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On the Fourier Spectra of Distributions in Clifford Analysis

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Abstract In recent papers by Brackx, Delanghe and Sommen, some fundamental higher dimensional distributions have been reconsidered in the framework of Clifford analysis, eventually leading to the introduction of four broad classes of new distributions in Euclidean space. In the current paper we continue the in-depth study of these distributions, more specifically the study of their behaviour in frequency space, thus extending classical results of harmonic analysis.

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1 Introduction

During the last fifty years, Clifford analysis has gained interest as a comprehensive function theory offering a direct, elegant and powerful generalization to higher dimension of the theory of holomorphic functions in the complex plane. In its most simple but still useful setting, flat m-dimensional Euclidean space, Clifford analysis is centered around so-called monogenic functions, i.e. null solutions of the Clifford-vector valued Dirac operator

$$\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j},$$

where $(e_1, \cdots, e_m)$ forms an orthogonal basis for the quadratic space $\mathbb{R}^m$ underlying the construction of the Clifford algebra $\mathbb{R}_{0,m}$. Monogenic functions have a special relationship with harmonic functions of several variables in that they are refining their properties. Note for instance that each harmonic function can be split into a so-called inner and an outer monogenic function, and that a real harmonic function is always the real part of a monogenic one, which does not need to be the case for a harmonic function of several complex variables. The reason is that, as does the Cauchy-Riemann operator in the complex plane, the rotation-invariant Dirac operator factorizes the m-dimensional Laplace operator. It hence is not surprising that Clifford analysis often leads to refinements or generalizations of classical results from harmonic analysis.

In [3] and [4] four broad families of distributions in Euclidean space $T_{\lambda,p}$, $U_{\lambda,p}$, $V_{\lambda,p}$ and $W_{\lambda,p}$, depending on parameters $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$, were introduced and studied in the framework of Clifford analysis. These distributions all spring from the already classically known distribution

$$T_\lambda = \mathrm{Fp}\, r_+^\lambda$$

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depending on the complex parameter λ. Here Fp is the fundamental distribution "finite parts" on the real line, and spherical co-ordinates have been used to convert an originally m-dimensional distribution into one acting on the real line. More precisely,

$$\langle T_\lambda, \phi \rangle = a_m \mathrm{Fp} \int_0^{+\infty} r^{\lambda+m-1} \Big[\frac{1}{a_m} \int_{S^{m-1}} \phi(\underline{x})\, dS(\underline{\omega}) \Big] dr = \langle \mathrm{Fp}\, r_+^{\lambda+m-1}, \Sigma^{(0)}[\phi] \rangle,$$

where a_m is the area of the unit sphere S^{m-1} in $\mathbb{R}^m$ and $\Sigma^{(0)}[\phi]$ denotes the so-called spherical mean of the testing function ϕ, obtained through integration over the unit sphere. In these, spherical co-ordinates not only reflect the "spherical" philosophy of the approach, encompassing all dimensions at once as opposed to a cartesian or tensorial approach with products of one-dimensional phenomena, but also enable to carry out the explicit calculations in one dimension after which they are exported again to the original setting of Euclidean space.

An analogous approach underlies the definition of the four families of distributions mentioned above, which respectively involve the inner spherical monogenics $P_p(\underline{\omega})$ ($p \in \mathbb{N}_0$), i.e., restrictions to the unit sphere of monogenic polynomials which are vector valued and homogeneous of degree p, and the related outer spherical monogenics $P_p(\underline{\omega})\underline{\omega}$, $\underline{\omega}P_p(\underline{\omega})$ and $\underline{\omega}P_p(\underline{\omega})\underline{\omega}$. Here, the spherical philosophy requires the introduction of generalized spherical means, where the involved spherical monogenic (inner or outer) appears in the integrand over the unit sphere. However, in view of the Fourier transformations aimed at, the distributions $T_{\lambda,p}$ and their normalized versions $T^*_{\lambda,p}$ are now reconsidered from a cartesian point of view in Section 3 of the paper.

Any of the distributions in these families may be considered as a kernel (K) for a convolution operator (L): $L[f] = K * f$. As is well known, see [20], such an operator may be realized in frequency space by a multiplication operator, its so-called Fourier symbol: $\mathcal{F}[L[f]] = \alpha\, \mathcal{F}[f]$, where $\alpha = \mathcal{F}[K]$. This underlines the importance of calculating the Fourier transforms of the distributions under consideration. It is worth mentioning that, for specific values of the parameters $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}_0$, those distributions turn into known kernel functions in harmonic and Clifford analysis: up to constants, $U_{-m,0}$ reduces to $Pv\frac{\underline{\omega}}{r^m}$, the higher dimensional analogue of the so–called "Principal Value" distribution on the real line which constitutes the convolution kernel for the higher dimensional Hilbert transform (see [5–8]); $U_{-m+1,0}$ reduces to the fundamental solution of the Dirac operator $\underline{\partial}$, while $T_{-m+2,0}$ is nothing but the fundamental solution of the Laplace operator; furthermore, for $\lambda = -p$ the inner and outer spherical monogenics $P_p(\underline{\omega})$, $P_p(\underline{\omega})\underline{\omega}$ and $\underline{\omega}P_p(\underline{\omega})$ are recovered; etc. Moreover when $\lambda = -m-p$ the four families provide examples of so-called principal value distributions (see [13, 20]), being tempered distributions obtained by a limiting process:

$$\langle K, \phi \rangle = \lim_{\underset{>}{\varepsilon \to 0}} \int_{\mathbb{R}^m \setminus B(0,\varepsilon)} K(\underline{x})\phi(\underline{x}) dV(\underline{x}), \quad \phi \in \mathcal{S}(\mathbb{R}^m).$$

In [13, 20] much attention is paid to the calculation of the Fourier transforms of the principal value distributions where the function K takes the form

$$K(\underline{x}) = \frac{k(\underline{\omega})}{r^m}, \quad \underline{\omega} \in S^{m-1}$$

with $k \in L_2(S^{m-1})$ such that

$$\int_{S^{m-1}} k(\underline{\omega}) dS(\underline{\omega}) = 0.$$

Such kind of principal value distributions lead to convolution operators in the following way:

$$(K * \phi)(\underline{y}) = \lim_{\varepsilon \underset{>}{\to} 0} \int_{\mathbb{R}^m \setminus B(0,\varepsilon)} K(\underline{y} - \underline{x})\phi(\underline{x}) dV(\underline{x}), \quad \phi \in \mathcal{S}(\mathbb{R}^m)$$

also known as singular integral operators (see [20, Theorem VI.3.1]). In this paper, we concentrate on the families $T_{\lambda,p}$, $U_{\lambda,p}$ and $V_{\lambda,p}$ introduced in [3, 4], for which we aim at calculating the Fourier spectra, thus generalizing the results obtained in [13, 20]. This is the subject of Sections 4 and 5.

In order to make the paper self-contained we recall in Section 2 some basic notions and results of Clifford analysis.

2 Clifford Analysis

Clifford analysis offers a function theory which is a higher dimensional analogue of the theory of holomorphic functions of one complex variable. For more details concerning this function theory and its applications (for instance to harmonic analysis) we refer the reader to [2, 9, 11, 12, 15–18].

Let, for $m \geq 2$, $\mathbb{R}^{0,m}$ be the real vector space $\mathbb{R}^m$, endowed with a non-degenerate quadratic form of signature $(0, m)$, let $(e_1, \cdots, e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$, and let $\mathbb{R}_{0,m}$ be the universal Clifford algebra constructed over $\mathbb{R}^{0,m}$. The non-commutative multiplication in $\mathbb{R}_{0,m}$ is then governed by the rules

$$e_i^2 = -1, \quad i = 1, 2, \cdots, m \quad \text{and} \quad e_i e_j + e_j e_i = 0, \quad i \neq j.$$

For a set $A = \{i_1, \cdots, i_h\} \subset \{1, \cdots, m\}$ with $1 \leq i_1 < i_2 < \cdots < i_h \leq m$, let $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$. Moreover, we put $e_\emptyset = 1$, the latter being the identity element; then $(e_A : A \subset \{1, \cdots, m\})$ is a basis for the Clifford algebra $\mathbb{R}_{0,m}$. Any $a \in \mathbb{R}_{0,m}$ may thus be written as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R}$$

or still as $a = \sum_{k=0}^{m} [a]_k$ where $[a]_k = \sum_{|A|=k} a_A e_A$ is a so-called k-vector ($k = 0, 1, \cdots, m$). If we denote the space of k-vectors by $\mathbb{R}_{0,m}^k$, then $\mathbb{R}_{0,m} = \bigoplus_{k=0}^{m} \mathbb{R}_{0,m}^k$, leading to the identification of $\mathbb{R}$ and $\mathbb{R}^{0,m}$ with respectively $\mathbb{R}_{0,m}^0$ and $\mathbb{R}_{0,m}^1$. We will also identify an element $\underline{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ with the one-vector (or vector for short) $\underline{x} = \sum_{j=1}^{m} x_j \, e_j$. The multiplication of any two vectors $\underline{x}$ and $\underline{y}$ is given by

$$\underline{x}\underline{y} = -\langle \underline{x}, \underline{y} \rangle + \underline{x} \wedge \underline{y}$$

with

$$\langle \underline{x}, \underline{y} \rangle = \sum_{j=1}^{m} x_j y_j = -\frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\underline{x}), \quad \underline{x} \wedge \underline{y} = \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\underline{x})$$

being a scalar and a 2-vector (also called bivector) respectively. In particular $\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2 = -\sum_{j=1}^{m} x_j^2$. Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution for which

$$\bar{e}_j = -e_j, \quad j = 1, \cdots, m.$$

In particular for a vector $\underline{x}$ we have $\bar{\underline{x}} = -\underline{x}$.

The Dirac operator in $\mathbb{R}^m$ is the first order vector valued differential operator

$$\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j},$$

its fundamental solution being given by

$$E_m(\underline{x}) = \frac{1}{a_m} \frac{\bar{\underline{x}}}{|\underline{x}|^m}.$$

For functions f defined in $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$, we say that f is left monogenic (respectively right monogenic) in the open region Ω of $\mathbb{R}^m$ iff f is continuously differentiable in Ω and satisfies in Ω the equation $\underline{\partial} f = 0$ (respectively the equation $f\underline{\partial} = 0$). As $\overline{\underline{\partial} f} = \bar{f}\ \bar{\underline{\partial}} = -\bar{f}\underline{\partial}$ a function f is left monogenic in Ω iff $\bar{f}$ is right monogenic in Ω. As moreover the Dirac operator factorizes the Laplace operator Δ, $-\underline{\partial}^2 = \underline{\partial}\ \bar{\underline{\partial}} = \bar{\underline{\partial}}\ \underline{\partial} = \Delta$, a (left or right) monogenic function in Ω is harmonic and hence C_∞ in Ω.

Introducing spherical co-ordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in S^{m-1}$, the Dirac operator $\underline{\partial}$ may be written as

$$\underline{\partial} = \underline{\omega}\partial_r + \frac{1}{r}\partial_{\underline{\omega}} = \underline{\omega}\Big(\partial_r - \frac{1}{r}\underline{\omega}\partial_{\underline{\omega}}\Big),$$

while the Laplace operator takes the form

$$\Delta = \partial_r^2 + \frac{m-1}{r}\partial_r + \frac{1}{r^2}\Delta^*,$$

Δ^* being the Laplace-Beltrami operator on S^{m-1}.

In the definition of our Clifford distributions a fundamental rôle is played by the so-called inner spherical monogenics. Starting from a homogeneous polynomial $P_p(\underline{x})$ of degree p which we take to be vector valued and left (and hence also right) monogenic, the following formulae are seen to hold in $\mathbb{R}^m$:

$$\begin{aligned}
&\underline{\partial} P_p(\underline{x}) = P_p(\underline{x})\underline{\partial} = 0,\\
&\underline{\partial}(\underline{x} P_p(\underline{x})) = (P_p(\underline{x})\underline{x})\underline{\partial} = -(m+2p)P_p(\underline{x}),\\
&\underline{\partial}(P_p(\underline{x})\underline{x}) = (\underline{x}P_p(\underline{x}))\underline{\partial} = (m-2)P_p(\underline{x}), \quad p \neq 0,\\
&\Delta P_p(\underline{x}) = \Delta(\underline{x}P_p(\underline{x})) \ = \ \Delta(P_p(\underline{x})\underline{x}) = 0.
\end{aligned}$$

By taking restrictions to the unit sphere S^{m-1} of the polynomials $P_p(\underline{x})$, we obtain so-called inner spherical monogenics $P_p(\underline{\omega})$. Conversely, given an inner spherical monogenic $P_p(\underline{\omega})$ then obviously

$$r^p P_p(\underline{\omega}) = P_p(\underline{x})$$

is a left and right monogenic homogeneous polynomial the restriction to the unit sphere of which is precisely $P_p(\underline{\omega})$. At the same time the functions

$$\frac{1}{r^{m+p-1}}\,\underline{\omega}P_p(\underline{\omega}) = \frac{1}{r^{m+2p}}\,\underline{x}P_p(\underline{x}) = Q_p^{(l)}(\underline{x}), \quad \frac{1}{r^{m+p-1}}\,P_p(\underline{\omega})\underline{\omega} = \frac{1}{r^{m+2p}}\,P_p(\underline{x})\underline{x} = Q_p^{(r)}(\underline{x})$$

are left, respectively right monogenic homogeneous functions of order $-(m+p-1)$ in the complement of the origin. Their restrictions to the unit sphere S^{m-1}, $\underline{\omega}P_p(\underline{\omega})$ and $P_p(\underline{\omega})\underline{\omega}$, are called outer spherical monogenics. Both the inner and the outer spherical monogenics are special cases of spherical harmonics.

Finally, in this paper we adopt the following definition of the Fourier transform

$$\mathcal{F}[f(\underline{x})](\underline{y}) = \hat{f}(\underline{y}) = \int_{\mathbb{R}^m} f(\underline{x}) \exp(-2\pi i \langle \underline{x}, \underline{y}\rangle) dV(\underline{x})$$

for which some well-known basic formulae hold:

$$\begin{aligned} &\mathcal{F}[\underline{\partial}\, f](\underline{y}) = 2\pi i\, \underline{y}\mathcal{F}[f](\underline{y}), \quad 2\pi i\, \mathcal{F}[\underline{x}\; f](\underline{y}) = -\underline{\partial}\mathcal{F}[f](\underline{y}), \\ &2\pi i\, \mathcal{F}[f\, \underline{x}](\underline{y}) = -\mathcal{F}[f](\underline{y})\underline{\partial}, \quad \mathcal{F}[\delta(\underline{x})] = 1, \quad \mathcal{F}[1](\underline{y}) = \delta(\underline{y}). \end{aligned} \tag{2.1}$$

3 Normalization of the Distributions $T_{\lambda,p}$

3.1 Definition of the distributions $T_{\lambda,p}$

We recall the definition of the family of distributions $T_{\lambda,p}$ as given in [3, 4].

First, let μ be a complex parameter, let x be a real variable and consider the function

$$x_+^\mu = \begin{cases} x^\mu, & x > 0, \\ 0, & x < 0, \end{cases}$$

which is a regular distribution for $\operatorname{Re}\mu > -1$. In addition, one defines, for $n \in \mathbb{N}$ and $\mu \in \mathbb{C}$ such that $-n-1 < \operatorname{Re}\mu < -n$, the classical one-dimensional "finite part" distribution $\operatorname{Fp} x_+^\mu$ by

$$\begin{aligned} \langle \operatorname{Fp} x_+^\mu, \phi\rangle &= \int_0^{+\infty} x^\mu \Big(\phi(x) - \phi(0) - \frac{\phi'(0)}{1!}x - \cdots - \frac{\phi^{(n-1)}(0)}{(n-1)!}x^{n-1}\Big)dx \\ &= \lim_{\substack{\varepsilon \to 0 \\ >}} \Big(\int_\varepsilon^{+\infty} x^\mu \phi(x)dx + \phi(0)\frac{\varepsilon^{\mu+1}}{\mu+1} + \cdots + \frac{\phi^{(n-1)}(0)}{(n-1)!}\frac{\varepsilon^{\mu+n}}{\mu+n}\Big). \end{aligned}$$

As a function of μ, x_+^μ is holomorphic in the half-plane $\operatorname{Re}\mu > -1$, and by analytic continuation $\operatorname{Fp} x_+^\mu$ is holomorphic in $\mathbb{C}\backslash\{-1,-2,-3,\cdots\}$, the singular points $\mu = -n$ ($n \in \mathbb{N}$) being simple poles with residue $\frac{(-1)^{n-1}}{(n-1)!}\,\delta_x^{(n-1)}$. This finite part distribution shows the following properties:

$$\frac{d}{dx}\operatorname{Fp} x_+^\mu = \mu \operatorname{Fp} x_+^{\mu-1}, \quad \mu \neq 0,-1,-2,-3,\cdots, \quad x \operatorname{Fp} x_+^\mu = \operatorname{Fp} x_+^{\mu+1}, \quad \mu \neq -1,-2,-3,\cdots.$$

Remark 3.1 By a slight change in the above expression for $\operatorname{Fp} x_+^\mu$ a definition may be given for negative entire exponents as well, through the so-called monomial pseudofunctions $\operatorname{Fp} x_+^{-n}$, $n \in \mathbb{N}$ (see e.g. [10, 19]):

$$\langle \operatorname{Fp} x_+^{-n},\, \phi(x)\rangle = \lim_{\substack{\varepsilon \to 0 \\ >}} \Big(\int_\varepsilon^{+\infty} x^{-n}\phi(x)\; dx + \phi(0)\frac{\varepsilon^{-n+1}}{-n+1} + \cdots + \frac{\phi^{(n-2)}(0)}{(n-2)!}\frac{\varepsilon^{-1}}{(-1)} + \frac{\phi^{(n-1)}(0)}{(n-1)!}\ln\varepsilon\Big)$$

with properties

$$\frac{d}{dx}\operatorname{Fp} x_+^{-n} = (-n)\operatorname{Fp} x_+^{-n-1} + (-1)^n \frac{1}{n!}\,\delta_x^{(n)} \quad \text{and} \quad x \operatorname{Fp} x_+^{-n} = \operatorname{Fp} x_+^{-n+1}.$$

However, in what follows, we have chosen to deal with the singularities of $\operatorname{Fp} x_+^\mu$ in another way.

Next we define the generalized spherical mean $\Sigma_p^{(0)}[\phi]$ (see also [21]), for a scalar valued testing function $\phi(\underline{x})$ in $\mathbb{R}^m$ and a vector valued, monogenic, homogeneous polynomial $P_p(\underline{x})$ of degree $p \neq 0$, as

(i) $\Sigma_{2k}^{(0)}[\phi] = \Sigma^{(0)}[P_{2k}(\underline{\omega})\phi(\underline{x})] = \dfrac{1}{a_m} \displaystyle\int_{S^{m-1}} P_{2k}(\underline{\omega})\phi(\underline{x})\, dS(\underline{\omega})$,

(ii) $\Sigma_{2k+1}^{(0)}[\phi] = \Sigma^{(0)}[r\, P_{2k+1}(\underline{\omega})\phi(\underline{x})] = \dfrac{r}{a_m} \displaystyle\int_{S^{m-1}} P_{2k+1}(\underline{\omega})\phi(\underline{x})\, dS(\underline{\omega})$.

Finally we define the distributions $T_{\lambda,p}$ where $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}_0$, as follows. Let ϕ be a scalar valued testing function, let $\mu = \lambda + m - 1$ and put $p_e = p$ if p is even, and $p_e = p - 1$ if p is odd; then

$$\langle T_{\lambda,p}, \phi \rangle = a_m \langle \mathrm{Fp}\, r_+^{\mu+p_e}, \Sigma_p^{(0)}[\phi] \rangle. \tag{3.1}$$

Let, for a moment, $\lambda \neq -m-n$ and $\lambda \neq -m-n-p_e$, $n = 0, 1, 2, \cdots$. Then the connection between $T_{\lambda,p}$ and $T_\lambda = T_{\lambda,0}$ is obtained in a natural way from

$$\begin{aligned}\langle T_\lambda P_p(\underline{x}), \phi(\underline{x}) \rangle &= \langle\, T_\lambda, P_p(\underline{x})\phi(\underline{x}) \rangle = a_m \langle \mathrm{Fp}\, r_+^{\mu}, \Sigma^{(0)}[P_p(\underline{x})\phi(\underline{x})] \rangle \\ &= a_m \langle \mathrm{Fp}\, r_+^{\mu+p_e}, \Sigma_p^{(0)}[\phi(\underline{x})] \rangle = \langle T_{\lambda,p}, \phi(\underline{x}) \rangle,\end{aligned}$$

leading, at least for the values of λ mentioned above, to

$$T_{\lambda,p} = T_\lambda P_p\,. \tag{3.2}$$

The other values of λ are not yet taken into account, as they seem to be simple poles of either the left- or the right-hand side of the relation (3.2). Further investigation of these assumed singularities is carried out in the next subsection.

3.2 The singularities of $T_{\lambda,p}$

As mentioned in the previous subsection it is clear from the definition itself that $T_{\lambda,p}$, considered as a function of $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$, inherits an infinite sequence of singularities (simple poles) from the finite part distribution, i.e. for $\mu + p_e = -n$, $n \in \mathbb{N}$, or equivalently, $\lambda = -m - p_e - n + 1$, $n \in \mathbb{N}$; the corresponding residue is given by

$$\mathrm{Res}_{\lambda=-m-p_e-n+1} \langle T_{\lambda,p}, \phi \rangle = a_m \Big\langle \frac{(-1)^{(n-1)}}{(n-1)!} \delta_r^{(n-1)}, \Sigma_p^{(0)}[\phi] \Big\rangle. \tag{3.3}$$

In this subsection we will examine these singularities more closely, revealing that in several subcases the residues turn out to be zero, on account of some specific properties of the generalized spherical mean operator $\Sigma_p^{(0)}$ and of the polynomial $P_p(\underline{x})$, respectively. Indeed, it has been shown in [3] that

Proposition 3.1 *The spherical mean $\Sigma_p^{(0)}[\phi]$ is an even testing function on the real r-axis. Its derivatives of odd order vanish at the origin $r = 0$, while for the derivatives of even order we have*

$$\{\partial_r^{2l} \Sigma_p^{(0)}[\phi]\}_{r=0} = \frac{(2l)!}{(p_e+2l)!} \frac{1}{C(\frac{p_e}{2}+l)} \{\Delta_m^{\frac{p_e}{2}+l}(\phi(\underline{x})P_p(\underline{x}))\}_{\underline{x}=\underline{0}} \tag{3.4}$$

with constants

$$C(l) = \frac{2^{2l} l!}{(2l)!} \Big(\frac{m}{2} + l - 1\Big) \cdots \Big(\frac{m}{2}\Big), \quad l \in \mathbb{N}_0.$$

In addition we may prove the following important results.

Proposition 3.2 *Let $P_p(\underline{x})$ be a vector valued, monogenic, homogeneous polynomial of degree p and let $r = |\underline{x}|$. Then for each $l \in \mathbb{N}_0$,*

$$P_p(\underline{\partial})r^{2l} = \begin{cases} 0, & \text{if } l < p, \\ 2^p \dfrac{l!}{(l-p)!}\ P_p(\underline{x})\ r^{2(l-p)}, & \text{if } l \geq p, \end{cases} \tag{3.5}$$

$$P_p(\underline{x})\underline{\partial}^{2l}\delta(\underline{x}) = \begin{cases} 0, & \text{if } l < p, \\ 2^p \dfrac{l!}{(l-p)!} P_p(\underline{\partial})\underline{\partial}^{2(l-p)}\delta(\underline{x}), & \text{if } l \geq p. \end{cases} \tag{3.6}$$

Proof The calculations being long and technical, we only sketch the main lines of the proof which proceeds in several steps.

Step 1 We write the vector valued monogenic homogeneous polynomial of degree p, $P_p(\underline{x})$, as

$$P_p(\underline{x}) \ = \ \sum_{i=1}^{m} e_i \Big(\sum_{|\underline{\alpha}|=p} b_{i,\underline{\alpha}} F(\underline{\alpha}) x_1^{\alpha_1} \cdots x_m^{\alpha_m} \Big) \tag{3.7}$$

with

$$F(\underline{\alpha}) = \frac{|\underline{\alpha}|!}{\alpha_1! \cdots \alpha_m!}.$$

Then its assumed monogenicity leads to the following conditions on its coefficients: for $i = 1, \cdots, m$, and for $\underline{\alpha} = (\alpha_1, \cdots, \alpha_m)$ with $|\underline{\alpha}| = p$ and $\alpha_i \geq 1$ one has

$$\begin{cases} \displaystyle\sum_{k=1}^{m} b_{k,\underline{\hat{\alpha}}^{k(1)}_{i(1)}} = 0, \\ b_{l,\underline{\alpha}} = b_{i,\underline{\hat{\alpha}}^{l(1)}_{i(1)}}, \quad l = 1, \cdots, m,\ l \neq i, \end{cases} \tag{3.8}$$

where

$$\underline{\hat{\alpha}}^{1(q_1)2(q_2)\cdots m(q_m)}_{1(s_1)2(s_2)\cdots m(s_m)} = (\alpha_1 + q_1 - s_1, \alpha_2 + q_2 - s_2, \cdots, \alpha_m + q_m - s_m) = \underline{\alpha} + \underline{q} - \underline{s}.$$

Step 2 As each term in the operator $P_p(\underline{\partial})$ is of the form $\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_m}^{\alpha_m}$, with $\alpha_i \in \mathbb{N}_0$, $i = 1, \cdots, m$, and $|\underline{\alpha}| = \sum_{i=1}^{m} \alpha_i = p$, we have explicitly calculated the action of such a term on r^{2l}, $l \in \mathbb{N}_0$, by a double induction argument both on the orders of derivation α_i and on the number of α_i's occurring (i.e., not being zero). The obtained result reads

$$\begin{aligned} &\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_m}^{\alpha_m} r^{2l} \\ &= \sum_{j=0}^{S_{\underline{\alpha}}} \Big(\sum_{|\underline{\beta}|=j} a_{\alpha_1,\beta_1} \cdots a_{\alpha_m,\beta_m} x_1^{\alpha_1 - 2\beta_1} \cdots x_m^{\alpha_m - 2\beta_m} \Big) [2l]_{2p-2j-2}\, r^{2l-2p+2j}\,, \end{aligned} \tag{3.9}$$

where

$$S_{\underline{\alpha}} = \sum_{i=1}^{m} \frac{(\alpha_i)_e}{2}, \quad [2l]_{2p-2j-2} = (2l)(2l-2)\cdots(2l-2p+2j+2),$$

$$a_{\alpha_i,\beta_i} = \begin{cases} \dfrac{1}{2^{\beta_i}} \dfrac{\alpha_i!}{\beta_i!(\alpha_i - 2\beta_i)!}, & \text{if } 0 \leq \beta_j \leq \frac{(\alpha_j)_e}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular note that

$$\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_m}^{\alpha_m} r^{2l} = 0, \quad \text{if } l < p - S_{\underline{\alpha}}. \tag{3.10}$$

Step 3 We now let $P_p(\underline{\partial})$ act on r^{2l} for $p \leq l$. On account of (3.9) and of the proposed form (3.7) of $P_p(\underline{x})$ this yields

$$\sum_{i=1}^{m} e_i \sum_{|\underline{\alpha}|=p} b_{i,\underline{\alpha}} F(\underline{\alpha}) \sum_{j=0}^{S_{\underline{\alpha}}} \Big(\sum_{|\underline{\beta}|=j} a_{\alpha_1,\beta_1}\cdots a_{\alpha_m,\beta_m} x_1^{\alpha_1-2\beta_1}\cdots x_m^{\alpha_m-2\beta_m}\Big)[2l]_{2p-2j-2} r^{2(l-p+j)},$$

which can be rewritten as

$$\begin{aligned} & 2^p \frac{l!}{(l-p)!} r^{2(l-p)} P_p(\underline{x}) \\ & + \sum_{i=1}^{m} e_i \sum_{|\underline{\alpha}|=p} b_{i,\underline{\alpha}} F(\underline{\alpha}) \sum_{j=1}^{S_{\underline{\alpha}}} \Big(\sum_{|\underline{\beta}|=j} a_{\alpha_1,\beta_1}\cdots a_{\alpha_m,\beta_m} x_1^{\alpha_1-2\beta_1}\cdots x_m^{\alpha_m-2\beta_m}\Big)[2l]_{2p-2j-2} r^{2(l-p+j)} \\ \equiv\ & 2^p \frac{l!}{(l-p)!} r^{2(l-p)} P_p(\underline{x}) + S_{p,l}(\underline{x}) \end{aligned} \tag{3.11}$$

by isolating the term for $j = 0$. As each of the terms in $S_{p,l}(\underline{x})$ may be proven to be zero, on account of the conditions (3.8) on the coefficients of $P_p(\underline{x})$, we are lead to the first part of (3.5).

Step 4 Next, we consider the case where $p > l$. First, let $l < p - \frac{p_e}{2}$. Then clearly, for each $\underline{\alpha}$ with $|\underline{\alpha}| = p$ we have $l < p - S_{\underline{\alpha}}$. Invoking (3.10) we thus have that

$$\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_m}^{\alpha_m} r^{2l} = 0, \quad \forall\, \underline{\alpha} = (\alpha_1, \cdots, \alpha_m),\ |\underline{\alpha}| = p,$$

yielding $P_p(\underline{\partial}) r^{2l} = 0$. Next, take $p - \frac{p_e}{2} \leq l < p$. In this case, the arguments of Step 3 may be rephrased quite literally, leading to an analogous result as in (3.11), however without the term for $j = 0$, since j will start from $p - l > 0$. So, also here $P_p(\underline{\partial}) r^{2l} = 0$, implying that the second part of (3.5) holds.

Step 5 Finally, expression (3.6) may be shown by conversion of (3.5) to frequency space and invoking properties (2.1) of the Fourier transform.

The proof is completed.

Returning to (3.3) for a more precise calculation of the residues, we will consider two distinct cases, according to the parity of n.

Case A $n = 2l + 2$, $l \in \mathbb{N}_0$.
In this case we rewrite (3.3) as

$$\mathrm{Res}_{\lambda=-m-p_e-2l-1} \langle T_{\lambda,p}, \phi\rangle = \frac{a_m}{(2l+1)!} \langle \delta_r, \partial_r^{(2l+1)} \Sigma_p^{(0)}[\phi]\rangle = \frac{a_m}{(2l+1)!} \{\partial_r^{2l+1} \Sigma_p^{(0)}[\phi]\}_{r=0},$$

the latter being zero on account of Proposition 3.1. Hence $T_{\lambda,p}$ shows no genuine poles whenever $n = 2l + 2$, or equivalently, $\lambda = -m - p_e - 2l - 1$, $l \in \mathbb{N}_0$.

Thus, the distributions $T_{-m-p_e-2l-1,p}$, $l \in \mathbb{N}_0$ can be defined by means of a limiting process:

$$\langle T_{-m-p_e-2l-1,p}, \phi\rangle = a_m \lim_{\mu\to -2l-2} \langle \mathrm{Fp}\, r_+^{\mu}, \Sigma_p^{(0)}[\phi]\rangle,$$

where the limit at the right-hand side exactly yields the monomial pseudofunction $\mathrm{Fp}\, r_+^{-2l-2}$ (see Remark 3.1).

Case B $n = 2l+1$, $l \in \mathbb{N}_0$.

Substitution of these values of n in (3.3) yields

$$\begin{aligned}\mathrm{Res}_{\lambda=-m-p_e-2l}\langle T_{\lambda,p}, \phi\rangle &= \frac{a_m}{(2l)!}\Big\langle \delta_r, p_r^{(2l)}\Sigma_p^{(0)}[\phi]\Big\rangle \\ &= a_m \frac{1}{(p_e+2l)!}\frac{1}{C(\frac{p_e}{2}+l)}\langle P_p(\underline{x})\Delta^{\frac{p_e}{2}+l}\delta(\underline{x}), \phi\rangle, \qquad (3.12)\end{aligned}$$

the last step holding on account of (3.4). Since, according to (3.6), the expression at the right-hand side of (3.12) equals zero for $p > \frac{p_e}{2} + l$, we conclude that $T_{\lambda,p}$ also has no genuine singularities in the case $\lambda = -m - p_e - 2l$ for $l = 0, 1, 2, \cdots, p - \frac{p_e}{2} - 1$; for this finite set of values, the distribution can be defined similarly as above by a limiting process, now involving the monomial pseudofunction $\mathrm{Fp}\, r_+^{-2l-1}$.

The results obtained are summarized in the following theorem.

Theorem 3.1 *Considered as a function of $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$, the distribution $T_{\lambda,p}$ shows simple poles at $\lambda = -m - 2p - 2l,\ l \in \mathbb{N}_0$, with residue*

$$\mathrm{Res}_{\lambda=-m-2p-2l}\, T_{\lambda,p} = a_m \frac{1}{(2p+2l)!}\frac{1}{C(p+l)} P_p(\underline{x})\Delta^{p+l}\delta(\underline{x}).$$

Remark 3.2 The above considerations lead to the conclusion that multiplication of T_λ with $P_p(\underline{x})$ in (3.2) causes the removal of its singularities $\lambda = -m - 2l$ for $l < p$. Hence, the equality (3.2) may be holomorphically extended to all couples (λ, p) which do not fulfill the relation $\lambda + 2p = -m - 2l,\ l \in \mathbb{N}_0$. This means that, whenever $T_{\lambda,p}$ is well-defined, we may rewrite it as $T_\lambda\, P_p$.

3.3 The distributions $T^*_{\lambda,p}$

In [7] the distributions T^*_λ are defined as normalizations of the distributions T_λ. This is done by removing the singularities of T_λ through the well-known technique of division by a deliberately chosen Γ-function. Here we generalize this normalization procedure to all distributions $T_{\lambda,p}$.

Noting that the function $\Gamma\big(\frac{\lambda+m+2p}{2}\big)$ shows exactly the same simple poles as $T_{\lambda,p}$, with residues

$$\mathrm{Res}_{\lambda=-m-2p-2l}\, \Gamma\Big(\frac{\lambda+m+2p}{2}\Big) = 2\frac{(-1)^l}{l!},$$

we are lead to the following definition of the so-called normalized distributions $T^*_{\lambda,p}$:

$$\begin{cases} T^*_{\lambda,p} = \pi^{\frac{\lambda+m}{2}+p}\dfrac{T_{\lambda,p}}{\Gamma\big(\frac{\lambda+m}{2}+p\big)}, & \lambda \neq -m-2p-2l, \\ T^*_{-m-2p-2l,p} = \dfrac{(-1)^l\, l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l}(p+l)!\,\Gamma\big(\frac{m}{2}+p+l\big)} P_p(\underline{x})\Delta^{p+l}\delta(\underline{x}), & l \in \mathbb{N}_0, \end{cases}$$

where, at the singularities of $T_{\lambda,p}$, the normalized distribution $T^*_{\lambda,p}$ is defined, up to constants, as the quotient of the residues involved.

According to the results of the previous subsection, in this definition, $T_{\lambda,p}$ should be interpreted in terms of the monomial pseudofunction $\mathrm{Fp}\, r_+^{-n}$ whenever $\lambda = -m - p_e - n + 1$, $n \in \mathbb{N}$, but $\lambda \neq -m - 2p - 2l$, $l \in \mathbb{N}_0$. Moreover, one can verify that for $p = 0$ this definition is in accordance with the definition of $T^*_\lambda = T^*_{\lambda,0}$ in [7].

4 The Fourier Spectra of the Distributions $T^*_{\lambda,p}$

For the calculation of the Fourier spectra of the distributions $T^*_{\lambda,p}$, we will start from the classical result (see [20, Theorem IV.4.1]): for those couples $(\lambda, p) \in \mathbb{C} \times \mathbb{N}_0$ for which $\mathrm{Re}\,\lambda$ is restricted to the strip $-m - p < \mathrm{Re}\,\lambda < -p$, the following formula holds

$$\mathcal{F}[T_\lambda P_p(\underline{x})](\underline{y}) = i^{-p}\pi^{-\frac{m}{2}-\lambda-p}\frac{\Gamma\left(\frac{\lambda+m}{2}+p\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}T_{-\lambda-m-2p}P_p(\underline{y})$$

or, following the results of the previous section,

$$\mathcal{F}[T_{\lambda,p}] = i^{-p}\pi^{-\frac{m}{2}-\lambda-p}\frac{\Gamma\left(\frac{\lambda+m}{2}+p\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}T_{-\lambda-m-2p,p}\,. \tag{4.1}$$

In [14, Lemma 2] it is shown that, by means of analytic continuation, the above formula also holds in the larger strip $-m-2p < \mathrm{Re}\,\lambda < 0$. However, both sides of (4.1) being meromorphic functions in the complex variable λ, through analytic continuation that equality is valid in each open connected area containing the strip $-m-p < \mathrm{Re}\,\lambda < -p$, and where the expression on both sides exist. Singularities occur in (4.1) when $\lambda = -m-2p-2l$, $l \in \mathbb{N}_0$, for the distribution at the left-hand side and when $\lambda = 2l$, $l \in \mathbb{N}_0$, for the one at the right-hand side. Naturally, the same singularities are also contained in the involved Γ-functions. Consequently, (4.1) is seen to hold for λ belonging to the set Ω, which is defined as

$$\Omega = \mathbb{C} \setminus (\{-m-2p-2l : l \in \mathbb{N}_0\} \cup \{2l : l \in \mathbb{N}_0\}).$$

This smoothens the path for the following fundamental result.

Theorem 4.1 *The Fourier transform of the distributions $T^*_{\lambda,p}$ is given by*

$$\mathcal{F}[T^*_{\lambda,p}] = i^{-p}T^*_{-\lambda-m-2p,p}, \quad \forall\,(\lambda,p) \in \mathbb{C}\times\mathbb{N}_0.$$

Proof Three cases have to be distinguished.

(i) $\lambda \in \Omega$

On account of (4.1) we indeed have

$$\mathcal{F}[T^*_{\lambda,p}] = \frac{\pi^{\frac{\lambda+m}{2}+p}}{\Gamma\left(\frac{\lambda+m}{2}+p\right)}\mathcal{F}[T_{\lambda,p}] = i^{-p}\frac{\pi^{-\frac{\lambda}{2}}}{\Gamma\left(-\frac{\lambda}{2}\right)}T_{-\lambda-m-2p,p} = i^{-p}T^*_{-\lambda-m-2p,p}.$$

(ii) $\lambda = -m-2p-2l$, $l \in \mathbb{N}_0$

Exploiting the definition of $T^*_{-\lambda-m-2p,p}$ and the properties of the Fourier transform we arrive at

$$\begin{aligned}\mathcal{F}[T^*_{-m-2p-2l,p}] &= \frac{(-1)^l l!\pi^{\frac{m}{2}-l}}{2^{2p+2l}(p+l)!\Gamma\left(\frac{m}{2}+p+l\right)}\mathcal{F}[P_p(\underline{x})\Delta^{p+l}\delta(\underline{x})]\\ &= \frac{(-1)^l l!\pi^{\frac{m}{2}-l}}{2^{2p+2l}(p+l)!\Gamma\left(\frac{m}{2}+p+l\right)}(-1)^p(2\pi i)^{2l+p}P_p(\underline{\partial})\rho^{2p+2l}.\end{aligned}$$

As $p \le p+l$, Proposition 3.2 leads to the desired result, i.e.

$$\mathcal{F}[T^*_{-m-2p-2l,p}] = i^{-p}\frac{\pi^{\frac{m}{2}+p+l}}{\Gamma\left(\frac{m}{2}+p+l\right)}\rho^{2l}P_p(\underline{y}) = i^{-p}T^*_{2l,p}. \tag{4.2}$$

In the above, we have used the notation $\rho = |\underline{y}|$.

(iii) $\lambda = 2l$, $l \in \mathbb{N}_0$

This case directly follows by the action of the Fourier operator on (4.2):

$$\mathcal{F}[T^*_{2l,p}](\underline{y}) = i^p\, T^*_{-m-2p-2l,p}(-\underline{y}) = i^{-p} T^*_{-m-2p-2l,p}(\underline{y}).$$

The proof is completed.

5 The Fourier Spectra of the Distributions $U^*_{\lambda,p}$ and $V^*_{\lambda,p}$

Along with the family of distributions $T_{\lambda,p}$ also two other families $U_{\lambda,p}$ and $V_{\lambda,p}$ have been defined, in which the higher dimensional "signum distribution" $\underline{\omega}$ plays an important rôle (see e.g. [3, 4]). While recalling their respective definitions we directly introduce the corresponding normalizations following the procedure used for the $T_{\lambda,p}$ in Section 3. To conclude the paper, the Fourier spectra of those normalizations $U^*_{\lambda,p}$ and $V^*_{\lambda,p}$ are calculated.

For a scalar valued testing function $\phi(\underline{x})$ in $\mathbb{R}^m$, and a vector valued, monogenic, homogeneous polynomial $P_p(\underline{x})$ of degree $p \neq 0$, the generalized spherical means $\Sigma^{(1)}_p[\phi]$ and $\Sigma^{(3)}_p[\phi]$ are defined as follows (see also [21]):

(i) $\Sigma^{(1)}_{2k}[\phi] = \Sigma^{(0)}[\underline{\omega} P_{2k}(\underline{\omega})\phi(\underline{x})] = \dfrac{1}{a_m}\displaystyle\int_{S^{m-1}} \underline{\omega} P_{2k}(\underline{\omega})\phi(\underline{x})dS(\underline{\omega}),$

(ii) $\Sigma^{(1)}_{2k+1}[\phi] = \Sigma^{(0)}[r\underline{\omega} P_{2k+1}(\underline{\omega})\phi(\underline{x})] = \dfrac{r}{a_m}\displaystyle\int_{S^{m-1}} \underline{\omega} P_{2k+1}(\underline{\omega})\phi(\underline{x})dS(\underline{\omega}),$

(iii) $\Sigma^{(3)}_{2k}[\phi] = \Sigma^{(0)}[P_{2k}(\underline{\omega})\underline{\omega}\phi(\underline{x})] = \dfrac{1}{a_m}\displaystyle\int_{S^{m-1}} P_{2k}(\underline{\omega})\underline{\omega}\phi(\underline{x})dS(\underline{\omega}),$

(iv) $\Sigma^{(3)}_{2k+1}[\phi] = \Sigma^{(0)}[rP_{2k+1}(\underline{\omega})\underline{\omega}\phi(\underline{x})] = \dfrac{r}{a_m}\displaystyle\int_{S^{m-1}} P_{2k+1}(\underline{\omega})\underline{\omega}\phi(\underline{x})dS(\underline{\omega}).$

Note that for $p = 0$ and $P_0(\underline{x}) = 1$ we have that $\Sigma^{(3)}_0[\phi] = \Sigma^{(1)}_0[\phi]$.

The definition of the distributions $U_{\lambda,p}$ and $V_{\lambda,p}$ then is similar to the one of the distributions $T_{\lambda,p}$ introduced above, however involving the newly introduced spherical means:

(i) $\langle U_{\lambda,p}, \phi\rangle = a_m \langle \mathrm{Fp}\, r_+^{\mu+p_e}, \Sigma^{(1)}_p[\phi]\rangle,$

(ii) $\langle V_{\lambda,p}, \phi\rangle = a_m \langle \mathrm{Fp}\, r_+^{\mu+p_e}, \Sigma^{(3)}_p[\phi]\rangle.$

Clearly, also these distributions show an infinite number of singularities, in view of which we will introduce their normalizations at once. The modus operandi from Subsection 3.3 is adopted, leading to the following definitions, with $l \in \mathbb{N}_0$:

$$\begin{cases} U^*_{\lambda,p} = \pi^{\frac{\lambda+m+1}{2}+p} \dfrac{U_{\lambda,p}}{\Gamma\left(\frac{\lambda+m+1}{2}+p\right)}, \quad \lambda \neq -m-2p-2l-1, \\ U^*_{-m-2p-2l-1,p} = \dfrac{(-1)^{p+1}\, l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l+1}(p+l)!\,\Gamma\left(\frac{m}{2}+p+l+1\right)} (\underline{\partial}^{2p+2l+1}\delta(\underline{x}))P_p(\underline{x}), \end{cases}$$

$$\begin{cases} V^*_{\lambda,p} = \pi^{\frac{\lambda+m+1}{2}+p} \dfrac{V_{\lambda,p}}{\Gamma\left(\frac{\lambda+m+1}{2}+p\right)}, \quad \lambda \neq -m-2p-2l-1, \\ V^*_{-m-2p-2l-1,p} = \dfrac{(-1)^{p+1}\, l!\, \pi^{\frac{m}{2}-l}}{2^{2p+2l+1}(p+l)!\,\Gamma\left(\frac{m}{2}+p+l+1\right)} P_p(\underline{x})(\underline{\partial}^{2p+2l+1}\delta(\underline{x})). \end{cases}$$

In order to calculate the Fourier spectra of the normalized distributions $U^*_{\lambda,p}$ and $V^*_{\lambda,p}$, note that they are interrelated with the "mother" family $T^*_{\lambda,p}$ by the multiplication with the

C_∞-function $\underline{x}$ as well as by the action of the Dirac operator $\underline{\partial}$ (see [1, Propositions 5.2 and 5.3]):

$$\underline{x}T^*_{\lambda,p} = \frac{\lambda+m+2p}{2\pi}U^*_{\lambda+1,p}\,, \quad \underline{\partial}T^*_{\lambda,p} = \lambda U^*_{\lambda-1,p}\,, \quad T^*_{\lambda,p}\underline{x} = \frac{\lambda+m+2p}{2\pi}V^*_{\lambda+1,p}\,, \quad T^*_{\lambda,p}\underline{\partial} = \lambda V^*_{\lambda-1,p}.$$

Hence it suffices to combine Theorem 4.1 with the properties (2.1) of the Fourier transform, to arrive at the following elegant results:

Proposition 5.1 *The Fourier transform of the distributions $U^*_{\lambda,p}$, respectively $V^*_{\lambda,p}$, is given by*

$$\mathcal{F}[U^*_{\lambda,p}] = i^{-p-1}\, U^*_{-\lambda-m-2p,p}, \quad \mathcal{F}[V^*_{\lambda,p}] = i^{-p-1}\, V^*_{-\lambda-m-2p,p}.$$

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