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Multivalued Stochastic Differential Equations with Non-Lipschitz Coefficients**

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Abstract The existence and uniqueness of solutions to the multivalued stochastic differential equations with non-Lipschitz coefficients are proved, and bicontinuous modifications of the solutions are obtained.

Keywords Multivalued stochastic differential equation, Maximal monotone operator, Non-Lipschitz, Bicontinuity
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1 Introduction

In this paper, we consider the following one-dimensional multivalued stochastic differential equation (MSDE in short):

$$\begin{cases} \mathrm{d}X_t + A(X_t)\mathrm{d}t \ni b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \\ X_0 = x \in \overline{D(A)}, \end{cases} \tag{1.1}$$

where A is a multivalued maximal monotone operator, W_t is a one-dimensional standard Brownian motion defined on some canonical probability space $(\Omega, \mathcal{F}, P)$, σ and b are continuous maps.

Except for the multivalued ordinary differential equations (see [1] or [2]), the multivalued stochastic differential equations (MSDEs) with Lipschitz coefficients have been considered recently (see [6, 3] among others). The MSDEs have a great deal of applications in many areas (see, e.g., [4]).

In practice, we often need to deal with equations with non-Lipschitz coefficients. But unfortunately, there are few papers to deal with the MSDEs with non-Lipschitz coefficients. In this paper, we prove the existence and uniqueness of the solution to equation (1.1) in the non-Lipschitz case. The existence of weak solution is obtained as [3]. For the uniqueness, we use the Tanaka's formula and Le Gall's method (see [5]). Moreover, we can have a bicontinuous modification for the solution. But since we are dealing with multivalued operators, the bicontinuity can not be obtained simply as in the case of SDEs. However, we can get the result under some conditions which are suggested in [7].

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The organization of this paper is as follows. In Section 2, we introduce notions and notations. In Section 3, we prove the existence and uniqueness. Finally, in Section 4, we give the bicontinuous modification.

2 Preliminaries

Given a multivalued operator A from $\mathbb{R}$ to $\mathbb{R}$, we define

$$\begin{aligned} D(A) &:= \{x \in \mathbb{R} : A(x) \neq \emptyset\}, \\ \mathrm{Im}(A) &:= \bigcup_{x \in D(A)} A(x), \\ \mathrm{Gr}(A) &:= \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R},\ y \in A(x)\}. \end{aligned}$$

A^{-1} is defined by: $y \in A^{-1}(x) \Leftrightarrow x \in A(y)$.

Definition 2.1 (see [1]) (1) *A multivalued operator A is called monotone if*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \quad \forall\, (x_1, y_1), (x_2, y_2) \in \mathrm{Gr}(A).$$

(2) *A monotone operator A is called maximal monotone if and only if*

$$(x_1, y_1) \in \mathrm{Gr}(A) \Leftrightarrow \{\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0,\ \forall\, (x_2, y_2) \in \mathrm{Gr}(A)\}.$$

We will need the following definition due to [3].

Definition 2.2 *A pair of continuous and $\mathcal{F}_t$-adapted processes (X, K) is called a strong solution of equation* (1.1) *if*

(i) $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ *with* $X_0 = x$ *and* $X_t \in \overline{D(A)}$, *a.s.,*

(ii) $K = \{K_t, \mathcal{F}_t; 0 \leq t < \infty\}$ *is of finite variation and* $K_0 = 0$, *a.s.,*

(iii) $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t - \mathrm{d}K_t$, $0 \leq t < \infty$, *a.s.,*

(iv) *for any continuous and $\mathcal{F}_t$-adapted functions (α, β), where $\alpha = \{\alpha_t, \mathcal{F}_t; 0 \leq t < \infty\}$ and $\beta = \{\beta_t, \mathcal{F}_t; 0 \leq t < \infty\}$, satisfying*

$$(\alpha_t, \beta_t) \in \mathrm{Gr}(A), \quad \forall\, t \in [0, +\infty),$$

the measure

$$\langle X_t - \alpha_t, \mathrm{d}K_t - \beta_t \mathrm{d}t \rangle \geq 0, \quad a.s.$$

We collect here some facts about the maximal monotone operator which will be needed in the sequel. For proofs we refer to [2].

Proposition 2.1 (1) *For each $x \in D(A)$, $A(x)$ is a closed and convex subset of $\mathbb{R}$. In particular, there is a unique $y \in A(x)$ such that $|y| = \inf\{|z| : z \in Ax\}$. $A^\circ(x) := y$ is called the minimal section of A, and we have*

$$x \in D(A) \Leftrightarrow |A^\circ(x)| < +\infty.$$

(2) *The resolvent operator* $J_\lambda := (1+\lambda A)^{-1}$ *is single-valued and Lipschitz continuous with Lipschitz constant* 1. *Moreover,* $\lim\limits_{\lambda\downarrow 0} J_\lambda x = x$ *for any* $x \in D(A)$.

(3) *The Yosida approximation* $A_\lambda := \lambda^{-1}(1-J_\lambda)$ *is monotone and Lipschitz continuous with Lipschitz constant* $\frac{1}{\lambda}$. *Moreover, as* $\lambda \downarrow 0$,

$$A_\lambda(x) \to A^\circ(x) \quad \text{and} \quad |A_\lambda(x)| \uparrow \begin{cases} |A^\circ(x)|, & \text{if } x \in D(A), \\ +\infty, & \text{if } x \notin D(A). \end{cases}$$

The following important proposition is taken from [3].

Proposition 2.2 *Let* A *be a multivalued maximal monotone operator,* $t \mapsto (X(t), K(t))$ *and* $t \mapsto (X'(t), K'(t))$ *be continuous functions with* $X(t),\ X'(t) \in \overline{D(A)}$, *and* $t \mapsto K(t),\ K'(t)$ *be of finite variation. Let* (α, β) *be continuous functions which satisfy*

$$(\alpha_t, \beta_t) \in \mathrm{Gr}(A), \quad \forall\, t \geq 0.$$

If

$$\langle X_t - \alpha_t, \mathrm{d}K_t - \beta_t \mathrm{d}t \rangle \geq 0, \quad \langle X'_t - \alpha_t, \mathrm{d}K'_t - \beta_t \mathrm{d}t \rangle \geq 0,$$

then

$$\langle X_t - X'_t, \mathrm{d}K_t - \mathrm{d}K'_t \rangle \geq 0.$$

The following example and three lemmas which will be needed are taken from [7].

Lemma 2.1 *Let* $\rho : \mathbb{R}^+ \mapsto \mathbb{R}^+$ *be a continuous and non-decreasing function. If* $g(s)$ *and* $q(s)$ *are two strictly positive functions on* $\mathbb{R}^+$ *such that*

$$g(t) \leq g(0) + \int_0^t q(s)\rho(g(s))\mathrm{d}s, \quad t \geq 0,$$

then

$$g(t) \leq f^{-1}\Big(f(g(0)) + \int_0^t q(s)\mathrm{d}s\Big), \tag{2.1}$$

where $f(x) := \int_{x_0}^x \frac{1}{\rho(y)}\mathrm{d}y$ *is well-defined for some* $x_0 > 0$.

Example 2.1 For $0 < \eta < \frac{1}{\mathrm{e}}$, define a concave function as

$$\rho_\eta(x) := \begin{cases} x \log x^{-1}, & x \leq \eta, \\ \eta \log \eta^{-1} + (\log \eta^{-1} - 1)(x - \eta), & x > \eta. \end{cases}$$

Choosing $x_0 = \eta$, we have

$$f(x) = \log\Big(\frac{\log \eta}{\log x}\Big), \quad 0 < x < \eta,$$
$$f^{-1}(x) = \exp\{\log \eta \cdot \exp\{-x\}\}, \quad x < 0.$$

If $g(0) < \eta$, substituting these into (2.1), we obtain

$$g(t) \leq (g(0))^{\exp\{-\int_0^t q(s)\mathrm{d}s\}}. \tag{2.2}$$

For $0<\eta<\frac{1}{\mathrm{e}}$, let $\rho_{1,\eta},\rho_{2,\eta}$ be two concave functions defined by

$$\rho_{j,\eta}(x):=\begin{cases} x[\log x^{-1}]^{\frac{1}{j}}, & x\le\eta,\\ \Big([\log\eta^{-1}]^{\frac{1}{j}}-\dfrac{1}{j}[\log\eta^{-1}]^{\frac{1}{j}-1}\Big)x+\dfrac{1}{j}[\log\eta^{-1}]^{\frac{1}{j}-1}\eta, & x>\eta,\end{cases}$$

where $j=1,2$.

Lemma 2.2 (1) *For $j=1,2$, $\rho_{j,\eta}$ is decreasing in η, i.e., $\rho_{j,\eta_1}\le\rho_{j,\eta_2}$ if $1>\eta_1>\eta_2$.*

(2) *For any $p\ge 0$ and η sufficiently small, we have*

$$x^p\rho_{j,\eta}^j(x)\le\frac{1}{j+p}\rho_{1,\eta^{j+p}}(x^{j+p}),\quad j=1,2.$$

Lemma 2.3 *Let $I_1,I_2\subset\mathbb{R}$ be two closed intervals and $X(s,t)$, $(s,t)\in I_1\times I_2$ a stochastically continuous process. For $n\in\mathbb{N}$, let*

$$X_n(s,t):=X(s,t)\wedge n\vee(-n).$$

If for every n there exist $p_n,C_n,\alpha_n>0$ such that

$$\mathbb{E}\Big[\sup_{s\in I_1}|X_n(s,t)-X_n(s,t')|^{p_n}\Big]\le C_n|t-t'|^{1+\alpha_n},\quad\forall\, t,t'\in I_2,$$

then X has a bicontinuous modification $\widetilde{X}$. In particular, if $p=p_n>1$ and $\alpha=\alpha_n>0$ are independent of n, then the paths $I_1\ni t\to\widetilde{X}(\,\cdot\,,t)\in C(I_1)$ are β-Hölder continuous for every $\beta<\alpha p^{-1}$.

Theorem 2.1 (Tanaka Formula) *If X is a continuous semimartingale, then for any real number a there exists an increasing continuous process $L_t^a(X)$, called the local time of X in a, such that*

$$\begin{aligned}|X_t-a|&=|X_0-a|+\int_0^t\operatorname{sgn}(X_s-a)\mathrm{d}X_s+L_t^a(X),\\ (X_t-a)^+&=(X_t-a)^++\int_0^t 1_{(X_s>a)}\mathrm{d}X_s+\frac12 L_t^a(X),\\ (X_t-a)^-&=(X_t-a)^--\int_0^t 1_{(X_s\le a)}\mathrm{d}X_s+\frac12 L_t^a(X).\end{aligned}$$

In particular, $|X-a|$, $(X-a)^+$ and $(X-a)^-$ are semimartingales.

3 Existence and Uniqueness

Assumption 3.1 *σ and b are continuous and bounded and satisfy*

(i) *there exists a strictly positive increasing continuous function ρ on $\mathbb{R}_+$, $\rho(0)=0$ and $\int_{0+}\rho^{-1}(u)\mathrm{d}u=\infty$, such that*

$$|\sigma(x)-\sigma(y)|^2\le\rho(|x-y|),\quad\forall\, x,y\in\mathbb{R},$$

(ii) *there exists a concave non-decreasing continuous function γ on $\mathbb{R}+$, $\gamma(0)=0$ and $\int_{0+}\gamma^{-1}(u)\mathrm{d}u=\infty$, such that*

$$|b(x)-b(y)|\le\gamma(|x-y|),\quad\forall\, x,y\in\mathbb{R}.$$

Lemma 3.1 (see [5]) *If X is a continuous semimartingale such that for every t,*

$$\int_0^t \frac{\mathrm{d}\langle X\rangle_s}{\rho(X_s)} 1_{(X_s>0)} < \infty, \quad a.s.,$$

then $L_t^0(X) = 0$, a.s.

We now have

Proposition 3.1 *Equation* (1.1) *has a unique solution under Assumption* 3.1.

Proof (Uniqueness) Let X^1 and X^2 be two solutions for equation (1.1) on the same probability space and with the same Brownian motion and initial value x. Then X^1 and X^2 are continuous semimartingales. By Theorem 2.1, we have

$$\begin{aligned} |X_t^1 - X_t^2| = L_t^0(X^1 - X^2) &+ \int_0^t \operatorname{sgn}(X_s^1 - X_s^2)(\sigma(X_s^1) - \sigma(X_s^2))\mathrm{d}W_s \\ &+ \int_0^t \operatorname{sgn}(X_s^1 - X_s^2)(b(X_s^1) - b(X_s^2))\mathrm{d}s - \int_0^t \operatorname{sgn}(X_s^1 - X_s^2)(\mathrm{d}K_s^1 - \mathrm{d}K_s^2). \end{aligned}$$

Set $X := X^1 - X^2$. Then

$$\int_0^t \frac{\mathrm{d}\langle X\rangle_s}{\rho(X_s)} 1_{(X_s>0)} = \int_0^t \frac{(\sigma(X_s^1) - \sigma(X_s^2))^2}{\rho(X_s^1 - X_s^2)} 1_{(X_s^1>X_s^2)}\mathrm{d}s \le t.$$

By Lemma 3.1,

$$L_t^0(X^1 - X^2) = 0.$$

By Proposition 2.2,

$$\int_0^t \operatorname{sgn}(X_s^1 - X_s^2)(\mathrm{d}K_s^1 - \mathrm{d}K_s^2) \ge 0.$$

Hence

$$\mathbb{E}|X_t^1 - X_t^2| \le \mathbb{E}\Big[\int_0^t |b(X_s^1) - b(X_s^2)|\mathrm{d}s\Big] \le \mathbb{E}\Big[\int_0^t \gamma(|X_s^1 - X_s^2|)\mathrm{d}s\Big] \le \int_0^t \gamma(\mathbb{E}|X_s^1 - X_s^2|)\mathrm{d}s.$$

Since $\gamma(0) = 0$ and $\int_{0+} \gamma^{-1}(u)\mathrm{d}u = \infty$, the equation $y' = \gamma(y)$, $y(0) = 0$ has a unique solution $y \equiv 0$. Let $v(t) := \int_0^t \gamma(\mathbb{E}|X_s^1 - X_s^2|)\mathrm{d}s$. The last inequality can be rewritten as $\mathbb{E}|X_t^1 - X_t^2| \le v(t)$. So $v'(t) = \gamma(\mathbb{E}|X_t^1 - X_t^2|) \le \gamma(v(t))$. By $v(0) = 0$ and the comparison theorem, $v(t) \le y(t) = 0$. That is

$$\mathbb{E}|X_t^1 - X_t^2| \equiv 0, \quad \forall\, t \ge 0.$$

(Existence) Consider the following equation

$$\mathrm{d}X_t^n = b(X_t^n)\mathrm{d}t - A_n(X_t^n)\mathrm{d}t + \sigma(X_t^n)\mathrm{d}W_t, \quad X_0^n = x. \tag{3.1}$$

Under Assumption 3.1, equation (3.1) has a unique solution X_t^n on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Since σ and b are bounded, the rest proof of the existence is straightforward as [3].

By Yamada-Watanabe's theorem, equation (1.1) admits a unique strong solution.

4 Bicontinuous Modification of the Solution

In the sequel, X_t^x denotes the unique solution of equation (1.1) with initial value x and C_p denotes a constant depending on p.

Assumption 4.1 *Suppose that σ and b are bounded and satisfy*

$$|\sigma(x)-\sigma(y)|^2 \le \rho_{2,\eta}^2(|x-y|) \quad \textit{and} \quad |b(x)-b(y)| \le \rho_{1,\eta}(|x-y|).$$

Lemma 4.1 *Under Assumption 4.1, equation (1.1) has a unique solution X_t^x. Moreover,*

$$x > y \textit{ implies } P(X_t^x \ge X_t^y, 0 < t < \infty) = 1. \tag{4.1}$$

Proof Obviously, σ and b satisfy Assumption 3.1, so there exists a unique solution X_t^x. Let $x > y$. By [8], the solution of the following equation

$$\begin{cases} \mathrm{d}X^{(n)}(t,x) = b(X^{(n)}(t,x))\mathrm{d}t - A_n(X^{(n)}(t,x))\mathrm{d}t + \sigma(X^{(n)}(t,x))\mathrm{d}W_t, \\ X^{(n)}(0,x) = x \end{cases}$$

has the property

$$P(X^{(n)}(t,x) > X^{(n)}(t,y), 0 < t < \infty) = 1, \quad \forall\, n.$$

Since $X^{(n)}(t,x) \to X_t^x$ as $n \to \infty$, (4.1) holds.

Lemma 4.2 *Under Assumption 4.1, for any $p \ge 2$ and $t \ge 0$, we have*

$$\mathbb{E}\Big[\sup_{0\le s\le t} |X_s^x - X_s^y|^p\Big] \le C_p|x-y|^{p\cdot\exp\{-C_p t\}}. \tag{4.2}$$

Proof Let $Z_s := X_s^x - X_s^y$, $x > y$. By Lemma 4.1, $Z_s \ge 0$. Since

$$Z_s = x - y + \int_0^s [b(X_u^x) - b(X_u^y)]\mathrm{d}u + \int_0^s [\sigma(X_u^x) - \sigma(X_u^y)]\mathrm{d}W_u - \int_0^s [\mathrm{d}K_u^x - \mathrm{d}K_u^y],$$

applying Itô's formula to Z_s^p, we have

$$\begin{aligned} Z_s^p = Z_0^p &+ p\int_0^s Z_u^{p-1}[b(X_u^x) - b(X_u^y)]\mathrm{d}u + p\int_0^s Z_u^{p-1}[\sigma(X_u^x) - \sigma(X_u^y)]\mathrm{d}W_u \\ &- p\int_0^s Z_u^{p-1}[\mathrm{d}K_u^x - \mathrm{d}K_u^y] + \frac{1}{2}p(p-1)\int_0^s Z_u^{p-2}|\sigma(X_u^x) - \sigma(X_u^y)|^2\mathrm{d}u. \end{aligned}$$

By $Z_u \ge 0$ and Proposition 2.2,

$$\langle Z_u^{p-1}, \mathrm{d}K_u^x - \mathrm{d}K_u^y\rangle \ge 0.$$

Using Assumption 4.1, Burkholder-Davis-Gundy's inequality and Young's inequality, we get

$$\begin{aligned} \mathbb{E}\Big[\sup_{0\le s\le t} Z_s^p\Big] &\le Z_0^p + C_p\mathbb{E}\Big[\int_0^t Z_u^{p-1}\rho_{1,\eta}(Z_u)\mathrm{d}u\Big] + C_p\mathbb{E}\Big[\int_0^t Z_u^{p-2}\rho_{2,\eta}^2(Z_u)\mathrm{d}u\Big] \\ &\quad + C_p\mathbb{E}\Big[\sup_{0\le s\le t}\Big|\int_0^s Z_u^{p-1}[\sigma(X_u^x) - \sigma(X_u^y)]\mathrm{d}W_u\Big|\Big] \end{aligned}$$

$$
\begin{aligned}
&\le Z_0^p + C_p\mathbb{E}\Big[\int_0^t Z_u^{p-1}\rho_{1,\eta}(Z_u)\mathrm{d}u\Big] + C_p\mathbb{E}\Big[\int_0^t Z_u^{p-2}\rho_{2,\eta}^2(Z_u)\mathrm{d}u\Big]\\
&\quad + C_p\mathbb{E}\Big(\int_0^t Z_u^{2p-2}|\sigma(X_u^x)-\sigma(X_u^y)|^2\mathrm{d}u\Big)^{\frac{1}{2}}\\
&\le Z_0^p + C_p\mathbb{E}\Big[\int_0^t Z_u^{p-1}\rho_{1,\eta}(Z_u)\mathrm{d}u\Big] + C_p\mathbb{E}\Big[\int_0^t Z_u^{p-2}\rho_{2,\eta}^2(Z_u)\mathrm{d}u\Big]\\
&\quad + C_p\mathbb{E}\Big(\sup_{0\le s\le t} Z_s^p\int_0^t Z_u^{p-2}\rho_{2,\eta}^2(Z_u)\mathrm{d}u\Big)^{\frac{1}{2}}\\
&\le Z_0^p + C_p\mathbb{E}\Big[\int_0^t Z_u^{p-1}\rho_{1,\eta}(Z_u)\mathrm{d}u\Big] + C_p\mathbb{E}\Big[\int_0^t Z_u^{p-2}\rho_{2,\eta}^2(Z_u)\mathrm{d}u\Big]\\
&\quad + \frac{1}{2}\mathbb{E}\Big[\sup_{0\le s\le t} Z_s^p\Big] + C_p\mathbb{E}\Big[\int_0^t Z_u^{p-2}\rho_{2,\eta}^2(Z_u)\mathrm{d}u\Big].
\end{aligned}
$$

By Lemma 2.2,

$$
\mathbb{E}\Big[\sup_{0\le s\le t} Z_s^p\Big] \le 2Z_0^p + C_p\mathbb{E}\Big[\int_0^t \rho_{1,\eta^p}(Z_u^p)\mathrm{d}u\Big] \le 2Z_0^p + C_p\int_0^t \rho_{1,\eta^p}\Big(\mathbb{E}\Big[\sup_{0\le u\le s} Z_u^p\Big]\Big)\mathrm{d}s.
$$

Finally, by (2.2), we can get

$$
\mathbb{E}\Big[\sup_{0\le s\le t} Z_s^p\Big] \le C_p Z_0^{p\cdot\exp\{-C_p t\}}.
$$

Since X_t^x is continuous with respect to t, by the above lemma we obtain

Theorem 4.1 *Let $p\ge 2$. For every $t\in[0,\frac{\log p}{C_p})$, the mapping $x\mapsto X_t^x$ has a β-Hölder continuous modification for $\beta<\mathrm{e}^{-C_p t}-\frac{1}{p}$. Moreover, if $T<\frac{\log p}{C_p}$, then X_t^x has a bicontinuous modification in $(t,x)\in[0,T]\times\overline{D(A)}$.*

In order to get a bicontinuous modification of the solution on the whole space $(t,x)\in\mathbb{R}_+\times\overline{D(A)}$, we make the following assumption.

Assumption 4.2 *Let $\eta\in(0,\mathrm{e}^{-1})$ and γ be a continuous function of the form*

$$
\gamma(x) = xg(x),
$$

where g is a positive continuous function on $\mathbb{R}^+$, bounded in $[1,\infty)$, such that

$$
\lim_{x\downarrow 0}\frac{g(x)}{\log x} = 0.
$$

Besides, σ and b are bounded and satisfy

$$
\begin{aligned}
|\sigma(x)-\sigma(y)|^2 &\le C\rho_{2,\eta}^2(|x-y|),\\
|b(x)-b(y)| &\le C_1\gamma(|x-y|).
\end{aligned}
$$

Theorem 4.2 *Under Assumption 4.2, there exists a modification X_t^x such that for every $t>0$, $x\mapsto X_\cdot^x\in C[0,t]$ is β-Hölder continuous for every $\beta<\frac{1-\sqrt{1-\exp\{-\frac{C}{2}t\}}}{1+\sqrt{1-\exp\{-\frac{C}{2}t\}}}$.*

Proof For $\varepsilon\in(0,1)$, we let

$$T_\varepsilon:=-\frac{2\log(1-\varepsilon)}{C\varepsilon}.$$

Then

$$\lim_{\varepsilon\to 1}T_\varepsilon=\infty.$$

Set

$$\varepsilon':=(1-\varepsilon)\frac{C}{2C_1}.$$

Take $\delta_\varepsilon\in(0,\frac{1}{2\mathrm{e}})$ such that $g(x)\le\varepsilon'\log x^{-1}$ for $x\in(0,\delta_\varepsilon)$. Then there exists $C_\varepsilon>0$ such that

$$|b(x)-b(y)|\le\begin{cases}\varepsilon' C_1|x-y|\log|x-y|^{-1}, & |x-y|<\delta_\varepsilon,\\ C_\varepsilon|x-y|, & |x-y|\ge\delta_\varepsilon.\end{cases}\tag{4.3}$$

For every $T<T_\varepsilon$, set

$$p_T(t):=\varepsilon\Big(1-\exp\Big\{-\frac{\varepsilon C}{2}(T-t)\Big\}\Big)^{-1},\quad t\in[0,T).$$

Then $t\mapsto p_T(t)$ is increasing and

$$p_T(0)>\varepsilon\Big(1-\exp\Big\{-\frac{-\varepsilon CT_\varepsilon}{2}\Big\}\Big)^{-1}=1.$$

Moreover, a direct calculus gives

$$p_T'(t)=\frac{C}{2}p_T(t)(p_T(t)-\varepsilon)=\frac{C}{2}p_T(t)(p_T(t)-1)+\varepsilon' C_1p_T(t).\tag{4.4}$$

Let f_n be a smooth function from $\mathbb{R}^+$ to $\mathbb{R}^+$ satisfying

$$f_n(x)=x,\quad x<n,\quad f_n(x)=n+1,\quad x>n+1,\quad f_n'\ge 0,\quad f_n''\le 0.\tag{4.5}$$

Set

$$Z_t:=X_t^x-X_t^y+\varepsilon_0,\quad 0<\varepsilon_0<\frac{1}{2\mathrm{e}}.$$

Again let $x>y$. Then we obtain $Z_t>0$ by Lemma 4.1. Applying Itô's formula to $f_n(Z_t)^{p_T(t)}$, we have

$$\begin{aligned}f_n(Z_t)^{p_T(t)}&=f_n(Z_0)^{p_T(0)}+\text{a martingale}+\int_0^t p_T'(s)f_n(Z_s)^{p_T(s)}\log f_n(Z_s)\mathrm{d}s\\&\quad+\int_0^t p_T(s)f_n(Z_s)^{p_T(s)-1}f_n'(Z_s)[b(X_s^x)-b(X_s^y)]\mathrm{d}s\\&\quad+\frac12\int_0^t p_T(s)f_n(Z_s)^{p_T(s)-1}f_n''(Z_s)|\sigma(X_s^x)-\sigma(X_s^y)|^2\mathrm{d}s\\&\quad+\frac12\int_0^t p_T(s)(p_T(s)-1)f_n(Z_s)^{p_T(s)-2}f_n'(Z_s)^2|\sigma(X_s^x)-\sigma(X_s^y)|^2\mathrm{d}s\\&\quad-\int_0^t p_T(s)f_n(Z_s)^{p_T(s)-1}f_n'(Z_s)[\mathrm{d}K_s^x-\mathrm{d}K_s^y]\\&=:f_n(Z_0)^{p_T(0)}+\text{a martingale}+\sum_{i=1}^4\int_0^t\xi_i(s)\mathrm{d}s-g(t).\end{aligned}$$

If $|X_s^x - X_s^y| < \delta_\varepsilon$, then $0 < Z_s < \frac{1}{\mathrm{e}}$ and $\xi_3(s) = 0$.

By (4.3) and Assumption 4.2, we have

$$
\begin{aligned}
|b(X_s^x) - b(X_s^y)| &\le -\varepsilon' C_1 |X_s^x - X_s^y| \log |X_s^x - X_s^y| \le -\varepsilon' C_1 Z_s \log Z_s,\\
|\sigma(X_s^x) - \sigma(X_s^y)|^2 &\le -C|X_s^x - X_s^y|^2 \log |X_s^x - X_s^y| \le -C Z_s^2 \log Z_s.
\end{aligned}
$$

So $\xi_1(s) + \xi_2(s) + \xi_4(s) \le 0$ since (4.4).

If $|X_s^x - X_s^y| \ge \delta_\varepsilon$, by (4.5), it is easy to see that there exist constants $C_{n,\varepsilon}$ such that

$$
\sum_{i=1}^{4} \xi_i(s) \le C_{n,\varepsilon} h(s) f_n(Z_s)^{p_T(s)},
$$

where

$$
h(s) := p_T(s)(p_T(s) - 1) + p_T(s) + p_T'(s) > 0.
$$

Hence

$$
\begin{aligned}
f_n(Z_t)^{p_T(t)} \le f_n(Z_0)^{p_T(0)} &+ \text{a martingale} + C_{n,\varepsilon} \int_0^t h(s) f_n(Z_s)^{p_T(s)} \mathrm{d}s\\
&- \int_0^t p_T(s) f_n(Z_s)^{p_T(s)-1} f_n'(Z_s)[\mathrm{d}K_s^x - \mathrm{d}K_s^y].
\end{aligned}
$$

Taking expectation on both sides, we have

$$
\begin{aligned}
\mathbb{E}[f_n(Z_t)^{p_T(t)}] \le f_n(Z_0)^{p_T(0)} &+ C_{n,\varepsilon} \mathbb{E}\Big[\int_0^t h(s) f_n(Z_s)^{p_T(s)} \mathrm{d}s\Big]\\
&- \mathbb{E}\Big[\int_0^t p_T(s) f_n(Z_s)^{p_T(s)-1} f_n'(Z_s)(\mathrm{d}K_s^x - \mathrm{d}K_s^y)\Big].
\end{aligned}
$$

Obviously, $h(s) \le h(t)$, $p_T(s) \le p_T(t)$ and f_n is bounded. Letting $\varepsilon_0 \downarrow 0$, by dominated convergence theorem, and $\langle X_s^x - X_s^y, \mathrm{d}K_s^x - \mathrm{d}K_s^y \rangle \ge 0$, $p_T(s) > 0$, $f_n' \ge 0$, we obtain

$$
\mathbb{E}[f_n(X_t^x - X_t^y)^{p_T(t)}] \le f_n(X_0^x - X_0^y)^{p_T(0)} + C_{n,\varepsilon} \mathbb{E}\Big[\int_0^t h(s) f_n(X_s^x - X_s^y)^{p_T(s)} \mathrm{d}s\Big].
$$

Trivially by Gronwall's inequality, we get

$$
\mathbb{E}[f_n(X_t^x - X_t^y)^{p_T(t)}] \le f_n(x-y)^{p_T(0)} \exp\Big\{C_{n,\varepsilon} \int_0^t h(s) \mathrm{d}s\Big\}, \quad t \in [0, T).
$$

When n is sufficiently large,

$$
\mathbb{E}[f_n(X_t^x - X_t^y)^{p_T(t)}] \le (x-y)^{p_T(0)} \exp\{C_{n,\varepsilon,t}\}, \quad t \in [0, T), \tag{4.6}
$$

where

$$
C_{n,\varepsilon,t} := C_{n,\varepsilon} \int_0^t h(s) \mathrm{d}s.
$$

Now we look for the $T(t,\varepsilon) \in (t, T_\varepsilon)$ such that

$$
\frac{p_{T(t,\varepsilon)}(0) - 1}{p_{T(t,\varepsilon)}(t)} = \sup_{t < T < T_\varepsilon} \frac{p_T(0) - 1}{p_T(t)}.
$$

We find that

$$T(t,\varepsilon) = -\frac{2}{\varepsilon C}\log\Big(1 - \sqrt{\varepsilon\Big(1-\exp\Big\{-\frac{\varepsilon Ct}{2}\Big\}\Big)}\Big).$$

Fix $t \in (0, T_\varepsilon)$. For any $s \in (0,t)$, let $S_s := T(t,\varepsilon) - t + s$. Then $p_{T(t,\varepsilon)}(t) = p_{S_s}(s)$. Applying Itô's formula, we have

$$\begin{aligned}
f_n(Z_s)^{p_{T(t,\varepsilon)}(t)} = {}& f_n(Z_0)^{p_{T(t,\varepsilon)}(t)} + \int_0^s p_{T(t,\varepsilon)}(t) f_n(Z_u)^{p_{T(t,\varepsilon)}(t)-1} f_n'(Z_u)[b(X_u^x) - b(X_u^y)]\mathrm{d}u \\
&+ \frac{1}{2}\int_0^s p_{T(t,\varepsilon)}(t) f_n(Z_u)^{p_{T(t,\varepsilon)}(t)-1} f_n''(Z_u)|\sigma(X_u^x) - \sigma(X_u^y)|^2\mathrm{d}u \\
&+ \frac{1}{2}\int_0^s p_{T(t,\varepsilon)}(t)(p_{T(t,\varepsilon)}(t)-1) f_n(Z_u)^{p_{T(t,\varepsilon)}(t)-2} f_n'(Z_u)^2|\sigma(X_u^x) - \sigma(X_u^y)|^2\mathrm{d}u \\
&+ \int_0^s p_{T(t,\varepsilon)}(t) f_n(Z_u)^{p_{T(t,\varepsilon)}(t)-1} f_n'(Z_u)[\sigma(X_u^x) - \sigma(X_u^y)]\mathrm{d}W_u \\
&- \int_0^s p_{T(t,\varepsilon)}(t) f_n(Z_u)^{p_{T(t,\varepsilon)}(t)-1} f_n'(Z_u)[\mathrm{d}K_u^x - \mathrm{d}K_u^y] \\
:= {}& f_n(Z_0)^{p_{T(t,\varepsilon)}(t)} + \sum_{i=1}^3 \int_0^s \eta_i(u)\mathrm{d}u + \int_0^s \eta_4(u)\mathrm{d}W_u - m(t).
\end{aligned}$$

When x is sufficiently small, $x^{p_{T(t,\varepsilon)}(t)}\log x^{-1} < C_\alpha x^{p_{T(t,\varepsilon)+\alpha}(t)}$ since $p_{T(t,\varepsilon)+\alpha}(t) < p_{T(t,\varepsilon)}(t)$ for every $\alpha > 0$.

If $|X_s^x - X_s^y| < \delta_\varepsilon$, then $f_n(Z_u)^{p_{T(t,\varepsilon)}(t)}\log f_n(Z_u)^{-1} < C_{\alpha,\varepsilon,t} f_n(Z_u)^{p_{T(t,\varepsilon)+\alpha}(t)}$. By (4.5),

$$\sum_{i=1}^3 \int_0^s \eta_i(u)\mathrm{d}u \le C_{\alpha,\varepsilon,t}\int_0^s f_n(Z_u)^{p_{T(t,\varepsilon)+\alpha}(t)}\mathrm{d}u.$$

If $|X_s^x - X_s^y| \ge \delta_\varepsilon$, there exist $C_{\alpha,n,\varepsilon,t}$ such that

$$\sum_{i=1}^3 \int_0^s \eta_i(u)\mathrm{d}u \le C_{\alpha,n,\varepsilon,t}\int_0^s f_n(Z_u)^{p_{T(t,\varepsilon)+\alpha}(t)}\mathrm{d}u.$$

By Burkholder-Davie-Gundy's inequality and Young's inequality, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le t} f_n(Z_s)^{p_{T(t,\varepsilon)}(t)}\Big] \\
&\le f_n(Z_0)^{p_{T(t,\varepsilon)}(t)} + C_{\alpha,n,\varepsilon,t}\mathbb{E}\Big[\int_0^t f_n(Z_u)^{p_{T(t,\varepsilon)+\alpha}(t)}\mathrm{d}u\Big] \\
&\quad + C_{\varepsilon,t}\mathbb{E}\Big[\sup_{0\le s\le t}\Big|\int_0^s f_n(Z_u)^{p_{T(t,\varepsilon)}(t)-1} f_n'(Z_u)[\sigma(X_u^x) - \sigma(X_u^y)]\mathrm{d}W_u\Big|\Big] \\
&\quad - \mathbb{E}\Big[\sup_{0\le s\le t}\int_0^s p_{T(t,\varepsilon)}(t) f_n(Z_u)^{p_{T(t,\varepsilon)}(t)-1} f_n'(Z_u)(\mathrm{d}K_u^x - \mathrm{d}K_u^y)\Big] \\
&\le f_n(Z_0)^{p_{T(t,\varepsilon)}(t)} + C_{\alpha,n,\varepsilon,t}\mathbb{E}\Big[\int_0^t f_n(Z_u)^{p_{T(t,\varepsilon)+\alpha}(t)}\mathrm{d}u\Big] \\
&\quad + C_{\alpha,n,\varepsilon,t}\mathbb{E}\Big(\sup_{0\le u\le t} f_n(Z_u)^{p_{T(t,\varepsilon)}(t)}\int_0^t f_n(Z_u)^{p_{T(t,\varepsilon)+\alpha}(t)}\mathrm{d}u\Big)^{\frac{1}{2}} \\
&\quad - \mathbb{E}\Big[\sup_{0\le s\le t}\int_0^s p_{T(t,\varepsilon)}(t) f_n(Z_u)^{p_{T(t,\varepsilon)}(t)-1} f_n'(Z_u)(\mathrm{d}K_u^x - \mathrm{d}K_u^y)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le f_n(Z_0)^{p_{T(t,\varepsilon)}(t)} + C_{\alpha,n,\varepsilon,t}\mathbb{E}\Big[\int_0^t f_n(Z_u)^{p_{T(t,\varepsilon)+\alpha}(t)}\mathrm{d}u\Big] \\
&\quad + \frac{1}{2}\mathbb{E}\Big[\sup_{0\le u\le t} f_n(Z_u)^{p_{T(t,\varepsilon)}(t)}\Big] + C_{\alpha,n,\varepsilon,t}\mathbb{E}\Big[\int_0^t f_n(Z_u)^{p_{T(t,\varepsilon)+\alpha}(t)}\mathrm{d}u\Big] \\
&\quad - \mathbb{E}\Big[\sup_{0\le s\le t}\int_0^s p_{T(t,\varepsilon)}(t) f_n(Z_u)^{p_{T(t,\varepsilon)}(t)-1} f_n'(Z_u)(\mathrm{d}K_u^x - \mathrm{d}K_u^y)\Big].
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{0\le s\le t} f_n(Z_s)^{p_{T(t,\varepsilon)}(t)}\Big] &\le 2f_n(Z_0)^{p_{T(t,\varepsilon)}(t)} + C_{\alpha,n,\varepsilon,t}\mathbb{E}\Big[\int_0^t f_n(Z_u)^{p_{T(t,\varepsilon)+\alpha}(t)}\mathrm{d}u\Big] \\
&\quad - 2\mathbb{E}\Big[\sup_{0\le s\le t}\int_0^s p_{T(t,\varepsilon)}(t) f_n(Z_u)^{p_{T(t,\varepsilon)}(t)-1} f_n'(Z_u)(\mathrm{d}K_u^x - \mathrm{d}K_u^y)\Big].
\end{aligned}
$$

Letting $\varepsilon_0 \downarrow 0$, by Proposition 2.2 and (4.6), we get

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{0\le s\le t} f_n(X_s^x - X_s^y)^{p_{T(t,\varepsilon)}(t)}\Big] &\le 2f_n(x-y)^{p_{T(t,\varepsilon)}(t)} \\
&\quad + C_{\alpha,n,\varepsilon,t}\mathbb{E}\Big[\int_0^t f_n(X_u^x - X_u^y)^{p_{T(t,\varepsilon)+\alpha}(t)}\mathrm{d}u\Big] \\
&= 2f_n(x-y)^{p_{T(t,\varepsilon)}(t)} + C_{\alpha,n,\varepsilon,t}\mathbb{E}\Big[\int_0^t f_n(X_u^x - X_u^y)^{p_{S_u+\alpha}(u)}\mathrm{d}u\Big] \\
&\le 2f_n(x-y)^{p_{T(t,\varepsilon)}(t)} + C_{\alpha,n,\varepsilon,t}\int_0^t f_n(x-y)^{p_{S_u+\alpha}(0)}\mathrm{d}u \\
&= 2f_n(x-y)^{p_{T(t,\varepsilon)}(t)} + C_{\alpha,n,\varepsilon,t}\int_0^t f_n(x-y)^{p_{T(t,\varepsilon)+\alpha}(t-u)}\mathrm{d}u \\
&\le C_{\alpha,n,\varepsilon,t}|x-y|^{p_{T(t,\varepsilon)+\alpha}(0)}.
\end{aligned}
$$

Consequently,

$$
\mathbb{E}\Big[\sup_{0\le s\le t}|X_n(s,x) - X_n(s,y)|^{p_{T(t,\varepsilon)}(t)}\Big] \le C_{\alpha,n,\varepsilon,t}|x-y|^{p_{T(t,\varepsilon)+\alpha}(0)},
$$

where

$$
X_n(s,x) := (-n)\vee X_s^x \wedge n.
$$

Since

$$
\lim_{\alpha\to 0} p_{T(t,\varepsilon)+\alpha}(0) = p_{T(t,\varepsilon)}(0),
$$

by [9, Theorem 2.1], X_n has a modification such that for every $\beta \in (0, (p_{T(t,\varepsilon)}(0)-1)p^{-1}_{T(t,\varepsilon)}(t))$, $x \mapsto X_n(\,\cdot\,, x) \in C[0,t]$ are β-Hölder continuous.

Furthermore, letting $\varepsilon \to 1$, we have $\lim_{\varepsilon\to 1} p_{T(t,\varepsilon)}(0) = p_{T(t,1)}(0)$. Set

$$
\beta_t := (p_{T(t,1)}(0) - 1)p^{-1}_{T(t,1)}(t) = \frac{1-\sqrt{1-\exp\{-\frac{C}{2}t\}}}{1+\sqrt{1-\exp\{-\frac{C}{2}t\}}}.
$$

By Lemma 2.3, $x \mapsto X_\cdot^x \in C[0,t]$ is β-Hölder continuous for $\beta < \beta_t$.

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