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On the Kähler-Ricci Flow on Projective Manifolds of General Type

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(*Dedicated to the memory of Shiing-Shen Chern*)

Abstract This note concerns the global existence and convergence of the solution for Kähler-Ricci flow equation when the canonical class, K_X, is numerically effective and big. We clarify some known results regarding this flow on projective manifolds of general type and also show some new observations and refined results.

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0 Introduction

Let X be an n-dimensional ($n \geq 2$) projective manifold of general type, i.e., its canonical divisor K_X is big. This is our main interest in this note.

We denote a Kähler metric by its Kähler form ω, in local complex coordinates $z^1, \cdots, z^n$,

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where we use the standard convention for summation and $(g_{i\bar{j}})$ is the positive Hermitian matrix valued function given by $g_{i\bar{j}} = g\big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\big)$.

Consider the following Kähler-Ricci flow

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t) - \widetilde{\omega}_t, \quad \widetilde{\omega}_0 = \omega_0, \tag{0.1}$$

where ω_0 is any given Kähler metric and $\mathrm{Ric}(\omega)$ denotes the Ricci form of ω, i.e., in the complex coordinates above, $\mathrm{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j$ where $(R_{i\bar{j}})$ is the Ricci tensor of ω.

It is natural to study properties of solutions for this flow when K_X is big and show how they are related to geometry of the underlying manifold X.

The main purpose of this note is to examine the global existence and convergence for the solution of this evolution equation when K_X is also numerically effective. On one hand, we clarify the situation regarding the Kähler-Ricci flow on projective manifolds of general type; on the other hand, we show some new observations and results.

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Theorem 0.1 *Let X be a projective manifold with its canonical divisor K_X big and numerically effective (i.e., nef.). Then for any initial Kähler metric ω_0, the flow* (0.1) *has a global solution $\widetilde{\omega}_t$ for all time $t \in [0, \infty)$ satisfying*

(1) *$\widetilde{\omega}_t$ converges to a positive current $\widetilde{\omega}_\infty$ representing $-c_1(X)$ as t tends to infinity;*

(2) *this limiting current is actually a smooth Kähler-Einstein metric outside a subvariety $S \subset X$ and $\widetilde{\omega}_t|_{X \setminus S}$ converges to $\widetilde{\omega}_\infty|_{X \setminus S}$ locally in C^∞- topology;*

(3) *in any local complex coordinate chart, $\widetilde{\omega}_\infty = \sqrt{-1}\partial\bar{\partial}\rho$ for some locally bounded plurisubharmonic function ρ;* [1]

(4) *$\widetilde{\omega}_\infty$ is canonical, that is, independent of the choice of the initial metric ω_0.*

Remark 0.2 We further expect that the local potential in (3) of the above theorem is continuous in general. It is very much likely that the continuity can be proved by a more delicate extension of Kolodziej's results in [13]. We hope to address this question in a forthcoming paper.

This theorem was proved in [2] in the case when K_X is ample and the initial Kähler class coincides with the canonical class itself. When X is given as above and the initial metric ω_0 is sufficiently positive, H. Tsuji proved in [17] the first two statements in the above theorem, that is, (0.1) has a global solution $\widetilde{\omega}_t$ and $\widetilde{\omega}_t$ converges to a positive current which is actually a smooth Kähler-Einstein metric outside a subvariety as t tends to infinity.[2] But we noticed that his basic arguments can still go through even after the extra assumption on ω_0 is removed. Our new observations are that the limiting current is in fact canonical and has bounded local potentials. This last property was proved by using results in the second named author's thesis which extends the potential theory developed by Bedford-Taylor [1] and Kolodziej [13] to singular varieties. Theorem 0.1 also gives a partial answer to the following conjecture (cf. [16]): *For any initial metric ω_0, the flow* (0.1) *has a* (*possibly singular*) *solution $\widetilde{\omega}_t$ which converges to a* (*possibly singular*) *metric in a suitable sense as $t \to \infty$. Moreover, this limiting metric may be singular but should be independent of the choice of the initial metric.* In fact, it was further expected that *all singularities of this limiting metric are of rational type.*

When X is a minimal complex surface of general type, by classification theory of complex surfaces, K_X is numerically effective and its canonical model $\overline{X}$ is a Kähler orbifold. It is known (cf. [19, 12]) that $\overline{X}$ admits a Kähler-Einstein orbifold metric which pulls back to a current with locally continuous potential and representing $-c_1(X)$. Then by an easy uniqueness result, one can deduce

Corollary 0.3 *If X is a minimal complex surface of general type, then the global solution of the Kähler-Ricci flow converges to a positive current $\widetilde{\omega}_\infty$ which descends to the Kähler-Einstein orbifold metric on its canonical model. In particular, $\widetilde{\omega}_\infty$ is smooth outside finitely many rational curves and has local continuous potential.*

In order to prove Theorem 0.1, we first reduce (0.1) in a standard way to a scalar parabolic equation.

Formally taking cohomology classes on both sides of (0.1), we can easily show that the cohomology class of $\widetilde{\omega}_t$ is equal to that of

$$\omega_t := -\mathrm{Ric}(\Omega) + e^{-t}(\omega_0 + \mathrm{Ric}(\Omega)),$$

[1] In this sense, we may refer $\widetilde{\omega}_\infty$ as a positive current with locally bounded potential. The complete proof of this part will appear in the thesis of the second author [21].

[2] The proof for convergence in [17] contains some unjustified statements. A uniqueness result was also claimed there.

where Ω is any fixed volume form on X and in local coordinates $z^1, \cdots, z^n$,

$$\mathrm{Ric}(\Omega) := -\sqrt{-1}\partial\bar{\partial}\log\Big(\frac{\Omega}{(\sqrt{-1})^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n}\Big),$$

which is formally denoted by $-\sqrt{-1}\partial\bar{\partial}\log\Omega$. This form actually represents the first Chern class of X.

Write $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ for a smooth function u on X and t. We have

$$\frac{\partial\widetilde{\omega}_t}{\partial t} = \frac{\partial\omega_t}{\partial t} + \sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t} = e^{-t}(-\mathrm{Ric}(\Omega) - \omega_0) + \sqrt{-1}\partial\bar{\partial}\frac{\partial u}{\partial t}.$$

Plugging this into (0.1), we can derive a scalar equation

$$\frac{\partial u}{\partial t} = \log\frac{\widetilde{\omega}_t^n}{\Omega} - u, \qquad u(0,\cdot) = 0. \tag{0.2}$$

Clearly, given a solution of this equation, we can easily construct a solution of (0.1) by reversing the steps above. Since this equation is parabolic, one can immediately get the short time existence and uniqueness of its solution. It also follows the equivalence of (0.1) and (0.2).[3] Theorem 0.1 will be proved by studying (0.2).

The organization of this note is as follows. In Section 1, we show that (0.2) has a solution for a maximal time interval $[0, T)$, where T is determined by K_X and the initial Kähler metric. In Section 2, we will study the convergence of the solution as $t \to T$. In Section 3, when K_X is numerically effective, we will show that the limiting current is independent of the choice of the initial metric. In Section 4, we will show that the limiting current has locally bounded potential. Some applications and generalizations will be discussed in the last sections.

Cascini and La Nave informed us that they independently proved Proposition 1.1 and a weaker version of Theorem 0.1 in [3].

1 Existence of Maximal Solutions

In this section, we will prove that (0.2) has a solution in the maximal time interval $[0, T)$, where $T := \sup\{t \,|\, (e^{-t} - 1)\, c_1(X) + e^{-t}[\omega_0] \text{ is ample}\}$. If K_X is numerically effective, then $T = \infty$.

For any small $\epsilon > 0$, we can choose $T_\epsilon > 0$ such that $T_\epsilon + \epsilon < T$ and a real closed $(1,1)$ form ψ_ϵ such that $[\psi_\epsilon] = K_X$ and $\psi_\epsilon + a_\epsilon \cdot \omega_0 > 0$, where $a_\epsilon = \frac{1}{e^{T_\epsilon+\epsilon}-1}$.[4] Choose a smooth volume form Ω_ϵ such that $\mathrm{Ric}(\Omega_\epsilon) = -\psi_\epsilon$. This Ω_ϵ is unique up to multiplication by a positive constant.

Set $\omega_t = \psi_\epsilon + e^{-t}(\omega_0 - \psi_\epsilon)$ and $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$. Then u can be chosen to satisfy (0.2) with Ω replaced by Ω_ϵ:

$$\frac{\partial u}{\partial t} = \log\frac{\widetilde{\omega}_t^n}{\Omega_\epsilon} - u, \qquad u(0,\cdot) = 0. \tag{1.1}$$

We will first show the solution for (1.1) exists for $t \in [0, T_\epsilon]$.

First we observe that ω_t is a Kähler metric for $t \in [0, T_\epsilon]$ with uniformly bounded geometry.

The parabolicity of this equation assures the local existence and uniqueness of solutions. In order to prove the existence of solutions for $t \in [0, T_\epsilon]$, it only remains to get uniform estimates of u for $t \in [0, T_\epsilon]$.

[3] Clearly, all the above discussions still hold if X is only a closed Kähler manifold.

[4] The notations, ≥ 0 and > 0, indicate the semi-positivity and positivity of forms in this note.

Applying maximum principle to (1.1), we can easily have $|u| \leq C_\epsilon$.[5]

Taking derivative of (1.1) with respect to t, we get

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t}\Big) = \Delta_{\widetilde{\omega}_t}\Big(\frac{\partial u}{\partial t}\Big) - e^{-t}\langle \widetilde{\omega}_t, \omega_0 - \psi_\epsilon\rangle - \frac{\partial u}{\partial t}.$$

Here Δ_ω denotes the Laplacian of a Kähler metric ω and $\langle \omega, F\rangle$ means the trace of F with respect to ω, where F is a real $(1,1)$-form.

The equation above can be rewritten as follows:

$$\frac{\partial}{\partial t}\Big(e^t\frac{\partial u}{\partial t}\Big) = \Delta_{\widetilde{\omega}_t}\Big(e^t\frac{\partial u}{\partial t}\Big) - \langle \widetilde{\omega}_t, \omega_0 - \psi_\epsilon\rangle, \tag{1.2}$$

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t} + u\Big) = \Delta_{\widetilde{\omega}_t}\Big(\frac{\partial u}{\partial t} + u\Big) - n + \langle \widetilde{\omega}_t, \psi_\epsilon\rangle. \tag{1.3}$$

The difference of these two gives

$$\frac{\partial}{\partial t}\Big(e^t\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - u\Big) = \Delta_{\widetilde{\omega}_t}\Big(e^t\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - u\Big) + n - \langle \widetilde{\omega}_t, \omega_0\rangle. \tag{1.4}$$

There is also a slightly modified difference version $(1 + a_\epsilon)\cdot(1.3) - a_\epsilon \cdot (1.2)$:

$$\begin{aligned}&\frac{\partial}{\partial t}\Big((1+a_\epsilon)\Big(\frac{\partial u}{\partial t} + u\Big) - \epsilon e^t\frac{\partial u}{\partial t}\Big)\\ &= \Delta_{\widetilde{\omega}_t}\Big((1+a_\epsilon)\Big(\frac{\partial u}{\partial t} + u\Big) - \epsilon e^t\frac{\partial u}{\partial t}\Big) - (1+a_\epsilon)n + \langle \widetilde{\omega}_t, \psi_\epsilon + a_\epsilon\omega_0\rangle.\end{aligned} \tag{1.5}$$

From (1.4), noticing $\langle \widetilde{\omega}_t, \omega_0\rangle > 0$ and using maximum principle, we see that the maximum of $e^t\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - u - nt$ is non-increasing, so we have

$$e^t\frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - u - nt \leq 0.$$

Now we combine it with local existence for small time and the uniform upper bound for u to conclude that $\frac{\partial u}{\partial t} < C_\epsilon$.

From (1.5), noticing $\langle \widetilde{\omega}_t, \psi_\epsilon + a_\epsilon\omega_0\rangle > 0$, by maximum principle, we see that minimum of $(1+a_\epsilon)\big(\frac{\partial u}{\partial t} + u\big) - a_\epsilon e^t\frac{\partial u}{\partial t} + (1+a_\epsilon)nt$ is non-decreasing. It follows

$$(1+a_\epsilon)\Big(\frac{\partial u}{\partial t} + u\Big) - a_\epsilon e^t\frac{\partial u}{\partial t} + (1+a_\epsilon)nt \geq \min_{t=0}\frac{\partial u}{\partial t} = -C_\epsilon,$$

which is $(1 + a_\epsilon - a_\epsilon e^t)\frac{\partial u}{\partial t} \geq -C_\epsilon - (1+a_\epsilon)u - (1+a_\epsilon)nt > -C_\epsilon^{'}$.

As $1 + a_\epsilon - a_\epsilon e^t \geq 1 + a_\epsilon - a_\epsilon e^{T_\epsilon} > 0$ for $t \in [0, T_\epsilon]$, we can conclude that

$$\frac{\partial u}{\partial t} > -C_\epsilon.$$

Until now we have got all the C^0-estimates needed. The existence of solution for (1.1) for $t \in [0, T_\epsilon]$ follows from the standard argument using second and higher order estimates. Hence we get the existence of solution in $[0, T_\epsilon]$.

[5] The constant C can be different at places with possibly some lower indices (as ϵ here) indicating the dependence of it on other constants. At time when confusion is likely to occur, we will use different C's to clarify. In fact, here the upper bound of u can be uniform for all time $t \in [0, T)$.

The desired existence of the solution for (0.2) is easy to see by considering the relations between all the equations as (1.1) for different ϵ's as follows.[6]

Consider (1.1) for some $\delta > 0$. Assume $\psi_\delta = \psi_\epsilon + \sqrt{-1}\partial\bar\partial f$ for some smooth real function f over X. Since $-\mathrm{Ric}(\Omega_\epsilon) = \psi_\epsilon$, we have $-\mathrm{Ric}(e^f\Omega_\epsilon) = \psi_\delta$. Thus one can take $\Omega_\delta = e^f\Omega_\epsilon$. Now the new "$\omega_t$" is

$$\eta_t = \psi_\delta + e^{-t}(\omega_0 - \psi_\delta) = \omega_t + (1 - e^{-t})\sqrt{-1}\partial\bar\partial f.$$

The equation (1.1) for δ is

$$\frac{\partial v}{\partial t} = \log\frac{(\eta_t + \sqrt{-1}\partial\bar\partial v)^n}{e^f\Omega_\epsilon} - v, \quad v(0,\cdot) = 0.$$

Define $\tilde u = v + (1 - e^{-t})f$. We have $\tilde u(0,\cdot) = v(0,\cdot) = 0$ and

$$\begin{aligned}\frac{\partial \tilde u}{\partial t} &= \frac{\partial v}{\partial t} + e^{-t}f = \log\frac{(\eta_t + \sqrt{-1}\partial\bar\partial v)^n}{e^f\Omega_\epsilon} - v + e^{-t}f \\ &= \log\frac{(\omega_t + \sqrt{-1}\partial\bar\partial \tilde u)^n}{\Omega_\epsilon} - v - f + e^{-t}f = \log\frac{(\omega_t + \sqrt{-1}\partial\bar\partial \tilde u)^n}{\Omega_\epsilon} - \tilde u. \end{aligned} \tag{1.6}$$

From uniqueness of the solution for (1.1), $\tilde u$ is just the original solution u.

This actually gives the explicit relation between solutions of (1.1) associated to different ϵ's and would allow us to glue together all these solutions for (1.1) associated to different ϵ's to get a global solution of (0.2) until the time T. In fact, (0.2) can be solved in the maximal time interval $[0, T)$ no matter which Ω is chosen. Notice that Ω is also involved in the definition of ω_t there without affecting the cohomological information and so the definition of T. We can summarize the above discussion in the following.[7]

Proposition 1.1 *Let X be a closed Kähler manifold. Then the Kähler-Ricci flow* (0.1) (*or* (0.2)) *with initial metric ω_0 has a unique smooth solution on $[0, T)$, where T is the maximum of t such that $(1 - e^{-t})K_X + e^{-t}[\omega_0]$ is a Kähler class. In particular, if K_X is numerically effective, the solution exists for all the time.*

There is another observation which will be useful later. First recall the following equation

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t}\Big) = \Delta_{\widetilde\omega_t}\Big(\frac{\partial u}{\partial t}\Big) - e^{-t}\langle\widetilde\omega_t, \omega_0 - \omega_\infty\rangle - \frac{\partial u}{\partial t}.$$

Taking another t-derivative on both sides, we get

$$\frac{\partial}{\partial t}\Big(\frac{\partial^2 u}{\partial t^2}\Big) = \Delta_{\widetilde\omega_t}\Big(\frac{\partial^2 u}{\partial t^2}\Big) + e^{-t}\langle\widetilde\omega_t, \omega_0 - \omega_\infty\rangle - \Big(\frac{\partial\widetilde\omega_t}{\partial t}, \frac{\partial\widetilde\omega_t}{\partial t}\Big)_{\widetilde\omega_t} - \frac{\partial^2 u}{\partial t^2}.$$

Sum up these two equation to get

$$\frac{\partial}{\partial t}\Big(\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t} + u\Big)\Big) \le \Delta_{\widetilde\omega_t}\Big(\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t} + u\Big)\Big) - \frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t} + u\Big),$$

which is just: $\frac{\partial}{\partial t}\big(e^t\frac{\partial}{\partial t}\big(\frac{\partial u}{\partial t} + u\big)\big) \le \Delta_{\widetilde\omega_t}\big(e^t\frac{\partial}{\partial t}\big(\frac{\partial u}{\partial t} + u\big)\big)$. Maximum principle then gives

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t} + u\Big) \le Ce^{-t},$$

which tells the essential decrease of $\frac{\partial u}{\partial t} + u$ and also of the volume form of $\widetilde\omega_t$.

[6]The equivalence between potential flow and metric flow gives a quicker proof.

[7]This result below gives an affirmative answer to one of the problems listed in [6].

2 Convergence Result

In this section we discuss the convergence of the Kähler metrics along the flow. We will still work on the level of potential and adopt the notations in the last section. Assume that X is a projective manifold with K_X being big. It is known from the previous section that the solution exists in the maximal time interval $[0,T)$. Because the limiting class $[\omega_T]$ may not be positive in general, we can not expect that the limit is a smooth metric. Actually, the idea of proving the convergence in such a situation was already used by H. Tsuji in [17]. The idea is to explore the bigness of K_X to get local estimates for the flow.

Recall the equation (0.2):

$$\frac{\partial u}{\partial t} = \log\frac{\widetilde{\omega}_t^n}{\Omega} - u, \quad u(0,\cdot) = 0.$$

We have shown in Section 1 by using maximum principle that u itself is uniformly bounded from above and

$$(e^t - 1)\frac{\partial u}{\partial t} \le u + nt.$$

It follows that $\frac{\partial u}{\partial t}$ is uniformly bounded from above by Cte^{-t} for some uniform constant (after some small time).

Using (1.2) and (1.3), we can get

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t}\Big) = \Delta_{\widetilde{\omega}_t}\Big(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t}\Big) - n + \langle\widetilde{\omega}_t, \omega_T\rangle, \tag{2.1}$$

where $\omega_T = \omega_\infty + e^{-T}(\omega_0 - \omega_\infty)$ ($T = \infty$ is allowed). Since (0.2) implies

$$\frac{\partial u}{\partial t} + u = \log\frac{\widetilde{\omega}_t^n}{\Omega},$$

the above equation determines how the volume form changes along the flow.

Now we want to use the bigness of K_X as Tsuji did in [17]. The following lemma can be found in [8] and the proof is essentially contained in [10].

Lemma 2.1 *Let L be a divisor in a projective manifold X. If L is nef. and big, then there is an effective divisor E and a number $a > 0$ such that $L - \epsilon E$ is Kähler for any $\epsilon \in (0, a)$.*

The proof essentially makes use of the openness of the big cone for the projective manifold X which clearly contains the positive cone and the fact that L should be in the closure of positive cone. In fact one can choose E to be big. Actually, if L is not nef., one can still use the openness of the cone for big divisors to get a similar result, however, the constant ϵ may not be as close to 0 as one wants. This result will be applied later and is called Kodaira's Lemma as in [18].

Lemma 2.2 *Let L be a divisor in a projective manifold X. If L is big, then there is an effective divisor E such that $L - \epsilon E$ is Kähler for $\epsilon \in (a, b)$ where $0 \le a < b < \infty$.*

In our situation, the cohomology $[\omega_T]$ may not be rational. However, one can check that the arguments for proving the above lemmae still work, more precisely, there is a divisor E and a Hermitian metric $h_{E,\epsilon}$ such that $\omega_T + \epsilon\sqrt{-1}\partial\bar{\partial}\log h_{E,\epsilon} > 0$ for $\epsilon \in (0, a)$. Let σ be a defining holomorphic section for E. Then we have

$$\omega_T + \epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 > 0,$$

where $|\cdot|_\epsilon$ denotes the norm induced by $h_{E,\epsilon}$.[8] Notice that $E=\{\sigma=0\}$ and the function $\log|\sigma|^2$ is only well-defined and smooth outside $E\subset X$.

We can reformulate (2.1) on $X\setminus\{\sigma=0\}$ as

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(\log\Big(\frac{\widetilde{\omega}_t^n}{|\sigma|^{2\epsilon}\Omega}\Big)-e^{t-T}\frac{\partial u}{\partial t}\Big)\\ &=\Delta_{\widetilde{\omega}_t}\Big(\log\Big(\frac{\widetilde{\omega}_t^n}{|\sigma|^{2\epsilon}\Omega}\Big)-e^{t-T}\frac{\partial u}{\partial t}\Big)-n+\langle\widetilde{\omega}_t,\omega_T+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2\rangle. \qquad (2.2)\end{aligned}$$

For any $t<T$, $\log\big(\frac{\widetilde{\omega}_t^n}{|\sigma|^{2\epsilon}\Omega}\big)$ blows up to $+\infty$ along $\{\sigma=0\}$. Hence, the minimum of $\log\big(\frac{\widetilde{\omega}_t^n}{|\sigma|^{2\epsilon}\Omega}\big)-e^{t-T}\frac{\partial u}{\partial t}$ is attained inside $X\backslash E$. At such a minimum point which might have time 0, we have

$$\langle\widetilde{\omega}_t,\omega_T+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2\rangle\le C_\epsilon.$$

It follows that $(\omega_T+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2)^n\le C_\epsilon\widetilde{\omega}_t^n$. Hence, we have

$$\log\Big(\frac{\widetilde{\omega}_t^n}{|\sigma|^{2\epsilon}\Omega}\Big)-e^{t-T}\frac{\partial u}{\partial t}\ge\log\Big(\frac{(\omega_T+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2)^n}{|\sigma|^{2\epsilon}\Omega}\Big)-C>-C_\epsilon.$$

Here we have used that facts that $\frac{(\omega_T+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2)^n}{|\sigma|^{2\epsilon}\Omega}$ is uniformly bounded from below over $X\backslash E$ and $e^{t-T}\frac{\partial u}{\partial t}$ is bounded from above by C. Therefore, on $X\backslash E$, we have

$$\log\Big(\frac{\widetilde{\omega}_t^n}{|\sigma|^{2\epsilon}\Omega}\Big)-e^{t-T}\frac{\partial u}{\partial t}>-C_\epsilon.$$

This implies $(1-e^{t-T})\frac{\partial u}{\partial t}+u>-C_\epsilon+\epsilon\log|\sigma|^2$. Since both u and $\frac{\partial u}{\partial t}$ are bounded from above, we deduce from the above

$$u>-C_\epsilon+\epsilon\log|\sigma|^2.$$

Moreover, if $T=\infty$, then we also have $\frac{\partial u}{\partial t}>-C_\epsilon+\epsilon\log|\sigma|^2$.

In order to get a lower bound for $\frac{\partial u}{\partial t}$ when $T<\infty$, we apply Lemma 2.2 to $K_X=[\omega_\infty]$. It is easy to see that considering the equation (1.3) and by the same arguments using maximum principle as above, we can have a similar lower bound for $\frac{\partial u}{\partial t}$ with a constant ϵ which may not be as close to 0 as we want. Also for the choice of the divisor E, it is more restrictive for $[\omega_\infty]$ than for $[\omega_T]$ when $T<\infty$, which is very clear from the geometry of the cones and the positions of $[\omega_T]$ and $K_X=[\omega_\infty]$.

Anyway, this ends our searching for C^0-estimates, the lower bound of which are locally uniform out of $\{\sigma=0\}$.

Now we present a modified second order estimate following arguments in [19] and [17]. For any $\epsilon\in(0,a)$ small enough, we set

$$\omega_{t,\epsilon}=\omega_\infty+\epsilon\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2+e^{-t}(\omega_0-\omega_\infty).$$

Then for any $t\in[0,T]$, $\omega_{t,\epsilon}$ is a smooth Kähler metric, in particular, its curvature is uniformly bounded by a constant which may depend on ϵ. Also we have $\widetilde{\omega}_t=\omega_{t,\epsilon}+\sqrt{-1}\partial\bar{\partial}(u-\epsilon\log|\sigma|^2)$. Notice that the function $u-\epsilon\log|\sigma|^2$ is defined only outside E.

[8] For simplicity, if there is no possible confusion, we will drop the subscripts E and ϵ in the norm later. The dependence will not affect the estimates below since we are going to work with at most two of them simultaneously and the difference is just a smooth nowhere 0 function.

On $X \setminus \{\sigma = 0\}$, (0.2) can be rewritten as

$$(\omega_{t,\epsilon} + \sqrt{-1}\partial\bar{\partial}(u - \epsilon\log|\sigma|^2))^n = e^{\frac{\partial u}{\partial t} + u + \log\frac{{\omega_0}^n}{{\omega_{t,\epsilon}}^n}} {\omega_{t,\epsilon}}^n.$$

Using the bounds on $\frac{\partial u}{\partial t}$ and u and the curvature of $\omega_{t,\epsilon}$, one can get as Tsuji did in [17] by using Yau's computation in [19] that[9]

$$\begin{aligned} & e^{C_\epsilon(u-\epsilon\log|\sigma|^2)}\Big(\Delta_{\widetilde{\omega}_t} - \frac{\partial}{\partial t}\Big)(e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle) \\ & > -C_\epsilon + \Big(C_\epsilon\frac{\partial u}{\partial t} - C_\epsilon\Big)\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle + C_\epsilon\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle^{\frac{n}{n-1}} \\ & > -C_\epsilon + (C_\epsilon\log|\sigma|^2 - C_\epsilon)\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle + C_\epsilon\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle^{\frac{n}{n-1}}. \end{aligned} \tag{2.3}$$

Unfortunately, the coefficients in the last inequality are not bounded, so one has to take some extra care of using maximum principle.

The maximum of $e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle$ must be attained inside $X \setminus \{\sigma = 0\}$. At such a maximum point, we have

$$\begin{aligned} 0 & > -C_\epsilon + (C_\epsilon\log|\sigma|^2 - C_\epsilon)\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle + C_\epsilon\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle^{\frac{n}{n-1}} \\ & = -C_\epsilon + C_\epsilon\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle(\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle^{\frac{1}{n-1}} + C_\epsilon\log|\sigma|^2 - C_\epsilon). \end{aligned}$$

We can derive from this that $\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle \le (C_\epsilon - C_\epsilon\log|\sigma|^2)^{n-1}$ as $|\sigma|$ is bounded. It follows that at a maximum,

$$e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle \le (C_\epsilon - C_\epsilon\log|\sigma|^2)^{n-1}e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}.$$

Recall that for any $\delta \in (0, a)$, we can choose C_δ such that $u > -C_\delta + \delta\log|\sigma|^2$. Choose $\delta = \frac{\epsilon}{2}$, then we have

$$u - \epsilon\log|\sigma|^2 > -C'_\epsilon - \frac{\epsilon}{2}\log|\sigma|^2.$$

Hence, at the maximum point considered above,

$$e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}\langle\omega_{t,\epsilon}, \widetilde{\omega}_t\rangle \le (C_\epsilon - C_\epsilon\log|\sigma|^2)^{n-1}|\sigma|^{\frac{\epsilon C_\epsilon}{2}} \le C''_\epsilon.$$

Then we get the second order estimate $e^{-C_\epsilon(u-\epsilon\log|\sigma|^2)}\langle\omega_0, \widetilde{\omega}_t\rangle \le C''_\epsilon$, that is,

$$\langle\omega_0, \widetilde{\omega}_t\rangle \le C''_\epsilon|\sigma|^{-2\epsilon C_\epsilon}. \tag{2.4}$$

Combining this with the known volume estimate

$$\widetilde{\omega}_t^n > C_\epsilon|\sigma|^{2\epsilon}{\omega_0}^n,$$

we have a uniform bound on $\widetilde{\omega}_t$ (in terms of a fixed metric) in any given compact subset of $X \setminus \{\sigma = 0\}$. Hence, we have a locally uniform second order derivative estimate of u. In general, when K_X is not ample, the constant C_ϵ may blow up as ϵ tends to 0, so we may not get a uniform bound on second derivatives of u on the whole manifold X.[10] The higher order derivative estimates for u outside $\{\sigma = 0\}$ follow from the standard theory on Monge-Ampere equations or Calabi's third order estimates as shown in [19].

[9]Here a lot of constants C_ϵ come up. Let us emphasize again that they do not have to be the same (and can not be the same sometimes).

[10]See [21] for more discussions about this.

Now it is routine to conclude that $u(t,\cdot)$ converges in C^∞-topology for any compact subset out of $\{\sigma=0\}$ as t goes to T as following. First, by the Ascoli-Arzela Theorem, any sequence $u(t_i,\cdot)$ $\big(\lim_{i\to\infty} t_i = T\big)$ has a convergent subsequence in local C^∞-topology in $X\backslash\{\sigma=0\}$. But u is essentially decreasing since $\frac{\partial u}{\partial t} < Cte^{-t}$ (for $t>\delta>0$) as shown in Section 1. Therefore, $u(t,\cdot)$ converges to a function u_T in C^0-norm along the Ricci flow as t tends to T.[11] Then the higher order derivative estimates of u imply that u converges to u_T in the C^∞-topology locally outside $\{\sigma=0\}$ as t tends T.[12] In particular, u_T is smooth outside $\{\sigma=0\}$. Moreover, it follows from the flow equation

$$(\omega_T + \sqrt{-1}\partial\bar{\partial}u_T)^n = e^{u_T + \frac{\partial u}{\partial t}|_T}\Omega, \qquad \text{on } X\setminus\{\sigma=0\}, \tag{2.5}$$

where $\frac{\partial u}{\partial t}|_T$ denotes the limit of $\frac{\partial u}{\partial t}$. The positive limiting current $\omega_T+\sqrt{-1}\partial\bar{\partial}u_T$ is actually a Kähler metric in $X\setminus\{\sigma=0\}$ by the above estimates for u.

If K_X is also numerically effective, then $T=\infty$ and $\frac{\partial u}{\partial t}|_T$ actually must vanish.[13] This implies that the limiting metric is actually Kähler-Einstein in $X\setminus\{\sigma=0\}$.[14]

Remark 2.3 E may not be unique. We can choose different E's to study (0.2). However, the limit $u(T,\cdot)$ is unique for this equation. This implies that u_T is smooth outside the intersection of all such E's. Remember the E's should be for K_X as we need the lower bound for $\frac{\partial u}{\partial t}$. In the terminology of algebraic geometry, such an intersection is called as the stable base locus of K_X. When $T=\infty$, it follows from the above that if this set is empty, then we actually have a smooth Kähler metric as the limit, so K_X is ample. So it can be taken as a characterization for the failure of K_X to be ample. In fact, the above discussion is still valid for general holomorphic line bundles over a projective manifold as shown later.

Now let us summarize the above discussion in the following.

Theorem 2.4 *Suppose X is a projective manifold with big canonical bundle K_X and ω_0 is a given Kähler metric. Let T be defined as in Proposition* 1.1. *Then the Kähler-Ricci flow* (0.1) *has a unique solution with initial data ω_0 on $[0,T)$ which converges to a current as $t\to T$ satisfying: this limiting current*[15] *is a smooth Kähler metric outside the stable base locus set of K_X and the solution of* (0.1) *converges to this limiting metric in the local C^∞-topology in this open subset. Moreover, in a suitable sense, the flow can be extended to the time T and we have the pointwise convergence of the flow on the level of potentials.*[16] *If K_X is also nef., then $T=\infty$ and the limiting current is Kähler-Einstein in its regular part.*

3 Uniqueness of Limit

In this section, assuming that K_X is nef. and big, we will prove that (0.1) has a unique Kähler-Einstein as its limit at infinity, that is, the limit is independent of the choice of the

[11] This would imply that u_T is plurisubharmonic with respect to ω_T (see in [4] for example).
[12] Interpolation inequalities as in [7] will be used.
[13] In fact, it is quite easy to get a degenerated exponential lower bound for $\frac{\partial u}{\partial t}$.
[14] The limiting metric is just the singular metric constructed in [15]. See [21] for more detail about different constructions.
[15] This current represents the cohomology class of K_X.
[16] Though the limiting current may be singular along the stable base locus of K_X, its Lelong number vanishes everywhere from our estimates and the potential function for the limiting current lies in any L^p-spaces for $p<\infty$ at this moment.

initial metric.[17]

If $\Omega_2 = e^f\Omega_1$ is another volume form, we have $u_2 = u_1 - (1-e^{-t})f$ where u_1 and u_2 are the solutions for (0.2) corresponding to Ω_1 and Ω_2, respectively. Then $u_{2,\infty} = u_{1,\infty} - f$ and consequently, the limiting metrics for Ω_1 and Ω_2 are the same.

Now we fix Ω and check the dependence on different initial Kähler metrics.

Recall that for any $\epsilon \in (0, a)$, we have

$$-C_\epsilon + \epsilon \log|\sigma|^2 < u < C, \quad -C_\epsilon + \epsilon \log|\sigma|^2 < \frac{\partial u}{\partial t} < C.$$

It follows that $e^{u+\frac{\partial u}{\partial t}}$ is bounded from above. Moreover, $\frac{\partial u}{\partial t}$ converges to zero outside $\{\sigma = 0\}$ as t tends to ∞. Hence, we have

$$\int_X e^{u_\infty}\Omega = \lim_{t\to\infty}\int_X e^{u+\frac{\partial u}{\partial t}}\Omega = \lim_{t\to\infty}\int_X \widetilde{\omega}_t^n = \lim_{t\to\infty}\int_X {\omega_t}^n = \int_X \omega_\infty^n. \tag{3.1}$$

To prove the uniqueness, suppose that ω_0 and ω are two initial Kähler metrics on X. Without loss of generality, we may assume that $\omega > \omega_0$.[18] The correspondent equations for potentials are

$$\begin{aligned} \frac{\partial u}{\partial t} &= \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, & u(0,\cdot) = 0,\\ \frac{\partial v}{\partial t} &= \log\frac{(\omega_t + e^{-t}(\omega-\omega_0) + \sqrt{-1}\partial\bar{\partial}v)^n}{\Omega} - v, \quad & v(0,\cdot) = 0, \end{aligned}$$

where $\omega_t = -\mathrm{Ric}(\Omega) + e^{-t}(\omega_0 + \mathrm{Ric}(\Omega))$. Taking their difference, we get

$$\frac{\partial(u-v)}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{(\omega_t + \sqrt{-1}\partial\bar{\partial}u + e^{-t}(\omega-\omega_0) + \sqrt{-1}\partial\bar{\partial}(v-u))^n} - (u-v),$$

and $(u-v)(0,\cdot) = 0$.

At a maximum point of $(u-v)$, we see $u-v \le 0$. Hence we have $u_\infty \le v_\infty$ over X. However, since $\int_X e^{u_\infty}\Omega = \int_X e^{v_\infty}\Omega$, we can conclude $u_\infty = v_\infty$.

This proves the uniqueness of limiting solutions for (0.2) with a fixed Ω but possibly different initial metrics. Then (4) in Theorem 0.1 follows.

4 Bounding Limiting Potentials

In this section, we will finish proving Theorem 0.1. What is left to show is that u_∞ is bounded. The proof is based on an extension of L^∞-estimate in [13] for complex Monge-Ampere equations in [21]. We refer the readers to [21] for detail.

Theorem 4.1 *Let $F : X \to \mathbb{C}P^N$ be a holomorphic map such that its image is a subvariety of the same dimension. Let ω be any Kähler form on $\mathbb{C}P^N$ and u is a weak solution in $\mathrm{PSH}_{F^*\omega}(X) \cap L^\infty(X)$ of the equation*

$$(F^*\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f\Omega, \tag{4.1}$$

[17] When K_X is not nef., generally speaking, the limit may depend on the initial metric even if initial metrics stay in the same cohomology class.

[18] The comparison between any two metrics ω_1 and ω_2 can be seen by considering $\omega_1 + \omega_2$.

where Ω is a fixed smooth volume form over X and f is a non-negative function in L^p for some $p > 1$. We normalize u such that $\sup_X u = 0$, then there is a constant C_p such that $\|u\|_{L^\infty} \leq C_p \|f\|^n_{L^p}$. Here C_p may depend on F, ω and p. Moreover, there would always be such a bounded solution for the above equation. And if F is locally blowing down with the image $F(X)$ having an orbifold structure, then any bounded solution is actually the unique continuous solution.

If F is an embedding, this estimate was proved in [13]. The detailed proof of this theorem will appear in [21]. We only explain why it[19] implies (3) of Theorem 0.1. Using a result in [9], a nef. and big canonical divisor K_X is generated by global holomorphic sections. So we can have a holomorphic map from X to $\mathbb{CP}^N$ with the (singular in general) image having the same dimension as that of X. Then we can take ω_∞ to be $F^*\omega_{FS}$ up to a positive constant, where ω_{FS} is the Fubini-Study metric $\mathbb{C}P^n$.

We can rewrite (0.2) as following:

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\frac{\partial u}{\partial t} + u}\Omega.$$

From the discussion before, we see the right-hand side has a uniform L^p-bound for all t. The results in [13] gives a L^∞-bound on u for each time t. However, this bound may depend on t.

Recall $\omega_t = \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ which represents a Kähler class for each finite t. For simplicity, assume $\omega_\infty \geq 0$ and $\omega_0 - \omega_\infty > 0$, where the first one is possible by the semi-ampleness of K_X. In order to have the limit u_∞ bounded, we only need to show that there is a uniform L^∞ -bound on u for all t. For this purpose, we need an extension of the result in [13] since the convexity of ω_t is not uniform in t. This will be done in [21] by exploring properties of the map from X to $\mathbb{CP}^N$. So we can have a solution of

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial}u)^n = e^u\Omega$$

in $\mathrm{PSH}_{\omega_\infty}(X) \cap L^\infty(X)$ [20] from the flow method.

Likewise, we can also deal with the limit for the finite time blow-up case (i.e., $T < \infty$). In order to do similar arguments, we only need $[\omega_T] = \eta \cdot A$ where A is a semi-ample class and $C > 0$. Since $[\omega_T] = (1 - e^{-T})\big(K_X + \frac{e^{-T}}{1-e^{-T}}[\omega_0]\big)$, whenever $K_X + \frac{e^{-T}}{1-e^{-T}}[\omega_0]$ is rational, Kawamata's result in [9] can give us the semi-ampleness.

In the case of complex dimension 2, clearly we should have

$$\Big(K_X + \frac{e^{-T}}{1 - e^{-T}}[\omega_0]\Big) \cdot C = 0,$$

where C is a complex curve. If $[\omega_0]$ is a rational Kähler class, then the coefficient $\frac{e^{-T}}{1-e^{-T}}$ is rational and so is the whole class. Hence we have the boundedness of the limit as $t \to T < \infty$. For general dimension, we need proper assumption to carry through the arguments above.

In fact, under the assumption above, we can have the boundedness of the solution more directly. Recall the equation

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t}\Big) = \Delta_{\widetilde{\omega}_t}\Big(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t}\Big) - n + \langle \widetilde{\omega}_t, \omega_T \rangle.$$

[19] More precisely, it is the proof of this theorem.

[20] We actually have the uniqueness of such a solution for this equation. See [21] for related results.

The assumption above tells $\omega_T + \sqrt{-1}\partial\bar{\partial} f \geq 0$. Thus we can easily get

$$\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t} + nt - f\Big) \geq \Delta_{\widetilde{\omega}_t}\Big(\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t} + nt - f\Big).$$

Then maximum principle gives $\frac{\partial u}{\partial t} + u - e^{t-T}\frac{\partial u}{\partial t} + nt - f \geq -C$. Since $t \in [0, T)$ where $T < \infty$, this gives $u \geq (e^{t-T} - 1)\frac{\partial u}{\partial t} - Ct - C \geq -C$.

There is another interesting observation from the boundedness of limiting potential for the case $T = \infty$. We have the essential decrease of volume form along the flow. Now in this case we know the limiting volume is bounded away from 0, so the volume form along the flow is uniformly bounded (also is $\frac{\partial u}{\partial t}$). The situation is different when $T < \infty$.

5 More in Complex Dimension Two

In this section, assume that X is a minimal complex surface of general type. Then K_X is nef. and big. It is well known that a basis of sections of mK_X for some $m > 0$ gives rise to a holomorphic map $P : X \to \mathbb{CP}^N$. The map P will contract finitely many rational curves to points (see for example in the appendix of [20] by Mumford) and the image $\overline{X} = P(X)$ is a Kähler orbifold (see for example in [5]). Set $m\omega_\infty = P^\star\omega_{\text{FS}}$, where ω_{FS} is the standard Fubini-Study metric on $\mathbb{CP}^N$. From discussions in previous sections, the limiting metric of the Kähler-Ricci flow we got before is smooth outside those rational curves contracted by P. In the following, we will show that this limiting metric coincides with the pull-back of the unique Kähler-Einstein orbifold metric on $\overline{X}$ with $K_{\overline{X}}$ as its Kähler class.

Let $\overline{\omega} = \frac{1}{m}\omega_{\text{FS}}|_{\overline{X}}$. Since it represents $K_{\overline{X}}$, there is a volume form $\overline{\Omega}$ on $\overline{X}$ such that $\text{Ric}(\overline{\Omega}) = -\overline{\omega}$. Moreover, this form pulls back to a smooth volume form Ω on X such that $\text{Ric}(\Omega) = -\omega_\infty$. Write the Kähler-Einstein orbifold metric as $\overline{\omega} + \sqrt{-1}\partial\bar{\partial} v$. Then v is a smooth function in the sense of orbifolds, particularly, v is continuous on $\overline{X}$. Furthermore, on $\overline{X}$, we have the Monge-Ampere equation

$$(\overline{\omega} + \sqrt{-1}\partial\bar{\partial} v)^2 = e^v\overline{\Omega}.$$

This equation pulls back to an equation on X:

$$(\omega_\infty + \sqrt{-1}\partial\bar{\partial} u)^2 = e^u\Omega,$$

where $u = P^\star v$ clearly belongs to $\text{PSH}_{\omega_\infty}(X) \cap C^0(X)$.

By the uniqueness of such a solution[21], we know that this (singular) metric $\omega_\infty + \sqrt{-1}\partial\bar{\partial} u$ has to be the same one as the limiting metric coming from the Kähler-Ricci flow. So we just prove that *the Kähler-Ricci flow on X has a global solution starting at any initial Kähler metric which converges to the pull-back of the unique Kähler-Einstein orbifold metric on its canonical model.*

Remark 5.1 Of course the consideration above also works for higher dimension with similar picture. But it is of course much more restrictive there.

[21] Indeed we know the solution from flow is also continuous here and then comparison principle will tell us that these two solutions are the same. But in fact, we do not even need the boundedness of the solution from flow to conclude this. See [21] for more detail.

6 A Final Remark

In this section, we discuss the extension of Theorem 0.1 to a generalized flow introduced by Tsuji in [18]. Consider the following evolution equation over a closed Kähler manifold X:

$$\frac{\partial \widetilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\widetilde{\omega}_t) - \widetilde{\omega}_t + S, \quad \widetilde{\omega}_0 = \omega_0, \tag{6.1}$$

where S is a fixed smooth real closed $(1,1)$-form.

The cohomology class $[\widetilde{\omega}_t]$ varies according to the following formula:

$$e^t[\widetilde{\omega}_t] - [\omega_0] = -(1-e^t)([S] - [\mathrm{Ric}(\Omega)]).$$

As before, we set $\omega_t = (S - \mathrm{Ric}(\Omega)) - e^{-t}(S - \mathrm{Ric}(\Omega) - \omega_0)$. Then we can write $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$. The same arguments as before reduce (6.1) to the following scalar flow:

$$\frac{\partial u}{\partial t} = \log \frac{\widetilde{\omega}_t^n}{\Omega} - u, \qquad u(0,\cdot) = 0. \tag{6.2}$$

The limiting equation (as $t \to \infty$) is

$$(L + \sqrt{-1}\partial\bar{\partial}u_\infty)^n = e^{u_\infty}\Omega, \tag{6.3}$$

where $L = S - \mathrm{Ric}(\Omega)$. Now we just observe that the arguments in previous sections can be applied to this equation almost directly. Let us briefly describe main outputs. Now $[L]$ takes the place of K_X before. First the arguments in Sections 1, 2 and 3 are not affected. We can conclude all the results in the following proposition.

Proposition 6.1 *Let X be a closed Kähler manifold. Then for any initial Kähler metric ω_0, the generalized flow* (6.2) *exists uniquely as long as $[\omega_t]$ is a Kähler class* ($t < T \leq \infty$ *where T is defined as before*). *In particular, if $[L]$ is numerically effective, the solution exists for all the time. Furthermore, assume that X is projective and $[L]$ is big, then as $t \to T$, the flow converges locally outside the stable base locus of $[L]$ to a smooth metric in C^∞-topology. This metric extends to a positive* $(1,1)$*-current over X which represents the cohomology class of $[\omega_T]$. If $[L]$ is also nef., then the limiting metric satisfies the limiting equation* (6.3) *in the regular part.*

Remark 6.2 It was proved in [14] that if $[L]$ is nef. and big, then the stable base locus is the union of the varieties V satisfying $[V]\cdot[L]^{\dim_{\mathbb{C}} V} = 0$. Notice that the later set is the classic characteristic set for the ampleness of $[L]$ (cf. [11]). As mentioned before, the flow can provide a direct proof that the stable base locus is a characteristic set for the ampleness of $[L]$.

There is a big issue for the discussion in Section 4 from our application of Kawamata's result. Though the canonical class K_X (or related ones as in Section 4) would be semi-ample if it is nef. and big, this is not the case for a general line bundle $[L]$ above. The result there makes heavy use of the map from X to $\mathbb{CP}^N$ which comes from the semi-ampleness of the class. So for a general nef. and big class $[L]$ [22], we have to assume for now that it is also semi-ample in order to get the result in Section 4.

Furthermore, for the case of $T < \infty$, the problem would become more serious as the class $[\omega_T]$ may not even be rational for a general chosen ω_0. Even if we consider the surface case and take $[\omega_0]$ to be rational as discussed before, the semi-ampleness is not shown in [9].

[22] The maximum existence time $T = \infty$.

Moreover, when $T < \infty$, the limit of $\frac{\partial u}{\partial t}$ is not 0 in the limiting equation. So it is favorable to continue the flow (to infinity). In order to get through this time T, one may expect to construct some weak solution for the equation to continue the flow and prove some kind of convergence as $t \to \infty$ (cf. [16]).

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