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Generalized Liouville Theorem in Nonnegatively Curved Alexandrov Spaces

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Abstract In this paper, Yau's conjecture on harmonic functions in Riemannian manifolds is generalized to Alexandrov spaces. It is proved that the space of harmonic functions with polynomial growth of a fixed rate is finite dimensional and strong Liouville theorem holds in Alexandrov spaces with nonnegative curvature.

Keywords Alexandrov space, Harmonic function, Harnack inequality
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1 Introduction

In 1975's, Yau proved strong Liouville theorem of harmonic functions on open (complete and noncompact) manifolds with nonnegative Ricci curvature in [17], i.e., any positive harmonic function on such manifolds must be constant. In addition, he raised in [18, 19] the following

Conjecture 1.1 *For an open manifold M with nonnegative Ricci curvature, the space of harmonic functions with polynomial growth of a fixed rate is finite dimensional.*

In the case of $\dim M = 2$, the conjecture was done by Li and Tam [12], Donnelly and Fefferman [6] early. The general case was solved by Colding and Minicozzi II [4] in 1997. The optimal dimension estimate was proved by Colding and Minicozzi II [5] and Li [11]. In the present paper, we generalize Yau's conjecture to Alexandrov spaces.

Assumption 1.1. Throughout the present paper, we always denote by X an n-dimensional Alexandrov space with nonnegative curvature, which is a complete, locally compact length space satisfying convexity condition in Alexandrov sense (see [1, 2]). In addition, X is connected, noncompact and without boundary.

For any $m > 0$ and fixed $p \in X$, set

$$H_m(X) = \{u \text{ is a harmonic function on } X \mid |u(x)| \leq C(d^m(p,x) + 1)\}.$$

Then the main results in the present paper are as follows.

Theorem 1.1 *Let Assumption* 1.1 *be fulfilled. Then* $\forall\, m > 0$*, we have*

$$\dim H_m(X) < \infty.$$

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Using the Moser iteration method (see [8, 13]), we obtain the Harnack inequality, which implies the strong Liouville theorem in Alexandrov spaces.

Theorem 1.2 (Liouville) *Let Assumption* 1.1 *be fulfilled. Then any positive harmonic function on X must be constant.*

The key point of the Moser iteration lies in establishing the uniform Poincaré inequality (see Theorem 3.1) and Sobolev inequality (see Theorem 3.2) in Alexandrov spaces with nonnegative curvature. Kuwae, Machigashira and Shioya [9] obtained the local weak Poincaré inequality for Alexandrov spaces with any curvature lower bound. Refining their arguments in nonnegative curvature case, we get the uniform Poincaré inequality. According to Saloff-Coste [16], by means of volume growth condition and pseudo-Poincaré technique, the Sobolev inequality can be established in Alexandrov spaces with nonnegative curvature.

Then we can carry out the standard Moser iteration to get the Harnack inequality in nonnegatively curved Alexandrov spaces. Another difficulty is that we cannot use the scaling method in Alexandrov spaces, so we must make estimates in geodesic ball with any radius, which causes the calculation more complicated.

2 Preliminaries

For notions and notations related to Alexandrov spaces with curvature bounded below, we refer readers to two references (see [1, 2]) and only recall some important facts used here.

(X, d) is an Alexandrov space with curvature $\geq \kappa$, for $\kappa \in \mathbb{R}$, its Hausdorff dimension coincides with the topological dimension and must be an integer or infinity. Here we only consider finite dimensional Alexandrov spaces. For any $p \in X$, the tangent cone at p, T_pX, which is defined by the Euclidean cone over direction space Σ_pX, coincides with the Gromov-Hausdorff limit of pointed rescaling spaces, that is,

$$T_pX = \lim_{\lambda\to\infty}(X, \lambda d, p).$$

So we get $\operatorname{curv} T_pX \geq 0$. Hence $\operatorname{curv} \Sigma_pX \geq 1$, and Σ_pX is a compact $(n-1)$ dimensional Alexandrov space. Then we can use induction on dimension of Alexandrov spaces with curvature bounded below. In addition, we can define natural semi-scalar product on T_pX, for any (t, ζ), $(s, \eta) \in T_pX$, in which $t, s > 0$ and $\zeta, \eta \in \Sigma_pX$,

$$(t, \zeta) \cdot (s, \eta) = \langle (t, \zeta), (s, \eta)\rangle = t^2 + s^2 - 2ts \cos d_\Sigma(\zeta, \eta),$$

where $d_\Sigma(\zeta, \eta)$ means the angular metric in Σ_pX.

We refer to Petrunin [15] for the definition of harmonic functions on Alexandrov spaces. For any domain $\Omega \subset X$, by $\mathrm{Lip}(\Omega)$, we mean the set of all Lipschitz functions on Ω, and by $\mathrm{Lip}_0(\Omega)$, we mean the set of Lipschitz functions with compact support in Ω. The gradient of Lipschitz function can be defined almost everywhere and the Sobolev space is well defined (see [3]).

Definition 2.1 *By Sobolev space $W^{1,2}(\Omega)$, we mean the closure of* $\mathrm{Lip}(\Omega) \cap L^2(\Omega)$ *with* *respect to the norm*

$$\|u\|^2_{W^{1,2}(\Omega)} = \int_\Omega (u^2 + |\nabla u|^2),$$

and by $W^{1,2}_0(\Omega)$*, we mean the closure of* $\mathrm{Lip}_0(\Omega)$ *with respect to the same norm, where* $|\nabla u| = \sqrt{\nabla u \cdot \nabla u}$*, the norm of the gradient of the Lipschitz function* u*. By* $u \in \mathrm{Lip}_{\mathrm{loc}}(X)$ *and* $u \in W^{1,2}_{\mathrm{loc}}(X)$*, we mean for any compact domain* $\Omega \subset X$*,* $u \in \mathrm{Lip}(\Omega)$ *and* $u \in W^{1,2}(\Omega)$*.*

Definition 2.2 *By a harmonic (subharmonic, superharmonic) function* u *on* $\Omega \subset X$*, we mean that* $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *and,* $\forall\, \varphi \in W^{1,2}_0(\Omega)$*,* $\varphi \geq 0$*, we have*

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0 \ (\leq 0, \geq 0). \tag{2.1}$$

Remark 2.1 In the sequel, without loss of generality, we may assume that X is connected and without boundary. Otherwise, we consider a connected component of X and doubling $\widetilde{X}$ of X according to Perelman's doubling theorem (see [14]), and define a harmonic function u on X as the restriction to X of a harmonic function $\widetilde{u}$ on $\widetilde{X}$. In addition, compact Alexandrov spaces admit only constant harmonic functions, which easily follows from (2.1) with density arguments. Thus we make Assumption 1.1 as before.

Without any confusion, we denote by $H^n(B(x,r))$ or $|B(x,r)|$ an n-dimensional Hausdorff measure of geodesic ball centered at x with radius r.

Recall the relative volume comparison theorem in Alexandrov spaces (see [1]).

Theorem 2.1 (Bishop-Gromov) *Let* X *be an* n*-dimensional Alexandrov space with curvature* $\geq \kappa$ *for some* $\kappa \in \mathbb{R}$*. Then* $\forall\, x \in X$*,* $\forall\, r > 0$*, we have that*

$$\frac{|B(x,r)|}{V^\kappa_r}$$

is nonincreasing in r*, where* V^κ_r *is the volume of* r*-ball in an* n*-dimensional simply connected complete space with constant curvature* κ*. Moreover,* $|B(x,r)| \leq V^\kappa_r$*.*

Therefore, under Assumption 1.1, we obtain from Bishop-Gromov volume comparison theorem that $\forall\, x \in X$, $\forall\, 0 < r < r'$,

$$\frac{|B(x,r')|}{|B(x,r)|} \leq \Big(\frac{r'}{r}\Big)^n, \tag{2.2}$$

$$|B(x,2r)| \leq 2^n |B(x,r)|, \tag{2.3}$$

and $|B(x,r)| \leq \omega_n r^n$, where ω_n denotes the volume of unit ball in $\mathbb{R}^n$. We call (2.3) the volume doubling property.

3 Analytic Tools: Poincaré and Sobolev Inequality

With the aid of classical works on the Poincaré inequality (see [7]) and the Sobolev inequality (see [16]), we prove the following corresponding results in Alexandrov spaces with nonnagative curvature.

Theorem 3.1 (Uniform Poincaré Inequality) *Let Assumption* 1.1 *be fulfilled. Then there exists a constant* $C = C(n)$*, such that for any* $u \in W^{1,2}_{\mathrm{loc}}(X)$, $\forall\, p \in X$, $\forall\, r > 0$,

$$\int_{B(p,r)} |u - u_B|^2 \le Cr^2 \int_{B(p,r)} |\nabla u|^2, \tag{3.1}$$

where $u_B = \frac{1}{|B(p,r)|} \int_{B(p,r)} u$.

Theorem 3.2 (Sobolev Inequality) *Let Assumption* 1.1 *be fulfilled and* $\dim X = n \ge 3$. *Then there exists a constant* $C = C(n)$*, such that for* $\forall\, p \in X$, $\forall\, r > 0$, $B := B(p,r)$ *and* $u \in W_0^{1,2}(B)$*, we have*

$$\|u\|_{\frac{2n}{n-2}} \le C \frac{r}{|B|^{\frac{1}{n}}} \Big(\int_B |\nabla u|^2 \Big)^{\frac{1}{2}}, \tag{3.2}$$

i.e.,

$$\Big(\fint_B u^{2\chi} \Big)^{\frac{1}{\chi}} \le Cr^2 \fint_B |\nabla u|^2, \tag{3.3}$$

where $\chi = \frac{n}{n-2}$ *and* $\fint_B u = \frac{1}{|B|} \int_B u$.

Remark 3.1 We can state a more general theorem on the Poincaré and Sobolev inequalities with any p-norm instead of 2-norm in Theorems 3.1 and 3.2, which are standard in [7, 16]. But in the present paper, 2-norm case is sufficient for our application and carrying out the Moser iteration.

We need a lemma to prove the Poincaré and Sobolev inequalities, which was stated by Kuwae et al. (see [9, Lemma 4.2]).

First we denote by $\gamma_{xy}(t)$, $t \in [0,1]$ the minimal geodesic joining x and y with parameter proportional to the arclength. In [9], Kuwae et al. dealt with the general curvature lower bound κ.

Lemma 3.1 *Let Assumption* 1.1 *be fulfilled. For any* $x \in X$ *and* $r > 0$, $u : B(x,r) \to \mathbb{R}^+$ *is a nonnegative function and for any given* $t \in (0,1]$*, we have*

$$\int_{B(x,r)} u(\gamma_{xy}(t)) \mathrm{d}H^n(y) \le \frac{1}{t^n} \int_{B(x,tr)} u(z) \mathrm{d}H^n(z), \tag{3.4}$$

where H^n *denotes an* n*-dimensional Hausdorff measure on* X*. From now on, we always denote* $\mathrm{d}y = \mathrm{d}H^n(y)$.

The weak uniform 1-Poincaré inequality follows from the previous lemma.

Lemma 3.2 *Let Assumption* 1.1 *be fulfilled. For any* $u \in W^{1,2}_{\mathrm{loc}}(X)$, $\forall\, x \in X$, $\forall\, r > 0$, *we have*

$$\int_{B(x,r)} |u - u_B| \le 2^{n+1} r \int_{B(x,3r)} |\nabla u|, \tag{3.5}$$

where $u_B = \frac{1}{|B(x,r)|} \int_{B(x,r)} u$.

Proof Using density arguments, we choose $u \in \mathrm{Lip}_{\mathrm{loc}}(X)$. We have

$$\begin{aligned}\int_{B(x,r)} |u-u_B| &\le \frac{1}{|B(x,r)|}\int_B\int_B |u(y)-u(z)|\mathrm{d}y\mathrm{d}z\\ &\le \frac{2r}{|B(x,r)|}\int_B\int_B\int_0^1 |\nabla u(\gamma_{yz}(t))|\mathrm{d}t\mathrm{d}y\mathrm{d}z\\ &= \frac{4r}{|B(x,r)|}\int_B\int_B\int_{\frac12}^1 |\nabla u(\gamma_{yz}(t))|\mathrm{d}t\mathrm{d}y\mathrm{d}z.\end{aligned}$$

To obtain the last equality, noting that $\gamma_{yz}(t)=\gamma_{zy}(1-t)$ and integrals of y and z are symmetric, we can change the variable $t'=1-t$ on $[0,\frac12]$. This is the crucial trick in our proof, which is due to Korevaar and Scheon [10] and also mentioned in [16]. Then by Lemma 3.1,

$$\begin{aligned}\int_{B(x,r)} |u-u_B| &\le \frac{4r}{|B(x,r)|}\int_{B(x,r)}\mathrm{d}y\int_{\frac12}^1\mathrm{d}t\int_{B(y,2r)} |\nabla u(\gamma_{yz}(t))|\mathrm{d}z\\ &\le \frac{4r}{|B(x,r)|}\int_{B(x,r)}\mathrm{d}y\int_{\frac12}^1\frac{1}{t^n}\mathrm{d}t\int_{B(y,2tr)} |\nabla u(w)|\mathrm{d}w\\ &\le \frac{4r}{|B(x,r)|}\int_{B(x,r)}\mathrm{d}y\int_{\frac12}^1\frac{1}{t^n}\mathrm{d}t\int_{B(x,3r)} |\nabla u(w)|\mathrm{d}w\\ &\le \frac{2^{n+1}r}{|B(x,r)|}\int_{B(x,r)}\mathrm{d}y\int_{B(x,3r)} |\nabla u(w)|\mathrm{d}w\\ &= 2^{n+1}r\int_{B(x,3r)}|\nabla u|.\end{aligned}$$

Hence the weak uniform 1-Poincaré inequality follows.

Proof of Theorem 3.1 According to standard techniques of the Poincaré inequality in metric space (see [7, Theorem 5.1 and Corollary 9.8]), since X is a connected length space and satisfies volume doubling property (2.3), combining with the weak uniform 1-Poincaré inequality in Lemma 3.2, we can soon get the uniform Poincaré inequality and the corresponding embedding theorem.

Next, we use the pseudo-Poincaré technique to obtain the Sobolev inequality (see [16]).

For any function on X, $u: X\to\mathbb{R}$, $\forall\, r>0$, we define $u_r: X\to\mathbb{R}$, such that $u_r(x)=\frac{1}{|B(x,r)|}\int_{B(x,r)}u$.

Lemma 3.3 *Let Assumption* 1.1 *be fulfilled. For any* $u\in\mathrm{Lip}_0(X)$, $\forall\, r>0$, *we have*

$$\|u-u_r\|_1 \le 2^{3n+1}r\|\nabla u\|_1. \tag{3.6}$$

Proof Using the same trick in Lemma 3.2, we have

$$\|u-u_r\|_1=\int_X |u(x)-u_r(x)|\le\int_X\int_X |u(x)-u(y)|\frac{\mathbf{1}_{B(x,r)}(y)}{|B(x,r)|}\mathrm{d}x\mathrm{d}y,$$

where $\mathbf{1}_{B(x,r)}$ denotes the characteristic function on $B(x,r)$.

To make it more symmetric in x and y, we note that

$$\frac{\mathbf{1}_{B(x,r)}(y)}{|B(x,r)|}\le 2^{\frac n2}\frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{|B(x,r)||B(y,r)|}},$$

which follows from $|B(y,r)| \le |B(y,2r)| \le 2^n |B(x,r)|$ for $y \in B(x,r)$. Hence

$$\begin{aligned}\|u-u_r\|_1 &\le 2^{\frac{n}{2}} \int_X\int_X |u(x)-u(y)| \frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{|B(x,r)||B(y,r)|}} \mathrm{d}x\mathrm{d}y \\ &\le 2^{\frac{n}{2}} \cdot 2r \int_X\int_X\int_0^1 |\nabla u(\gamma_{xy}(t))| \frac{\mathbf{1}_{B(x,r)}(y)\mathbf{1}_{B(y,r)}(x)}{\sqrt{|B(x,r)||B(y,r)|}} \mathrm{d}t\mathrm{d}x\mathrm{d}y \\ &\le 2^{\frac{n}{2}} \cdot 4r \int_X\int_X\int_{\frac{1}{2}}^1 |\nabla u(\gamma_{xy}(t))| \frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{|B(x,r)||B(y,r)|}} \mathrm{d}t\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Using the fact $|B(x,r)| \le |B(x,2r)| \le 2^n |B(y,r)|$, where $d(x,y) \le r$, we get

$$\frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{|B(x,r)||B(y,r)|}} \le 2^{\frac{n}{2}} \frac{\mathbf{1}_{B(x,r)}(\gamma_{xy}(t))}{|B(x,r)|}.$$

It follows that

$$\begin{aligned}\|u-u_r\|_1 &\le 2^{n+2} r \int_X\int_X\int_{\frac{1}{2}}^1 |\nabla u(\gamma_{xy}(t))| \frac{\mathbf{1}_{B(x,r)}(\gamma_{xy}(t))}{|B(x,r)|} \mathrm{d}t\mathrm{d}x\mathrm{d}y \\ &\le 2^{n+2} r \int_X \frac{1}{|B(x,r)|}\mathrm{d}x \int_{\frac{1}{2}}^1 \mathrm{d}t \int_{B(x,r)} |\nabla u(\gamma_{xy}(t))| \mathbf{1}_{B(x,r)}(\gamma_{xy}(t))\mathrm{d}y \\ &\le 2^{n+2} r \int_X \frac{1}{|B(x,r)|}\mathrm{d}x \int_{\frac{1}{2}}^1 \mathrm{d}t \int_{B(x,\frac{r}{t})} |\nabla u(\gamma_{xy}(t))|\mathrm{d}y \\ &\le 2^{n+2} r \int_X \frac{1}{|B(x,r)|}\mathrm{d}x \int_{\frac{1}{2}}^1 \frac{1}{t^n}\mathrm{d}t \int_{B(x,r)} |\nabla u(w)|\mathrm{d}w \\ &\le 2^{2n+1} r \int_X\int_X |\nabla u(w)| \frac{\mathbf{1}_{B(x,r)}(w)}{|B(x,r)|}\mathrm{d}x\mathrm{d}w \\ &\le 2^{3n+1} r \int_X |\nabla u(w)|\mathrm{d}w \int_X \frac{\mathbf{1}_{B(w,r)}(x)}{|B(w,r)|}\mathrm{d}x \\ &\le 2^{3n+1} r \int_X |\nabla u(w)|\mathrm{d}w \\ &= 2^{3n+1} r \|\nabla u\|_1. \end{aligned} \tag{3.7}$$

To obtain inequality (3.7), we use the volume doubling property again.

Proof of Theorem 3.2 By means of the standard pseudo-Poincaré technique (see [16, Theorems 5.2.3, 3.2.9]) and volume growth condition (2.2), we obtain the Sobolev inequality from Lemma 3.3.

4 Yau's Conjecture

In this section, we use the method in [4] to show that Yau's Conjecture 1.1 holds in Alexandrov spaces with nonnegative curvature. We show the classical Caccioppoli inequality for subharmonic function in Alexandrov spaces as follows.

Lemma 4.1 *Let u be a subharmonic function on X. Then for any $\eta \in W_0^{1,2}(X)$, we have*

$$\int_X \eta^2 |\nabla u|^2 \le 4 \int_X |\nabla\eta|^2 u^2. \tag{4.1}$$

Proof Set the test function in (2.1) $\varphi = \eta^2 u$. Then

$$\int \nabla u \cdot (2\eta u \nabla\eta + \eta^2 \nabla u) \le 0.$$

The Hölder inequality yields

$$\int \eta^2 |\nabla u|^2 \le -\int 2\eta u \nabla\eta \cdot \nabla u \le 2\Big(\int \eta^2 |\nabla u|^2\Big)^{\frac{1}{2}} \Big(\int |\nabla\eta|^2 u^2\Big)^{\frac{1}{2}}.$$

Hence, the lemma follows.

Proof of Theorem 1.1 In order to prove Yau's conjecture, it suffices to check the following three conditions of the underlying manifold according to [4]. Here we consider Alexandrov space X with nonnegative curvature.

Firstly, the volume doubling property follows from Bishop-Gromov volume comparison theorem and Assumption 1.1 (see (2.3)).

Secondly, the uniform Poincaré inequality has been proved in Theorem 3.1.

Last, the reverse Poincaré inequality for harmonic functions is stated in the following:

Suppose that u is a harmonic function on X. For any $\Omega > 1$, there exists a constant $C = C(\Omega)$, such that for any $x \in X$ and $r > 0$, we have

$$r^2 \int_{B(x,r)} |\nabla u|^2 \le C \int_{B(x,\Omega r)} u^2. \tag{4.2}$$

Set η as follows:

$$\eta(y) = \begin{cases} 1, & d(y,x) < r, \\ \dfrac{\Omega r - d(y,x)}{\Omega r - r}, & r \le d(y,x) < \Omega r, \\ 0, & \Omega r \le d(y,x), \end{cases}$$

where $d(\,\cdot\,, x)$ is the distance function with respect to x. It is easy to check $|\nabla\eta| \le \frac{1}{(\Omega-1)r}$. By the Caccioppoli inequality (4.1), we obtain

$$r^2 \int_{B(x,r)} |\nabla u|^2 \le \frac{4}{(\Omega-1)^2} \int_{B(x,\Omega r)} u^2.$$

Then Theorem 1.1 follows immediately, and hence Yau's conjecture is proved in Alexandrov spaces.

5 Liouville Theorem

In the case of $\dim X = 2$, we show a strong Liouville theorem for nonnegative superharmonic function, which automatically implies strong Liouville theorem for nonnegative harmonic function.

Theorem 5.1 *Let Assumption* 1.1 *be fulfilled and* $\dim X = 2$*, and* u *be a nonnegative superharmonic function on* X*. Then* u *must be constant.*

Proof Take $v = u + \epsilon \geq 0$ for some $\epsilon > 0$. Then it suffices to prove the theorem for v. Set $w = \log v$, and test function $\varphi = \frac{\eta^2}{v}$, where $\eta \in \mathrm{Lip}_0(X)$ and $\eta \geq 0$. Noting that $\nabla w = \frac{\nabla v}{v}$, we have

$$0 \leq \int \nabla v \cdot \Big(\frac{2\eta\nabla\eta}{v} - \frac{\eta^2 \nabla v}{v^2}\Big) = \int (2\eta\nabla\eta \cdot \nabla w - |\nabla w|^2\eta^2).$$

Hence

$$\int \eta^2 |\nabla w|^2 \leq \int 2\eta\nabla\eta \cdot \nabla w \leq 2\Big(\int \eta^2|\nabla w|^2\Big)^{\frac{1}{2}}\Big(\int |\nabla\eta|^2\Big)^{\frac{1}{2}}.$$

So we get

$$\int \eta^2|\nabla w|^2 \leq 4\int |\nabla \eta|^2. \tag{5.1}$$

Choose η as follows. For some $k > 1$,

$$\eta(y) = \begin{cases} 1, & d(y,x) < r, \\ 1 + \dfrac{\log r - \log d(y,x)}{\log k}, & r \leq d(y,x) < kr, \\ 0, & kr \leq d(y,x). \end{cases}$$

Easy calculation shows that

$$|\nabla\eta|(y) = \frac{|\nabla d|}{d(y,x)\log k} \leq \frac{1}{d(y,x)\log k},$$

which holds almost everywhere in $y \in B(x,kr) \setminus B(x,r)$, otherwise equals zero. So it follows from (5.1) that

$$\int_{B(x,r)} |\nabla w|^2 \leq 4\int_{B(x,kr)\setminus B(x,r)} \frac{1}{d^2(y,x)\log^2 k} = 4\int_r^{kr} \mathrm{d}\tau \int_{\partial B(x,\tau)} \frac{1}{\tau^2 \log^2 k}.$$

Since $\mathrm{curv}\, X \geq 0$, using the volume comparison in surface (see [1, Corollary 10.6.10]), we have

$$H^1(\partial B(x,\tau)) \leq 2\pi\tau.$$

Hence

$$\int_{B(x,r)} |\nabla w|^2 \leq \frac{8\pi}{\log^2 k}\int_r^{kr} \frac{1}{\tau}\mathrm{d}\tau = \frac{8\pi}{\log k}.$$

By letting $k \to \infty$, we get $|\nabla w| = 0$ a.e., which implies $w = \text{const.}$ and $v = \text{const.}$

Remark 5.1 Usually we call the manifolds (Alexandrov spaces) satisfying Theorem 5.1 parabolic manifolds. But the previous proof for Theorem 5.1 does not work in higher dimension. In higher dimensional case, we have to show Liouville theorem by the Moser iteration.

Since the uniform Poincaré inequality and Sobolev inequality have been proved as before, the standard Moser iteration can be carried out (see [8, 13]). Our proofs are almost the same, except that we shall carry out all the estimates in geodesic ball with any radius because the scaling technique can not be applied in Alexandrov spaces. Proofs of the following theorems are included in Appendices (see Theorems 6.1 and 6.2).

Theorem 5.2 *Let Assumption* 1.1 *be fulfilled and* $\dim X = n \geq 3$*. For any subharmonic function* u *on* X *and any* $p > 0$*,* $0 < \theta < \tau \leq 1$*, there exists a constant* $C = C(n, p, \theta, \tau)$*, such that for any* $B_R := B(x, R)$*,*

$$\sup_{B_{\theta R}} u \leq C\Big(\fint_{B_{\tau R}} |u|^p\Big)^{\frac{1}{p}}. \tag{5.2}$$

Theorem 5.3 *Let Assumption* 1.1 *be fulfilled and* $\dim X = n \geq 3$*. For any nonnegative superharmonic function* u *on* X *and any* $0 < \theta < \tau < 1$*,* $0 < p < \frac{n}{n-2}$*, there exists a constant* $C = C(n, p, \theta, \tau)$*, such that for any* $B_R := B(x, R)$*,*

$$\inf_{B_{\theta R}} u \geq C\Big(\fint_{B_{\tau R}} |u|^p\Big)^{\frac{1}{p}}. \tag{5.3}$$

Now, the Harnack inequality follows immediately from Theorems 5.2 and 5.3.

Theorem 5.4 (Harnack Inequality) *Let Assumption* 1.1 *be fulfilled and* $\dim X = n \geq 3$*. For any nonnegative harmonic function* u *on* X *and* $r > 0$*, there exists a constant* $C = C(n)$ *such that*

$$\sup_{B_r} u \leq C \inf_{B_r} u. \tag{5.4}$$

Proof We only need to take $\theta = \frac{1}{4}$, $\tau = \frac{1}{2}$ and $p = 1$.

According to [16], the Harnack inequality (5.4) implies Liouville theorem.

Proof of Theorem 1.2 For $u \geq 0$, we know $\inf_X u \geq 0$. Applying the Harnack inequality to $\big(u - \inf_X u\big)$, we have

$$\sup_{B_r} \Big(u - \inf_X u\Big) \leq C \inf_{B_r} \Big(u - \inf_X u\Big)$$

for any $r > 0$, and C does not depend on r. By letting $r \to \infty$, we observe that the right-hand side of the inequality tends to zero. Hence $u = \inf_X u = \text{const}$.

6 Appendices

Theorem 6.1 *Let Assumption* 1.1 *be fulfilled and* $\dim X = n \geq 3$*. For any subharmonic function* u *on* X *and any* $p > 0$*,* $0 < \theta < \tau \leq 1$*, there exists a constant* $C = C(n, p, \theta, \tau)$*, such that for any* $B_R := B(x, R)$*,*

$$\sup_{B_{\theta R}} u \leq C\Big(\fint_{B_{\tau R}} |u|^p\Big)^{\frac{1}{p}}. \tag{6.1}$$

Proof Step 1 We prove the theorem for $p \geq 2$. Set $u^+ = \max\{u, 0\}$, $\overline{u} = u^+ + k$ for some $k > 0$, and for some $m > 0$,

$$\overline{u}_m = \begin{cases} \overline{u}, & u < m, \\ k + m, & u \geq m. \end{cases}$$

Note that $\nabla \overline{u}_m = \nabla \overline{u} = \nabla u$ a.e., $0 < u < m$, otherwise $\nabla \overline{u}_m = 0$ a.e.

Set the test function $\varphi = \eta^2(\overline{u}_m^\beta \overline{u} - k^{\beta+1}) \in W_0^{1,2}(B_R)$ for any $\beta \geq 0$, $\eta \in \mathrm{Lip}_0(B_R)$ and $\eta \geq 0$. A direct calculation shows

$$\begin{aligned}
0 &\geq \int \nabla u \cdot \nabla \varphi \\
&= \int \nabla u \cdot \{2\eta\nabla\eta(\overline{u}_m^\beta \overline{u} - k^{\beta+1}) + \eta^2(\beta \overline{u}_m^{\beta-1}\overline{u}\nabla\overline{u}_m + \overline{u}_m^\beta \nabla \overline{u})\} \\
&= \int \nabla \overline{u} \cdot 2\eta\nabla\eta(\overline{u}_m^\beta \overline{u} - k^{\beta+1}) + \int \beta\eta^2\overline{u}_m^\beta|\nabla\overline{u}_m|^2 + \int \eta^2\overline{u}_m^\beta|\nabla\overline{u}|^2 \\
&\geq -\int 2\eta|\nabla\eta||\nabla\overline{u}|\overline{u}_m^\beta\overline{u} + \int \beta\eta^2\overline{u}_m^\beta|\nabla\overline{u}_m|^2 + \int \eta^2\overline{u}_m^\beta|\nabla\overline{u}|^2 \\
&\geq \frac{1}{2}\int \eta^2\overline{u}_m^\beta|\nabla\overline{u}|^2 + \beta\int \eta^2\overline{u}_m^\beta|\nabla\overline{u}_m|^2 - 2\int \overline{u}_m^\beta|\overline{u}|^2|\nabla\eta|^2.
\end{aligned}$$

Hence, we have

$$\frac{1}{2}\int \eta^2\overline{u}_m^\beta|\nabla\overline{u}|^2 + \beta\int \eta^2\overline{u}_m^\beta|\nabla\overline{u}_m|^2 \leq 2\int \overline{u}_m^\beta|\overline{u}|^2|\nabla\eta|^2. \tag{6.2}$$

Set $w = \overline{u}_m^{\frac{\beta}{2}}\overline{u}$. Then

$$\nabla w = \frac{\beta}{2}\overline{u}_m^{\frac{\beta}{2}-1}\overline{u}\nabla\overline{u}_m + \overline{u}_m^{\frac{\beta}{2}}\nabla\overline{u} = \frac{\beta}{2}\overline{u}_m^{\frac{\beta}{2}}\nabla\overline{u}_m + \overline{u}_m^{\frac{\beta}{2}}\nabla\overline{u},$$

and so

$$|\nabla w|^2 \leq \overline{u}_m^\beta\Big(\frac{\beta^2}{2}|\nabla\overline{u}_m|^2 + 2|\nabla\overline{u}|^2\Big) \leq 4(\beta+1)\overline{u}_m^\beta\Big(\beta|\nabla\overline{u}_m|^2 + \frac{1}{2}|\nabla\overline{u}|^2\Big).$$

From (6.2), we obtain

$$\begin{aligned}
\int |\nabla w|^2\eta^2 &\leq 8(\beta+1)\int w^2|\nabla\eta|^2, \\
\int |\nabla(\eta w)|^2 &\leq 18(\beta+1)\int w^2|\nabla\eta|^2.
\end{aligned} \tag{6.3}$$

For given $\theta_0 \leq \theta$ and any $\theta_0 \leq a < b \leq 1$, we choose $0 \leq \eta \in \mathrm{Lip}_0(B_{bR})$, $\eta|_{B_{aR}} = 1$ and $|\nabla\eta| \leq \frac{1}{(b-a)R}$. It follows from the Sobolev inequality that

$$\Big(\fint_{B_{bR}} |(\eta w)|^{2\chi}\Big)^{\frac{1}{\chi}} \leq C(n)(\beta+1)b^2R^2\fint_{B_{bR}} w^2|\nabla\eta|^2,$$

where $\chi = \frac{n}{n-2}$. Thanks to $\frac{|B_{bR}|}{|B_{aR}|} \leq \frac{b^n}{a^n} \leq \frac{1}{\theta_0^n}$, we obtain

$$\Big(\fint_{B_{aR}} |w|^{2\chi}\Big)^{\frac{1}{\chi}} \leq \frac{C(n,\theta_0)(\beta+1)}{(b-a)^2}\fint_{B_{bR}} w^2.$$

Recalling the definition of w, we have

$$\Big(\fint_{B_{aR}} \overline{u}_m^{\beta\chi}\overline{u}^{2\chi}\Big)^{\frac{1}{\chi}} \leq \frac{C(\beta+1)}{(b-a)^2}\fint_{B_{bR}} \overline{u}_m^\beta\overline{u}^2.$$

Set $\gamma = \beta + 2 \geq 2$. Then

$$\Big(\fint_{B_{aR}} \overline{u}_m^{\gamma\chi}\Big)^{\frac{1}{\gamma\chi}} \leq \Big(\frac{C(\gamma-1)}{(b-a)^2}\Big)^{\frac{1}{\gamma}}\Big(\fint_{B_{bR}} \overline{u}^\gamma\Big)^{\frac{1}{\gamma}}.$$

If the right-hand side of last inequality is bounded, by letting $m \to \infty$, we conclude that

$$\Big(\fint_{B_{aR}} \overline{u}^{\gamma\chi}\Big)^{\frac{1}{\gamma\chi}} \le \Big(\frac{C(\gamma-1)}{(b-a)^2}\Big)^{\frac{1}{\gamma}}\Big(\fint_{B_{bR}} \overline{u}^{\gamma}\Big)^{\frac{1}{\gamma}}. \tag{6.4}$$

We start the Moser iteration as follows. Set $r_i = \theta R + \frac{\tau-\theta}{2^{i-1}}R$ and $\gamma_i = p\chi^{i-1}$ for $i = 1, 2, \cdots$, and denote $\mathrm{I}_i = (\fint_{B_{r_i}} |\overline{u}|^{\gamma_i})^{\frac{1}{\gamma_i}}$. Then

$$\mathrm{I}_{i+1} \le \Big(\frac{C4^{i-1}(\gamma_i-1)}{(\tau-\theta)^2}\Big)^{\frac{1}{\gamma_i}} \mathrm{I}_i \le \Big(\frac{C}{(\tau-\theta)^2}\Big)^{\sum\frac{1}{\gamma_j}} 4^{\sum\frac{j-1}{\gamma_j}} \prod(\gamma_j-1)^{\frac{1}{\gamma_j}} \mathrm{I}_1, \tag{6.5}$$

where $\sum\frac{1}{\gamma_j} \le \frac{n}{2p}$, $\sum\frac{j-1}{\gamma_j} \le C(n,p)$ and

$$\prod(\gamma_j-1)^{\frac{1}{\gamma_j}} \le \prod(\gamma_j)^{\frac{1}{\gamma_j}} \le p^{\sum\frac{1}{\gamma_j}}\chi^{\sum\frac{j-1}{\gamma_j}} \le C(n,p).$$

Hence, by letting $i \to \infty$ in (6.5), we get

$$\sup_{B_{\theta R}} \overline{u} \le C(n,p,\theta_0)\Big(\frac{1}{(\tau-\theta)^n}\fint_{B_{\tau R}} |\overline{u}|^p\Big)^{\frac{1}{p}}.$$

At last, by letting $k \to 0$, we have

$$\sup_{B_{\theta R}} u^+ \le C(n,p,\theta_0)\Big(\frac{1}{(\tau-\theta)^n}\fint_{B_{\tau R}} (u^+)^p\Big)^{\frac{1}{p}}.$$

Step 2 For the case $p < 2$, we have

$$\begin{aligned}\sup_{B_{\theta R}} u^+ &\le C(n,\theta_0)\Big(\frac{1}{(\tau-\theta)^n}\fint_{B_{\tau R}} (u^+)^2\Big)^{\frac{1}{2}} \\ &\le C\frac{1}{(\tau-\theta)^{\frac{n}{2}}}\Big(\sup_{B_{\tau R}} u^+\Big)^{1-\frac{p}{2}}\Big(\fint_{B_{\tau R}} (u^+)^p\Big)^{\frac{1}{2}} \\ &\le \frac{1}{2}\sup_{B_{\tau R}} u^+ + \frac{C(n,p,\theta_0)}{(\tau-\theta)^{\frac{n}{p}}}\Big(\fint_{B_{\tau R}} (u^+)^p\Big)^{\frac{1}{p}}.\end{aligned} \tag{6.6}$$

The last inequality follows from Young's inequality.

We recall a useful lemma (see [8, Lemma 4.3]).

Lemma 6.1 *Let f be a nonnagetive and bounded function on $[\tau_0, \tau_1]$ with $\tau_0 \ge 0$. Suppose that for $\tau_0 \le t < s \le \tau_1$, we have*

$$f(t) \le \theta f(s) + \frac{A}{(s-t)^\alpha} + B$$

for some $\theta \in [0, 1)$. Then for any $\tau_0 \le t < s \le \tau_1$, there holds

$$f(t) \le c(\alpha,\theta)\Big\{\frac{A}{(s-t)^\alpha} + B\Big\}.$$

Using this lemma and estimate (6.6), we conclude that

$$\sup_{B_{\theta R}} u \le C(n,p,\theta_0)\Big(\frac{1}{(\tau-\theta)^n}\fint_{B_{\tau R}} |u|^p\Big)^{\frac{1}{p}}.$$

By choosing $\theta = \theta_0$, we have

$$\sup_{B_{\theta R}} u \le C(n,p,\theta,\tau)\Big(\fint_{B_{\tau R}} |u|^p\Big)^{\frac{1}{p}}.$$

Theorem 6.2 *Let Assumption* 1.1 *be fulfilled and* $\dim X = n \geq 3$. *For any nonnegative superharmonic function* u *on* X *and any* $0 < \theta < \tau < 1$, $0 < p < \frac{n}{n-2}$, *there exists a constant* $C = C(n, p, \theta, \tau)$, *such that for any* $B_R := B(x, R)$,

$$\inf_{B_{\theta R}} u \geq C\Big(\fint_{B_{\tau R}} |u|^p \Big)^{\frac{1}{p}}. \tag{6.7}$$

Proof Step 1 The theorem holds for some $p_0 > 0$.

Set $\overline{u} = u + k > 0$ for some $k > 0$. By letting $k \to 0$, it suffices to prove the theorem for $\overline{u}$. Set $v = \overline{u}^{-1}$, the test function $\varphi = \frac{\phi}{\overline{u}^2}$ for any $\phi \in W_0^{1,2}(X)$ and $\phi \geq 0$. Direct calculation shows

$$\int \nabla v \cdot \nabla \phi \leq -\int \frac{2\phi |\nabla \overline{u}|^2}{\overline{u}^3} \leq 0,$$

i.e., v is a subharmonic function on X. According to Theorem 6.1, for any $p > 0$, $0 < \theta < \tau \leq 1$, there exists a $C = C(n, p, \theta, \tau)$, such that

$$\sup_{B_{\theta R}} v \leq C\Big(\fint_{B_{\tau R}} |v|^p \Big)^{\frac{1}{p}}.$$

That is,

$$\inf_{B_{\theta R}} \overline{u} \geq C\Big(\fint_{B_{\tau R}} |\overline{u}|^{-p} \Big)^{-\frac{1}{p}} = C\Big(\fint_{B_{\tau R}} |\overline{u}|^{-p} \fint_{B_{\tau R}} |\overline{u}|^{p} \Big)^{-\frac{1}{p}} \Big(\fint_{B_{\tau R}} |\overline{u}|^{p} \Big)^{\frac{1}{p}}. \tag{6.8}$$

It suffices to prove that for some $p_0(n, \tau) > 0$, we have

$$\fint_{B_{\tau R}} |\overline{u}|^{-p_0} \fint_{B_{\tau R}} |\overline{u}|^{p_0} \leq C(n, \tau). \tag{6.9}$$

Let $w := \log \overline{u} - \mu$, where $\mu = \fint_{B_{\tau R}} \log \overline{u}$. In order to prove inequality (6.9), it suffices to show

$$\fint_{B_{\tau R}} \mathrm{e}^{p_0 |w|} \leq C(n, \tau). \tag{6.10}$$

Note that

$$\mathrm{e}^{p_0|w|} = 1 + p_0|w| + \frac{(p_0|w|)^2}{2} + \cdots + \frac{(p_0|w|)^\alpha}{\alpha!} + \cdots,$$

where $\alpha \in \mathbb{N}$. Hence we should estimate every term of the expression

$$\fint_{B_{\tau R}} \frac{(p_0|w|)^\alpha}{\alpha!}.$$

We derive the inequality for w as follows. For any $\varphi \in W_0^{1,2}(X)$ and $\varphi \geq 0$, set the test function $\varphi \overline{u}^{-1}$. A direct calculation shows

$$0 \leq \int \nabla \overline{u} \cdot \Big(\frac{\nabla \varphi}{\overline{u}} - \frac{\varphi \nabla \overline{u}}{\overline{u}^2} \Big),$$

that is,

$$0 \leq \int \nabla w \cdot \nabla \varphi - \varphi |\nabla w|^2. \tag{6.11}$$

Substituting φ with φ^2, we get

$$\int \varphi^2|\nabla w|^2 \le 2\int \varphi\nabla\varphi\cdot\nabla w \le 2\Big(\int \varphi^2|\nabla w|^2\Big)^{\frac{1}{2}}\Big(\int |\nabla\varphi|^2\Big)^{\frac{1}{2}}.$$

Hence

$$\int \varphi^2|\nabla w|^2 \le 4\int |\nabla\varphi|^2. \tag{6.12}$$

Choosing $\varphi|_{B_{\tau R}} = 1$, $\operatorname{supp}\varphi \subset B_R$ and $|\nabla\varphi| \le \frac{1}{(1-\tau)R}$, we have

$$\int_{B_{\tau R}} |\nabla w|^2 \le \frac{4}{(1-\tau)^2R^2}|B_R|.$$

Noting that $\int_{B_{\tau R}} w = 0$, we get by the Poincaré inequality that

$$\fint_{B_{\tau R}} |w|^2 \le C(n)\tau^2R^2\fint_{B_{\tau R}}|\nabla w|^2 \le C(n,\tau)\frac{|B_R|}{|B_{\tau R}|} \le C\frac{1}{\tau^n} \le C(n,\tau), \tag{6.13}$$

which is the required estimate for $\alpha = 2$.

Claim 6.1 *For any $\tau' \in (\tau, 1)$, we have*

$$\fint_{B_{\tau' R}} |w|^2 \le C(n,\tau,\tau'). \tag{6.14}$$

Proof Choosing $\varphi|_{B_{\tau' R}} = 1$, $\operatorname{supp}\varphi \subset B_R$ and $|\nabla\varphi| \le \frac{1}{(1-\tau')R}$, we have

$$\int_{B_{\tau' R}} |\nabla w|^2 \le \frac{4}{(1-\tau')^2R^2}|B_R|.$$

The Poincaré inequality yields

$$\fint_{B_{\tau' R}} |w - w_{B_{\tau' R}}|^2 \le C(n)(\tau' R)^2\fint_{B_{\tau' R}}|\nabla w|^2 \le C(n,\tau').$$

Hence

$$\begin{aligned}
\fint_{B_{\tau' R}} |w|^2 &\le C(\epsilon)\fint_{B_{\tau' R}} |w - w_{B_{\tau' R}}|^2 + (1+\epsilon)|w_{B_{\tau' R}}|^2 \\
&\le C(n,\tau',\epsilon) + (1+\epsilon)\Big(\fint_{B_{\tau' R}} w\Big)^2 \\
&= C(n,\tau',\epsilon) + (1+\epsilon)\frac{1}{|B_{\tau' R}|^2}\Big(\int_{B_{\tau' R}\setminus B_{\tau R}} w\Big)^2 \\
&\le C(n,\tau',\epsilon) + (1+\epsilon)\frac{|B_{\tau' R}\setminus B_{\tau R}|}{|B_{\tau' R}|}\fint_{B_{\tau' R}} w^2.
\end{aligned} \tag{6.15}$$

To obtain (6.15), note that $\fint_{B_{\tau R}} w = 0$. So

$$\Big(1 - (1+\epsilon)\frac{|B_{\tau' R}\setminus B_{\tau R}|}{|B_{\tau' R}|}\Big)\fint_{B_{\tau' R}} w^2 \le C(n,\tau',\epsilon).$$

Since we have

$$\frac{|B_{\tau' R}\setminus B_{\tau R}|}{|B_{\tau' R}|} = 1 - \frac{|B_{\tau R}|}{|B_{\tau' R}|} \le 1 - \Big(\frac{\tau}{\tau'}\Big)^n,$$

it follows that

$$\Big\{1-(1+\epsilon)\Big(1-\Big(\frac{\tau}{\tau'}\Big)^n\Big)\Big\}\fint_{B_{\tau' R}} w^2 \le C(n,\tau',\epsilon). \tag{6.16}$$

We choose $\epsilon=\epsilon(\tau,\tau')$, such that $(1+\epsilon)(1-(\frac{\tau}{\tau'})^n)<1$. Then the claim follows from (6.16) that

$$\fint_{B_{\tau' R}} |w|^2 \le C(n,\tau,\tau').$$

Next, we estimate $\fint_{B_{\tau R}} |w|^\alpha$ for any $\alpha\ge 2$. Set test function $\varphi=\zeta^2|w_m|^{2\beta}$ in (6.11), where $\beta\ge 1$, $\zeta\in \mathrm{Lip}_0(X)$ and $\zeta\ge 0$,

$$w_m=\begin{cases} m, & w>m,\\ w, & |w|\le m,\\ -m, & w<-m.\end{cases}$$

A direct calculation shows

$$0\le \int \nabla w(2\zeta\nabla\zeta|w_m|^{2\beta}+2\beta\zeta^2|w_m|^{2\beta-1}\nabla|w_m|)-|\nabla w|^2\zeta^2|w_m|^{2\beta}.$$

So

$$\begin{aligned}\int |\nabla w|^2\zeta^2|w_m|^{2\beta} &\le \int 2\zeta|w_m|^{2\beta}\nabla w\cdot\nabla\zeta+2\beta\zeta^2|w_m|^{2\beta-1}\nabla w_m\cdot\nabla|w_m|\\ &\le \int 2\zeta|w_m|^{2\beta}|\nabla w||\nabla\zeta|+2\beta\zeta^2|w_m|^{2\beta-1}|\nabla w_m|^2.\end{aligned}$$

Young's inequality yields

$$2\beta|w_m|^{2\beta-1}\le \frac{2\beta-1}{2\beta}|w_m|^{2\beta}+\frac{1}{2\beta}(2\beta)^{2\beta}=\Big(1-\frac{1}{2\beta}\Big)|w_m|^{2\beta}+(2\beta)^{2\beta-1}.$$

Hence

$$\begin{aligned}\int |\nabla w|^2\zeta^2|w_m|^{2\beta} &\le 4\beta\int \zeta|w_m|^{2\beta}|\nabla w||\nabla\zeta|+(2\beta)^{2\beta}\int\zeta^2|\nabla w_m|^2\\ &\le 2\beta\Big(\frac{1}{4\beta}\int\zeta^2|w_m|^{2\beta}|\nabla w|^2+4\beta\int|w_m|^{2\beta}|\nabla\zeta|^2\Big)+(2\beta)^{2\beta}\int\zeta^2|\nabla w_m|^2.\end{aligned}$$

Then we get

$$\int |\nabla w|^2\zeta^2|w_m|^{2\beta}\le 16\beta^2\int|w_m|^{2\beta}|\nabla\zeta|^2+2(2\beta)^{2\beta}\int\zeta^2|\nabla w_m|^2.$$

By letting $m\to\infty$, an inequality about w follows. Then using Young's inequality again, we obtain

$$\begin{aligned}|\nabla(\zeta|w|^\beta)|^2 &\le 2|\nabla\zeta|^2|w|^{2\beta}+2\beta^2\zeta^2|w|^{2\beta-2}|\nabla w|^2\\ &\le 2|\nabla\zeta|^2|w|^{2\beta}+2\zeta^2|\nabla w|^2\Big(\frac{\beta-1}{\beta}|w|^{2\beta}+\frac{1}{\beta}\beta^{2\beta}\Big).\end{aligned}$$

Hence

$$\begin{aligned}\int|\nabla(\zeta|w|^\beta)|^2 &\le 32\Big\{(2\beta)^{2\beta}\int\zeta^2|\nabla w|^2+\beta^2\int|w|^{2\beta}|\nabla\zeta|^2\Big\}\\ &\le 128\Big\{(2\beta)^{2\beta}\int|\nabla\zeta|^2+\beta^2\int|w|^{2\beta}|\nabla\zeta|^2\Big\}.\end{aligned} \tag{6.17}$$

For any $\tau \le a < b \le 1$, choose $\zeta|_{B_{aR}} = 1$, $\operatorname{supp}\zeta \subset B_{bR}$ and $|\nabla\zeta| \le \frac{1}{(b-a)R}$. It follows from the Sobolev inequality that

$$\Big(\fint_{B_{bR}} (\zeta|w|^{\beta})^{2\chi}\Big)^{\frac{1}{\chi}} \le C(n)b^2R^2\Big\{(2\beta)^{2\beta}\fint_{B_{bR}}|\nabla\zeta|^2 + \beta^2\fint_{B_{bR}}|w|^{2\beta}|\nabla\zeta|^2\Big\},$$

where $\chi = \frac{n}{n-2}$. Hence

$$\Big(\fint_{B_{aR}}|w|^{2\beta\chi}\Big)^{\frac{1}{\chi}} \le \frac{C(n,\tau)(2\beta)^2}{(b-a)^2}\Big\{(2\beta)^{2\beta} + \fint_{B_{bR}}|w|^{2\beta}\Big\}.$$

Now we start the Moser iteration as follows. Set $\beta_i = 2\chi^{i-1}$, $r_i = (\tau + \frac{1-\tau}{2^i})R$ for $i = 1, 2, \cdots$. Then we have

$$\Big(\fint_{B_{r_{i+1}}}|w|^{\beta_{i+1}}\Big)^{\frac{1}{\beta_{i+1}}} \le \Big(\frac{C(n,\tau)4^{i+1}(\beta_i)^2}{(1-\tau)^2}\Big)^{\frac{1}{\beta_i}}\Big\{\beta_i + \Big(\fint_{B_{r_i}}|w|^{\beta_i}\Big)^{\frac{1}{\beta_i}}\Big\}. \tag{6.18}$$

Taking a notation $\mathrm{I}_i = (\fint_{B_{r_i}}|w|^{\beta_i})^{\frac{1}{\beta_i}}$, we get

$$\mathrm{I}_{i+1} \le C^{\frac{1}{\beta_i}}4^{\frac{i}{\beta_i}}\beta_i^{\frac{2}{\beta_i}}(\beta_i + \mathrm{I}_i) \le C^{\sum\frac{i}{\beta_i}}\Big(\prod\beta_i^{\frac{1}{\beta_i}}\Big)^2\Big(\sum^{i}\beta_j + \mathrm{I}_1\Big).$$

Note that $\sum\frac{i}{\beta_i} < C$, $\prod\beta_i^{\frac{1}{\beta_i}} < C$ and $\sum^{i}\beta_j < C\beta_{i+1}$. Hence

$$\mathrm{I}_{i+1} \le C(n,\tau)(\beta_{i+1} + \mathrm{I}_1).$$

For any integer $\alpha \ge 2$ (the estimate of $\alpha = 0, 1$ is trivial), there exists $i \ge 1$, such that $\beta_i \le \alpha < \beta_{i+1}$. Then

$$\Big(\fint_{B_{\tau R}}|w|^{\alpha}\Big)^{\frac{1}{\alpha}} \le \Big(\fint_{B_{\tau R}}|w|^{\beta_{i+1}}\Big)^{\frac{1}{\beta_{i+1}}} \le C\mathrm{I}_{i+1} \le C(\beta_{i+1} + \mathrm{I}_1) \le C(\alpha + \mathrm{I}_1) \le C_0(n,\tau)\alpha, \tag{6.19}$$

where the Bishop-Gromov volume comparison is used intrinsically, and the last step follows from $\mathrm{I}_1 = (\fint_{B_{\tau' R}}|w|^2)^{\frac{1}{2}} \le C(n,\tau)$, in which $\tau' = \frac{1+\tau}{2}$.

Hence, for any $\alpha \ge 2$, we have

$$\fint_{B_{\tau R}}\frac{(p_0|w|)^{\alpha}}{\alpha!} \le \frac{(p_0C_0\alpha)^{\alpha}}{\alpha!} \le (p_0C_0\mathrm{e})^{\alpha},$$

where we use the Sterling's formula. Choosing $p_0 = (2C_0\mathrm{e})^{-1}$, we draw the following conclusion

$$\fint_{B_{\tau R}}\mathrm{e}^{p_0|w|} \le C\Big(1 + \frac{1}{2} + \frac{1}{4} + \cdots\Big) \le C(n,\tau).$$

Step 2 In order to prove the theorem for any $0 < p < \frac{n}{n-2}$, by iteration it suffices to prove the following claim.

Claim 6.2 *For any $\theta \le l_1 < l_2 < 1$ and $0 < p_2 < p_1 < \frac{n}{n-2}$, there exists a constant $C = C(n, l_1, l_2, p_1, p_2, \theta)$, such that*

$$\Big(\fint_{B_{l_1 R}}\overline{u}^{p_1}\Big)^{\frac{1}{p_1}} \le C\Big(\fint_{B_{l_2 R}}\overline{u}^{p_2}\Big)^{\frac{1}{p_2}}. \tag{6.20}$$

Proof Set the test function $\varphi = \overline{u}^{-\beta}\zeta^2$ for $\beta \in (0,1)$. Then

$$0 \le \int \nabla \overline{u} \cdot (-\beta \overline{u}^{-\beta-1}\zeta^2 \nabla \overline{u} + 2\overline{u}^{-\beta}\zeta\nabla\zeta).$$

Hence

$$\int \overline{u}^{-\beta-1}\zeta^2|\nabla \overline{u}|^2 \le \frac{2}{\beta}\int \overline{u}^{-\beta}|\zeta||\nabla\zeta||\nabla\overline{u}|.$$

By Hölder's inequality, we obtain

$$\int \overline{u}^{-\beta-1}\zeta^2|\nabla \overline{u}|^2 \le \frac{4}{\beta^2}\int \overline{u}^{1-\beta}|\nabla\zeta|^2.$$

Set $\gamma = 1-\beta$, $w = \overline{u}^{\frac{\gamma}{2}}$. Then

$$\nabla w = \frac{\gamma}{2}\overline{u}^{\frac{\gamma}{2}-1}\nabla\overline{u}.$$

Hence, we get

$$\int |\nabla w|^2\zeta^2 \le \frac{\gamma^2}{(1-\gamma)^2}\int w^2|\nabla\zeta|^2$$

and

$$\int |\nabla(\zeta w)|^2 \le \frac{4}{(1-\gamma)^2}\int w^2|\nabla\zeta|^2.$$

For any $\theta \le a < b < 1$, choose $\zeta|_{B_{aR}} = 1$, $\operatorname{supp}\zeta \subset B_{bR}$ and $|\nabla\zeta| \le \frac{1}{(b-a)R}$. The Sobolev inequality yields

$$\Big(\fint_{B_{bR}} (\zeta w)^{2\chi}\Big)^{\frac{1}{\chi}} \le \frac{C(n)b^2R^2}{(1-\gamma)^2}\fint_{B_{bR}} w^2|\nabla\zeta|^2.$$

Then

$$\Big(\fint_{B_{aR}} w^{2\chi}\Big)^{\frac{1}{\chi}} \le \frac{C(n,\theta)}{(1-\gamma)^2(b-a)^2}\fint_{B_{bR}} w^2.$$

Recalling the definition of w, we have

$$\Big(\fint_{B_{aR}} \overline{u}^{\gamma\chi}\Big)^{\frac{1}{\gamma\chi}} \le \Big(\frac{C(n,\theta)}{(1-\gamma)^2(b-a)^2}\Big)^{\frac{1}{\gamma}}\Big(\fint_{B_{bR}} \overline{u}^{\gamma}\Big)^{\frac{1}{\gamma}} \tag{6.21}$$

for any $\gamma \in (0,1)$.

We start the Moser iteration as follows.

Set $\gamma_i = l_1 + \frac{l_2-l_1}{2^{i-1}}$, $r_i = p_2\chi^{i-1}$, $i = 1, 2, \cdots$. Then

$$\Big(\fint_{B_{r_{i+1}R}} \overline{u}^{\gamma_{i+1}}\Big)^{\frac{1}{\gamma_{i+1}}} \le \Big(\frac{C4^i}{(1-\gamma_i)^2(l_2-l_1)^2}\Big)^{\frac{1}{\gamma_i}}\Big(\fint_{B_{r_iR}} \overline{u}^{\gamma_i}\Big)^{\frac{1}{\gamma_i}}.$$

Introducing a notation $\mathrm{I}_i = (\fint_{B_{r_iR}} \overline{u}^{\gamma_i})^{\frac{1}{\gamma_i}}$, we have

$$\mathrm{I}_{i+1} \le \frac{C^{\sum\limits^{i} \frac{j}{\gamma_j}}}{(l_2-l_1)^{2\sum\limits^{i}\frac{1}{\gamma_j}}(\prod\limits^{i}(1-\gamma_i)^{\frac{1}{\gamma_i}})^2}\mathrm{I}_1,$$

whenever $\gamma_i < 1$. For $0 < p_2 < p_1 < \frac{n}{n-2}$, we can assume $p_2 < 1$. Then there exists $i \geq 1$, such that $\gamma_i \leq p_1 < \gamma_{i+1}$. Hence, by the Bishop-Gromov volume comparison theorem, we obtain

$$\Big(\fint_{B_{l_1 R}} \overline{u}^{p_1}\Big)^{\frac{1}{p_1}} \leq \Big(\fint_{B_{l_1 R}} \overline{u}^{\gamma_{i+1}}\Big)^{\frac{1}{\gamma_{i+1}}} \leq C\mathrm{I}_{i+1} \leq \frac{C^{\sum\limits^{i} \frac{j}{\gamma_j}}}{(l_2 - l_1)^{2\sum\limits^{i} \frac{1}{\gamma_j}} \Big(\prod\limits^{i} (1-\gamma_j)^{\frac{1}{\gamma_j}}\Big)^2} \mathrm{I}_1.$$

Note that

$$\sum^{i} \frac{j}{\gamma_j} < C(p_1, p_2) \quad \text{and} \quad \prod^{i} (1-\gamma_j)^{\frac{1}{\gamma_j}} \geq \prod^{i} (1-p_1)^{\frac{1}{\gamma_j}} \geq (1-p_1)^{\frac{n}{2p_2}}.$$

Hence, we prove the claim

$$\Big(\fint_{B_{l_1 R}} \overline{u}^{p_1}\Big)^{\frac{1}{p_1}} \leq C(n, l_1, l_2, p_1, p_2, \theta)\Big(\fint_{B_{l_2 R}} \overline{u}^{p_2}\Big)^{\frac{1}{p_2}}.$$

By combining Step 1 with Step 2, the conclusion follows immediately. For any $0 < \theta < \tau < 1$, $0 < p < \frac{n}{n-2}$, by using Step 1 for $\tau' = \frac{1+\tau}{2}$, there exists $p_0 = p_0(n, \tau) > 0$, for which we can assume $p_0 < p$, and constant $C = C(n, \theta, \tau)$, such that

$$\inf_{B_{\theta R}} u \geq C\Big(\fint_{B_{\tau' R}} |u|^{p_0}\Big)^{\frac{1}{p_0}}.$$

Then Step 2 implies

$$\Big(\fint_{B_{\tau' R}} |u|^{p_0}\Big)^{\frac{1}{p_0}} \geq C(n, p, \theta, \tau)\Big(\fint_{B_{\tau R}} |u|^{p}\Big)^{\frac{1}{p}},$$

which proves the theorem.

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