) = (\boldsymbol{R}_0(\varepsilon A_1\phi_m,\phi_m),\boldsymbol{v}), \quad \forall \boldsymbol{v}\in\mathbb{V}. \tag{3.17}$$

Problem $\boldsymbol{P}_K^m$ has clearly a maximal solution $(\boldsymbol{u}_m,\phi_m)\in C^0([\tau,T_m);\mathbb{Y}_K)$ on some time interval $[\tau,T_m)$, $T_m\in(\tau,T]$.

Let us now take the scalar product in $\mathbb{H}$ of equation (3.13) with $\frac{2}{\kappa}\boldsymbol{u}_m$. Then the scalar product in $L^2(\Omega)$ of equations (3.14) and (3.15) with $2\partial_t\phi_m$ and $2\mu_m$, respectively, and add the resulting relations. Then, observing in particular that

$$(P_{1,m}B_1(\boldsymbol{u}_m,\phi_m),\mu_m)_{L^2} = (P_{0,m}\boldsymbol{R}_0(\mu_m,\phi_m),\boldsymbol{u}_m),$$

we obtain the following energy equality:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{\kappa}|\boldsymbol{u}_m(t)|^2 + \varepsilon|\nabla\phi_m(t)|_{L^2}^2 + \alpha\gamma|\phi_m(t)|_{L^2}^2 + 2\alpha(F_\gamma(\phi_m(t)),1)_{L^2}\Big]\\ &+\frac{2}{\kappa}\big|\sqrt{\nu(\phi_m(t))}\,D\boldsymbol{u}_m(t)\big|^2 + 2(A_K(\phi_m)\mu_m(t),\mu_m(t))_{L^2} = \frac{2}{\kappa}(\boldsymbol{u}_m(t),\boldsymbol{g}_m(t)).\end{aligned} \tag{3.18}$$

Note that, setting

$$L_K(\mu(t)) = \begin{cases}|\mu(t)|_{L^2}^2, & \text{if } K = AC,\\ \big|\sqrt{\mathcal{M}(\phi(t))}\,\nabla\mu(t)\big|_{L^2}^2, & \text{if } K = CH,\end{cases} \tag{3.19}$$

we have

$$(A_K\mu_m(t),\mu_m(t))_{L^2} = L_K(\mu(t)). \tag{3.20}$$

Thus, recalling (1.15), on account of (3.1) and (3.18), we obtain the following energy inequality, for almost any $t\in[\tau,T_m)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_K(\boldsymbol{u}_m(t),\phi_m(t)) + \frac{\nu_0}{\kappa}\|\boldsymbol{u}_m(t)\|^2 + 2L_K(\mu_m(t)) \le \frac{\kappa}{\nu_0}|\boldsymbol{g}_m(t)|_{\mathbb{V}^*}^2. \tag{3.21}$$

This inequality immediately implies that the maximal solution $(\boldsymbol{u}_m, \phi_m)$ can be defined up to the given final time T.

In the case $K = CH$, we need to recover the H^1-norm of the chemical potential μ_m. To do that, from equation (3.14) we easily deduce, for all $t \in [\tau, T]$,

$$|\langle \mu_m(t) \rangle_\Omega| = |\alpha \langle P_{1,m} f(\phi_m(t)) \rangle_\Omega| \leq c(1 + |\phi_m(t)|^3_{L^3}), \tag{3.22}$$

where c is a positive constant that depends at most on Ω, α, ε, ν_0, but is independent of time, initial data and m. From now on c will denote a positive constant of this kind. Such a constant may vary even from line to line.

Integrating both sides of (3.21) between τ and t, we deduce that $\{\boldsymbol{u}_m\}$ is bounded in $L^\infty([\tau, T]; \mathbb{H}) \cap L^2([\tau, T]; \mathbb{V})$ and $\{\phi_m\}$ is bounded in $L^\infty([\tau, T]; H^1(\Omega))$. Moreover, if $K = AC$, on account of (3.20) and (3.21), we have that $\{\mu_m\}$ is bounded in $L^2([\tau, T]; L^2(\Omega))$. Thus, recalling (3.14), we also have that $\{\phi_m\}$ is bounded in $L^2([\tau, T]; D(A_1))$. In the case $K = CH$, due to (3.20)–(3.22), we have that $\{\mu_m\}$ is bounded in $L^2([\tau, T]; H^1(\Omega))$. Thus (3.14) entails that $\{\phi_m\}$ is bounded in $L^2([\tau, T]; D(A_1) \cap H^3(\Omega))$. In addition from (3.13) (see also (2.6)–(2.7)), we also recover that $\{\partial_t \boldsymbol{u}_m\}$ is bounded in $L^{\frac{4}{3}}([\tau, T]; \mathbb{V}^*)$ and, from (3.15), we infer that $\{\partial_t \phi_m\}$ is bounded in $L^2([\tau, T]; (H^1(\Omega))^*)$.

Summing up, using well-known compactness arguments, we can find a pair $(\boldsymbol{u}, \phi)$ such that, up to subsequences,

$$\boldsymbol{u}_m \to \boldsymbol{u} \quad \text{weakly star in } L^\infty([\tau, T]; \mathbb{H}), \text{ weakly in } L^2([\tau, T]; \mathbb{V}), \tag{3.23}$$

$$\partial_t \boldsymbol{u}_m \to \partial_t \boldsymbol{u} \quad \text{weakly in } L^{\frac{4}{3}}([\tau, T]; \mathbb{V}^*), \tag{3.24}$$

$$\boldsymbol{u}_m \to \boldsymbol{u} \quad \text{strongly in } L^2([\tau, T]; \mathbb{H}), \tag{3.25}$$

$$\phi_m \to \phi \quad \text{weakly star in } L^\infty([\tau, T]; H^1(\Omega)) \text{ and weakly in } L^2([\tau, T]; D(A_1)), \tag{3.26}$$

$$\mu_m \to \mu \quad \text{weakly in } L^2([\tau, T] \times \Omega), \tag{3.27}$$

$$\partial_t \phi_m \to \partial_t \phi \quad \text{weakly in } L^2([\tau, T]; (H^1(\Omega))^*), \tag{3.28}$$

$$\phi_m \to \phi \quad \text{strongly in } L^2([\tau, T]; H^1(\Omega)) \cap C([\tau, T]; L^2(\Omega)). \tag{3.29}$$

If $K = CH$, we also have

$$\mu_m \to \mu \quad \text{weakly in } L^2([\tau, T]; H^1(\Omega)), \quad \phi_m \to \phi \quad \text{weakly in } L^2([\tau, T]; H^3(\Omega)). \tag{3.30}$$

Observe that $(\boldsymbol{u}, \phi)$ satisfies all the regularity properties listed in (2.10)–(2.12). Now, employing standard techniques, and using the above convergence properties, we can now show that (up to subsequences)

$$B_0(\boldsymbol{u}_m, \boldsymbol{u}_m) \rightharpoonup B_0(\boldsymbol{u}, \boldsymbol{u}), \quad \boldsymbol{R}_0(\mu_m, \phi_m) \rightharpoonup \boldsymbol{R}_0(\mu, \phi), \quad \text{in } L^{\frac{4}{3}}([\tau, T]; \mathbb{V}^*),$$

whereas $f_\gamma(\phi_m)$ converges strongly to $f_\gamma(\phi)$ in $C([\tau, T]; L^2(\Omega))$, as $m \to \infty$. Consequently, we can pass to the limit in (3.13)–(3.15) and find that $(\boldsymbol{u}, \phi)$ solves (2.13)–(2.15). It is also standard to recover initial conditions (1.8) (see also Remark 2.2).

Let us now prove inequality (3.10) first (i.e., $K = AC$). On account of (3.18), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}_{AC}(\boldsymbol{u}_m(t), \phi_m(t)) + \rho \mathcal{L}_{AC}(\boldsymbol{u}_m(t), \phi_m(t)) \leq \Lambda^1_m(t), \tag{3.31}$$

where $\rho = \rho_{AC}$ is given by (3.7) and

$$\begin{aligned}\Lambda_m^1 := &-\frac{2\nu_0}{\kappa}\|\boldsymbol{u}_m\|^2 + \rho|\boldsymbol{u}_m|^2 - 2|\mu_m|_{L^2}^2 - \varepsilon(2-\rho)|\nabla\phi_m|_{L^2}^2 \\ &+ 2\alpha[\rho(F_\gamma(\phi_m) - f_\gamma(\phi_m)\phi_m, 1)_{L^2} - (1-\rho)(f_\gamma(\phi_m)\phi_m, 1)_{L^2}] \\ &+ \frac{2}{\kappa}(\boldsymbol{u}_m, \boldsymbol{g}_m) + \rho|\phi_m|_{L^2}^2 + \rho c_{F_{AC}}. \end{aligned} \tag{3.32}$$

Then, from (3.3)–(3.4), it follows

$$\begin{aligned}\Lambda_m^1 \le &-\Big(\frac{2\nu_0}{\kappa} - \rho C_\Omega^2\Big)\|\boldsymbol{u}_m\|^2 - 2|\mu_m|_{L^2}^2 \\ &- \varepsilon(2-\rho)|\nabla\phi_m(t)|_{L^2}^2 - [c_{f_\gamma}\alpha(1-\rho) - \rho - 2\alpha c_f'\rho]|\phi_m|_{L^2}^2 \\ &- \alpha(1-\rho)(|F_\gamma(\phi_m)|, 1)_{L^2} + \frac{2}{\kappa}(\boldsymbol{u}_m, \boldsymbol{g}_m) \\ &+ \alpha|\Omega|(c_{f_\gamma} + 2c_f'')) + \rho c_{F_{AC}},\end{aligned}$$

and, thanks to (3.7), we deduce

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{AC}(\boldsymbol{u}_m, \phi_m) + \rho\mathcal{L}_{AC}(\boldsymbol{u}_m, \phi_m) + \frac{\nu_0}{\kappa}\|\boldsymbol{u}\|^2 + 2|\mu|_{L^2}^2 \\ &\le \frac{2}{\kappa}(\boldsymbol{u}_m, \boldsymbol{g}_m) + \alpha|\Omega|(c_{f_\gamma} + 2c_f'')) + \rho c_{F_{AC}}.\end{aligned} \tag{3.33}$$

Thus, if we fix a time $T > \tau$ and take $\Lambda \in C_0^\infty((\tau, T); \mathbb{R}_+)$, from (3.33) we infer

$$\begin{aligned}&-\int_\tau^T \mathcal{L}_{AC}(\boldsymbol{u}_m(s), \phi_m(s))\Lambda'(s)\mathrm{d}s + \rho_{AC}\int_\tau^T \mathcal{L}_{AC}(\boldsymbol{u}_m(s), \phi_m(s))\Lambda(s)\mathrm{d}s \\ &+ \int_\tau^T \Big(\frac{\nu_0}{\kappa}\|\boldsymbol{u}_m(s)\|^2 + 2|\mu_m(s)|_{L^2}^2\Big)\Lambda(s)\mathrm{d}s \\ \le &\int_\tau^T \Big(\frac{2}{\kappa}(\boldsymbol{g}_m(s), \boldsymbol{u}_m(s)) + \alpha|\Omega|(c_{f_\gamma} + 2c_f'') + \rho_{AC}c_{F_{AC}}\Big)\Lambda(s)\mathrm{d}s.\end{aligned} \tag{3.34}$$

Observe now that (3.25) and (3.29) imply

$$\|(\boldsymbol{u}_m, \phi_m)\|_{L^2([\tau,T];\mathbb{Y}_K)} \to \|(\boldsymbol{u}, \phi)\|_{L^2([\tau,T];\mathbb{Y}_K)}. \tag{3.35}$$

Hence, up to subsequences, we also have $\|(\boldsymbol{u}_m(s), \phi_m(s))\|_K \to \|(\boldsymbol{u}(s), \phi(s))\|_K$ as $m \to +\infty$, almost everywhere in $[\tau, T]$.

On the other hand, on account of (1.14), we have that

$$\begin{aligned}&\int_\tau^T [(F_\gamma(\phi_{m_k}(s)) - F_\gamma(\phi(s)), 1)_{L^2}]^2\mathrm{d}s \\ &\le Q_\gamma(|\phi|_{L^\infty([\tau,T];H^1(\Omega))})(T-\tau)|\phi_m - \phi|_{C([\tau,T];L^2(\Omega))}\end{aligned} \tag{3.36}$$

for some nonnegative increasing continuous function Q_γ. Then, up to subsequences, we infer that (see (3.29)) $(F_\gamma(\phi_m), 1)_{L^2} \to (F_\gamma(\phi), 1)_{L^2}$ strongly in $L^2([\tau, T])$ and almost everywhere on $[\tau, T]$. On the other hand, for any functions $\Lambda \in C_0^\infty((\tau, T); \mathbb{R}_+)$, it is not difficult to see, from (3.21), that $\mathcal{L}_{AC}(\boldsymbol{u}_m(s), \phi_m(s))\Lambda'(s)$ attains its supremum on $[\tau, T]$. Thus, the Lebesgue dominated convergence theorem implies that (see (2.6))

$$\int_\tau^T \mathcal{L}_{AC}(\boldsymbol{u}_m(s), \phi_m(s))\Lambda'(s)\mathrm{d}s \to \int_\tau^T \mathcal{L}_{AC}(\boldsymbol{u}(s), \phi(s))\Lambda'(s)\mathrm{d}s, \quad \text{as } m \to +\infty. \tag{3.37}$$

Moreover, we have (see (3.23))

$$\begin{aligned}
&\boldsymbol{u}_m\Lambda^{\frac{1}{2}} \to \boldsymbol{u}\Lambda^{\frac{1}{2}}, \quad \text{weakly in } L^2([\tau,T];\mathbb{V}),\\
&\mu_m\Lambda^{\frac{1}{2}} \to \mu\Lambda^{\frac{1}{2}}, \quad \text{weakly in } L^2([\tau,T];L^2(\Omega)).
\end{aligned}$$

Hence, we deduce, for all $t \in (\tau, T]$,

$$\int_\tau^t \|\boldsymbol{u}(s)\|^2\Lambda(s)\mathrm{d}s \le \liminf_{m\to\infty}\int_\tau^t \|\boldsymbol{u}_m(s)\|^2\Lambda(s)\mathrm{d}s, \tag{3.38}$$

$$\int_\tau^t |\mu(s)|^2_{L^2}\Lambda(s)\mathrm{d}s \le \liminf_{m\to\infty}\int_\tau^t |\mu_m(s)|^2_{L^2}\Lambda(s)\mathrm{d}s. \tag{3.39}$$

Therefore, we can pass to the limit in (3.34) thanks to (3.37)–(3.39), and inequality (3.10) follows.

It remains to prove (3.11) (i.e., $K = CH$). Recalling (3.18), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{CH}(\boldsymbol{u}_m(t),\phi_m(t)) + \rho\mathcal{L}_{CH}(\boldsymbol{u}_m(t),\phi_m(t)) \le \Lambda_m^2(t), \tag{3.40}$$

where $\rho = \rho_{CH}$ is given by (3.8) and

$$\begin{aligned}
\Lambda_m^2 = &-\frac{2\nu_0}{\kappa}\|\boldsymbol{u}_m\|^2 + \rho|\boldsymbol{u}_m|^2 - 2|\sqrt{\mathcal{M}(\phi_m)}\nabla\overline{\mu}_m|^2_{L^2}\\
&+ 2\zeta(\overline{\mu}_m,\overline{\phi}_m)_{L^2} - \varepsilon(2\zeta-\rho)|\nabla\overline{\phi}_m|^2_{L^2}\\
&+ 2\alpha[\rho(F(\phi_m) - f(\phi_m)\overline{\phi}_m, 1)_{L^2} - (\zeta-\rho)(f(\phi_m)\overline{\phi}_m,1)_{L^2}]\\
&+ \frac{2}{\kappa}(\boldsymbol{u}_m,\boldsymbol{g}_m) + \rho M_{0,m} + \rho c_{F_{CH}},
\end{aligned} \tag{3.41}$$

where

$$\overline{\phi}_m(t) := \phi_m(t) - M_{0,m}, \quad \overline{\mu}_m(t) := \mu_m(t) - \langle\mu_m(t)\rangle_\Omega$$

with

$$M_{0,m} := \langle\phi_{0,m}\rangle_\Omega.$$

Observe that $|M_{0,m}| \le M$ and $\langle\overline{\phi}_m(t)\rangle_\Omega = \langle\overline{\mu}_m(t)\rangle_\Omega = 0$ for all $t \in [\tau,T]$.

Using (3.5), (3.6), (3.9) and recalling (1.15), from (3.41), we infer

$$\begin{aligned}
\Lambda_m^2 \le &-\Big(\frac{2\nu_0}{\kappa} - \rho C_\Omega^2\Big)\|\boldsymbol{u}_m\|^2 - (2m_0 - \zeta\widetilde{C}_\Omega^2\varepsilon^{-1})|\nabla\overline{\mu}_m|^2_{L^2}\\
&- \varepsilon(\zeta - \rho - 2\alpha\rho c_f'\widetilde{C}_\Omega^2\varepsilon^{-1})|\nabla\overline{\phi}_m|^2_{L^2} - \alpha(\zeta-\rho)|F(\phi_m)|_{L^1}\\
&+ \frac{2}{\kappa}(\boldsymbol{u}_m,\boldsymbol{g}_m) + 2\alpha\rho c_f''|\Omega| + \alpha(\zeta-\rho)c_{M_0}|\Omega| + \rho c_{F_{CH}}.
\end{aligned} \tag{3.42}$$

Let us now choose $\zeta = \varepsilon\widetilde{C}_\Omega^{-2}m_0$ and recall (3.8) so that from (3.40) and (3.42), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{CH}(\boldsymbol{u}_m,\phi_m) + \rho\mathcal{L}_{CH}(\boldsymbol{u}_m,\phi_m) + \frac{\nu_0}{\kappa}\|\boldsymbol{u}_m\|^2 + m_0|\nabla\mu_m|^2_{L^2}\\
&\le \frac{2}{\nu_0}(\boldsymbol{u}_m,\boldsymbol{g}_m) + 2\alpha\rho c_f''|\Omega| + \alpha(\varepsilon\widetilde{C}_\Omega^{-2} - \rho)c_{M_0}|\Omega| + \rho c_{F_{CH}}.
\end{aligned} \tag{3.43}$$

Arguing as in the previous case $K = AC$, from (3.43) we infer (3.11).

Following [19], we define the trajectory space for P_K.

Definition 3.1 *The trajectory space $\mathcal{TR}(\boldsymbol{g},K)$ for a given $\boldsymbol{g}\in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*)$ consists of all the pairs $(\boldsymbol{u},\phi)$ satisfying (2.10)–(2.15) and either (3.10), if $K=AC$, or (3.11), if $K=CH$, on any time interval $[\tau,T]\subset\mathbb{R}_+$. Similarly, we can define $\mathcal{TR}(\boldsymbol{g},K)$ for a given $\boldsymbol{g}\in L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{V}^*)$, by letting $[\tau,T]\subset\mathbb{R}$.*

The following result is a direct consequence of Theorem 3.1.

Corollary 3.1 *Let (1.14) hold and let $\boldsymbol{g}\in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*)$. Then, for any $(\boldsymbol{u}_0,\phi_0)\in\mathbb{Y}_K$ there exists a trajectory $(\boldsymbol{u},\phi)\in\mathcal{TR}(\boldsymbol{g},K)$ such that $\boldsymbol{u}(0)=\boldsymbol{u}_0$ and $\phi(0)=\phi_0$.*

Moreover, arguing as in [19, Chapter XV, Corollary 1.5], we obtain an inequality which is equivalent to (3.10)–(3.11), namely,

Corollary 3.2 *Suppose that f satisfies (1.14) and let $\boldsymbol{g}\in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*)$. Any $(\boldsymbol{u},\phi)\in\mathcal{TR}(\boldsymbol{g},K)$ is such that*

$$\begin{aligned}
&\mathcal{L}_K(\boldsymbol{u}(t),\phi(t))-\mathcal{L}_K(\boldsymbol{u}(\tau),\phi(\tau))\\
&+\int_\tau^t\Big(\rho_K\mathcal{L}_K(\boldsymbol{u}(s),\phi(s))+\frac{\nu_0}{\kappa}\|\boldsymbol{u}(s)\|^2+\sigma_K L_K(\mu(s))\Big)\mathrm{d}s\\
&\le\frac{2}{\kappa}\int_\tau^t\langle\boldsymbol{g}(s),\boldsymbol{u}(s)\rangle\mathrm{d}s+\Theta_K(t-\tau)
\end{aligned}\tag{3.44}$$

for almost all $t,\tau\in\mathbb{R}_+$ with $t\ge\tau$. Here $\sigma_{AC}=2$ and $\sigma_{CH}=1$.

Similarly, we also get the following results (see [19, Chapter XV, Corollary 1.6]).

Corollary 3.3 *Suppose that f satisfies (1.14) and let $\boldsymbol{g}\in L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*)$. Any $(\boldsymbol{u},\phi)\in\mathcal{TR}(\boldsymbol{g},K)$ is such that*

$$\begin{aligned}
&\mathcal{L}_K(\boldsymbol{u}(t),\phi(t))-\mathcal{L}_K(\boldsymbol{u}(\tau),\phi(\tau))+\rho_K\int_\tau^t\mathcal{L}_K(\boldsymbol{u}(s),\phi(s))\mathrm{d}s\\
&+\int_\tau^t\Big(\frac{\nu_0}{2\kappa}\|\boldsymbol{u}(s)\|^2+\sigma_K L_K(\mu(s))\Big)\mathrm{d}s\\
&\le\int_\tau^t\frac{4\kappa}{\nu_0}|\boldsymbol{g}(s)|^2_{\mathbb{V}^*}\mathrm{d}s+\Theta_K(t-\tau)
\end{aligned}\tag{3.45}$$

for almost all $t,\tau\in\mathbb{R}_+$ with $t\ge\tau$.

Remark 3.1 Inequality (3.45) holds for any $t\ge\tau$. Indeed, define the function (see [19, Theorem II.1.7])

$$\begin{aligned}
\mathcal{G}(t):=\mathcal{L}_K(\boldsymbol{u}(t),\phi(t))&+\int_0^t\Big(\rho_K\mathcal{L}_K(\boldsymbol{u}(t),\phi(t))+\frac{\nu_0}{\kappa}\|\boldsymbol{u}(s)\|^2+\sigma_K L_K(\mu(s))\Big)\mathrm{d}s\\
&-\frac{1}{\kappa}\int_0^t\langle\boldsymbol{g}(s),\boldsymbol{u}(s)\rangle\mathrm{d}s-\Theta_K t,
\end{aligned}$$

and note that $\mathcal{G}$ is lower semicontinuous, since $(\boldsymbol{u},\phi)\in C_w([0,T];\mathbb{Y}_K)$ for all $T>0$ implies that $t\mapsto\|(\boldsymbol{u}(t),\phi(t))\|_K$ is lower semicontinuous. The other summands are continuous (see also (2.6)).

Let us recall, for the reader's convenience, a weak formulation of Gronwall's inequality taken from [19, Chapter XV, Lemma 1.2].

Proposition 3.1 *Let $y, a \in L^1_{\rm loc}(\mathbb{R}_+)$ and suppose that the following inequality holds:*

$$-\int_0^\infty y(s)\Lambda'(s)\mathrm{d}s + \eta\int_0^\infty y(s)\Lambda(s)\mathrm{d}s \le \int_0^\infty a(s)\Lambda(s)\mathrm{d}s \tag{3.46}$$

for any $\Lambda \in C_0^\infty(\mathbb{R}_+;\mathbb{R}_+)$ and for some $\eta \in \mathbb{R}$. Then

$$y(t)\mathrm{e}^{\eta t} \le y(\tau)\mathrm{e}^{\eta\tau} + \int_\tau^t a(s)\mathrm{e}^{\eta s}\mathrm{d}s \tag{3.47}$$

for almost all $t, \tau \in \mathbb{R}_+$ such that $t \ge \tau$.

In order to establish the existence of the trajectory attractor, the following uniform estimate plays an essential role.

Proposition 3.2 *Suppose that f satisfies* (1.14) *and let $\boldsymbol{g} \in L^2_{\rm loc}(\mathbb{R}_+;\mathbb{V}^*)$. If $(\boldsymbol{u},\phi) \in \mathcal{TR}(\boldsymbol{g},K)$, then the following estimate holds:*

$$\begin{aligned}&\mathcal{L}_K(\boldsymbol{u}(t),\phi(t))\mathrm{e}^{\rho_K t} - \mathcal{L}_K(\boldsymbol{u}(\tau),\phi(\tau))\mathrm{e}^{\rho_K\tau} + \int_\tau^t \Big(\frac{\nu_0}{2\kappa}\|\boldsymbol{u}(s)\|^2 + \sigma_K L_K(\mu(s))\Big)\mathrm{e}^{\rho_K s}\mathrm{d}s\\ &\le \int_\tau^t \Big(\frac{4\kappa}{\nu_0}|\boldsymbol{g}(s)|^2_{\mathbb{V}^*} + \Theta_K\Big)\mathrm{e}^{\rho_K s}\mathrm{d}s\end{aligned} \tag{3.48}$$

for almost all t, $\tau \in \mathbb{R}_+ \setminus Q_{\boldsymbol{u},\phi}$, $t \ge \tau$, where $Q_{\boldsymbol{u},\phi}$ has zero Lebesgue measure.

The proof follows from estimates (3.10)–(3.11) and Proposition 3.1.

Remark 3.2 In order to prove estimate (3.48), we have employed both assumptions of (1.14). However, the first assumption of (1.14) can be replaced by a weaker one. Indeed, we can only assume that $f \in C(\mathbb{R};\mathbb{R})$ satisfies, for any $y \in \mathbb{R}$, the following inequalities

$$\begin{cases}|F(y)| \le 2f(y)(y-\lambda_K) + c_{1,K},\\ F(y) - f(y)(y-\lambda_K) \le c_{2,K}(y-\lambda_K)^2 + c_{3,K},\\ |f(y)| \le c_{4,K}(1+|y|^3),\end{cases}$$

where $c_{i,K}$, $i = 1,\cdots,4$, are nonnegative constants and $\lambda_{AC} = 0$ or $\lambda_{CH} = M$. Note that, in the case $K = CH$, we can no longer recovery the regularity $\phi \in L^2_{\rm loc}(\mathbb{R};H^3(\Omega))$ (see (2.12)). Indeed, we cannot have $f(\phi) \in H^1(\Omega)$ so that from (2.14) we cannot deduce the H^3-regularity of $\phi(t)$.

We conclude this section with a basic dissipative estimate which holds under an additional assumption on $\boldsymbol{g}$. Let us recall the definition of the Banach space of translation-bounded functions in $L^p_{\rm loc}(\mathbb{R}_+;X)$, X being a real Banach space and $p > 1$, i.e.,

$$L^p_b(\mathbb{R}_+;X) := \Big\{g \in L^p_{\rm loc}(\mathbb{R}_+;X) \,:\, \|g\|^p_{L^p_b(\mathbb{R}_+;X)} = \sup_{\tau\in\mathbb{R}_+}\int_\tau^{\tau+1}\|g(s)\|^p_X\mathrm{d}s < +\infty\Big\}. \tag{3.49}$$

For any $\boldsymbol{g} \in L^2_b(\mathbb{R}_+;\mathbb{V}^*)$ and any given $\rho > 0$, we set

$$\boldsymbol{\beta}_\rho(\boldsymbol{g}) = \sup_{h\in[1,2]}\sup_{t\ge0}\Big(\rho\int_0^h \|\boldsymbol{g}(t+s)\|^2_{\mathbb{V}^*}\mathrm{e}^{\rho s}\mathrm{d}s\frac{1}{\mathrm{e}^{\rho h}-1}\Big), \tag{3.50}$$

and it is easy to show that (see [19, Chapter V])

$$\frac{\rho}{\mathrm{e}^{\rho}-1}\|\boldsymbol{g}\|^2_{L^2_b(\mathbb{R}_+;\mathbb{V}^*)} \le \boldsymbol{\beta}_\rho(\boldsymbol{g}) \le \frac{2\rho}{\mathrm{e}^{\rho}-1}\|\boldsymbol{g}\|^2_{L^2_b(\mathbb{R}_+;\mathbb{V}^*)}. \tag{3.51}$$

We can now prove the following proposition.

Proposition 3.3 *If f satisfies (1.14) and $\boldsymbol{g} \in L^2_b(\mathbb{R}_+;\mathbb{V}^*)$, then any $(\boldsymbol{u},\phi) \in \mathcal{TR}(\boldsymbol{g},K)$ satisfies the estimate*

$$\begin{aligned}&\mathcal{L}_K(\boldsymbol{u}(t),\phi(t))\mathrm{e}^{\rho_K t} - \mathcal{L}_K(\boldsymbol{u}(\tau),\phi(\tau))\mathrm{e}^{\rho_K \tau} + \int_\tau^t \Big(\frac{\nu_0}{2\kappa}\|\boldsymbol{u}(s)\|^2 + \sigma_K L_K(\mu(s))\Big)\mathrm{e}^{\rho_K s}\mathrm{d}s\\ &\le \Big(\frac{4\kappa}{\rho_K\nu_0}\boldsymbol{\beta}_{\rho_K}(\boldsymbol{g}) + \Theta_K\Big)(\mathrm{e}^{\rho_K t} - \mathrm{e}^{\rho_K \tau})\end{aligned} \tag{3.52}$$

for almost all $t,\ \tau \in \mathbb{R}_+ \setminus Q_{\boldsymbol{u},\phi}$, $t+1 \ge \tau$, where $Q_{\boldsymbol{u},\phi}$ has zero Lebesgue measure.

Proof To get (3.52), we need to estimate the last term on the right-hand side of (3.48). Let $t,\tau \in \mathbb{R}_+$ such that $t \ge \tau + 1$. Take an integer $m \in \mathbb{N}$ such that $h := \frac{t-\tau}{m} \in [1,2]$ and define $t_j = \tau + jh$, $j = 0,1,\cdots,m-1$. A simple calculation yields the following inequality:

$$\begin{aligned}\int_\tau^t \Big(\frac{4\kappa}{\nu_0}\|\boldsymbol{g}(s)\|^2_{\mathbb{V}^*} + \Theta_K\Big)\mathrm{e}^{\rho_K s}\mathrm{d}s &= \sum_{j=0}^{m-1}\int_{t_j}^{t_{j+1}} \Big(\frac{4\kappa}{\nu_0}\|\boldsymbol{g}(s)\|^2_{\mathbb{V}^*} + \Theta_K\Big)\mathrm{e}^{\rho s}\mathrm{d}s\\ &\le \Big[\frac{4\kappa}{\nu_0}\boldsymbol{\beta}_\rho(\boldsymbol{g}) + \Theta_K\rho_K\Big]\rho_K^{-1}(\mathrm{e}^{\rho_K t} - \mathrm{e}^{\rho_K \tau}),\end{aligned}$$

and (3.52) follows from (3.48).

4 Trajectory Attractors

In order to define the dynamical system, we need to introduce first the appropriate functional framework following [19] (see also [18]).

Let us consider the space

$$\begin{aligned}\mathcal{Z}_K[\tau,T] := \{(\boldsymbol{v},\psi) \in L^\infty([\tau,T];\mathbb{Y}_K) \cap L^2([\tau,T];\mathbb{V}\times D(A_1)) :\\ \partial_t\boldsymbol{v} \in L^{\frac43}([\tau,T];\mathbb{V}^*),\ \partial_t\psi \in L^2([\tau,T];(H^1(\Omega))^*)\\ \text{and}\ \ \psi \in L^2([\tau,T];H^3(\Omega))\ \text{if}\ \ K = CH\}\end{aligned} \tag{4.1}$$

for any fixed $T > \tau \ge 0$, endowed with the following norm:

$$\begin{aligned}\|(\boldsymbol{v},\psi)\|^2_{\mathcal{Z}_K[\tau,T]} &= \|(\boldsymbol{v},\psi)\|^2_{L^\infty([\tau,T];\mathbb{Y}_K)} + \|(\boldsymbol{v},\psi)\|^2_{L^2([\tau,T];\mathbb{V}\times D(A_1))}\\ &\quad + \|\partial_t\boldsymbol{v}\|^2_{L^{\frac43}([\tau,T];\mathbb{V}^*)} + \|\partial_t\psi\|^2_{L^2([\tau,T];(H^1(\Omega))^*)} + \delta_K\|\psi\|^2_{L^2([\tau,T];H^3)},\end{aligned}$$

where $\delta_{AC} = 0$ and $\delta_{CH} = 1$. It is easy to see that $\mathcal{Z}_K[\tau,T]$ is Banach space. Moreover, we also introduce the Banach space $\mathcal{Z}^+_{b,K}$ defined by

$$\begin{aligned}\mathcal{Z}^+_{b,K} := \{(\boldsymbol{v},\psi) \in L^\infty(\mathbb{R}_+;\mathbb{Y}_K) \cap L^2_b(\mathbb{R}_+;\mathbb{V}\times D(A_1)) :\\ \partial_t\boldsymbol{v} \in L_b^{\frac43}(\mathbb{R}_+;\mathbb{V}^*),\ \partial_t\psi \in L^2_b(\mathbb{R}_+;(H^1(\Omega))^*)\\ \text{and}\ \ \psi \in L^2_b(\mathbb{R}_+;H^3(\Omega))\ \text{if}\ \ K = CH\},\end{aligned} \tag{4.2}$$

and the function space $\mathcal{Z}^+_{\mathrm{loc},K}$ defined as follows

$$\begin{aligned}\mathcal{Z}^+_{\mathrm{loc},K} := \{(\boldsymbol{v},\psi) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{Y}_K) \cap L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}\times D(A_1)) :\\ \partial_t \boldsymbol{v} \in L^{\frac{4}{3}}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*),\ \partial_t\psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+;(H^1(\Omega))^*)\\ \text{and } \psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+;H^3(\Omega)) \text{ if } K = CH\}.\end{aligned} \tag{4.3}$$

This space $\mathcal{Z}^+_{\mathrm{loc},K}$ is the inductive limit of the topological spaces $\{\Xi_K[0,T]\}_{T>0}$ defined here below (see [19, Chapter XII, Definition 1.3]).

Definition 4.1 *We denote by $\Xi_K[0,T]$ the space $\mathcal{Z}_K[0,T]$ endowed with the convergence topology: a sequence $(\boldsymbol{v}_n,\psi_n) \in \mathcal{Z}_K[0,T]$ converge to $(\boldsymbol{v},\psi)$ if and only if $(\boldsymbol{v}_n,\psi_n) \to (\boldsymbol{v}_n,\psi_n)$ $*$-weakly in $L^\infty([0,T];\mathbb{Y}_K)$ and weakly in $L^2([0,T];\mathbb{V}\times D(A_1))$, $\partial_t\boldsymbol{v}_n \to \partial_t\boldsymbol{v}$ weakly in $L^{\frac{4}{3}}([0,T];\mathbb{V}^*)$, $\partial_t\psi_n \to \partial_t\psi$ weakly in $L^2([0,T];(H^1(\Omega))^*)$, and, if $K = CH$, $\psi_n \to \psi$ weakly in $L^2([0,T];H^3(\Omega))$.*

We recall that $\mathcal{Z}^+_{\mathrm{loc},K}$ is a Hausdorff and Fréchet-Urysohn space with a countable topology base (see [19, Chapter XII]).

Observe now that a function $\boldsymbol{g}_0 \in L^2_b(\mathbb{R}_+;\mathbb{V}^*)$ if and only if is translation-compact in $L^2_{w,\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*)$ (see [19, Chapter V, Proposition 4.1]). Thus, if we consider the translation semigroup

$$(\mathbb{T}(t)\boldsymbol{g}_0)(s) = \boldsymbol{g}_0(s+t),$$

and we define the following hulls of $\boldsymbol{g}_0$ as

$$\mathcal{H}^+(\boldsymbol{g}_0) = \overline{\{\mathbb{T}(t)\boldsymbol{g}_0, t\geq 0\}}^{L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*)}, \quad \mathcal{H}^+_w(\boldsymbol{g}_0) = \overline{\{\mathbb{T}(t)\boldsymbol{g}_0, t\geq 0\}}^{L^2_{w,\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*)},$$

it turns out that $\mathcal{H}^+(\boldsymbol{g}_0) \equiv \mathcal{H}^+_w(\boldsymbol{g}_0)$. Moreover, we have that $\mathcal{H}^+_w(\boldsymbol{g}_0)$ is metrizable and complete and $\mathbb{T}(t)\mathcal{H}^+_w(\boldsymbol{g}_0) = \mathcal{H}^+_w(\boldsymbol{g}_0)$, for all $t \geq 0$, with attractor $\omega(\mathcal{H}^+_w(\boldsymbol{g}_0))$ (see [19, Chapter V, Lemma 4.1 and Proposition 4.2]).

Remark 4.1 If $\boldsymbol{g}_0$ is periodic, quasi-periodic or almost periodic, then $\boldsymbol{g}_0$ is translation-compact in the space $L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*)$ (see [19, Chapter V]).

Recalling Definition 3.1, from Corollary 3.1 and Proposition 3.3, we deduce the following proposition.

Proposition 4.1 *Suppose that f satisfies* (1.14) *and assume $\boldsymbol{g}_0 \in L^2_b(\mathbb{R}_+;\mathbb{V}^*)$. Then*

$$\mathcal{TR}(\boldsymbol{g},K) \subset \mathcal{Z}^+_{b,K} \subset \mathcal{Z}^+_{\mathrm{loc},K}, \quad \forall\, \boldsymbol{g} \in \mathcal{H}^+(\boldsymbol{g}_0). \tag{4.4}$$

We can now define the trajectory dynamical system. Indeed, let us consider the family of trajectory spaces

$$\{\mathcal{TR}(\boldsymbol{g},K) \mid \boldsymbol{g} \in \mathcal{H}^+(\boldsymbol{g}_0)\},$$

where $\boldsymbol{g}_0 \in L^2_b(\mathbb{R}_+;\mathbb{V}^*)$ is fixed. Then set

$$\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K) = \bigcup_{\boldsymbol{g}\in\mathcal{H}^+(\boldsymbol{g}_0)} \mathcal{TR}(\boldsymbol{g},K).$$

This is a topological space endowed with the topology of $\mathcal{Z}^+_{\mathrm{loc},K}$. However, in order to prove that it is closed in $\mathcal{Z}^+_{\mathrm{loc},K}$, we must require more on the symbol $\boldsymbol{g}_0$. Indeed, we have the following lemma.

Lemma 4.1 *Suppose that f satisfies* (1.14) *and let $\boldsymbol{g}_0$ be translation-compact either in $L^2_{w,\mathrm{loc}}(\mathbb{R}_+;\mathbb{H})$ or in $L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*)$. Then $\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K) \subset \mathcal{Z}^+_{b,K}$ is closed in $\mathcal{Z}^+_{\mathrm{loc},K}$.*

The proof goes essentially as in [18, Proposition 8.3].

We now define the translation semigroup $\{\mathbb{T}_K(t), t \geq 0\}$ acting on $\mathcal{Z}^+_{\mathrm{loc},K}$ by the formula

$$\mathbb{T}_K(t)(\boldsymbol{u}(\,\cdot\,),\phi(\,\cdot\,)) = (\boldsymbol{u}(t+\,\cdot\,),\phi(t+\cdot\,)), \quad t \geq 0, \tag{4.5}$$

where the pair on the right-hand side is a solution with symbol $\mathbb{T}(t)\boldsymbol{g}_0$.

Arguing as in [19, Propsositions 1.1 and 1.3], we can prove the following theorem.

Theorem 4.1 *Let the assumptions of Proposition* 4.1 *hold. Then $\{\mathbb{T}_K(t)\}$ is continuous in the topological space $\mathcal{Z}^+_{\mathrm{loc},K}$ and*

$$\mathbb{T}_K(t)(\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)) \subseteq \mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K), \quad \forall\, t \geq 0.$$

We recall the definition of trajectory attractor in our case (see [19, Chapter XII, Definition 2.2]).

Definition 4.2 *The trajectory attractor of the semigroup $\{\mathbb{T}_K(t), t \geq 0\}$ on $\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$ is a set $\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K) \subset \mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$ such that:*

(i) *$\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$ is compact in $\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$;*

(ii) *$\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$ is strictly invariant, that is,*

$$\mathbb{T}_K(t)\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K) = \mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K), \quad \forall\, t \geq 0; \tag{4.6}$$

(iii) *$\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$ is a uniformly attracting set for the semigroup $\{\mathbb{T}_K(t), t \geq 0\}$, that is, for every neighborhood $\mathcal{O} = \mathcal{O}(\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K))$ in the topology of $\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$, there exists $t^+ \geq 0$ such that*

$$\mathbb{T}_K(t)(\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)) \subseteq \mathcal{O}, \quad \forall\, t \geq t^+. \tag{4.7}$$

On account of Definition 3.1, observe that we can also define the spaces $\mathcal{Z}_K[\tau,T]$ on a given bounded interval $[\tau,T] \subset \mathbb{R}$ and then introduce $\mathcal{Z}_{b,K}$ and $\mathcal{Z}_{\mathrm{loc},K}$ taking $\mathbb{R}$ in place of $\mathbb{R}_+$. Let us indicate by $G(\boldsymbol{g}_0)$ the set of all complete symbols in $\mathcal{H}^+_w(\boldsymbol{g}_0)$, that is, the functions $\boldsymbol{f} \in L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{V}^*)$ such that $\boldsymbol{f}_t(s) = \Pi_+\boldsymbol{f}(t+s)$, $s \geq 0$, belongs to $\mathcal{H}^+_w(\boldsymbol{g}_0)$ for all $t \in \mathbb{R}$. We can now give the following definition (see [18, Section 4]).

Definition 4.3 *For any given $\boldsymbol{f} \in G(\boldsymbol{g}_0)$, the kernel $\mathbb{K}_{K,\boldsymbol{f}}$ of system* (2.13)–(2.15)*, with $\boldsymbol{f}$ in place of $\boldsymbol{g}$, consists of all solutions to P_K on $\mathbb{R}$ which satisfy either* (3.10)*, if $K = AC$, or* (3.11)*, if $K = CH$, on any time interval $[\tau,T] \subset \mathbb{R}$, and that are bounded in $\mathcal{Z}_{b,K}$.*

We are finally ready to state the main result of this paper.

Theorem 4.2 *Suppose that f satisfies* (1.14) *and let $\boldsymbol{g}_0$ be translation-compact either in $L^2_{w,\mathrm{loc}}(\mathbb{R}_+;\mathbb{H})$ or in $L^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{V}^*)$. Then the translation semigroup $\{\mathbb{T}_K(t),\, t \geq 0\}$ acting on $\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$ admits a uniform* (*with respect to $\boldsymbol{g} \in \mathcal{H}^+(\boldsymbol{g}_0)$*) *trajectory attractor $\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$ that satisfies properties* (i)–(iii) *of Definition* 4.2*. The set $\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$ is bounded in $\mathcal{Z}^+_{b,K}$ and compact in $\mathcal{Z}^+_{\mathrm{loc},K}$. Moreover, we have*

$$\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K) = \mathcal{A}_{\omega(\mathcal{H}^+(\boldsymbol{g}_0))}(K) = \Pi_+\mathbb{K}_{K,G(\boldsymbol{g}_0)} = \Pi_+ \bigcup_{\boldsymbol{f}\in G(\boldsymbol{g}_0)} \mathbb{K}_{K,\boldsymbol{f}}.$$

The proof of Theorem 4.2 is based on the following lemma.

Lemma 4.2 *Suppose that f satisfies* (1.14) *and let $\boldsymbol{g}_0 \in L^2_b(\mathbb{R}_+;\mathbb{V}^*)$. Then, there exists $R_K > 0$, depending only on ν_0, ν_1, κ, ε, α, $|\Omega|$, and $\|\boldsymbol{g}\|_{L^2_b(\mathbb{R}_+;\mathbb{V}^*)}$, such that*

$$\|\mathbb{T}_K(t)(\boldsymbol{u},\phi)\|^2_{\mathcal{Z}^b_K} \leq Q_K(\|(\boldsymbol{u},\phi)\|^2_{L^\infty([0,1];\mathbb{Y}_K)})\mathrm{e}^{-\rho_K t} + R_K, \quad \forall\, t \geq 1, \tag{4.8}$$

for any $\boldsymbol{g} \in \mathcal{H}^+(\boldsymbol{g}_0)$ and any trajectory $(\boldsymbol{u},\phi) \in \mathcal{TR}(\boldsymbol{g},K)$. Here Q_K is a monotone, positive non-decreasing function that is independent of time and $(\boldsymbol{u},\phi)$.

Proof Using estimate (3.52) and recalling (3.1)–(3.2), we deduce

$$\begin{aligned}
&\|\mathbb{T}_K(t)(\boldsymbol{u},\phi)\|^2_{L^\infty(\mathbb{R}_+;\mathbb{Y}_K)} \\
&\leq (\|(\boldsymbol{u},\phi)\|^2_{L^\infty([0,1];\mathbb{Y}_K)} + 2\alpha\|F_K(\phi)\|_{L^\infty([0,1];L^1(\Omega))} + c_{F_K})\mathrm{e}^{-\rho_K t} \\
&\quad + \frac{8\kappa}{\rho_K\nu_0}2(1+\rho_K)\|\boldsymbol{g}_0\|^2_{L^2_b(\mathbb{R}_+;\mathbb{V}^*)} + \Theta_K, \quad \forall\, t \geq 1.
\end{aligned} \tag{4.9}$$

Here we have also used (3.51) and the fact that $\boldsymbol{\beta}_\rho(\boldsymbol{g}) \leq \boldsymbol{\beta}_\rho(\boldsymbol{g}_0)$ for all $\boldsymbol{g} \in \mathcal{H}^+(\boldsymbol{g}_0)$ (see [19, Chapter XV, Remark 1.3]). Integrating (3.52) from t to $t+1$ and using (4.9), we deduce, for all $t \geq 1$,

$$\int_t^{t+1} \Big(\frac{\nu_0}{2\kappa}\|\boldsymbol{u}(s)\|^2 + \sigma_K L_K(\mu(s))\Big)\mathrm{d}s \leq Q_K(\|(\boldsymbol{u},\phi)\|^2_{L^\infty([0,1];\mathbb{Y}_K)})\mathrm{e}^{-\rho_K t} + R_K. \tag{4.10}$$

Here Q_K stands for a monotone, positive non-decreasing function that is independent of time and $(\boldsymbol{u},\phi)$, while R_K denotes a positive constant depending on ν_0, ν_1, κ, ε, α, $|\Omega|$, and $\|\boldsymbol{g}\|_{L^2_b(\mathbb{R}_+;\mathbb{V}^*)}$. Both $Q_K(\,\cdot\,)$ and R_K may vary from line to line in the course of this proof.

For estimating the remaining norms (see (4.2)) we have to consider the cases $K = AC$ and $K = CH$ separately. In the former one, recalling (1.14) and (3.19) and using (4.9)–(4.10), from (2.14), we deduce that

$$\begin{aligned}
\int_t^{t+1} |A_1\phi(s)|^2_{L^2}\mathrm{d}s &\leq \varepsilon^{-2}\int_t^{t+1}[|\mu(s)|^2_{L^2} + \alpha^2(1+|\phi(s)|^6_{L^6})]\mathrm{d}s \\
&\leq Q_{AC}(\|(\boldsymbol{u},\phi)\|^2_{L^\infty([0,1];\mathbb{Y}_K)})\mathrm{e}^{-\rho_{AC} t} + R_{AC}
\end{aligned} \tag{4.11}$$

for all $t \geq 1$. In the case $K = CH$, we have to control the chemical potential μ in $L^2([t,t+1];H^1(\Omega))$. Recalling (3.19), it suffices to estimate the spatial average of μ. From (2.14), we deduce (see (1.14))

$$\int_t^{t+1} \langle\mu(s)\rangle^2_\Omega\mathrm{d}s \leq c\int_t^{t+1}(1+|\phi(s)|^6_{L^6})\mathrm{d}s$$

for all $t \geq 1$. The above inequality, together with (1.15) and estimates (4.10)–(4.11), yields

$$\int_t^{t+1} |\mu(s)|^2_{H^1}\mathrm{d}s \leq Q_{CH}(\|(\boldsymbol{u},\phi)\|^2_{L^\infty([0,1];\mathbb{Y}_K)})\mathrm{e}^{-\rho_{CH} t} + R_{CH} \tag{4.12}$$

for all $t \geq 1$. Taking advantage of estimate (4.12), it is not difficult to show first an estimate similar to (4.11). Then, using well-known elliptic regularity results, we infer

$$\int_t^{t+1} |\phi(s)|^2_{H^3}\mathrm{d}s \leq Q_{CH}(\|(\boldsymbol{u},\phi)\|^2_{L^\infty([0,1];\mathbb{Y}_K)})\mathrm{e}^{-\rho_{CH} t} + R_{CH}. \tag{4.13}$$

Using well-known inequalities to estimate $B_0(\boldsymbol{u},\boldsymbol{u})$ and $\boldsymbol{R}_0(A_1\phi,\phi)$ in $\mathbb{V}^*$ (see (2.7)), and exploiting the above estimates, it is easy to show that

$$\begin{aligned}&\|\boldsymbol{R}_0(A_1\phi,\phi)\|_{L^{\frac{4}{3}}([t,t+1];\mathbb{V}^*)}+\|B_0(\boldsymbol{u},\boldsymbol{u})\|_{L^{\frac{4}{3}}([t,t+1];\mathbb{V}^*)}+\|B_1(\boldsymbol{u},\phi)\|_{L^2([t,t+1];(H^1(\Omega))^*)}\\&\le Q_K(\|(\boldsymbol{u},\phi)\|^2_{L^\infty([0,1];\mathbb{Y}_K)})\mathrm{e}^{-\rho_K t}+R_K,\quad \forall\, t\ge 1.\end{aligned}\tag{4.14}$$

Then, by comparison in equations (2.13) and (2.15), thanks to (4.11) and (4.13), we get

$$\|\partial_t\boldsymbol{u}\|_{L^{\frac{4}{3}}([t,t+1];\mathbb{V}^*)}+\|\partial_t\phi\|_{L^2([t,t+1];(H^1(\Omega))^*)}\le Q_K(\|(\boldsymbol{u},\phi)\|^2_{L^\infty([0,1];\mathbb{Y}_K)}\mathrm{e}^{-\rho t})+R_K.\tag{4.15}$$

Collecting estimates (4.9)–(4.11), (4.13), (4.15), we obtain (4.8). The lemma is proven.

Proof of Theorem 4.2 On account of (4.8), we consider, for instance, the set

$$B_R(\mathcal{Z}^+_{b,K}):=\{(\boldsymbol{v},\psi)\in\ \mathcal{Z}^+_{b,K}\cap\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K):\|(\boldsymbol{v},\psi)\|_{\mathcal{Z}^+_{b,K}}\le 2R_K\},\tag{4.16}$$

which is bounded and absorbing for the semigroup $\{\mathbb{T}_K(t),\ t\ge 0\}$ in the space $\mathcal{Z}^+_{b,K}$. This set is precompact and also closed by Lemma 4.1. Moreover, it is metrizable in $\mathcal{Z}^+_{\mathrm{loc},K}$ (see, e.g., [19, Chapter XI, Theorem 1.7]). The proof thus follows from [19, Chapter XIV, Theorem 2.1].

Due to compactness arguments, the trajectory attractor attracts the bounded subsets of $\mathcal{Z}^+_{b,K}$ in some strong topologies. Indeed, Theorem 4.2 implies the following corollary.

Corollary 4.1 *Let the assumptions of Theorem* 4.2 *hold. For any* $B_K\subset\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$, *bounded in* $\mathcal{Z}^+_{b,K}$, *for every* $T>0$ *and each* $\delta\in(0,1]$, *we have*

$$\lim_{t\to+\infty}\mathrm{dist}_{C([0,T];\mathbb{W}^{-\frac{\delta}{2}}\times H^{1-\delta}(\Omega))}(\mathbb{T}_K(t)B_{K|[0,T]},\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)_{|[0,T]})=0,\tag{4.17}$$

$$\lim_{t\to+\infty}\mathrm{dist}_{L^2([0,T];\mathbb{W}^{\frac{1-\delta}{2}}\times H^{2-\delta})}(\mathbb{T}_{AC}(t)B_{AC|[0,T]},\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(AC)_{|[0,T]})=0,\tag{4.18}$$

$$\lim_{t\to+\infty}\mathrm{dist}_{L^2([0,T];\mathbb{W}^{\frac{1-\delta}{2}}\times H^{3-\delta})}(\mathbb{T}_{CH}(t)B_{CH|[0,T]},\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(CH)_{|[0,T]})=0,\tag{4.19}$$

where $\mathbb{W}^k=(\mathbb{W}^{-k})^*$, *for* $k\le 0$ *and* $\mathbb{W}^s:=\mathbb{H}^s(\Omega)\cap\mathbb{H}$ *for* $s\ge 0$.

Let us now define, for any $\mathbb{B}_K\in\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$, the sections

$$\mathbb{B}_K(t)=\{(\boldsymbol{v}(t),\psi(t))\mid(\boldsymbol{v},\psi)\in\mathbb{B}_K\}\subset\mathbb{W}^{-\frac{\delta}{2}}\times H^{1-\delta}(\Omega),\quad t\ge 0.$$

Similarly, if we set

$$\begin{aligned}\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)(t)&=\{(\boldsymbol{v}(t),\psi(t))\mid(\boldsymbol{v},\psi)\in\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)\}\subset\mathbb{W}^{-\frac{\delta}{2}}\times H^{1-\delta}(\Omega),\quad t\ge 0,\\ \mathbb{K}_{K,G(\boldsymbol{g}_0)}(t)&=\{(\boldsymbol{v}(t),\psi(t))\mid(\boldsymbol{v},\psi)\in\mathbb{K}_{K,G(\boldsymbol{g}_0)}\}\subset\mathbb{W}^{-\frac{\delta}{2}}\times H^{1-\delta}(\Omega),\qquad t\in\mathbb{R},\end{aligned}$$

then a further consequence of Theorem 4.2 is as follows (see [19, Chapter XIV, Theorem 2.2, Definition 2.6, Corollary 2.2]).

Corollary 4.2 *Let the assumptions of Theorem* 4.2 *hold. Then the bounded set*

$$\mathcal{A}_{gl}(K)=\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)(0)=\mathbb{K}_{K,G(\boldsymbol{g}_0)}\subset\mathbb{Y}_K$$

is the uniform (*with respect to* $\boldsymbol{g}\in\mathcal{H}^+(\boldsymbol{g}_0)$) *global attractor in* $\mathbb{W}^{-\frac{\delta}{2}}\times H^{1-\delta}(\Omega)$, $\delta\in(0,1]$, *of the trajectory space* $\mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$, *that is*,

(I) $\mathcal{A}_{gl}(K)$ *is compact in* $\mathbb{W}^{-\frac{\delta}{2}} \times H^{1-\delta}(\Omega)$;

(II) $\mathcal{A}_{gl}(K)$ *satisfies the attracting property*

$$\lim_{t\to+\infty} \mathrm{dist}_{\mathbb{W}^{-\frac{\delta}{2}}\times H^{1-\delta}(\Omega)}(\mathbb{B}_K(t), \mathcal{A}_{gl}(K)) = 0$$

for any bounded set $\mathbb{B}_K \subset \mathcal{T}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$;

(III) $\mathcal{A}_{gl}(K)$ *is the minimal set satisfying* (I) *and* (II).

Remark 4.2 Theorem 4.2 and its corollaries also hold in the case $\boldsymbol{g}_0 \in L^2_b(\mathbb{R}_+; \mathbb{V}^*)$, provided that a different definition of solution is adopted. This new definition is based on inequality (3.52) instead of (3.10) or (3.11) (see [18, Definition 8.4]). We recall that, by using Definition 3.1, it is not possible to prove Lemma 4.1 in the sole assumption $\boldsymbol{g}_0 \in L^2_b(\mathbb{R}_+; \mathbb{V}^*)$.

We end this section with some additional remarks on the nature of external forces $\boldsymbol{g} = \boldsymbol{g}(t, x)$ acting on the binary fluid mixture. Suppose that they have the form

$$\boldsymbol{g}_0(t, x) = \boldsymbol{g}_+(x) + \boldsymbol{g}_1(t, x), \tag{4.20}$$

when $\boldsymbol{g}_+ \in \mathbb{H}$ does not depend on t and $\boldsymbol{g}_1$ satisfies the following condition:

$$(\mathbb{T}(h)\boldsymbol{g}_0)(s) = \boldsymbol{g}_1(s + h) \to 0, \quad \text{as } h \to +\infty, \tag{4.21}$$

strongly in $L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{V}^*)$ or weakly in $L^2_{w,\mathrm{loc}}(\mathbb{R}_+; \mathbb{H})$. It is not difficult to check that, in the first case, $\boldsymbol{g}$ is either translation-compact in $L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{V}^*)$ or in $L^2_{w,\mathrm{loc}}(\mathbb{R}_+; \mathbb{H})$. Thus assumptions of Theorem 4.2 on the external forces are satisfied and we can prove the following theorem.

Theorem 4.3 *Suppose that f satisfies* (1.14) *and let* (4.20)–(4.21) *hold. Then the trajectory attractor coincides with the attractor of the corresponding autonomous system* (*i.e., the one obtained by replacing $\boldsymbol{g}_0$ with $\boldsymbol{g}_+$*). *More precisely, we have*

$$\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K) = \mathcal{A}_{\boldsymbol{g}_+}(K). \tag{4.22}$$

The existence of the attractors follows from Theorem 4.2, whereas (4.22) is a consequence of [19, Chapter XVII, Theorem 1.1].

Let us report a couple of examples of perturbations terms $\boldsymbol{g}_1$ that satisfy condition (4.21).

Example 4.1 Let

$$\boldsymbol{g}_1(t, x) = \boldsymbol{\varphi}_0(x)\theta(t), \tag{4.23}$$

where $\boldsymbol{\varphi}_0 \in \mathbb{V}^*$ and

$$\theta(t) = \frac{\sin(t^2)}{1 + t^2}.$$

Then $\boldsymbol{g}_1$ satisfies (4.21) in the strong topology of $L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{V}^*)$.

Example 4.2 Let the function $\boldsymbol{g}_1$ have the form (4.23) with $\boldsymbol{\varphi}_0 \in \mathbb{H}$ and

$$\theta(t) = \sin(t^3).$$

It is known that $\boldsymbol{g}_1(t + h) \to 0$ as $h \to +\infty$ in the weak topology of $L^2([t, t + 1]; \mathbb{H})$, for every $t \in \mathbb{R}_+$. Hence $\boldsymbol{g}_1$ satisfies (4.21) in the weak topology of $L^2_{w,\mathrm{loc}}(\mathbb{R}_+; \mathbb{H})$.

Remark 4.3 Theorem 4.2 clearly applies to the 2D case which was analyzed in [32, 31] with a time-independent external force. In the 2D case, thanks to uniqueness, the trajectories can be described through a family of processes $\{U_{\boldsymbol{g},K}(t,\tau)\}$ acting on the phase-space $\mathbb{Y}_K$, with symbol $\boldsymbol{g}$ in the hull of some given $\boldsymbol{g_0} \in L^2_b(\mathbb{R}_+;\mathbb{V}^*)$ (see [49, 50] for possible generalizations). In this case, the uniform global attractor $\mathcal{A}_{gl}(K)$ is compact in $\mathbb{Y}_K$ and, supposing $\boldsymbol{g_0}$ quasiperiodic with respect to time, one should also prove that $\mathcal{A}_{gl}(K)$ has finite fractal dimension. With such a symbol, the existence of exponential attractors can also be proven (see [27] for 2D incompressible Navier-Stokes equations). Thus Theorem 4.2 allows us to generalize some of the results contained in [32, 31] to nonautonomous external forces.

Finally, suppose that f satisfies (1.14). If $\boldsymbol{g}_0$ is translation-compact in $L^2_{w,\mathrm{loc}}(\mathbb{R}_+;\mathbb{H})$, then, setting $\boldsymbol{g}_{0,m} = P_{0,m}\boldsymbol{g}_0$, we have that $\boldsymbol{g}_{0,m}$ is also translation-compact in $L^2_{w,\mathrm{loc}}(\mathbb{R}_+;\mathbb{H})$. Thus, following [19, Chapter XVI], the Galerkin approximation problems $\boldsymbol{P}^m_K$ (see (3.13)–(3.16)) can be shown to have the uniform trajectory attractor $\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_{0,m})}(K)$ which is bounded in $\mathcal{Z}^+_{b,K}$ and compact in $\mathcal{Z}^+_{\mathrm{loc},K}$. Consequently, arguing as in the proof of [19, Chapter XVI, Theorem 3.1], it is possible to prove

Theorem 4.4 *For any neighborhood $\mathcal{O}$ of the trajectory attractor $\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)$ in $\mathcal{Z}^+_{\mathrm{loc},K}$, there exists $m_0 = m_0(\mathcal{O}) \in \mathbb{N}$ such that*

$$\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_{0,m})}(K) \subset \mathcal{O}, \quad \forall\, m \geq m_0.$$

In addition, for every $T > 0$ and $\delta \in (0,1]$, we have

$$\begin{aligned}
\lim_{m\to+\infty} \mathrm{dist}_{C([0,T];\mathbb{W}^{-\frac{\delta}{2}}\times H^{1-\delta}(\Omega))}(\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_{0,m})}(K)_{|[0,T]}, \mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(K)_{|[0,T]}) = 0,\\
\lim_{m\to+\infty} \mathrm{dist}_{L^2([0,T];\mathbb{W}^{\frac{1-\delta}{2}}\times H^{2-\delta})}(\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_{0,m})}(AC)_{|[0,T]}, \mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(AC)_{|[0,T]}) = 0,\\
\lim_{m\to+\infty} \mathrm{dist}_{L^2([0,T];\mathbb{W}^{\frac{1-\delta}{2}}\times H^{3-\delta})}(\mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_{0,m})}(CH)_{|[0,T]}, \mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_0)}(CH)_{|[0,T]}) = 0.
\end{aligned}$$

We conclude with a related result for the (uniform) global attractor (see [19, Chapter XVI, Corollary 2.1]).

Corollary 4.3 *Setting* (*see Corollary* 4.2)

$$\mathcal{A}^m_{gl}(K) := \mathcal{A}_{\mathcal{H}^+(\boldsymbol{g}_{0,m})}(K)(0),$$

we have, for all $\delta \in (0,1]$,

$$\lim_{m\to+\infty} \mathrm{dist}_{\mathbb{W}^{-\frac{\delta}{2}}\times H^{1-\delta}(\Omega)}(\mathcal{A}^m_{gl}(K), \mathcal{A}_{gl}(K)) = 0.$$

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