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Geometry of Ricci Solitons**

Huai-Dong CAO*

(*Dedicated to the memory of Shiing-Shen Chern*)

Abstract Ricci solitons are natural generalizations of Einstein metrics on one hand, and are special solutions of the Ricci flow of Hamilton on the other hand. In this paper we survey some of the recent developments on Ricci solitons and the role they play in the singularity study of the Ricci flow.

Keywords Ricci soliton, Singularity of Ricci flow, Stability, Gaussian density
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The concept of *Ricci solitons* was introduced by Hamilton [43]. They are natural generalizations of Einstein metrics, which have been a significant subject of intense study in differential geometry and geometric analysis. Ricci solitons also correspond to special solutions of Hamilton's Ricci flow (see [41]) and often arise as limits of dilations of singularities in the Ricci flow (see [45, 12, 21, 57]). They can be viewed as fixed points of the Ricci flow, as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings, and Perelman's $\mathcal{F}$ and $\mathcal{W}$ functionals (see [54]) are of Lyapunov type for this dynamical system. Ricci solitons are of interests to physicists as well and are called *quasi-Einstein* in physics literature (see [32, 19]). For above reasons, it is very important to understand the geometry of Ricci solitons and try to classify them both topologically and geometrically. A lot of work has been done in these directions during the past twenty years. In this paper, we will survey some of the recent development on Ricci solitons and the role they play in the singularity study of the Ricci flow.

1 Ricci Solitons

1.1 Ricci solitons

Recall that a Riemannian metric g_{ij} is *Einstein* if its Ricci tensor $R_{ij} = \rho g_{ij}$ for some constant ρ. A smooth n-dimensional manifold M^n with an Einstein metric g is an *Einstein manifold*. Ricci solitons, introduced by Hamilton [43], are natural generalizations of Einstein metrics.

Definition 1.1 *A complete Riemannian metric g_{ij} on a smooth manifold M^n is called a Ricci soliton if there exist a vector field $V = (V^i)$ and a constant ρ such that the Ricci tensor*

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*Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA. E-mail: huc2@lehigh.edu
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R_{ij} of the metric g_{ij} satisfies the equation

$$2R_{ij} + \nabla_i V_j + \nabla_j V_i = 2\rho g_{ij}. \tag{1}$$

Moreover, if V is the gradient vector field of a function f on M, then we have a gradient Ricci soliton, satisfying the equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}. \tag{2}$$

For $\rho = 0$ the Ricci soliton is steady (or translating), for $\rho > 0$ it is shrinking and for $\rho < 0$ expanding. The function f is called a potential function of the Ricci soliton.

Since $\nabla_i V_j + \nabla_j V_i = L_V g_{ij}$ is the Lie derivative of the metric g in the direction of V, we can also write the Ricci soliton equations (1) and (2) as

$$2\mathrm{Ric} + L_V g = 2\rho g \quad \text{and} \quad \mathrm{Ric} + \nabla^2 f = \rho g$$

respectively.

When the underlying manifold is a complex manifold, we have the following corresponding notion of Kähler-Ricci solitons.

Definition 1.2 *A complete Kähler metric $g_{\alpha\bar{\beta}}$ on a complex manifold X^n of complex dimension n is called a Kähler-Ricci soliton if there exists a real number ρ and a holomorphic vector field $V = (V^\alpha)$ on X such that the Ricci tensor $R_{\alpha\bar{\beta}}$ of the metric $g_{\alpha\bar{\beta}}$ satisfies the equation*

$$2R_{\alpha\bar{\beta}} + \nabla_{\bar{\beta}} V_\alpha + \nabla_\alpha V_{\bar{\beta}} = 2\rho g_{\alpha\bar{\beta}}. \tag{3}$$

It is called a gradient Kähler-Ricci soliton if the holomorphic vector field V is the gradient vector field of a real-valued function f on X^n so that

$$R_{\alpha\bar{\beta}} + \nabla_\alpha \nabla_{\bar{\beta}} f = \rho g_{\alpha\bar{\beta}} \quad \textit{and} \quad \nabla_\alpha \nabla_\beta f = 0. \tag{4}$$

Again, for $\rho = 0$ the soliton is steady, for $\rho > 0$ it is shrinking and for $\rho < 0$ expanding.

Note that the case $V = 0$, or f being a constant function, is an Einstein (or Kähler-Einstein) metric. Thus Ricci solitons are natural extensions of Einstein metrics. In fact, we will see later that there are no non-Einstein compact steady or expanding Ricci solitons.

Lemma 1.1 (See [46]) *Let g_{ij} be a complete gradient Ricci soliton with potential function f. Then we have*

$$R + |\nabla f|^2 - 2\rho f = C$$

for some constant C. Here R denotes the scalar curvature.

Proof Let g_{ij} be a complete gradient Ricci soliton on a manifold M^n so that there exists a potential function f such that the soliton equation (2) holds. Taking the covariant derivatives and using the commutating formula for covariant derivatives, we obtain

$$\nabla_i R_{jk} - \nabla_j R_{ik} + R_{ijkl} \nabla_l f = 0.$$

Taking the trace on j and k, and using the contracted second Bianchi identity

$$\nabla_j R_{ij} = \frac{1}{2} \nabla_i R,$$

we get

$$\nabla_i R - 2R_{ij}\nabla_j f = 0.$$

Thus

$$\nabla_i(R + |\nabla f|^2 - 2\rho f) = 2(R_{ij} + \nabla_i\nabla_j f - \rho g_{ij})\nabla_j f = 0.$$

Therefore

$$R + |\nabla f|^2 - 2\rho f = C$$

for some constant C.

Proposition 1.1 (See [46]) *On a compact manifold M^n, a gradient steady or expanding Ricci soliton is necessarily an Einstein metric.*

Proof By Lemma 1.1, we know that

$$R + |\nabla f|^2 - 2\rho f = C \tag{5}$$

for some constant C. On the other hand, taking the trace in equation (2), we get

$$R + \Delta f = n\rho. \tag{6}$$

Taking the difference of (5) and (6), we get

$$\Delta f - |\nabla f|^2 + 2\rho f = n\rho - C. \tag{7}$$

When M is compact and $\rho \leq 0$, it is elementary to check that f must be a constant and hence g_{ij} is an Einstein metric.

1.2 Examples of nontrivial Ricci solitons

Proposition 1.1 says that there exist no nontrivial compact gradient steady or expanding solitons. What about compact gradient shrinking solitons? In real dimension $n = 2$, Hamilton [43] showed that any compact shrinking soliton on a Riemann surface must be of constant positive Gaussian curvature. For $n = 3$, Ivey [48] proved a similar result, namely any shrinking soliton on a compact 3-manifold must have constant positive sectional curvature. However, when $n \geq 4$ there do exist nontrivial compact gradient shrinking solitons. Also, there exist noncompact Ricci solitons (steady, shrinking and expanding) that are not Einstein. It turns out that all the known examples, with one exception, are rotationally symmetric and found by solving certain nonlinear ODE (system). Moreover, all the known examples of nontrivial compact shrinking solitons so far are Kähler.

Example 1.1 For $n = 4$, the first example of compact shrinking soliton was constructed independently by Koiso [49] and the author [11] on compact complex surface $\mathrm{CP}^2\#(-\mathrm{CP}^2)$, where $(-\mathrm{CP}^2)$ means the complex projective space with the opposite orientation. This is a gradient Kähler-Ricci soliton and has U(2) symmetry and positive Ricci curvature. More generally, they found $\mathrm{U}(n)$-invariant Kähler-Ricci solitons on twisted projective line bundle over CP^{n-1} for all $n \geq 2$.

Example 1.2 Recently, Wang-Zhu [61] found another gradient Kähler-Ricci soliton on $\mathrm{CP}^2\#2(-\mathrm{CP}^2)$ which has $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry. More generally, they found gradient Kähler-Ricci solitons on certain toric varieties.

Example 1.3 Also recently, Feldman-Ilmanen-Knopf [31] found complete noncompact U(n)-invariant shrinking gradient Kähler-Ricci solitons, which are conelike at infinity and satisfy quadratic decay for the curvature.

Example 1.4 In two dimensions Hamilton [43] wrote down the first example of a complete noncompact steady soliton on $\mathbb{R}^2$, called the *cigar soliton*, where the metric is given by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

with potential function

$$f = -\log(1 + x^2 + y^2).$$

The cigar soliton has positive curvature and is asymptotic to a cylinder of finite circumference 2π at ∞.

Example 1.5 In the Riemannian case, higher dimensional examples of noncompact steady solitons were found by Robert Bryant [6] on $\mathbb{R}^n$ ($n \geq 3$) which have positive sectional curvature.

Example 1.6 In the Kähler case, the author [11] found noncompact gradient steady Kähler-Ricci solitons on $\mathbb{C}^n$. These examples are complete, rotationally symmetric, of positive curvature. It is interesting to point out that the geodesic sphere S^{2n-1} of radius s is an S^1-bundle over CP^{n-1} where the diameter of S^1 is on the order 1, while the diameter of CP^{n-1} is on the order $\sqrt{s}$. In addition, the author [11] found another example on $\mathbb{C}^n$ with blow-up at the origin.

Example 1.7 Also the author [12] constructed a family of complete noncompact expanding solitons on $\mathbb{C}^n$. These expanding Kähker-Ricci solitons all have U(n) symmetry and positive sectional curvature and are conelike at infinity.

Example 1.8 Additional examples of complete noncompact Kähler-Ricci expanding solitons were found more recently by Feldman-Ilmanen-Knopf [31] (essentially on $\mathbb{C}^n$ with blow-up at the origin). These examples are also U(n)-invariant and conelike at infinity.

1.3 Compact shrinking Kähler-Ricci solitons

We commented in last subsection that all known compact shrinking Ricci solitons are Kähler, so the Kähler case is very special. Let X^n be a compact Kähler manifold with Kähler metric $g_{\alpha\bar{\beta}}$. Then the Kähler form

$$\omega = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^\beta$$

is a closed real (1,1)-form. Let $[\omega] \in H^2(M, \mathbb{R})$ be the *Kähler class* of $g_{\alpha\bar{\beta}}$. The Ricci form

$$\mathrm{Rc} = \frac{\sqrt{-1}}{2} R_{\alpha\bar{\beta}} dz^\alpha \wedge dz^\beta$$

is also a closed real (1,1)-form. In fact, it is well known that the *first Chern class* $c_1(X)$ is represented by the Ricci form Rc:

$$c_1(X) = \frac{1}{\pi}[\mathrm{Rc}] \in H^2(M, \mathbb{Z}).$$

If the first Chern class $c_1(X) = 0$, then by Yau's celebrated solution (see [62]) to the Calabi conjecture in each Kähler class on X there exists a unique Calabi-Yau metric (i.e., Ricci flat

Kähler metric). Also, by the work of Aubin [2] and Yau [62], if $c_1(X) < 0$ then there exists a unique Kähler-Einstein metric $g_{\alpha\bar{\beta}}$ with $R_{\alpha\bar{\beta}} = -g_{\alpha\bar{\beta}}$. On the other hand, from previous discussions, we know that only possible nontrivial Kähler-Ricci solitons on X^n are shrinking gradient solitons. If X admits such a gradient soliton g, then it would follow that

$$\mathrm{Ric} - \rho g + \partial\bar{\partial} f = 0$$

for some real-valued potential function f on X and some positive constant $\rho > 0$. Hence $c_1(X) > 0$ is a necessary condition for X to admit a gradient shrinking soliton.

Now assume that X^n is a compact Kähler manifold with $c_1(X) > 0$ and that the Kähler form ω is cohomologous to the Ricci form Rc. Let us first recall the definition of the Futaki invariant (see [33]). Since ω and Rc represent the same cohomology class, by Hodge theory, we have

$$R_{\alpha\bar{\beta}} - g_{\alpha\bar{\beta}} = \partial_\alpha \bar{\partial}_\beta f, \quad \text{or in short} \quad \mathrm{Rc} - \omega = \frac{\sqrt{-1}}{2}\partial\bar{\partial} f,$$

for some real-valued function f on X. Let $\mathfrak{h}(X)$ denote the Lie algebra of holomorphic vector fields on X. For any $V \in \mathfrak{h}(X)$, define

$$F(V) = \int_X Vf = \int_X \nabla f \cdot V.$$

Futaki [33] showed that the definition of $F : \mathfrak{h}(X) \to \mathbb{C}$ is independent of the choice of Kähler metrics in the Kähler class $[\omega] = \pi c_1(X)$. As a consequence, a necessary condition for X to admit a Kähler-Einsten metric is Futaki invariant $F = 0$. Also, if X admits a nontrivial gradient shrinking Kähler-Ricci soliton with potential function f, then $F(\nabla f) = \int_X |\nabla f|^2 > 0$. Thus the existence of a Kähler-Einstein metric and the existence of a nontrivial gradient shrinking soliton is mutually exclusive. We would like to point out that in [59, 60], Tian-Zhu obtained certain generalization of the Futaki invariant whose vanishing is a necessary condition for the existence of gradient shrinking solitons. They also extended the uniqueness theorem of Bando-Mabuchi [3] on Kähler-Einstein metrics to the case of gradient shrinking Kähler-Ricci solitons. Very recently, a compactness theorem for compact shrinking Kähler-Ricci solitons has been proved by Sesum and the author [16], extending the compactness theorem of Anderson [1], Bando-Kasue-Nakajima [4], and Tian [58] for Kähler-Einstein metrics.

2 Variational Structures

In this section we consider Perelman's $\mathcal{F}$-functional and $\mathcal{W}$-functional and the related λ-energy and ν-energy respectively. We will see that critical points of the λ-energy (respectively ν-energy) are precisely given by gradient steady (respectively shrinking) Ricci solitons. Throughout this section we assume that M^n is a compact smooth manifold.

2.1 Perelman's $\boldsymbol{F}$-functional and $\boldsymbol{\lambda}$-energy

In [54] Perelman considered the functional

$$\mathcal{F}(g_{ij}, f) = \int_M (R + |\nabla f|^2) e^{-f} dV$$

defined on the space of Riemannian metrics and smooth functions on M. Here R is the scalar curvature of the metric g_{ij} and f is a smooth function on M^n.

Lemma 2.1 (See [54, §1.1], also [17, Lemma 1.5.2]) *If $\delta g_{ij} = v_{ij}$ and $\delta f = \phi$ are variations of g_{ij} and f respectively, then the first variation of $\mathcal{F}$ is given by*

$$\delta\mathcal{F}(v_{ij}, \phi) = \int_M \Big[-v_{ij}(R_{ij} + \nabla_i\nabla_j f) + \Big(\frac{v}{2} - \phi\Big)(2\Delta f - |\nabla f|^2 + R)\Big] e^{-f} dV,$$

where $v = g^{ij}v_{ij}$.

Next we consider the associated energy introduced by Perelman:

$$\lambda(g_{ij}) = \inf\Big\{\mathcal{F}(g_{ij}, f) : f \in C^\infty(M), \int_M e^{-f}dV = 1\Big\}.$$

Clearly $\lambda(g_{ij})$ is invariant under diffeomorphisms. If we set $u = e^{-\frac{f}{2}}$, then the functional $\mathcal{F}$ can be expressed as

$$\mathcal{F} = \int_M (Ru^2 + 4|\nabla u|^2)dV.$$

Thus

$$\lambda(g_{ij}) = \inf\Big\{\int_M (Ru^2 + 4|\nabla u|^2)dV : \int_M u^2 dV = 1\Big\}.$$

Hence $\lambda(g_{ij})$ is just the first eigenvalue of the operator $-4\Delta + R$. Let $u_0 > 0$ be the first eigenfunction of the operator $-4\Delta + R$ so that

$$-4\Delta u_0 + Ru_0 = \lambda(g_{ij})u_0.$$

Then $f_0 = -2\log u_0$ is a minimizer of $\lambda(g_{ij})$:

$$\lambda(g_{ij}) = \mathcal{F}(g_{ij}, f_0).$$

Note that f_0 satisfies the equation

$$-2\Delta f_0 + |\nabla f_0|^2 - R = \lambda(g_{ij}). \tag{8}$$

Let $h = h_{ij}$ be a symmetric 2-tensor and consider variations $g_{ij}(s) = g_{ij} + sh_{ij}$. Then it is an easy consequence of Lemma 2.1 and equation (8), that the first variation $\mathcal{D}_g\lambda(h)$ of $\lambda(g_{ij})$ is given by

$$\frac{d}{ds}\Big|_{s=0}\lambda(g_{ij}(s)) = \int -h_{ij}(R_{ij} + \nabla_i\nabla_j f)e^{-f}dV, \tag{9}$$

where f is a minimizer $\lambda(g_{ij})$. In particular, the critical points of λ are precisely steady gradient Ricci solitons (which are necessarily Ricci flat by Proposition 1.1). Note, by diffeomorphism invariance of λ, $\mathcal{D}_g\lambda$ vanishes on any Lie derivative $h_{ij} = L_V g_{ij}$, and hence on $2\nabla_i\nabla_j f = L_{\nabla f}g_{ij}$. Thus, by inserting $h = -2(\mathrm{Ric} + \nabla^2 f)$ in (9) one recovers the following result of Perelman [54].

Proposition 2.1 (See [54, §2.2], also [17, Corollary 1.5.4]) *$\lambda(g_{ij}(t))$ is nondecreasing along the Ricci flow and the monotonicity is strict unless we are on a steady gradient soliton. In particular, any steady Ricci soliton is necessarily a gradient soliton.*

We remark that by considering the quantity

$$\bar{\lambda}(g_{ij}) = \lambda(g_{ij})(\mathrm{Vol}(g_{ij}))^{\frac{2}{n}},$$

which is a scale invariant version of $\lambda(g_{ij})$, Perelman [54] also showed the following result.

Proposition 2.2 (See [54, §2.3], also [17, Corollary 1.5.5]) *$\bar\lambda(g_{ij})$ is nondecreasing along the Ricci flow whenever it is nonpositive; moreover, the monotonicity is strict unless we are on a gradient expanding soliton. In particular, any expanding Ricci soliton is necessarily a gradient soliton.*

Combining with Proposition 1.1, we immediately get

Proposition 2.3 *On a compact manifold, a steady or expanding Ricci soliton is necessarily an Einstein metric.*

See [28] for an alternative proof of Proposition 2.3.

2.2 Perelman's $\mathcal{W}$-functional and ν-energy

In order to study shrinking Ricci solitons, we consider the $\mathcal{W}$-functional of Perelman [54] defined by

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV,$$

where g_{ij} is a Riemannian metric, f a smooth function on M^n, and τ a positive scale parameter. Clearly the functional $\mathcal{W}$ is invariant under simultaneous scaling of τ and g_{ij} (or equivalently the parabolic scaling), and invariant under diffeomorphism. Namely, for any positive number a and any diffeomorphism φ, we have

$$\mathcal{W}(a\varphi^* g_{ij}, \varphi^* f, a\tau) = \mathcal{W}(g_{ij}, f, \tau).$$

Lemma 2.2 (See [54], also [17, Lemma 1.5.7]) *If $v_{ij} = \delta g_{ij}$, $\phi = \delta f$, and $\eta = \delta\tau$, then*

$$\begin{aligned}\delta\mathcal{W}(v_{ij}, \phi, \eta) = &\int_M -\tau v_{ij}\Big(R_{ij} + \nabla_i f \nabla_j f - \frac{1}{2\tau} g_{ij}\Big)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \\ &+ \int_M \Big(\frac{v}{2} - \phi - \frac{n}{2\tau}\eta\Big)[\tau(R + 2\Delta f - |\nabla f|^2) + f - n - 1](4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \\ &+ \int_M \eta\Big(R + |\nabla f|^2 - \frac{n}{2\tau}\Big)(4\pi\tau)^{-\frac{n}{2}} e^{-f} dV.\end{aligned}$$

Here $v = g^{ij} v_{ij}$ as before.

Similarly to the λ-energy, we can consider

$$\mu(g_{ij}, \tau) = \inf\Big\{\mathcal{W}(g_{ij}, f, \tau) : f \in C^\infty(M),\ \frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int_M e^{-f} dV = 1\Big\}.$$

Note that if we let $u = e^{-\frac{f}{2}}$, then the functional $\mathcal{W}$ can be expressed as

$$\mathcal{W}(g_{ij}, f, \tau) = \int_M [\tau(Ru^2 + 4|\nabla u|^2) - u^2 \log u^2 - nu^2](4\pi\tau)^{-\frac{n}{2}} dV,$$

and the constraint $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV = 1$ becomes $\int_M u^2 (4\pi\tau)^{-\frac{n}{2}} dV = 1$.

Therefore $\mu(g_{ij}, \tau)$ corresponds to the best constant of a logarithmic Sobolev inequality. Since the nonquadratic term is subcritical (in view of Sobolev exponent), it is rather straightforward to show that

$$\inf\Big\{\int_M [\tau(4|\nabla u|^2 + Ru^2) - u^2 \log u^2 - nu^2](4\pi\tau)^{-\frac{n}{2}} dV \,\Big|\, \int_M u^2 (4\pi\tau)^{-\frac{n}{2}} dV = 1\Big\}$$

is achieved by some nonnegative function $u \in H^1(M)$ which satisfies the Euler-Lagrange equation

$$\tau(-4\Delta u + Ru) - 2u\log u - nu = \mu(g_{ij}, \tau)u.$$

One can further show that u is positive. Then the standard regularity theory of elliptic PDE shows that u is smooth. We refer the reader to Rothaus [56] for more details. To summarize, $\mu(g_{ij}, \tau)$ is achieved by a minimizer f satisfying the nonlinear equation

$$\tau(2\Delta f - |\nabla f|^2 + R) + f - n = \mu(g_{ij}, \tau).$$

Proposition 2.4 (See [54], also [17, Corollary 1.5.9]) *$\mu(g_{ij}(t), \tau - t)$ is nondecreasing along the Ricci flow; moveover, the monotonicity is strict unless we are on a shrinking gradient soliton. In particular, any shrinking Ricci soliton is necessarily a gradient soliton.*

Finally, we define the ν-energy

$$\nu(g_{ij}) = \inf\Big\{\mathcal{W}(g, f, \tau) : f \in C^\infty(M), \tau > 0, \ \frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int e^{-f} dV = 1\Big\}.$$

One checks that $\nu(g_{ij})$ is realized by a pair (f, τ) that solve the equations

$$\tau(-2\Delta f + |Df|^2 - R) - f + n + \nu = 0, \quad \frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int fe^{-f} dV = \frac{n}{2} + \nu. \tag{10}$$

Consider variations $g_{ij}(s) = g_{ij} + sh_{ij}$ as before. Using Lemma 2.2 and the equation (10), one calculates the first variation $\mathcal{D}_g\nu(h)$ to be

$$\frac{d}{ds}\Big|_{s=0}\nu(g_{ij}(s)) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int -h_{ij}\Big[\tau(R_{ij} + \nabla_i\nabla_j f) - \frac{1}{2}g_{ij}\Big]e^{-f} dV.$$

A stationary point of ν thus satisfies

$$R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij} = 0,$$

which says that g_{ij} is a gradient shrinking Ricci soliton.

As before, $\mathcal{D}_g\nu(h)$ vanishes on Lie derivatives. By scale invariance it vanishes on multiplies of the metric. Inserting $h_{ij} = -2(R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij})$, one recovers Perelman's formula that finds that $\nu(g_{ij}(t))$ is monotone on a Ricci flow, and constant if and only if $g_{ij}(t)$ is a gradient shrinking Ricci soliton.

3 Ricci Solitons and the Ricci Flow

3.1 Ricci solitons as special solutions of the Ricci flow

Let us first examine how Einstein metrics behave under Hamilton's Ricci flow

$$\frac{\partial g_{ij}(t)}{\partial t} = -2R_{ij}(t). \tag{11}$$

If the initial metric is Ricci flat, so that $R_{ij} = 0$ at $t = 0$, then clearly the metric does not change under the Ricci flow. Hence any Ricci flat metric is a stationary solution. This happens, for example, on a flat torus or on any $K3$-surface with a Calabi-Yau metric.

If the initial metric is Einstein with positive scalar curvature, then the metric will shrink under the Ricci flow by a time-dependent factor. Indeed, since the initial metric is Einstein, we have

$$R_{ij}(0) = \rho g_{ij}(0)$$

for some $\rho > 0$. Let

$$g_{ij}(x,t) = a(t) g_{ij}(x,0).$$

Then it is easy to check that in this case the Ricci flow corresponds to the ODE

$$a'(t) = -2\rho$$

for the conformal factor $a(t)$ whose solution is given by $a(t) = 1 - 2\rho t$. Thus the evolving metric

$$g_{ij}(t) = (1 - 2\rho t) g_{ij}(0)$$

shrinks homothetically to a point as $t \to T = \frac{1}{2\rho}$, while the scalar curvature becomes infinite like $\frac{1}{T-t}$.

By contrast, if the initial metric is an Einstein metric of negative scalar curvature, the metric will expand homothetically for all times. Suppose

$$R_{ij}(0) = -\rho g_{ij}(0)$$

at $t = 0$ with $\rho > 0$. Then the solution to the Ricci flow is given by

$$g_{ij}(t) = (1 + 2\rho t) g_{ij}(0).$$

Hence the evolving metric $g_{ij}(t)$ exists and expands homothetically for all time, and the curvature will fall back to zero like $-\frac{1}{t}$. Note that now the evolving metric $g_{ij}(t)$ only goes back in time to $-\frac{1}{2\rho}$, when the metric explodes out of a single point in a "big bang".

Now suppose that we have a steady Ricci soliton $\hat{g}_{ij}$ on a smooth manifold M^n so that

$$2R_{ij} + \nabla_i V_j + \nabla_j V_i = 0, \tag{12}$$

and suppose that the vector field $V = (V^i)$ generates a one-parameter group of diffeomorphisms φ_t of M (this is always the case when M is compact). Then clearly

$$g_{ij}(t) = \varphi_t^* \hat{g}_{ij}$$

is a solution to the Ricci flow with $\hat{g}_{ij}$ as the initial metric, since the time derivative of $g_{ij}(t)$ is given by the Lie derivative $\mathcal{L}_V g_{ij}$ of the evolving metric $g_{ij}(t)$ which is equal to $-2R_{ij}(t)$ in the case of a steady soliton. Conversely, if φ_t is a one-parameter group of diffeomorphisms generated by a vector field V on M and

$$g_{ij}(t) = \varphi_t^* \hat{g}_{ij}$$

is a solution to the Ricci flow with the initial metric $\hat{g}_{ij}$, then the Ricci term $-2\mathrm{Ric}$ in the RHS of the Ricci flow equation is equal to the Lie derivative $\mathcal{L}_V g$ of the evolving metric $g_{ij}(t)$. In particular, the initial metric $g_{ij}(0) = \hat{g}_{ij}$ satisfies the steady Ricci soliton equation (12). Thus steady Ricci solitons are one-to-one correspondent to Ricci flow solutions they generate. For this reason we often do not distinguish a steady Ricci soliton with the Ricci flow solution it generates. More generally, we can consider a solution to the Ricci flow which moves by diffeomorphisms and also shrinks or expands by a (time-dependent) factor at the same time. Such a solution corresponds to either shrinking or expanding Ricci soliton. For example, a shrinking gradient Ricci soliton satisfying the equation

$$R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij} = 0$$

corresponds to its Ricci flow solution $g_{ij}(t)$ of the form

$$g_{ij}(t) := (T - t)\varphi_t^*(g_{ij}), \quad t < T,$$

where φ_t are the diffeomorphisms generated by the gradient vector field of f, and $\tau = T - t$.

3.2 Ricci solitons and singularity models of the Ricci flow

Consider a solution $g_{ij}(t)$ to the Ricci flow on $M^n \times [0, T)$, $T \le +\infty$, where either M^n is compact or at each time t the metric is complete and has bounded curvature. We say that $g_{ij}(t)$ is a *maximal* solution of the Ricci flow if either $T = +\infty$ or $T < +\infty$ and the norm of its curvature tensor $|Rm|$ is unbounded as $t \to T$. In the latter case, we say $g_{ij}(t)$ is a singular solution to the Ricci flow.

Clearly, a round sphere S^3 will shrink to a point under the Ricci flow in some finite time, so this gives rise to one type of singularities in the Ricci flow. On the other hand, if we take a dumbbell metric on S^3 with a neck like $S^2 \times B^1$, we expect the neck will shrink because the positive curvature in the S^2 direction will dominate the slightly negative curvature in the B^1 direction. In some finite time we expect the neck will pinch off. If we dilate in space and time at the maximal curvature point, then we expect the limit of dilations converge to the round infinite cylinder $S^2 \times \mathbb{R}$. As in the minimal surface theory and harmonic map theory, one usually tries to understand the structure of a singularity by rescaling the solution (or blow up) to obtain a sequence of solutions and study its limit. For the Ricci flow, the theory was first developed by Hamilton in [46].

Denote by

$$K_{\max}(t) = \sup_{x \in M} |Rm(x,t)|_{g_{ij}(t)}.$$

According to [46], one can classify maximal solutions into three types; every maximal solution is clearly of one and only one of the following three types:

Type I $\quad T < +\infty$ and $\sup (T - t) K_{\max}(t) < +\infty$;

Type II(a) $\quad T < +\infty$ but $\sup (T - t) K_{\max}(t) = +\infty$;

Type II(b) $\quad T = +\infty$ but $\sup t K_{\max}(t) = +\infty$;

Type III $\quad T = +\infty$, $\sup t K_{\max}(t) < +\infty$.

For each type of maximal solution, Hamilton defines a corresponding type of limiting singularity model.

Definition 3.1 *A solution $g_{ij}(x,t)$ to the Ricci flow on the manifold M, where either M is compact or at each time t the metric $g_{ij}(\cdot,t)$ is complete and has bounded curvature, is called a singularity model if it is not flat and of one of the following three types:*

Type I *The solution exists for $t \in (-\infty, \Omega)$ for some constant Ω with $0 < \Omega < +\infty$, and*

$$|Rm| \le \Omega/(\Omega - t)$$

everywhere with equality somewhere at $t = 0$;

Type II *The solution exists for* $t \in (-\infty, +\infty)$, *and*

$$|Rm| \leq 1$$

everywhere with equality somewhere at $t = 0$;

Type III *The solution exists for* $t \in (-A, +\infty)$ *for some constant* A *with* $0 < A < +\infty$, *and*

$$|Rm| \leq A/(A+t)$$

everywhere with equality somewhere at $t = 0$.

Definition 3.2 *A solution of the Ricci flow is said to satisfy the injectivity radius condition if for every sequence of (almost) maximum points* $\{(x_k, t_k)\}$, *there exists a constant* $c_2 > 0$ *independent of* k *such that*

$$\operatorname{inj}(M, x_k, g_{ij}(t_k)) \geq \frac{c_2}{\sqrt{K_{\max}(t_k)}} \quad \text{for all } \ k.$$

Here, by a sequence of (almost) maximum points, we mean $\{(x_k, t_k) \in M \times [0, T)\}$, $k = 1, 2, \cdots$, *has the following property: there exist positive constants* c_1 *and* $\alpha \in (0, 1]$ *such that*

$$|Rm(x_k, t_k)| \geq c_1 K_{\max}(t), \quad t \in \left[t_k - \frac{\alpha}{K_{\max}(t_k)}, t_k\right]$$

for all k.

In [54], Perelman proved an important result, called no local collapsing theorem (see also Theorem 3.3.3 in [17]) for solutions to the Ricic flow on a compact manifold. Combining this theorem of Perelman with the local injectivity radius estimate of Cheng-Li-Yau [25] and Cheeger-Gromov-Taylor [20] immediately yields the following result, which is conjectured by Hamilton in [46].

Theorem 3.1 (Little Loop Lemma, see [17, Theorem 4.2.4]) *Let* $g_{ij}(t)$, $0 \leq t < T < +\infty$, *be a solution of the Ricci flow on a compact manifold* M^n. *Then there exists a constant* $\delta > 0$ *having the following property: if at a point* $x_0 \in M$ *and a time* $t_0 \in [0, T)$,

$$|Rm|(\cdot, t_0) \leq r^{-2}, \quad \text{on } \ B_{t_0}(x_0, r)$$

for some $r \leq \sqrt{T}$, *then the injectivity radius of* M *with respect to the metric* $g_{ij}(t_0)$ *at* x_0 *is bounded from below by*

$$\operatorname{inj}(M, x_0, g_{ij}(t_0)) \geq \delta r.$$

Clearly by the above Little Loop Lemma a maximal solution on a compact manifold with the maximal time $T < +\infty$ always satisfies the injectivity radius condition. Also, by the Gromoll-Meyer injectivity radius estimate (see [39]), a solution on a complete noncompact manifold with positive sectional curvature also satisfies the injectivity radius condition. We refer the reader to Chapeter 4 of [17] for more detailed discussions.

Theorem 3.2 (See [46, Theorem 16.2]) *For any maximal solution to the Ricci flow which satisfies the injectivity radius condition and is of Type* I, II, *or* III, *there exists a sequence of dilations of the solution which converges in* C^∞_{loc} *topology to a singularity model of the corresponding type.*

In the case of manifolds with nonnegative curvature operator, or Kähler metrics with nonnegative holomorphic bisectional curvature, we can bound the Riemannian curvature by the scalar curvature R upto a constant factor depending only on the dimension. Then we can slightly modify the statements in the previous theorem as follows.

Theorem 3.3 (See [46, Theorem 16.3]) *For any complete maximal solution to the Ricci flow with bounded and nonnegative curvature operator on a Riemannian manifold, or on a Kähler manifold with bounded and nonnegative holomorphic bisectional curvature, there exists a sequence of dilations which converges to a singular model.*

For Type I *solutions: the limit model exists for* $t \in (-\infty, \Omega)$ *with* $0 < \Omega < +\infty$ *and has*

$$R \leq \Omega/(\Omega - t)$$

everywhere with equality somewhere at $t = 0$*;*

For Type II *solutions: the limit model exists for* $t \in (-\infty, +\infty)$ *and has*

$$R \leq 1$$

everywhere with equality somewhere at $t = 0$*;*

For Type III *solutions: the limit model exists for* $t \in (-A, +\infty)$ *with* $0 < A < +\infty$ *and has*

$$R \leq A/(A + t)$$

everywhere with equality somewhere at $t = 0$*.*

The following results obtained by Hamilton [45], the author [12], and Chen-Zhu [21] characterize the Type II and Type III singular models of the Ricci flow and the Kähler-Ricci flow with nonnegative curvature respectively.

Theorem 3.4 (See [45]) *Any Type* II *singularity model of the Ricci flow with nonnegative curvature operator and positive Ricci curvature must be a steady Ricci soliton.*

Theorem 3.5 (See [12]) (i) *Any Type* II *singularity model on a Kähler manifold with nonnegative holomorphic bisectional curvature and positive Ricci curvature must be a steady Kähler-Ricci soliton;*

(ii) *Any Type* III *singularity model on a Kähler manifold with nonnegative holomorphic bisectional curvature and positive Ricci curvature must be a shrinking Kähler-Ricci soliton.*

Theorem 3.6 (See [21]) *Any Type* III *singularity model of the Ricci flow with nonnegative curvature operator and positive Ricci curvature must be a homothetically expanding Ricci soliton.*

We remark that the basic idea in proving the above theorems is to apply the Li-Yau-Hamilton estimates for the Ricci flow (see [44]) and Kähler-Ricci flow (see [10]), and the strong maximum principle type arguments.

By exploring Perelman's μ-entropy, Sesum studied compact Type I singularity model and obtained the following

Theorem 3.7 (See [57]) *Let* $(M, g_{ij}(t))$ *be a Type* I *singularity model obtained as a rescaling limit of a Type* I *maximal solution. Suppose* M *is compact. Then* $(M, g_{ij}(t))$ *must be a gradient shrinking Ricci soliton.*

It is possible that the compactness assumption of the rescaling limit in the above theorem may not be needed and it is desirable to remove this assumption.

4 Geometry of Complete Noncompact Ricci Solitons

We will now examine the structure of steady or expanding Ricci solitons of the sort we get as a Type II or Type III limit.

In [46], Hamilton proved the following

Proposition 4.1 (See [46, Theorem 20.1]) *Suppose that we have a complete noncompact gradient steady Ricci soliton g_{ij} with bounded curvature so that*

$$R_{ij} = \nabla_i \nabla_j F$$

for some potential function F on M. Assume that the Ricci curvature is strictly positive and the scalar curvature R attains its maximum $R_{\max}$ at a point $x_0 \in M^n$. Then

$$|\nabla F|^2 + R = R_{\max}$$

everywhere on M^n. Furthermore, F is a convex exhaustion function and attains its minimum at x_0. As a consequence, the underlying manifold M^n must be diffeomorphic to the Euclidean space $\mathbb{R}^n$.

Remark 4.1 Similar conclusions hold for a complete noncompact expanding gradient Ricci soliton with positive Ricci curvature such that the scalar curvature R attains its maximum $R_{\max}$. Namely, the potential function F is an exhausting and convex function and the underlying manifold is diffeomorphic to the Euclidean space $\mathbb{R}^n$.

The following result is observed in [14] and [13].

Proposition 4.2 (See [14, 13]) *Suppose that we have a complete noncompact gradient steady Kähler-Ricci soliton satisfying the same assumptions of Proposition* 4.1*. Then the underlying complex manifold X^n is Stein and diffeomorphic to $\mathbb{R}^{2n}$.*

Very recently, Bryant [7] and Chau-Tam [18] have shown

Theorem 4.1 (See [7, 18]) *Under the same assumption as in Proposition* 4.2*, X^n is in fact biholomorphic to the complex Euclidean space $\mathbb{C}^n$.*

Now we turn our attention to more geometric aspects of Ricci solitons. Suppose that M^n is an n-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature. First let us recall two geometric concepts, the *asymptotic scalar curvature ratio* and the *asymptotic volume ratio*, both defined by Hamilton [46].

Let O be a fixed point on a Riemannian manifold M^n, s the distance to the fixed point O, and R the scalar curvature. The asymptotic scalar curvature ratio is defined by

$$A = \limsup_{s \to +\infty} R s^2.$$

The definition is independent of the choice of the fixed point O and invariant under dilation.

Consider the geodesic ball $B(O, r)$ centered at O of radius r. The well-known Bishop-Gromov volume comparison theorem tells us that the ratio $\mathrm{Vol}(B(O,r))/r^n$ is nonincreasing in $r \in [0, +\infty)$. Thus there exists a limit

$$\nu_M = \lim_{r \to +\infty} \frac{\mathrm{Vol}(B(O,r))}{r^n},$$

which is called the asymptotic volume ratio of the Riemannian manifold M. ν_M is invariant under dilation and is independent of the choice of the origin.

We remark that the concept of asymptotic scalar curvature ratio is particular useful on manifolds with positive sectional curvature. The well-known gap theorems established by Greene-Wu [38], Eschenberg-Shrader-Strake [30] and Drees [29] show that any complete noncompact n-dimensional (except $n = 4$ or 8) Riemannian manifold of positive sectional curvature must have $A > 0$. Similar results on complete noncompact Kähler manifolds of positive holomorphic bisectional curvature were obtained by Mok-Siu-Yau [50], Chen-Zhu [22] and Ni-Tam [52].

Proposition 4.3 (See [46, Theorem 20.2]) *For a complete noncompact gradient steady Ricci soliton with bounded curvature and positive sectional curvature of dimension $n \geq 3$ where the scalar curvature assumes its maximum at a point $O \in M$, the asymptotic scalar curvature ratio is infinite, i.e.,*

$$A = \limsup_{s\to+\infty} Rs^2 = +\infty,$$

where s is the distance to the point O.

In fact, one has the stronger conclusion that

$$\limsup_{s\to+\infty} Rs^{1+\varepsilon} = +\infty$$

for arbitrarily small $\varepsilon > 0$.

A solution $g_{ij}(t)$ to the Ricci flow is called *ancient* if it is defined for $-\infty < t < T$. By definition, Type I and Type II singularity models are ancient, and so is a steady Ricci soliton. In [54], Perelman proves a more general result for ancient solutions.

Theorem 4.2 (See [54, Proposition 11.4], also [17, Lemma 6.3.1]) *Let $g_{ij}(t)$ be a complete non-flat ancient solution to the Ricci flow with bounded and nonnegative curvature operator on a noncompact manifold M^n. Then the asymptotic volume ratio with respect to $g_{ij}(t)$ is zero: $\nu_M(t) = 0$ for all t.*

In the Kähler case, the same result is obtained independently by Chen-Zhu [23]. Moreover, we have the following improved result, by Chen-Zhu [24] and the author [13] for $n = 2$ and Ni [51] for all n, in which the assumption of nonnegativity of curvature operator is replaced by the weaker assumption of nonnegative holomorphic bisectional curvature.

Theorem 4.3 (See [24], and [13] for $n = 2$, [51] for all n) *Let $g_{\alpha\bar\beta}(t)$ be a complete non-flat ancient solution to the Kähler-Ricci flow with bounded and nonnegative holomorphic bisectional curvature on a noncompact complex manifold X^n. Then the asymptotic volume ratio ν_X with respect to $g_{\alpha\bar\beta}(t)$ is zero for all t.*

For complete noncompact ancient Type I-like solution, Chow and Lu [26] proved the following result in dimensions three.

Proposition 4.4 (See [26]) *If $g_{ij}(t)$ is a complete noncompact ancient Type* I*-like solution $\big($i.e., $\sup_{t\in(-\infty,T)} (T-t)\, K_{\max}(t) < +\infty\big)$ to the Ricci flow with bounded and positive sectional curvature on an orientable three-manifold, then the asymptotic scalar curvature ratio in $g_{ij}(t)$ is infinite for all t: $A = \infty$.*

In fact, the above result of Chow and Lu can be extended to all dimensions. Namely,

Proposition 4.5 *Any complete noncompact ancient Type* I*-like solution to the Ricci flow with bounded and positive curvature operator on an n-dimensional manifold must have infinite asymptotic scalar curvature ratio.*

Indeed one can argue by contradiction. Suppose a complete noncompact ancient Type I-like solution to the Ricci flow with bounded and positive curvature operator has finite asymptotic scalar curvature ratio. Then it follows from Theorem 19.2 of [46] that its asymptotic volume ratio would be positive. This is then a contradiction to Theorem 4.2.

Note that, by the Hamilton-Ivey curvature pinching theorem and Perelman's no local collapsing theorem, the Type I singularity model of a Type I compact maximal solution is necessarily κ-nocollapsed and of nonnegative sectional curvature.

Proposition 4.6 (See [55, Lemma 1.2]) *There does not exist a three-dimensional complete noncompact κ-noncollapsed gradient shrinking soliton with positive sectional curvature.*

Based on the above Proposition 4.6 (see also the proof of Lemma 6.4.1 in [17]), Perelman [55] obtained the following classification result (see also [17, Lemma 6.4.1]), which is an improvement of a result of Hamilton (see [46, Theorem 26.5]).

Theorem 4.4 (Classification of Three-Dimensional Shrinking Solitons) *Let $g_{ij}(t)$ be a nonflat gradient shrinking soliton to the Ricci flow on a three-manifold M^3. Suppose that $g_{ij}(t)$ has bounded and nonnegative sectional curvature and is κ-noncollapsed on all scales for some $\kappa > 0$. Then $(M, g_{ij}(t))$ is one of the following:*

(i) *the round three-sphere $\mathbb{S}^3$, or one of its metric quotients;*

(ii) *the round infinite cylinder $\mathbb{S}^2 \times \mathbb{R}$, or its $\mathbb{Z}_2$ quotient.*

In the Kähler case, we have the following result of Ni [51].

Proposition 4.7 (See [51]) *In any complex dimension, there is no complete noncompact gradient shrinking Kähler-Ricci soliton with positive holomorphic bisectional curvature.*

We end this section by stating the following important uniqueness result of Hamilton for 2-dimensional complete steady Ricci solitons.

Theorem 4.5 (See [43]) *The only complete steady Ricci soliton on a two-dimensional manifold with bounded curvature which assumes its maximum* 1 *at an origin is the "cigar" soliton on the plane $\mathbb{R}^2$ with the metric*

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}.$$

5 Stability of Ricci Solitons

In this section we describe the second variation formulas for Perelman's λ-energy and ν-energy due to Hamilton, Ilmanen and the author [15].

5.1 Second variation of λ-energy

Recall that the λ-energy is defined by

$$\lambda(g_{ij}) = \inf\left\{\mathcal{F}(g_{ij}, f) : f \in C^\infty(M),\ \int_M e^{-f} dV = 1\right\}$$

and its first variation is given by

$$\frac{d}{ds}\Big|_{s=0}\lambda(g(s)) = \int -h_{ij}(R_{ij} + \nabla_i\nabla_j f)e^{-f}dV,$$

where f is the minimizer.

For any symmetric 2-tensor $h = h_{ij}$ and 1-form $\omega = \omega_i$, we write $Rm(h,h) := R_{ijkl}h_{ik}h_{jl}$, $\operatorname{div}\omega := \nabla_i\omega_i$, $(\operatorname{div} h)_i := \nabla_j h_{ji}$, $(\operatorname{div}^*\omega)_{ij} = -\frac{1}{2}(\nabla_i\omega_j + \nabla_j\omega_i) = -\frac{1}{2}L_{\omega^\#}g_{ij}$.

Theorem 5.1 (See [15]) *Let (M^n, g) be a compact Ricci flat manifold and consider variations $g(s) = g + sh$. Then the second variation $\mathcal{D}^2_g\lambda(h,h)$ of λ at g is given by*

$$\frac{d^2}{ds^2}\Big|_{s=0}\lambda(g(s)) = \int \langle Lh, h\rangle dV,$$

where

$$Lh := \frac{1}{2}\Delta h + \operatorname{div}^*\operatorname{div} h + \frac{1}{2}\nabla^2 v_h + Rm(h,\cdot),$$

and v_h satisfies

$$\Delta v_h = \operatorname{div}\operatorname{div} h.$$

Note that if we decompose $C^\infty(\mathrm{Sym}^2(T^*M))$ as

$$\ker\operatorname{div} \oplus \operatorname{im}\operatorname{div}^*,$$

one verifies that L vanishes on $\operatorname{im}\operatorname{div}^*$, that is, on Lie derivatives. On $\ker\operatorname{div}$ one has

$$L = \frac{1}{2}\Delta_L,$$

where

$$\Delta_L h := \Delta h + 2Rm(h,\cdot) - \mathrm{Rc}\cdot h - h\cdot\mathrm{Rc}$$

is the Lichnerowicz Laplacian on symmetric 2-tensors. We call a critical point g of λ linearly stable if $L \le 0$.

Example 5.1 A Calabi-Yau $K3$ surface and more generally, any manifold with a parallel spinor has $\Delta_L \le 0$ (see [40, 27]). So these manifolds are linearly stable in the sense presented here.

Example 5.2 Let g be compact and Ricci flat. Following [9, 40] we examine conformal variations. It is convenient to replace ug by

$$h = Su := (\Delta u)g - D^2u,$$

which differs from the conformal direction only by a Lie derivative and is divergence free. We have

$$\Delta_L Su = (S\Delta u)g,$$

so Δ_L has the same eigenvalues as Δ. In particular, $N \le 0$ in the conformal direction. This contrasts with the Einstein functional.

5.2 Second variation of ν-energy

Recall that the ν-energy is defined by

$$\nu(g_{ij}) = \inf \Big\{ \mathcal{W}(g, f, \tau) : f \in C^\infty(M), \tau > 0, \ \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int e^{-f} dV = 1 \Big\},$$

and its first variation is given by

$$\frac{d}{ds}\Big|_{s=0} \nu(g_{ij}(s)) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int -h_{ij} \Big[\tau (R_{ij} + \nabla_i \nabla_j f) - \frac{1}{2} g_{ij} \Big] e^{-f} dV.$$

In [15], we stated the second variation of the ν-energy for general shrinking Ricci solitons. Here, for simplicity, we only mention the formula for Einstein metrics, which are trivial shrinking gradient solitons with $f \equiv \frac{n}{2}$, normalized by $\mathrm{Ric} = \frac{g}{2\tau}$.

Theorem 5.2 (See [15]) *Let (M, g) be an Einstein manifold of positive scalar curvature and consider variations $g(s) = g + sh$. Then the second variation $\mathcal{D}_g^2 \nu(h, h)$ is given by*

$$\frac{d^2}{ds^2}\Big|_{s=0} \nu(g(s)) = \frac{\tau}{\mathrm{vol}(g)} \int \langle Nh, h \rangle,$$

where

$$Nh := \frac{1}{2} \Delta h + \mathrm{div}^* \,\mathrm{div}\, h + \frac{1}{2} \nabla^2 v_h + Rm(h, \cdot) - \frac{g}{2n\tau \,\mathrm{vol}(g)} \int \mathrm{tr}_g h,$$

and v_h is the unique solution of

$$\Delta v_h + \frac{v_h}{2\tau} = \mathrm{div}\,\mathrm{div}\, h, \qquad \int v_h = 0.$$

As in the previous case, N is degenerate negative elliptic and vanishes on $\mathrm{im}\,\mathrm{div}^*$. Write

$$\ker \mathrm{div} = (\ker \mathrm{div})_0 \oplus \mathbb{R} g,$$

where $(\ker \mathrm{div})_0$ is defined by $\int \mathrm{tr}_g h = 0$. Then on $(\ker \mathrm{div})_0$ we have

$$N = \frac{1}{2} \Big(\Delta_L - \frac{1}{\tau} \Big),$$

where Δ_L is the Lichnerowicz Laplacian. So the linear stability of a shrinker comes down to the (divergence free) eigenvalues of the Lichnerowicz Laplacian. Let us write μ_L for the maximum eigenvalue of Δ_L on symmetric 2-tensors and μ_N for the maximum eigenvalue of N on $(\ker \mathrm{div})_0$,

Example 5.3 The round sphere is linearly stable: $\mu_N = -\frac{2}{(n-1)\tau} < 0$. In fact, it is *geometrically stable* (i.e. nearby metrics are attracted to it up to scale and gauge) by the results of Hamilton [41, 42, 43] and Huisken [47].

Example 5.4 For complex projective space $\mathbb{CP}^m$, the maximum eigenvalue of Δ_L on $(\ker \mathrm{div})_0$ is $\mu_L = \frac{1}{\tau}$ by work of Goldschmidt [37], so $\mathbb{CP}^m$ is neutrally linearly stable, i.e. the maximum eigenvalue of m on $(\ker \mathrm{div})_0$ is $\mu_N = 0$.

Example 5.5 Any product of two Einstein manifolds $M = M_1^{n_1} \times M_2^{n_2}$ is linearly unstable, with $\mu_N = \frac{1}{2\tau}$. The destabilizing direction $h = \frac{g_1}{n_1} - \frac{g_2}{n_2}$ corresponds to a growing discrepancy in the size of the factors.

Example 5.6 Any compact Kähler-Einstein manifold X^n of positive scalar curvature with $\dim H^{1,1}(X) \geq 2$ is linearly unstable. Indeed, we can compute μ_N as follows. Let σ be a harmonic 2-form and h be the corresponding metric perturbation; then $\Delta_L h = 0$, and if σ is chosen perpendicular to the Kähler form, then as above we obtain $\mu_N = \frac{1}{2\tau}$.

Example 5.7 Let Q^m denote the complex hyperquadric in $\mathbb{CP}^{m+1}$ defined by

$$\sum_{i=0}^{m+1} z_i^2 = 0,$$

a Hermitian symmetric space of compact type, hence a Kähler-Einstein manifold of positive scalar curvature.

(a) Q^2 is isometric to $\mathbb{CP}^1 \times \mathbb{CP}^1$, the simplest example of the above instability phenomenon.

(b) Q^3 has $\dim H^{1,1}(Q^3) = 1$, so the above discussion does not apply. But the maximum eigenvalue of Δ_L on $(\ker \operatorname{div})_0$ is $\mu_L = -\frac{2}{3\tau}$ by work of Gasqui and Goldschmidt [35] (or see [36]). The proximate cause is a representation that appears in the sections of the symmetric tensors but not in scalars or vectors. Therefore, Q^3 is linearly unstable with

$$\mu_N = \frac{1}{6\tau}.$$

See further exploration of this example in [8].

(c) For Q^4, the maximum eigenvalue of Δ_L on symmetric tensors is $\mu_L = -\frac{1}{\tau}$ by work of Gasqui and Goldschmidt [34] (or see [36]). So Q^4 is neutrally linearly stable: $\mu_N = 0$.

6 The Gaussian Density of Shrinking Ricci Solitons

In [15], the notion of Gaussian density (or central density) of a shrinking Ricci soliton is introduced. In the case of a compact shrinking soliton (M^n, g), the Gaussian density is simply given by

$$\Theta(M) = \Theta(M, g) := e^{\nu(M,g)},$$

where $\nu(M, g) = \nu(g_{ij})$ is the ν-energy of the compact shrinking Ricci soliton (M, g).

In the following discussion, we will normalize Einstein manifolds of positive scalar curvature by $\mathrm{Ric} = \frac{g}{2\tau}$, $\tau = \frac{1}{2(n-1)}$, so that the round sphere S^n has radius 1. As shown in [15], we have the following facts.

(1) $\Theta(S^n) = \Big(\dfrac{n-1}{2\pi e}\Big)^{n/2} \operatorname{vol}(S^n)$.

(2) If M is an Einstein manifold of positive scalar curvature, then

$$\Theta(M) = \Big(\frac{1}{4\pi\tau e}\Big)^{\frac{n}{2}} \operatorname{vol}_\tau(M) \leq \Theta(S^n),$$

with equality if and only if $M = S^n$.

(3) $\Theta(\mathbb{CP}^m) = \Big(\dfrac{m+1}{\pi e}\Big)^N \dfrac{\operatorname{vol}(S^{2m+1})}{2\pi}$.

(4) The Kähler-Einstein manifold $M = \mathbb{CP}^2 \# k(-\mathbb{CP}^2)$, $k = 0, 3, \cdots, 8$, has $\Theta(M) = \frac{9-k}{2e^2}$.

(5) $\Theta(M_1 \times M_2) = \Theta(M_1)\Theta(M_2)$.

As in [15], we say that one shrinking soliton *decays* to another if there is a small perturbation of the first whose Ricci flow develops a singularity modelled on the second. Because the ν-invariant is monotone during the flow, decay can only occur from a shrinking soliton of lower density to one of higher density. This creates a "decay lowerarchy". (It should be a partial order.)

7 4-D Einstein Manifolds and Shrinking Ricci Solitons

In this section we collect information about stability and Gaussian density values of all known (orientable) positive Einstein 4-manifolds and 4-dimensional compact shrinking Ricci solitons. Below is a list containing all those information. Note that we knew the Θ values of all known examples except Wang-Zhu soliton (see [61]). We also remark that Koiso-Cao soliton metric (see [49, 11]) and the Page metric (see [53]) are both U(2)-invariant metrics on $\mathbb{CP}^2\#(-\mathbb{CP}^2)$. The former, but not the latter, is Kähler. At this point we do not know whether Koiso-Cao soliton, Page metric, and Wang-Zhu soliton are stable or not, though we suspect they are not. As pointed out in [15], the Page metric may well decay to the Koiso-Cao metric, and, by the discussion in [31], either metric might decay to $\mathbb{CP}^2$ via a $\mathbb{CP}^1$ pinches off.

Shrinking Solitons	**Type**	Θ	Θ	**Stability**
S^4	Einstein	$\frac{6}{e^2}$	.812	Stable
$\mathbb{CP}^2$	Einstein	$\frac{9}{2e^2}$	.609	Stable
$S^2 \times S^2$	Einstein product	$\frac{4}{e^2}$	.541	Unstable
$\mathbb{CP}^2\#(-\mathbb{CP}^2)$	Kähler-Ricci soltion (see [49, 11])	$\frac{3.826}{e^2}$	.518	Unknown
$\mathbb{CP}^2\#(-\mathbb{CP}^2)$	Einstein (Page metric) (see [53])	$\frac{3.821}{e^2}$	.517	Unknown
$\mathbb{CP}^2\#2(-\mathbb{CP}^2)$	Kähler-Ricci soltion (see [61])	$< \frac{7}{2e^2}$	???	Unknown
$\mathbb{CP}^2\#3(-\mathbb{CP}^2)$	Kähler-Einstein	$\frac{3}{e^2}$	.406	Unstable
$\mathbb{CP}^2\#4(-\mathbb{CP}^2)$	Kähler-Einstein	$\frac{5}{2e^2}$	.338	Unstable
$\mathbb{CP}^2\#5(-\mathbb{CP}^2)$	Kähler-Einstein	$\frac{2}{e^2}$	.271	Unstable
$\mathbb{CP}^2\#6(-\mathbb{CP}^2)$	Kähler-Einstein	$\frac{3}{2e^2}$	.203	Unstable
$\mathbb{CP}^2\#7(-\mathbb{CP}^2)$	Kähler-Einstein	$\frac{1}{e^2}$	.135	Unstable
$\mathbb{CP}^2\#8(-\mathbb{CP}^2)$	Kähler-Einstein	$\frac{1}{2e^2}$	.068	Unstable

Finally, we remark that on any compact Einstein 4-manifold M^4, the Hitchin-Thorpe inequality (see e.g., [5]) says that

$$2\chi(M) \geq 3|\tau(M)|,$$

where $\chi(M)$ is the Euler characteristic and $\tau(M)$ is the signature of M^4. An interesting question is whether the Hitchin-Thorpe inequality holds for 4-dimensional compact shrinking Ricci soliton.

8 Open Problems

In conclusion, we collect the following open problems:

(i) Is it true that linearly stable compact 4-dimensional shrinking solitons are necessarily Einstein?

(ii) Show that the only linearly stable Einstein 4-manifolds are either the round sphere S^4 or the complex projective space $\mathbb{CP}^2$ with the Fubini-Study metric.

(iii) Does the Hitchin-Thorpe inequality hold for compact 4-dimensional shrinking solitons?

(iv) Compute the precise value of the Gaussian density Θ for the Wang-Zhu soliton metric on $\mathbb{CP}^2\#2(-\mathbb{CP}^2)$.

(v) For $n \geq 4$, find a non-product, nontrivial, purely Riemannian, compact (or complete noncompact) shrinking soliton. A lot of techniques have been developed in the last couple of decades to construct Einstein metrics. It would be very interesting to explore and extend some of these techniques to construct shrinking solitons.

(vi) Show that for $n \geq 4$ there are no complete noncompact κ-noncollapsed shrinking Ricci soliton with positive sectional curvature. For $n = 3$, this was proved by Perelman (see Proposition 4.3).

(vii) Show that any Type I singularity model obtained as a rescaling limit of a Type I maximal solution is necessarily a gradient shrinking soliton.

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