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Optimally Embedding 3-Ary n-Cubes into Grids

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Abstract The 3-ary n-cube, denoted as Q_n^3, is an important interconnection network topology proposed for parallel computers, owing to its many desirable properties such as regular and symmetrical structure, and strong scalability, among others. In this paper, we first obtain an exact formula for the minimum wirelength to embed Q_n^3 into grids. We then propose a load balancing algorithm for embedding Q_n^3 into a square grid with minimum dilation and congestion. Finally, we derive an $O(N^2)$ algorithm for embedding Q_n^3 into a gird with balanced communication, where N is the number of nodes in Q_n^3. Simulation experiments are performed to verify the total wirelength and evaluate the network cost of our proposed embedding algorithm.

Keywords 3-ary n-cube, embedding algorithm, grid, interconnection network

1 Introduction

Interconnection networks take an important role in parallel computer systems. Selecting an appropriate interconnection network is crucial because it can greatly affect the parallel computer's communication, fault tolerant capability, and hardware cost. The topology of an interconnection network specifies the way processors are connected in a parallel computer system. It determines the network's bandwidth, delay, reliability, scalability, and adaptability. Therefore, the selection of the interconnection network is a vital decision in parallel computer design. When evaluating the performance of an interconnection network, embeddability and fault-tolerability are two critical metrics.

The hypercube is one of the most popular interconnection networks for parallel computing systems[1] due to its many attractive properties, such as regularity, recursive structure, node symmetry and edge symmetry, and efficient routing and broadcasting. The 3-ary n-cube, denoted as Q_n^3, is proposed, and soon considered as an important extension of hypercube. Q_n^3 not only preserves the excellent properties of the hypercube, but also adds new desired properties, such as reduced message latency and ease of implementation[2,3]. Because of Q_n^3's excellent properties, it has attracted the interest of many researchers since its proposal. Hsieh *et al.* studied the embedding of paths and cycles into Q_n^3, and proved that Q_n^3 is edge-pancyclic[4]. Dong *et al.* studied the embedding of paths and cycles into 3-ary n-cubes with faulty nodes/links[5]. Lv *et al.* worked on Hamiltonian cycle/path embedding in 3-ary n-cubes with the fault of structure $K_{1,3}$[6]. Yuan *et al.* investigated the g-good-neighbor conditional diagnosability of Q_n^3 under the PMC and MM* models, which facilitated accurate reliability measurements in parallel systems using Q_n^3 as the underlying network[7].

Q_n^3 has not only attracted research interest in

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academic community, but also been practically used to build parallel computers. The Blue Gene/L and the Cray XT5 supercomputers are two prominent examples[8]. In Blue Gene/L computer, the processors are interconnected through five networks and constructed through a 3D torus topology. This super computer has the highest total bandwidth and manages a large quantity of communications. Moreover, Q_n^3 has also been used in the construction of data center networks such as CamCube[9] and NovaCube[10]. CamCube is designed to make it easier to develop services in data centers, which uses the 3D torus topology as an alternative to the traditional switch-based network, with each server directly connected to six other servers. CamCube solves the problem of building distributed programs running in data centers, and builds a much simpler platform to implement these applications. When multiple routing is performed on CamCube, the application is more efficient and the extra performance cost is extremely low. NovaCube is also a data center network topology based on Q_n^3, which has added the connection between servers on the basis of regular torus.

An interconnection network can generally be modeled by a graph in which vertices (nodes) represent processors, and edges represent communication links between processors. We usually denote a graph by $G = (V, E)$, where V is the vertex set and E the edge set. Graph embedding is the operation of mapping a guest graph into a host graph. Given a guest graph G and a host graph H, an embedding f from G to H can be defined as an injective mapping from $V(G)$ to $V(H)$. The quality of an embedding can be measured by certain cost criteria. Some common ones are congestion, dilation, expansion, and load. We will define these parameters in Section 2. In addition to these parameters, wirelength is another criterion for embedding, and is widely used in VLSI design[11]. The wirelength is the total wire length required to complete the entire VLSI layout.

Most researches on graph embedding consider paths, cycles, meshes and trees as guest graphs because these are the structures widely used in parallel computers[12–15]. In [16], Fan *et al.* studied the embedding of paths with all possible lengths between any two vertices into crossed cube. Fan *et al.* also proved that the cycles of all possible lengths can be embedded into the twisted cube[17]. Han *et al.* studied the embedding of three different types of special meshes into locally twisted cubes[18]. In all these embeddings, the guest graphs (paths and cycles) are less complex than the host.

Another set of embedding problems focus on embedding guest graphs into linear arrays and grids. That is, the guest graphs are more complex than the host. Embedding the graph into a linear array is also called linear layout (or linear arrangement) problem. The minimum linear layout problem was first stated by Harper in 1964 and has been proved to be NP-complete[19]. Nakano proposed a linear layout of generalized hypercube[20]. Fan *et al.* solved the minimum linear arrangement problem for exchanged hypercube in linear time[21]. Miller *et al.* studied the minimum linear arrangement of incomplete hypercubes[22]. Interconnection networks can also layout into optical linear arrays. In [23], Chen and Shen discussed embeddings of bidirectional and unidirectional hypercubes on a class of optical networks which include linear arrays. Yu *et al.* proposed an embedding of 3-ary n-cubes into optical linear arrays with minimum congestion[24]. In [25], Liu studied the embedding of exchanged hypercubes into optical linear arrays with optimal congestion.

The grid embedding is concerning not only the grid's ability to simulate other structures, but also different structures' layout on chips. Network-on-chip (NoC) is a new communication mode of system-on-chip (SoC)[26–28]. The topological structure of NoC largely refers to the structure of the macro network, that is, the interconnection network made into the chip. NoC topology can be classified into two categories. One is direct network topology, such as mesh[29] and torus[30]. The other is indirect network topology, such as fat-tree[31] and butterfly. Due to the restriction of the chip area, the embedded network's total wirelength becomes a crucial issue that affects the NoC's communication performance. In [11], Bezrukov *et al.* obtained approximate results and the lower bound estimate of wirelength for embedding hypercube into a grid. They also studied the exact congestion for embedding the hypercube into a rectangular grid[32]. Heckmann *et al.* stated an optimal embedding of complete binary trees into lines and grids with optimal dilation[33]. In [34], Manuel *et al.* proposed an embedding of hypercube into a grid with minimum wirelength. Wei *et al.* proposed a new distributed congestion control mechanism for NoC[35]. Experiments showed that their congestion control mechanism alleviated performance degradation for loads beyond saturation, and maintained adequate levels of throughput at high loads.

When laying out interconnection networks into square grids, smaller layout area means faster communication. Two important factors affect layout on chips: the number of tracks and the quality of communication. A track is a continuous horizontal or vertical line on which the wires are placed without overlapping any other wires[36]. For a given network, a good layout should minimize the number of tracks. In communication among components, load balancing improves the distribution of workloads across multiple computing resources. Load balancing aims to optimize resource use, maximize throughput, minimize response time, and avoid overload of any single resource.

Most existing embedding results focused on optimizing just one single parameter, without considering other parameters. In this paper, we try to achieve multiple optimization targets while embedding Q_n^3 into grids. We first investigate the embedding of Q_n^3 into a linear array (a special, 1-dimensional grid) and a grid, respectively, with minimum wirelength. We then propose a layout of Q_n^3 into a square grid. We will present an algorithm for embedding Q_n^3 into a grid with balanced communication while minimizing dilation and congestion. The major contributions of the paper are as follows.

1) We prove that the minimum wirelength of Q_n^3 into a linear array is $\frac{1}{2}(3^{2n}-3^n)(3^{n+1}-2n\times 3^{n-1})$, and the minimum wirelength of Q_n^3 layout into grid $M(3^{n_1},3^{n_2})$.

2) We prove that Q_n^3 can be embedded into a 2-dimensional square grid with dilation $2\times 3^{\frac{n}{2}-1}$ for even n, $3^{\lceil\frac{n}{2}\rceil-1}$ for odd n, and with congestion

$$\begin{cases} \dfrac{3^{\lceil n/2\rceil+1}-1}{8}, & \text{if } \lceil n/2\rceil \text{ is odd},\\ \dfrac{3^{\lceil n/2\rceil+3}-3}{8}, & \text{if } \lceil n/2\rceil \text{ is even}.\end{cases}$$

3) We will present an $O(N^2)$ algorithm for embedding Q_n^3 into a grid with balanced communication, where N denotes the number of vertices in Q_n^3.

The rest of this paper is organized as follows. Section 2 gives definitions and notations used in the paper. Section 3 presents the embedding of Q_n^3 into a linear array and a grid, respectively, with minimum wirelength. Section 4 gives an embedding of Q_n^3 into a square grid, and proposes an embedding algorithm with balanced communication. Section 5 concludes the paper.

2 Preliminaries

In this section, we will give some definitions used in this paper. All graphs in this paper are simple undirected graphs. Given two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, if $V_2\subseteq V_1$ and $E_2\subseteq E_1$, G_2 is said to be a subgraph of G_1. The subgraph induced by $V^{'}\subseteq V_1$ in G_1 is denoted by $G_1[V^{'}]$, where $V^{'}\subseteq V_1$. Furthermore, we use $G-V'$ to denote $G[V(G)\backslash V']$. For a graph $G=(V,E)$, a (u,v)-path of length l from vertex u to vertex v is denoted by $P=(u_0,u_1,...,u_{l-1})$, where $u_0=u$ and $u_i=v$ are called the two end vertices of path P, and all the vertices $u_0,u_1,...,u_{l-1}$ are distinct. A Hamiltonian path is defined as a path which traverses each vertex of graph G exactly once. If there exists a Hamiltonian path between any two distinct vertices of graph G, we say that graph G is a Hamiltonian connected graph.

Graph embedding can be defined as: for two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, where G_1 represents the graph to be embedded, and G_2 represents the graph into which other graphs are to be embedded, an embedding from G_1 to G_2 is an injective mapping $\psi: V(G_1)\to V(G_2)$. There are four common parameters used to measure the quality of an embedding. The congestion of an embedding ψ is defined as $cong(G_1,G_2,\psi)=\max\{cong(e)|e\in E_2\}$, which measures queuing delay of messages, where $cong(e)$ denotes the number of edges of G_1 whose image paths in G_2 include the edge e.

Definition 1[11]. *Let $EC_f(e)$ denote the number of edges (u,v) of G such that e is in the path $P_f(u,v)$ between vertices $f(u)$ and $f(v)$ in H. The edge congestion of an embedding f of G into H is given by,*

$$EC_f(G,H)=\max\{EC_f(e)|e\in E(H)\}.$$

Then, the minimum edge congestion of G into H is defined as

$$EC(G,H))=\min\{EC_f(G,H)|f \textit{ is an embedding from } G \textit{ to } H\}.$$

The smaller the congestion of an embedding is, the lower the queuing delay that the graph G_2 simulates the graph G_1. The expansion of an embedding ψ of G_1 into G_2 is defined as $exp(G_1,G_2,\psi)=|V_1|/|V_2|$, which measures processor utilization. The smaller the expansion of an embedding is, the more efficient the processor utilization that the graph G_2 simulates the graph G_1. Obviously, the expansion of the embedding is at least 1. The dilation of embedding ψ is defined as: $dil(G_1,G_2,\psi)=\max\{\text{dist}(G_2,\psi(u),\psi(v))\,|(u,v)\in E_1\}$, which measures the communication delay, where $\text{dist}(G_2,\psi(u),\psi(v))$ denotes the distance between the two vertices $\psi(u)$ and

$\psi(v)$ in G_2. The smaller the dilation of an embedding is, the shorter the communication delay that the graph G_2 simulates the graph G_1. The processing time of tasks is another crucial factor to measure the communication performance, referred as to the load in the embedding. The load of an embedding ψ is denoted by $load(G_1, G_2, \psi) = \max\{load(v)|v = \psi(u), u \in V_1\}$, where $load(v)$ denotes the number of vertices of G_1 mapped on v. For graph G_1 with N vertices and G_2 with M vertices, we say an embedding has a balanced load when the load of every vertice of G_1 is at least $\lfloor \frac{N}{M} \rfloor$ and at most $\lceil \frac{N}{M} \rceil$.

Definition 2[34]. *The wirelength of an embedding f of G into H is given by*

$$WL_f(G, H) = \sum_{(u,v)\in G} d_H(f(u), f(v)),$$

where $d_H(f(u), f(v))$ denotes the length of the path $P_f(u, v)$ in H. Then, the minimum wirelength of G into H is defined as

$$WL(G, H) = \min\{WL_f(G, H)|f \text{ is an embedding from } G \text{ to } H\}.$$

The wirelength problem is to find an embedding of G into H that induces the minimum wirelength, and thought to be cost-effective.

A graph G_1 is isomorphic to another graph G_2 (represented by $G_1 \cong G_2$) if there exists a bijection $f : V(G_1) \to V(G_2)$, such that $(u, v) \in E(G_1)$ if and only if $(f(u), f(v)) \in E(G_2)$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, and a subset $S \subseteq V_1$, let f be a mapping from V_1 to V_2. Let $T = \{x \in V(G_2)|\text{there is } y \in S, \text{ such that } y = f(x)\}$. Then we write $T = f(S)$ and $S = f^{-1}(T)$. Given graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2), ..., G_k = (V_k, E_k)$, we define the cross product of $G_1, G_2, ..., G_k$, denoted by $G_1 \otimes G_2 \otimes ... \otimes G_k$, where $V = \{(v_1, v_2, ..., v_k)|v_i \in V_i, 1 \leqslant i \leqslant k\}$ and $E = \{((u_1, u_2, ..., u_k), (v_1, v_2, ..., v_k))|\text{such that } (u_i, v_i) \in E_i \text{ and } u_j = v_j \text{ for } 1 \leqslant i \leqslant k,\ j \neq i\}$. We define a k-ary cycle of length k, denoted by C_k, as a graph consisting of k vertices and k edges. Then we give the definition of 3-ary n-cube as below.

Definition 3. *The 3-ary n-cube can be seen as cross product of n 3-cycles*:

$$Q_n^3 = \underbrace{C_3 \otimes C_3 \otimes ... \otimes C_3}_{n}.$$

Therefore, Q_n^3 can also be defined as follows:

$$Q_n^3 = \begin{cases} C_3, & if\ n = 1, \\ C_3 \otimes Q_{n-1}^3, & if\ n \geqslant 2. \end{cases}$$

For any integer $n \geqslant 1$, a binary string x of length n will be written as $x_{n-1}x_{n-2}...x_1x_0$, where $x_i \in \{0, 1\}$ for any integer $i \in \{0, 1, ..., n-1\}$. Given any $x = x_{n-1}x_{n-2}...x_1x_0$, for any inter $i \in \{0, 1, ..., n-1\}$, x_i is said to be the i-th bit of x and $x_{n-1}x_{n-2}...x_k$ $(0 \leqslant k \leqslant n-1)$ is called a prefix of x. Besides, x_0 is called the first bit of x, and x_{n-1} is called the last bit of x. We have another definition of 3-ary n-cube as below.

Definition 4[4]. *The 3-ary n-cube Q_n^3 $(n \geqslant 1)$ has $N = 3^n$ vertices, each of the form $x = (x_{n-1}...x_1x_0)$, where $0 \leqslant x_i \leqslant 2$ for every $0 \leqslant i \leqslant n-1$. Two vertices $x = (x_{n-1}...x_1x_0)$ and $y = (y_{n-1}...y_1y_0)$ are adjacent if and only if there exists an integer j with $0 \leqslant j \leqslant n-1$, such that $x_j = y_j \pm 1 (\text{mod } 3)$ and $x_i = y_i$ for $i \in \{0, 1, 2, ..., n-1\} - \{j\}$.*

Furthermore, the i-th position, from the right to the left, of the n-bit string $x_nx_{n-1}...x_1$ is called the i-dimension. The edge (x, y) is called a j-dimensional edge or simply a j-edge. A vertex incident to a j-edge is called a j-dimensional vertex.

Let $Q_{n-1}^3(p)$ denote the subgraph of Q_n^3 induced by $\{(u_{n-1}u_{n-2}...u_i...u_0) \in V(Q_n^3)|u_i = p\}$, where $0 \leqslant p \leqslant 2$. We may divide Q_n^3 into three disjoint subgraphs: $Q_{n-1}^3(0)$, $Q_{n-1}^3(1)$, $Q_{n-1}^3(2)$ along dimension i for any i with $0 \leqslant i \leqslant 2$. By Definition 4, we have $Q_{n-1}^3(j) \cong Q_{n-1}^3$, for any integer j with $0 \leqslant j \leqslant 2$. According to the definition of Q_n^3, there are exactly 3^{n-1} edges, which form a perfect matching between $Q_{n-1}^3(j)$ and $Q_{n-1}^3(j+1)$ for $0 \leqslant j \leqslant 2$. We call $Q_{n-1}^3(j)$ and $Q_{n-1}^3(j+1)$ to be adjacent subcubes, and call the edges between two adjacent subcubes "bridges". Figs.1(a)–1(c) demonstrate Q_1^3, Q_2^3, and Q_3^3, respectively. Similar to the n-dimensional hypercube, the n-dimensional Q_n^3 is $2n$-regular.

3 Embedding the 3-Ary n-Cube into a Linear Array and a Grid

In this section, we propose embeddings of Q_n^3 into a linear array and a grid with minimum wirelength, respectively. Before discussing the issue, we first introduce the following definitions. The wirelength problem is solved by edge isoperimetric problem.

3.1 Edge Isoperimetric Problem for 3-Ary n-Cube

In this subsection, we investigate the optimal set and the edge isoperimetric problem of 3-ary n-cube.

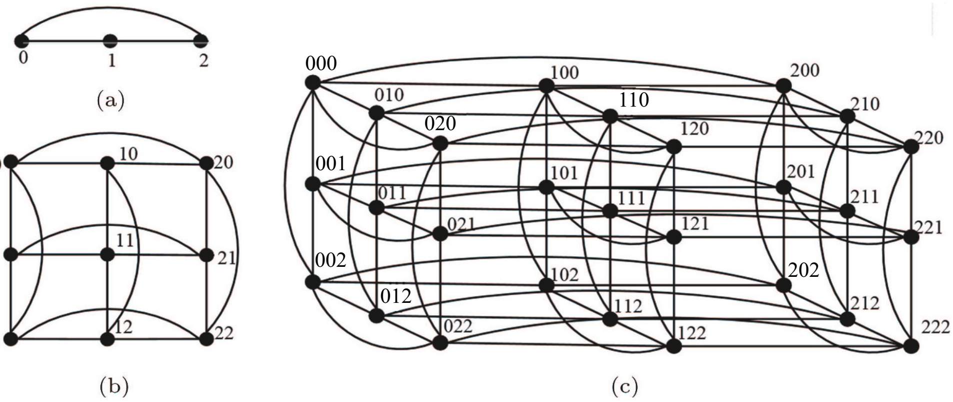

Fig.1. (a) 3-ary 1-cube Q_1^3. (b) 3-ary 2-cube Q_2^3. (c) 3-ary 3-cube Q_3^3.

The maximum induced subgraph of Q_n^3 is crucial for calculating the edge congestion. Therefore, the primary purpose in this subsection is to prove the maximum induced subgraph of Q_n^3.

The following two definitions of the edge isoperimetric problem of a graph $G = (V, E)$ have been studied in [37]. The first problem is to find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimum size among all subsets of the same cardinality. Mathematically, for a given positive integer m, if $\theta_G(m) = \min_{A\subseteq V,|A|=m}|[A, V - A]_G|$, where $[A, V - A]_G = \{(u, v) \in E|u \in A, v \in (V - A)\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $|[A, V - A]_G| = \theta_G(m)$, which is called an optimal set.

Another problem is called maximum induced subgraph problem[37], which is to find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximum among all induced subgraphs with the same number of vertices. Mathematically, for a given positive integer m, if $I_G(m) = \max_{A\subseteq V,|A|=m}|T_G(A)|$, where $T_G(A) = \{(u, v) \in E|u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $|A| = m$ and $T_G(A) = |I_G(m)|$. For regular graphs, the optimal set problem and the maximum subgraph problem are equivalent.

Lemma 1[38]. *Let V be the set of vertices of Q_n^3, and $Q_{n-1}^3(0)$, $Q_{n-1}^3(1)$ and $Q_{n-1}^3(2)$ are three disjoint subgraphs. Then* $|E(Q_n^3[V_{i+j}])| \leqslant \sum_{i=0}^{2} |E(Q_n^3[V_i])| + \sum_{0\leqslant i<j\leqslant 2} \min\{|V_i|, |V_j|\}$.

Definition 5. *For any integer $m \geqslant 1$ and $S \subseteq V(G)$ with $|S| = m$, if $G[S]$ is the subgraph with the maximum number of edges among all induced subgraphs with m vertices, then $G[S]$ is called the maximum induced graph with m vertices in G.*

Definition 6. *Let $f : V(Q_n^3) \to \{1, 2, ..., 3^n\}$ be a mapping, where for arbitrary vertex $u = u_{n-1}u_{n-2}...u_0$ in Q_n^3,*

$$lex(u) = \sum_{i=0}^{n-1} u_i 3^i + 1.$$

which is actually the decimal number of u.

Lemma 2. *Let IL_i denote the incomplete Q_n^3 on i vertices, and then L_i is isomorphic to IL_i for $1 \leqslant i \leqslant 3^n$.*

Proof. Let $f : L_i \to IL_i$ by $f(l) = 3^n - l - 1$. Therefore, if $(l_1l_2...l_n)$ is the ternary representation of l, then $(l_1'l_2'...l_n')$ is the ternary representation of $f(l)$, where $l_i' = 1 - l_i$. Then (x, y) is an edge in $L_i \Leftrightarrow$ the ternary representations of u and v differ in exactly one bit, and the same holds for $f(u)$ and $f(v)$. Thus (u, v) is an edge in L_i and $(f(u), f(v))$ is an edge in IL_i. □

Lemma 3. *Let K be a subgraph of Q_n^3 isomorphic to L_k where $k \leqslant 3^{n-1}$. Let K_1, K_2 and K_3 be disjoint segments induced by k_1, k_2 and k_3 consecutive vertices of $Q_{n-1}^3(0)$, $Q_{n-1}^3(1)$ and $Q_{n-1}^3(2)$, respectively such that $k_1 + k_2 + k_3 = k$. Then $|E(Q_n^3[K_1 \cup K_2 \cup K_3])| \leqslant |E(Q_n^3[K])|$.*

Proof. By the definition of Q_n^3, we can partition Q_n^3 into three disjoint subgraphs: $Q_{n-1}^3(0)$, $Q_{n-1}^3(1)$, $Q_{n-1}^3(2)$ along dimension i for any i with $0 \leqslant i \leqslant 2$. Let $E(Q_n^3[K_j \wedge K_{j+1}])$ denote the set of edges in Q_n^3 with one end in K_j and the other end in K_{j+1}. Without loss of generality, we assume that $k_3 \leqslant k_2 \leqslant k_1$. We have the following cases.

Case 1. $1 \leqslant |V| \leqslant 3^{n-1}$.

Case 1.1 $K_1 \subseteq Q_{n-1}^3(0)$.

Assume $k^{'} = l_1 + l_2$ be the number of non-consecutive vertices in $K^{'}$ that lie in $Q_{n-1}^3(0)$ where $k^{'} = k_1$. Let L_1 and L_2 be two disjoint segments induced by l_1 and l_2 consecutive vertices in $Q_{n-1}^3(0)$.

Clearly, $|E(Q_n^3[K'])| \leqslant |E(Q_n^3[L_{k'}])|$. This implies that $|E(Q_n^3[K'])| = |E(Q_n^3[l_1])| + |E(Q_n^3[l_2])| + |E(Q_n^3[l_1 \wedge l_2])| \leqslant |E(Q_n^3[L_{l_1}])| + |E(Q_n^3[L_{l_2}])| + l_1$. By Lemma 1, we get $|E(Q_n^3[L_1 \cup L_2])| \leqslant |E(Q_n^3[K'])|$.

Case 2. $3^{n-1} < |V| < 2 \times 3^{n-1}$.

Case 2.1 $K_1 \subset Q_n^3(i), 0 \leqslant i \leqslant 2$.

Let k_1 and k_2 be the vertices that lie in $Q_{n-1}^3(0)$ and $Q_{n-1}^3(1)$, respectively, inducing subgraphs K_1 and K_2, respectively. Since there is only one edge between K_1 and K_2, $|E(Q_n^3[K_1 \wedge K_2])| \leqslant k_2$. Let $H_1 = L_{k_1}$. Then $|E(Q_n^3[H_1])| = |E(Q_n^3[L_{k_1}])|$. Let H_2 be the subgraph of Q_n^3 induced by the vertices in $Q_{n-1}^3(1)$ labeled as $3^{n-1}-1, 3^{n-1}-2, ..., 3^{n-1}-k_2$. This implies that $|E(Q_n^3[K_1 \cup K_2])| = |E(Q_n^3[K_1])| + |E(Q_n^3[K_2])| + |E(Q_n^3[K_1 \wedge K_2])| \leqslant |E(Q_n^3[L_{k_1}])| + |E(Q_n^3[L_{k_2}])| + k_2$. By Lemma 1, we get $|E(Q_n^3[K_1 \cup K_2])| \leqslant |E(Q_n^3[L_{k_1} + L_{k_2}])| = |E(Q_n^3[H_1 \cup H_2])|$.

Case 2.2 $K_1 \subset Q_n^3(i) \cup Q_{n-1}^3(i+1), 0 \leqslant i \leqslant 2$.

Let k_1, k_2 be the number of consecutive vertices in K_1, K_2 that lie in $E(Q_n^3[K_1 \wedge K_2]$ respectively. Then $|E(Q_n^3[K_1])| \leqslant |E(Q_n^3[L_{k_1}])|$, $|E(Q_n^3[K_2])| \leqslant |E(Q_n^3[L_{k_2}])|$ and $|E(Q_n^3[K_1 \wedge K_2])| \leqslant 2k_2$. Hence $|E(Q_n^3[K_1 \cup K_2])| \leqslant |E(Q_n^3[L_{k_1}])| + |E(Q_n^3[L_{k_2}])| + 2k_2$. Let $H_1 = L_{k_1}$. Then $|E(Q_n^3[H_1])| = |E(Q_n^3[L_{k_1}])|$. Let H_2 be the subgraph of Q_n^3 induced by the vertices in Q_s^1 labeled as $3^{n-1}-1, 3^{n-1}-2, ..., 3^{n-1}-k_2$. This implies $|E(Q_n^3[H_2])| = |E(Q_n^3[L_{k_2}])|$ and $|E(Q_n^3[H_1 \wedge H_2])| \geqslant 2k_2$. Therefore $|E(Q_n^3[H_1 \wedge H_2])| \geqslant |E(Q_n^3[L_{k_1}])| + |E(Q_n^3[L_{k_2}])| + 2k_2$ and hence $|E(Q_n^3[K_1 \cup K_2])| \leqslant |E(Q_n^3[H_1 \cup H_2])|$. Let $k_2 \cup k_3 = p + q$ be the number of vertices in K_2 that lie in $Q_n^3 \setminus Q_{n-1}^3(i)$. Then $|E(Q_n^3[K_1 \wedge K_2])| \leqslant k_2$. But $|E(Q_n^3[K_1])| \leqslant |E(Q_n^3[L_{k_1}])|$. Similarly $|E(Q_n^3[K_2])| \leqslant |E(Q_n^3[L_{k_2}])|$. This implies that $|E(Q_n^3[K_1 \cup K_2])| = |E(Q_n^3[K_1])| + |E(Q_n^3[K_2])| + |E(Q_n^3[K_1 \wedge K_2])| \leqslant |E(Q_n^3[L_{k_1}])| + |E(Q_n^3[L_{k_2}])| + k_2$. By Lemma 1, we get $|E(Q_n^3[K_1 \cup K_2])| \leqslant |E(Q_n^3[H_1 \cup H_2])|$.

Case 3. $2 \times 3^{n-1} + 1 < |V| \leqslant 3^n$.

Let k_1, k_2 and k_3 be the number of consecutive vertices in K_1, K_2 and K_3 that lie in $Q_{n-1}^3(0)$, $Q_{n-1}^3(1)$ and $Q_{n-1}^3(2)$ respectively. Then $|Q_n^3[K_1])| \leqslant |E(Q_n^3[L_{k_1}])|$, $|E(Q_n^3[K_2])| \leqslant |E(Q_n^3[L_{k_2}])|$ and $|E(Q_n^3[K_3])| \leqslant |E(Q_n^3[L_{k_3}])|$. Then $|E(Q_n^3[K_1 \wedge K_2]| \leqslant 2k_2$, $|E(Q_n^3[K_2 \wedge K_3])| \leqslant 2k_3$ and $|E(Q_n^3[K_3 \wedge K_1])| \leqslant 2k_3$. Hence $|E(Q_n^3[K_1 \cup K_2])| \leqslant |E(Q_n^3[L_{k_1}])| + |E(Q_n^3[L_{k_2}])| + 2k_2$, $|E(Q_n^3[K_2 \cup K_3])| \leqslant |E(Q_n^3[L_{k_2}])| + |E(Q_n^3[L_{k_3}])| + 2k_3$, and $|E(Q_n^3[K_1 \cup K_3])| \leqslant |E(Q_n^3[L_{k_1}])| + |E(Q_n^3[L_{k_3}])| + 2k_3$. Let $H_1 = L_{k_1}$. Then $|E(Q_n^3[H_1])| = |E(Q_n^3[L_{k_1}])|$. Let H_2 be the subgraph of Q_n^3 induced by the vertices in $Q_{n-1}^3(1)$ labeled as $3^{n-1}-1, 3^{n-1}-2, ..., 3^{n-1}-k_2$, and H_3 be the subgraph of Q_n^3 induced by the vertices in $Q_{n-1}^3(2)$ labeled as $3^{n-1}-1, 3^{n-1}-2, ..., 3^{n-1}-k_3$. This implies $|E(Q_n^3[H_2])| = |E(Q_n^3[L_{k_2}])|$ and $|E(Q_n^3[H_1 \wedge H_2])| \geqslant 2k_2$, $|E(Q_n^3[H_3])| = |E(Q_n^3[L_{k_3}])|$ and $|E(Q_n^3[H_2 \wedge H_3])| \geqslant 2k_3$. Therefore $|E(Q_n^3[H_1 \cup H_2 \cup H_3])| \geqslant |E(Q_n^3[L_{k_1}])| + |E(Q_n^3[L_{k_2}])| + |E(Q_n^3[L_{k_3}])| + 2k_2 + 2k_3$ and hence $|E(Q_n^3[K_1 \cup K_2 \cup K_3])| \leqslant |E(Q_n^3[H_1 \cup H_2 \cup H_3])|$. □

Definition 7. *For any integer $m \geqslant 1$ and $S \subseteq V(G)$ with $|S| = m$, if $G[S]$ is the subgraph with the maximum number of edges among all induced subgraphs with m vertices, then $G[S]$ is called the maximum induced graph with m vertices in G.*

Lemma 4. *For any integer $1 \leqslant m \leqslant 3^n$, let $S \subseteq V(Q_n^3)$ with $S = \{x \in V(Q_n^3) | lex(x) \leqslant m\}$. Then $Q_n^3[S]$ is a maximum induced subgraph with m vertices.*

Proof. Let X be a set of m consecutive vertices on S. Let Y be a set of m non-consecutive vertices on S. Then $Y = \bigcup_{i=1}^{j} S_i$ where $j \geqslant 2$, S_i's are mutually disjoint and each S_i is a set of consecutive vertices such that $\sum_{i=1}^{j} |S_i| = k$. We claim that $|E(Q_n^3[Y])| \leqslant |E(Q_n^3[X])|$. We prove this claim by induction on τ. When $\tau = 2$, by Lemma 1, we get $|E(Q_n^3[Y])| \leqslant |E(Q_n^3[X])|$. Assume that the claim is true for $\tau - 1$. Then $|E(Q_n^3[\bigcup_{i=1}^{\tau} K_i])| \leqslant |E(Q_n^3[K])|$ where K is induced by $k - |K_\tau|$ consecutive vertices. And $|E(Q_n^3[\bigcup_{i=1}^{\tau} K_i])| = |E(Q_n^3[\bigcup_{i=1}^{\tau-1} K_i \cup K_p])| \leqslant |E(G[K \cup K_p])| \leqslant |E(G[X])|$. □

Lemma 5. *For $1 \leqslant i \leqslant 3^{n-1}$, L_i is an optimal set in Q_{n-1}^3.*

Proof. Let S be an induced subgraph of Q_n^3 which is isomorphic to L_j, $j \leqslant 3^{n-1}$. Let N be a set of k non-consecutive vertices in Q_n^3. Then $N = \bigcup_{i=1}^{p} A_i$ where $p \geqslant 2$, A_i's are equally disjoint and each A_i is a set of consecutive vertices in Q_n^3 such that $\sum_{i=1}^{p} |A_i| = s$. In case an A_i contains vertices labeled as $3^{n-1}-1$ and 3^{n-1}, then we split A_i into two sets such that one set ends with label $3^{n-1}-1$ and the other set begins with label 3^{n-1}. By induction and Lemma 4, we get $|E(Q_n^3[N])| \leqslant |E(Q_n^3[S])|$. Thus L_i is an optimal set in Q_{n-1}^3. □

Theorem 1. *For $1 \leqslant i \leqslant 3^n$, L_i is an optimal set in Q_n^3.*

Proof. By Definition 4, Q_n^3 can be partitioned into $Q_{n-1}^3[0]$, $Q_{n-1}^3[1]$ and $Q_{n-1}^3[2]$. By Lemma 5, L_i is an optimal set for $1 \leqslant i \leqslant 3^{n-1}$. Now let $i > 3^{n-1}$. Then we have $L_i' = Q_n^3 - L_i \cong L_{3^n - i}$. Since $3^n - i < 3^n - 1$, by Lemma 5, L_i' is an optimal set in Q_n^3. □

3.2 Embedding the 3-Ary n-Cube into a Linear Array

In this subsection, we will give an embedding of Q_n^3 into a linear array with minimum wirelength. When the host graph is a linear array, we call the wirelength of the embedding as linear wirelength, and the dilation of the embedding is most commonly called the bandwidth. The bandwidth problem, which is NP-complete[19], can be defined as follows.

Definition 8. *For any integer $n \geqslant 1$, the linear array of n vertices, denoted by L_n, is a graph such that $V(L_n) = \{1, 2, ..., n\}$ and where $E(L_n) = \{(i, i+1) | i \in [1, n-1]\}$.*

Definition 9. *Let $lex : V(Q_n^3) \rightarrow \{1, 2, ..., 3^n\}$ be a mapping, where for arbitrary vertex $u = u_{n-1}u_{n-2}...u_0$ in Q_n^3,*

$$lex(u) = \sum_{i=0}^{n-1} u_i 3^i + 1,$$

which is actually the decimal number of u.

Let G be a graph and L_n be a linear array with n vertices. Let f be an embedding from G to L_n. The bandwidth of the embedding f of G into L_n is defined as

$$B_f(G) = \max\{|f(v) - f(u)| (u, v) \in E(G)\}.$$

Furthermore, the minimum bandwidth from all embeddings from G to L_n is defined as

$$B(G) = \min\{B_f(G) | f \text{ is an embedding from } G \text{ to } L_n\}.$$

The bandwidth problem is to find an embedding of G into L_n such that it has the minimum bandwidth.

Theorem 2. *Q_n^3 can be embedded into L_{3^n} with dilation $2 \times 3^{n-1}$.*

Proof. Let $f = lex$. For an arbitrary vertex α_0 in $Q_{n-1}^3(0)$, let its incident edges be $(\alpha_0, \alpha_1), (\alpha_0, \alpha_2)$, $(\alpha_0, \alpha_3), (\alpha_0, \alpha_4)$, (α_0, β_0), and (α_0, γ_0), where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in V(Q_{n-1}^3(0))$, $\beta_0 \in V(Q_{n-1}^3(1))$ and $\gamma_0 \in V(Q_{n-1}^3(2))$ (see Fig.2). Clearly, $\max\{\text{dist}(L_{3^n}, f(x), f(y)) | x, y \in \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_0, \gamma_0\}\} = \max\{|f(\gamma_0) - f(\alpha_0)\ |(\alpha_0, \gamma_0) \in Q_n^3\} = 2 \times 3^{n-1}$. Therefore, the dilation of embedding Q_n^3 into L_{3^n} can be formulated as follows:

$$\begin{aligned} & dil(f, Q_n^3, L_{3^n}) \\ = & \max\{\text{dist}(L_{3^n}, f(u), f(v)) | (u, v) \in V(Q_n^3)\} \\ = & 2 \times 3^{n-1}. \end{aligned}$$ □

Lemma 6. *The lex embedding of Q_n^3 into a linear array L_{3^n} induces a minimum wirelength.*

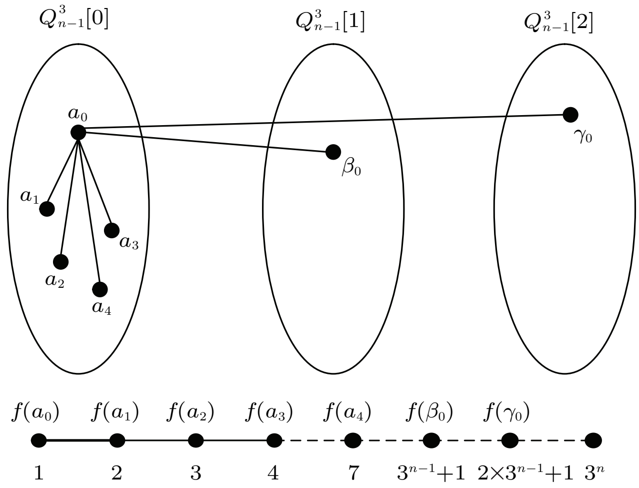

Fig.2. Adjacent vertices of α_0.

Proof. Let $f = lex$ and $G = Q_n^3$. For $1 \leqslant i \leqslant 3^n$, let S_i be the i-th edge of L_{3^n}. Removal of S_i leaves L_{3^n} into two components X_i and $X_i^{'}$ where $V(X_i) = \{0, 1, ..., i\}$ and $V(X_i^{'}) = \{i+1, i+2, ..., 3^n\}$. Then S_i partitions $E(L_{3^n}) - S_i, i = 1, 2$. Then S_1, S_2 are disjoint sets and $S = S_1 \cup S_2$ is an edge cut of L_{3^n}. For each j, $E(L_{3^n}) - S_j$ has three components H_{j1}, H_{j2} and H_{j3} induced by consecutive vertices on C_{3^n} with $|H_{j1}| = 3^{n-1}$, $|H_{j2}| = 3^{n-1}$ and $|H_{j3}| = 3^{n-1}$. Let G_i and $G_i^{'}$ be the inverse images of X_i and $X_i^{'}$ under f, respectively. By Lemma 4, $\bigcup_{i=1}^{3^i}[G_i]$ is isomorphic to Q_i^3 with $1 \leqslant i \leqslant n$. It can be further verified that $\{(i-1, i)\}$ satisfies Lemma 3, and the edge congestion $EC_f(S_i)$ is minimum under embedding lex for $i = 1, 2, ..., 3^n$. Thus the wirelength $WL_f(Q_n^3, L_{3^n})$ of embedding Q_n^3 into L_{3^n} is minimum. □

Lemma 7. *The minimum wirelength of Q_n^3 into L_{3^n} under f is:*

$$WL_f(Q_n^3, L_{3^n}) = \frac{1}{2}(3^{2n} - 3^n)(3^{n+1} - 2n \times 3^{n-1}).$$

Proof. Let $f = lex$ and $S_j = \{(j, j+1)\}$. Then S_j is an edge cut of L_{3^n}, $2 \leqslant j \leqslant 3^n - 2$, which disconnects L_{3^n} into two linear arrays L_j and $L_j^{'}$, where $V(L_j) = \{1, 2, ..., j\}$ and $V(L_j^{'}) = \{j+1, j+2, ..., 3^n - 2\}$. By Lemma 4, $f^{-1}(L_j)$ is a maximum subgraph with k vertices where $k = |V(f^{-1}(L_j))|$. Thus the edge congestion T_j^n for edge cut S_j is as below:

$$\begin{cases} T_j^{n-1} + 2j,\ 1 \leqslant j < 3^{n-1}, \\ 2 \times 3^{n-1},\ j = 3^{n-1}, j = 2 \times 3^{n-1}, \\ T_{j-3^{n-1}}^{n-1} + 2 \times 3^{n-1},\ 3^{n-1} + 1 \leqslant j < 2 \times 3^{n-1}, \\ T_{j-2\times3^{n-1}}^{n-1} + 2 \times (j - 2 \times 3^{n-1}), \\ 2 \times 3^{n-1} + 1 \leqslant j \leqslant 3^n. \end{cases}$$

It can be further verified that the edge congestion $EC_f(S_j)$ in $\{(i-1,i)\}$ is minimum under embedding *lex*. Thus the wirelength of embedding Q_n^3 into L_{3^n} is minimum. Thus $EC_f(S_j) = 3^{j+1} - 2j \times 3^{j-1}$. Let $S_1, S_2, ..., S_j$ be j edge cuts of L_{3^n}, $1 \leqslant j \leqslant 3^n - 1$. Therefore

$$\begin{aligned} & WL_f(Q_n^3, L_{3^n}) \\ = & \sum_{j=1}^{3^n-1} EC_f(S_j) \\ = & \frac{1}{2}(3^{2n} - 3^n)(3^{n+1} - 2n \times 3^{n-1}). \end{aligned}$$ □

3.3 Embedding the 3-Ary n-Cube into Grid $M(3^{\lfloor n/2 \rfloor}, 3^{\lceil n/2 \rceil})$

In this subsection, we propose an embedding of Q_n^3 into a grid with minimum wirelength. The proposed embedding of Q_n^3 into L_{3^n} in Subsection 3.1 is actually an embedding of Q_n^3 into the special grid, which is a $1 \times n$ grid. In the following, we will give an embedding of Q_n^3 into grid $M(3^{\lfloor n/2 \rfloor}, 3^{\lceil n/2 \rceil})$ with minimum wirelength. Firstly, the definition of grid is given as below.

Notation 1. *An $m \times n$ grid $M(m,n)$ is denoted by an $m \times n$ matrix*

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix},$$

where $V(M) = \{\alpha_{ij} \,|1 \leqslant i \leqslant m, \text{ and } 1 \leqslant j \leqslant n\}$, $(\alpha_{i,j}, \alpha_{i,j+1}) \in E(M)$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n-1$, and $(\alpha_{k,l}, \alpha_{k+1,l}) \in E(M)$ for $1 \leqslant k \leqslant m-1$ and $1 \leqslant l \leqslant n$. $\langle \alpha_{11}, \alpha_{12}, \cdots, \alpha_{1n} \rangle$ and $\langle \alpha_{m1}, \alpha_{m2}, \cdots, \alpha_{mn} \rangle$ are called the row-borders, while $\langle \alpha_{11}, \alpha_{21}, \cdots, \alpha_{m1} \rangle$ and $\langle \alpha_{1n}, \alpha_{2n}, \cdots, \alpha_{mn} \rangle$ are called the column-borders.

Definition 10. *Let $\pi : Q_n^3 \to M(3^{\lfloor n/2 \rfloor}, 3^{\lceil n/2 \rceil})$ be an embedding, which is defined as follows. The first column is labeled from 1 to $3^{\lceil n/2 \rceil}$ from top to bottom. The i-th column is labeled from 1 to $(i-1)3^{\lfloor n/2 \rfloor} + 1, (i-1)3^{\lfloor n/2 \rfloor} + 2, ..., i3^{\lfloor n/2 \rfloor}$ from top to bottom where $i = 1, 2, ..., 3^{\lceil n/2 \rceil}$. Then, for any $v \in V(Q_n^3)$, let $\pi(v) = lex(v)$.*

Then, we first prove the edge congestion problem and the wirelength problem of Q_n^3 into a grid can be solved by using the embedding π.

Lemma 8. *$R_i^{lex} = \{1, ..., i3^{\lceil \frac{n}{2} \rceil}\}$ is an optimal set in Q_n^3 for $i = 1, 2, ..., 3^{\lfloor \frac{n}{2} \rfloor}$ and $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$.*

Proof. This proof can be obtained directly from Theorem 1. □

Lemma 9. *For $j = 1, 2, ..., 3^{\lfloor \frac{n}{2} \rfloor}$,*

$$\begin{aligned} & C_j^{lex} \\ = & \left\{ \begin{array}{l} 1, 1 \times 3^{\lfloor \frac{n}{2} \rfloor},\ 2 \times 3^{\lfloor \frac{n}{2} \rfloor}, ..., 3^{\lceil \frac{n}{2} \rceil} \times 3^{\lfloor \frac{n}{2} \rfloor}, \\ 2, 1 \times 3^{\lfloor \frac{n}{2} \rfloor} + 1,\ 2 \times 3^{\lfloor \frac{n}{2} \rfloor} + 1, ..., \\ 3^{\lceil \frac{n}{2} \rceil} \times 3^{\lfloor \frac{n}{2} \rfloor} + 1, \\ \vdots \\ j, 1 \times 3^{\lfloor \frac{n}{2} \rfloor} + j - 1,\ 2 \times 3^{\lfloor \frac{n}{2} \rfloor} + j - 1, ..., \\ 3^{\lceil \frac{n}{2} \rceil} \times 3^{\lfloor \frac{n}{2} \rfloor} + j - 1 \end{array} \right\} \end{aligned}$$

is an optimal set in Q_n^3 where $3^{\lceil \frac{n}{2} \rceil} + 3^{\lfloor \frac{n}{2} \rfloor} = n$.

Proof. Let $f : C_j^{lex} \to L_{j \times 3^{\lfloor \frac{n}{2} \rfloor}}$ with $f(k \times 3^{\lceil \frac{n}{2} \rceil} + l) = l \times 3^{\lfloor \frac{n}{2} \rfloor} + k$. We use $u_1 u_2 ... u_n$ in C_j^{lex} to denote the ternary string of $l \times 3^{\lfloor \frac{n}{2} \rfloor} + k$. Since the ternary string representations of two numbers u and v differ in exactly one bit, the same holds for $f(u)$ and $f(v)$. Thus (u, v) is an edge in R_i and $(f(u), f(v))$ is an edge in L_{2^i}. Therefore, R_i is isomorphic to L_i. By Theorem 1, C_j^{lex} is an optimal set of Q_n^3. □

Next, we will give the minimum wirelength of embedding Q_n^3 into the grid $M(3^{n_1}, 3^{n_2})$, for $n_1 + n_2 = n$ and $n \geqslant 4$.

Theorem 3. *Let $G = Q_n^3$ and $H = M(3^{n_1}, 3^{n_2})$, where $n_1 + n_2 = n$. Let $S_1, S_2, ..., S_p$ be p edge cuts of $M(3^{n_1}, 3^{n_2})$, $1 \leqslant p \leqslant 3^{n_2 - 1}$, which consists of edges between the columns j and $j+1$ of $M(3^{n_1}, 3^{n_2})$, $1 \leqslant j \leqslant 3^{n_2 - 1}$. Furthermore, let $f = \pi$. Then*

$$\sum_{j=1}^{3^{n_2}-1} EC_f(S_j) = \frac{1}{2} 3^{n_1} (3^{2n_2} - 3^{n_2})(3^{n_2+1} - 2n_2 \times 3^{n_2-1}).$$

Proof. Let H_{j1} and H_{j2} denote two connected components of $M(3^{n_1}, 3^{n_2}) - S_j$, where $f(G_{j1}) = H_{j1}$ and $f(G_{j2}) = H_{j2}$, as depicted in Fig.3. According to Lemma 4, the subgraph induced by $V(G_{j_1})$ is maximum. Therefore, $EC_f(S_j)$ is minimum, $1 \leqslant j \leqslant 3^{n_2} - 1$. Thus we have:

$$\begin{aligned} & \sum_{j=1}^{3^{n_2}-1} EC_f(S_j) = \sum_{j=1}^{3^{n_2}-1} EC_f(S_j) \\ = & \sum_{j=1}^{3^{n_2}-1} \lambda_G(j \times 3^{n_1}) \\ = & \frac{1}{2} 3^{n_1} (3^{2n_2} - 3^{n_2})(3^{n_2+1} - 2n_2 \times 3^{n_2-1}). \end{aligned}$$ □

Theorem 4. *Let $G = Q_n^3$ and $H = M(3^{n_1}, 3^{n_2})$, where $n_1 + n_2 = n$. Let $f = \pi$ and S_1, S_2, ..., S_p be p edge cuts of $M(3^{n_1}, 3^{n_2})$, $1 \leqslant p \leqslant 3^{n_2 - 1}$. Furthermore, let H_{j1} and H_{j2} denote two connected components of $M(3^{n_1}, 3^{n_2}) - S_j$, where $f(G_{j1}) = H_{j1}$ and*

$f(G_{j2}) = H_{j2}$. For any $1 \leqslant j \leqslant p$, if $EC_f(H_{j1})$ is minimum, then $f^{-1}(H_{j1})$ is a maximum subgraph in G.

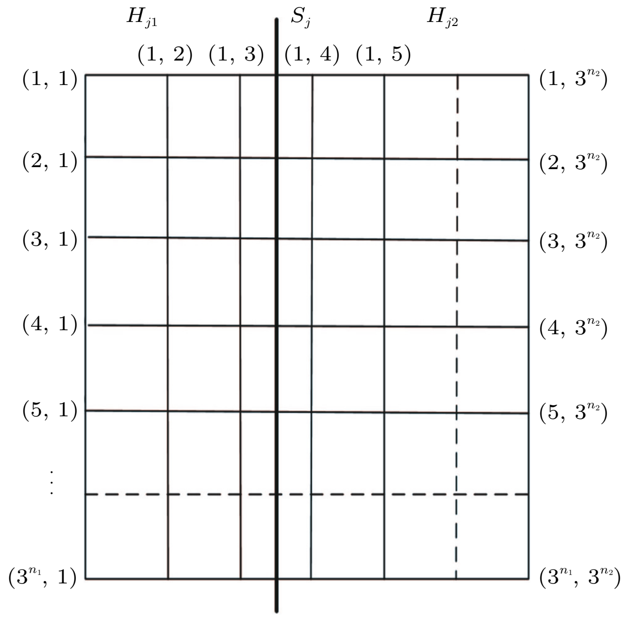

Fig.3. Edge cut of $M(3^{n_1}, 3^{n_2})$.

Proof. Suppose $EC_f(\theta_H(j_1))$ is minimum with $V(H_{j1}) = m$. We will prove that the subgraph induced by $G_{j1} = f^{-1}(H_{j1})$ is maximum in Q^3_n on m vertices. Otherwise, there exists $V(G'_{j1}) \subseteq V(Q^3_n)$ such that $|E(G_{j1})| < |E(G'_{j1})|$. Since Q^3_n is $2n$-regular, $EC_f(\theta_H(j_1)) = nm - 2|E(G_{j1})| > nm - 2|E(G'_{j1})| = EC_f(\theta_H(f(G'_{j1})))$, which is a contradiction to our assumption. Therefore, $f^{-1}(H_{j1})$ is a maximum induced subgraph of Q^3_n. □

Theorem 5. *The minimum wirelength of embedding Q^3_n into the grid $M(3^{n_1}, 3^{n_2})$ is*

$$\begin{aligned}&WL(Q^3_n, M(3^{n_1}, 3^{n_2}))\\ =&\frac{1}{2}3^{n_1}(3^{2n_2} - 3^{n_2})(3^{n_2+1} - 2n_2 \times 3^{n_2-1}) +\\ &\frac{1}{2}3^{n_2}(3^{2n_1} - 3^{n_1})(3^{n_1+1} - 2n_1 \times 3^{n_1-1}).\end{aligned}$$

Proof. Let $f : Q^3_n \to M(3^{n_1}, 3^{n_2})$ be the embedding π, where $n_1 + n_2 = n$ and $n_1 \leqslant n_2$. Let $C_i = \{(\alpha_{i,j}, \alpha_{i,j+1}) | 1 \leqslant j \leqslant 3^{n_2}\}$ and $R_j = \{(\alpha_{i,j}, \alpha_{i+1,j}) | 1 \leqslant i \leqslant 3^{n_1}\}$, where $1 \leqslant i \leqslant 3^{n_1}$ and $1 \leqslant j \leqslant 3^{n_2}$. Let H_{j1} and H_{j2} denote two connected components of $M(3^{n_1}, 3^{n_2}) - R_j$, where $f(G_{j1}) = H_{j1}$ and $f(G_{j2}) = H_{j2}$. Let H_{i1} and H_{i2} denote two connected components of $M(3^{n_1}, 3^{n_2}) - C_i$, where $f(G_{i1}) = H_{i1}$ and $f(G_{i2}) = H_{i2}$. Obviously, each edge of C_i has the same edge congestion. Thus the sum of edge congestion of each column is equal. By Lemma 7, the sum of edge congestion of each column of $M(3^{n_1}, 3^{n_2})$ is $\frac{1}{2}(3^{2n_1} - 3^{n_1})(3^{n_1+1} - 2n_2 \times 3^{n_1-1})$. Similarly, it is easy to verify the sum of edge congestion of each row is $3^{n_2}(3^{2n_1-1} - 3^{n_1})$. Let G_{j1} and G_{i1} be the inverse images of R_{j1} and C_{i1} under the embedding f respectively. Clearly, G_{i1} is a subgraph induced by $V(f^{-1}(H_{i1}))$. By Lemma 4, it is certain that G_{i1} is a maximum induced subgraph of Q^3_n. Thus $EC_f(C_i)$ is minimum for $i = 1, 2, ..., 3^{n_1}$. Therefore, G_{i1} is a maximum subgraph induced by R_{j1}. Thus $EC_f(R_j)$ is minimum, where $j = 1, 2, ..., 3^{n_2}$. Therefore, the wirelength of embedding Q^3_n into $M(3^{n_1}, 3^{n_2})$ is:

$$\begin{aligned}&WL(Q^3_n, M(3^{n_1}, 3^{n_2}))\\ =&\sum_{j=1}^{3^{n_1}} \lambda_G(j \times 3^{n_2}) + \sum_{i=1}^{3^{n_2}} \lambda_G(i \times 3^{n_1})\\ =&\frac{1}{2}3^{n_1}(3^{2n_2} - 3^{n_2})(3^{n_2+1} - 2n_2 \times 3^{n_2-1}) +\\ &\frac{1}{2}3^{n_2}(3^{2n_1} - 3^{n_1})(3^{n_1+1} - 2n_1 \times 3^{n_1-1}).\end{aligned}$$ □

Let $N = 3^n$ be the number of vertices of Q^3_n. By Theorem 5, the number of edge cuts is $(3^{n_2} - 1)$ and deleting each edge cut needs one time unit, and thus deleting all edge cuts takes $(3^{n_2} - 1)$ time units. Consequently, the total time for embedding Q^3_n into $M(3^{n_1}, 3^{n_2})$ with minimum wirelength is $O(N + 3^{n_2} - 1 + 1) \leqslant O(2N)$, $n_1 + n_2 = n$, which is linear.

4 Square Grid Layout of 3-Ary n-Cube

In this section, we discuss an embedding of Q^3_n into a square grid. Subsection 4.1 gives an embedding of Q^3_n into a square grid with a balanced load that minimizes the dilation and the congestion. In Subsection 4.2, an embedding algorithm of Q^3_n into a grid with balanced communication is proposed and the correctness of this algorithm is also analyzed.

4.1 Embedding Q^3_n into a Square Grid

In this subsection, we first propose an embedding of Q^3_n into a 2-dimensional square grid with minimum congestion, and then obtain the required number of tracks for implementing Q^3_n into a chip. It is different between the dilation problem and the wirelength problem to some extent that an embedding with the minimum dilation needs not have the minimum wirelength and vice versa. Since the wirelength problem itself is NP-complete[39], the question arises whether it is possible to obtain a lower bound for dilation without considering the wirelength of an embedding.

Theorem 6. *For any integer $n \geqslant 3$, Q_n^3 can be embedded into the square grid $M(h,h)$ with dilation*:

$$dil(Q_n^3; M(3^{\frac{n}{2}}, 3^{\frac{n}{2}})) = 2\times 3^{\frac{n}{2}-1}, n = 2k, k \geqslant 1,$$
$$dil(Q_n^3; M(\lceil\sqrt{3^n}\rceil, \lceil\sqrt{3^n}\rceil)) = 3^{\lceil\frac{n}{2}\rceil-1},$$
$$n = 2k+1, k \geqslant 1.$$

Proof. Let $f = \pi$, $G_1 = Q_{n-1}^3(0)$, $G_2 = Q_{n-1}^3(1)$, and $G_3 = Q_{n-1}^3(2)$. Furthermore, for any integers h and w, let h and w be the number of columns and rows of M respectively. Clearly, the grid $M(h,w)$ has $3^{\lfloor n/2\rfloor}$ columns. Each subcube $Q_{n-1}^3(k)(0 \leqslant k \leqslant 2)$ is embedded into its each column by using the method presented in Theorem 2. For any $(u,v) \in Q_n^3$, we have the following two cases in $M(h,w)$.

Case 1. n is even. The vertex number of Q_n^3 is a quadratic number. Thus Q_n^3 can be embedded into a grid $M(h,h)$, where $h = 3^{\frac{n}{2}}$. We have to estimate the distances between image vertices within the subcubes of Q_n^3 and have the following cases.

Case 1.1. $(u,v) \in E(Q_{n-1}^3(k))$, $0 \leqslant k \leqslant 2$. Let C_j be the set of vertices of the j-th column of $M(3^{\frac{n}{2}} \times 3^{\frac{n}{2}})$, $1 \leqslant j \leqslant 3^{n_1}$. Since the maximum value of the distance between $f(u)$ and $f(v)$ in M is equal to the dilation of embedding Q_{n-2}^3 into a linear array with 3^{n-2} vertices, the maximum value of the distance between $f(u)$ and $f(v)$ is $2\times 3^{\frac{n}{2}-1}$.

Case 1.2. $u \in V(Q_{n-1}^3(k))$ and $v \in V(Q_n^3 - Q_{n-1}^3(k))$, $0 \leqslant k \leqslant 2$. Let $E_j = \{((i,j),(i,j+1))|1 \leqslant i \leqslant 3^{\lfloor\frac{n}{2}\rfloor-1}\}, 1 \leqslant j \leqslant 3^{\lceil\frac{n}{2}\rceil}$. Clearly, the subcubes G_1 and G_3 are mapped to columns $1, 2, ..., 3^{\lfloor n/2\rfloor-1}$ and $2\times 3^{\lfloor n/2\rfloor-1}+1, 2\times 3^{\lfloor n/2\rfloor-1}+2, ..., 3^{\lfloor n/2\rfloor}$ in M, respectively. By Theorem 2, the maximum value of the distance between $f(u)$ and $f(v)$ in M is $2\times 3^{\lfloor\frac{n}{2}\rfloor}+1$.

Case 2. n is odd. We firstly embed Q_n^3 into a rectangular grid $M(h,3h)$ by using Theorem 5. Firstly, we apply the same embedding method as case 1 and make use of the result of Theorem 2. Secondly, we transform grid $M(h,3h)$ into a square grid $M' = (\lceil\sqrt{3}h\rceil, \lceil\sqrt{3}h\rceil)$ by compressing the columns of M. Algorithm 1 performs the process of compressing.

Fig.4 shows the transformation of grid $M(9,3)$ into grid $M(6,6)$. For embedding Q_n^3 into $M' = (\lceil\sqrt{3}h\rceil, \lceil\sqrt{3}h\rceil)$, we have the following two cases.

Case 2.1. $(u,v) \in E(Q_{n-1}^3(k)), 0 \leqslant k \leqslant 2$. For any vertex $(i,j) \in V(M)$, we embed it into one of columns $\lceil\sqrt{3}(j-1)\rceil-1$, $\lceil\sqrt{3}(j-1)\rceil$, $\lceil\sqrt{3}(j-1)\rceil+1$ or $\lceil\sqrt{3}(j-1)\rceil+2$ of M'. Then the maximum value of the distance between $f(u)$ and $f(v)$ is 3 in row direction. Since the vertex $(i,j) \in V(M)$ is embedded into one of the rows $\lceil(i-2)/\sqrt{3}\rceil,...,\lceil(i+4)/\sqrt{3}\rceil$ of M', $1 \leqslant i \leqslant h$, $1 \leqslant j \leqslant 2h$, then the maximum value of the distance between $f(u)$ and $f(v)$ is $3^{\lceil n/2\rceil}-1$.

Algorithm 1. Constructing a Square Grid $M(\sqrt{3}h, \sqrt{3}h)$ for Q_n^3

Input: grid $M(h,3h)$, where $h = 3^{\lfloor\frac{n}{2}\rfloor}$ and n is odd
Output: an embedding f of Q_n^3 into $M(\sqrt{3}h, \sqrt{3}h)$

for $i = 1$ to $3h$ **do**
  Let $(1,i)$ be a vertex in the 1st column;
  if $\lceil i/\sqrt{3}\rceil \neq \lceil(i-1)/\sqrt{3}\rceil$ **then**
    Embedding $(1,i)$ into $(1, \lceil i/\sqrt{3}\rceil) \in V(M')$
  else
    Embedding $(1,i)$ into $(2, \lceil i/\sqrt{3}\rceil) \in V(M')$
  end if
end for
for $j = 2$ to h **do**
  Label the j-th column of M' as $n_1(j), n_2(j), ..., n_{\sqrt{3}h}(j)$ from top to bottom, where $n_i(j) \in \{1,2\}$ is the number of vertices of the j-th column which are embedded into the i-th row of M'
  Embedding the $(j+1)$-th column of M into M' as $n_{((1+j)\bmod\lceil\sqrt{3}h\rceil)+1}(1), n_{((2+j)\bmod\lceil\sqrt{3}h\rceil)+1}(1), ..., n_{((\lceil\sqrt{3}h\rceil+j)\bmod\lceil\sqrt{3}h\rceil+1)}(1)$
  /*$(j+1)$-th is the 1st column cyclically shift by j rows.*/
end for
return f

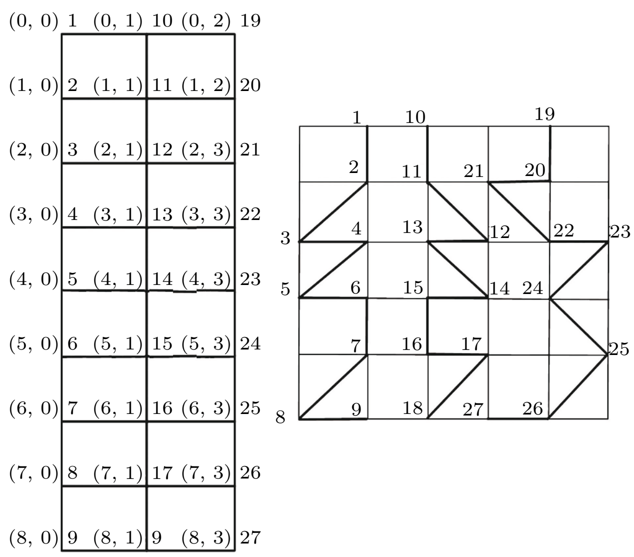

Fig.4. Transforming (a) grid $M(9,3)$ into (b) grid $M(6,6)$.

Case 2.2. $u \in V(Q_{n-1}^3(k))$ and $v \in V(Q_n^3 - Q_{n-1}^3(k))$, $0 \leqslant k \leqslant 2$. For any vertex $(i,j) \in V(M)$, we embed it into one of rows $\lceil(i-2)/\sqrt{3}\rceil,...,\lceil(i+4)/\sqrt{3}\rceil$ of M', $1 \leqslant i \leqslant h$, $1 \leqslant j \leqslant 2h$. Then the maximum value of the distance between $f(u)$ and $f(v)$ is $\lceil 7/\sqrt{3}\rceil = 4$ in column direction. Since the vertex (i,j) is embedded into one of columns $\lceil\sqrt{3}(j-1)\rceil-1$, $\lceil\sqrt{3}(j-1)\rceil$,

$\lceil\sqrt{3}(j-1)\rceil+1$ or $\lceil\sqrt{3}(j-1)\rceil+2$ of M', the maximum value of the distance between $f(u)$ and $f(v)$ is $3^{\lceil n/2\rceil}-1$.

Hence, the dilation of embedding Q_n^3 into M' is:

$$dil(Q_n^3; M(3^{\frac{n}{2}}, 3^{\frac{n}{2}})) = 2\times 3^{\frac{n}{2}-1}, n=2k, k\geqslant 1;$$
$$dil(Q_n^3; M(\lceil\sqrt{3^n}\rceil, \lceil\sqrt{3^n}\rceil)) = 3^{\lceil\frac{n}{2}\rceil-1},$$
$$n=2k+1, k\geqslant 1.$$ □

Lemma 10[37]. *Let G and H be two graphs with $V(G)=V(H)$. For any integer l, $0\leqslant l\leqslant |V|$,*

$$EC(G,H)\geqslant \max_{1\leqslant l\leqslant |V(G)|-1}\frac{\theta_G(l)}{\theta_H(l)}.$$

Theorem 7. *The minimum edge congestion cong $=(Q_n^3, M(3^{\lfloor n/2\rfloor}, 3^{\lceil n/2\rceil}))$ of embedding Q_n^3 into the grid $M(3^{\lfloor n/2\rfloor}, 3^{\lceil n/2\rceil})$ is*

$$\begin{cases}\dfrac{3^{\lceil n/2\rceil+1}-1}{8}, \text{if } \lceil n/2\rceil \text{ is odd},\\ \dfrac{3^{\lceil n/2\rceil+3}-3}{8}, \text{if } \lceil n/2\rceil \text{ is even}.\end{cases}$$

Proof. Let $f=\pi$ and $l=3^{n-1}-3^{n-2}+3^{n-3}-\ldots+(-1)^{n-\lfloor n/2\rfloor+1}\times 3^{\lfloor n/2\rfloor}$ with $3^{n-2}\leqslant l<3^{n-1}$. For any integer β, let $l=\beta\times 3^{\lfloor n/2\rfloor}$. Furthermore, let M_s denote the subgrid $M_s(3^{\lfloor n/2\rfloor},\beta)$. Then it can be obtained

$$\theta_{M(3^{\lfloor n/2\rfloor},\ 3^{\lceil n/2\rceil})}(l)\leqslant \theta_{M(3^{\lfloor n/2\rfloor},\ 3^{\lceil n/2\rceil})}(|V(M_s)|) = 3^{\lfloor n/2\rfloor}.$$

Also

$$\theta_{Q_n^3}(l)=\begin{cases}3^{n-1}+3^{n-2}+\ldots+3^{\lfloor n/2\rfloor},\\ \text{if } \lceil n/2\rceil \text{ is odd},\\ 3^{n-1}+3^{n-2}+\ldots+3^{\lfloor n/2\rfloor+1},\\ \text{if } \lceil n/2\rceil \text{ is even}.\end{cases}$$

Therefore

$$\begin{aligned}&\frac{\theta_{Q_n^3}(l)}{\theta_{M(3^{\lfloor n/2\rfloor},\ 3^{\lceil n/2\rceil})}(l)}\\ \geqslant\ &\frac{1}{3^{\lfloor n/2\rfloor}}\begin{cases}3^{n-1}+3^{n-3}+\ldots+3^{\lfloor n/2\rfloor},\\ \text{if } \lceil n/2\rceil \text{ is odd},\\ 3^{n-1}+3^{n-3}+\ldots+3^{\lfloor n/2\rfloor+1},\\ \text{if } \lceil n/2\rceil \text{ is even}\end{cases}\\ =\ &\begin{cases}1+3^2+\ldots+3^{\lceil n/2\rceil-1}, \text{if } \lceil n/2\rceil \text{ is odd},\\ 3+3^3+\ldots+3^{\lceil n/2\rceil+1}, \text{if } \lceil n/2\rceil \text{ is even}\end{cases}\\ =\ &\begin{cases}\dfrac{3^{\lceil n/2\rceil+1}-1}{8}, & \text{if } \lceil n/2\rceil \text{ is odd},\\ \dfrac{3^{\lceil n/2\rceil+3}-3}{8}, & \text{if } \lceil n/2\rceil \text{ is even}\end{cases}\end{aligned}$$

By Lemma 8, the minimum congestion of embedding Q_n^3 into $M(3^{\lfloor n/2\rfloor}, 3^{\lceil n/2\rceil})$ under f is

$$\begin{cases}\dfrac{3^{\lceil n/2\rceil+1}-1}{8}, & \text{if } \lceil n/2\rceil \text{ is odd},\\ \dfrac{3^{\lceil n/2\rceil+3}-3}{8}, & \text{if } \lceil n/2\rceil \text{ is even}.\end{cases}$$ □

To compute the required number of tracks, the parameter trisection width is considered. Trisection width is defined as the number of links interconnecting three subgraphs having the same number of vertices. We have the following theorem.

Theorem 8. *The required number of tracks for connecting an array of Q_n^3 with N vertices is $N-\log_3 N$.*

Proof. Let $N=3^n$ denote the vertex number of Q_n^3. Construct a hamiltonian path in Q_n^3, and let this path be a base track. By Definition 4, Q_n^3 can be divided into three subcubes $Q_n^3(0)$, $Q_n^3(1)$ and $Q_n^3(2)$ with $N/3$ vertices each. Considering that any vertex in one subcube has only one neighbor in the other subcube, there are $2N/3$ links between the two subcubes. Then the trisection width of the first partition is $2N/3$. We continue to divide each subcube into three equal sub-subcubes with $N/9$ vertices, and the trisection width of this division is $2N/9$. We repeat this division n times. The trisection width of Q_3^3 is illustrated in Fig.5. Let t_i denote the number of required tracks for Q_n^3, which can be obtained by summing the trisection width in each procedure. Based on the above division, it can be obtained as bellow. It needs one track for constructing the Hamiltonian path. Then the first trisection needs $2\times 3^{n-1}-1$ tracks, the second trisection needs $2\times 3^{n-2}-1$ tracks,..., the $(n-1)$-th trisection needs $2\times 3^1-1$ tracks, and the n-th trisection needs $2\times 3^0-1$ tracks.

Thus, the required number of tracks is,

$$\begin{aligned}t_i &= 2+(2\times 3^{n-1}-1)+\ldots+(2\times 3^1-1)\\ &= 2(3^{n-1}+3^{n-2}+\ldots+1)-(n-1)\\ &= 2\times\frac{1}{2}(3^n-1)-n+1\\ &= 3^n-n\\ &= N-\log_3 N.\end{aligned}$$ □

By Theorem 6 and Theorem 7, we can get Theorem 9 as below.

Theorem 9. *When n is odd with $n\geqslant 5$, Q_n^3 can be embedded into a square grid with balanced load and minimum congestion and dilation.*

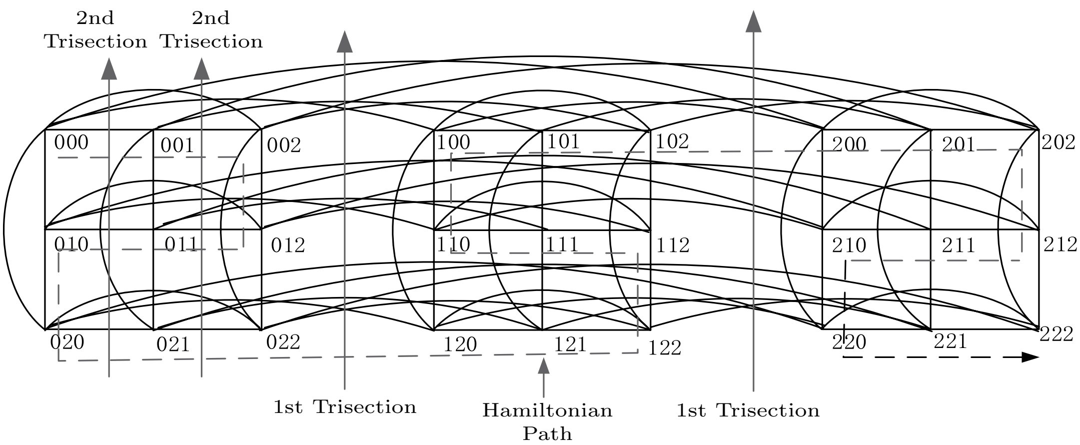

Fig.5. Trisection of Q_3^3.

4.2 Algorithm for Embedding 3-Ary n-Cube into a Grid

In this subsection, we first present an algorithm for embedding Q_n^3 into a grid with balanced communication, and then analyze the time complexity of this algorithm. We propose an embedding of $V(Q_n^3) \to V(M)$ considering communication volume, which keeps a load balancing communication among all processors.

Let $G = Q_n^3$ and $H = M(3^{\lfloor n/2 \rfloor}, 3^{\lceil n/2 \rceil})$. Let $f(u) = \sum_{(u,x) \in E(G)} w(u,x)$ for any $u \in V(G)$, where $w : E(Q_n^3) \to R^+$ denotes the weight function, and $w(u,v)$ represents the communication volume between u and x. Moreover, let $g(v) = \sum_{x \in V(H)} dist(v,x)$ for any $v \in V(H)$, where $dist(H,v,x)$ denotes the communication distance between v and x. An algorithm for embedding Q_n^3 into a grid with balanced communication is given below.

Theorem 10. *There exists an $O(N^2)$ algorithm for embedding Q_n^3 into a grid with balanced communication, where $N = 3^n$ is the number of vertices in Q_n^3.*

Proof. By Algorithm 2, an embedding considering communication of Q_n^3 into grid is proposed. We state our embedding and prove that this embedding has balanced communication performance.

Our algorithm has the following steps. 1) Let $G_0 = Q_n^3$, and suppose $f(u_0) = \max\{f(u) | u \in V(G_0)\}$, i.e., u_0 is a vertex that has maximum communication with neighbour vertices among all vertices in G_0. Let $H_0 = M(3^{\lfloor n/2 \rfloor}, 3^{\lceil n/2 \rceil})$, suppose $g(v_0) = \min\{g(v) | v \in V(H_0)\}$, i.e., v_0 is the vertex with the minimum sum of distances between it and all other vertices of H_0. Then we assign the vertex u_0 to v_0 in H_0. 2) For any integer i, with $1 \leqslant i \leqslant 3^n - 1$, let $G_i = Q_n^3 - \{u_0, u_1, ..., u_{i-1}\}$. Suppose $f(u_{i-1}) = \max\{f(u) | u \in V(G_{i-1})\}$, i.e., u_{i-1} is a vertex that has maximum communication with other vertices in G_i. Let $H_i = M(3^{\lfloor n/2 \rfloor}, 3^{\lceil n/2 \rceil}) - \{v_0, v_1, ..., v_{i-1}\}$. Suppose $g(v_{i-1}) = \min\{g(v) | v \in V(H_i)\}$, i.e., v_{i-1} is the vertex with the minimum sum of distances between it and all other vertices of H_i. At last, mapping the last vertex u_i to v_i, with $i = 3^n - 1$.

Algorithm 2 . Algorithm of Embedding Q_n^3 into Grid $M(p,q)$

Input: Q_n^3 and grid $M(p,q)$, with $p = 3^{\lfloor \frac{n}{2} \rfloor}, q = 3^{\lceil \frac{n}{2} \rceil}$
Output: an embedding h of Q_n^3 into $M(p,q)$ with balanced communication

Let $max = -1$;
Let $maxIndex = -1$;
Let $min = +\infty$;
Let $minIndex = -1$;
Choose $u_0 \in V(Q_n^3)$ with maximum $\sum_{(u_i,x) \in E(Q_n^3)} w(u_i,x)$;
$S_1 = \{u_0\}$;
Choose $v_0 \in V(M(p,q))$ with minimum $\sum_{(v_i,x) \in V(M(p,q))} dist(v_i,x)$;
$S_2 = \{v_0\}$;
for $i = 1$ to $3^n - 2$ **do**
  for all $(u_i,x) \in E(Q_n^3)$, with $x \in S_1$ and $u_i \in V(Q_n^3) - S_1$ **do**
    if $\sum_{(u_i,y) \in E(Q_n^3)} w(u_i,y) > max$ **then**
      $max = \sum_{(u_i,x) \in E(Q_n^3)} w(u_i,x)$;
      $maxIndex = i$;
      $S_1 = S_1 \cup \{u_i\}$;
    end if
  end for
  for all $(v_i,x) \in E(M(p,q))$, with $x \in S_2$ and $v_i \in V(M(p,q)) - S_2$ **do**
    if $\sum_{(v_i,y)} dist(v_i,y) < min$ **then**
      $min = \sum_{(v_i,y)} dist(v_i,y)$;
      $minIndex = i$;
      $S_2 = S_2 \cup \{v_i\}$;
    end if
  end for
  Let $h(u_i) = v_i$;
end for
return f

Let $t(N)$ denote the running time of Algorithm 2. It takes one time unit to traverse a weight edge, and thus the total number of time units is 3^n. Since the grid M and Q_n^3 has the same number of vertices, it takes 3^n time units for choosing the vertices with minimum sum distance. Therefore, it takes one time unit for mapping vertex u_i of Q_n^3 to vertex v_i of grid M; thus the total execution time of Algorithm 2 is $t = O(3^n(2 \times 3^n + 1) = O(3^{2n})) = O(N^2)$. □

5 Simulation and Experiments

With the increase of the interconnection network scale, the delay of message passing seriously affects the communication efficiency between nodes. Network cost is the most crucial factor to measure an interconnection network. Especially the redundant search messages will increase exponentially, which would seriously influence the efficiency of the interconnection network search schemes. Congestion and dilation directly affect the queuing delay of messages and communication delay in the embedding process.

We perform the embedding schemes with experiments on a server. The configuration of the server is as follows: NVIDIA GTX 1060 GPU, Intel® Xeon® E5-2670 CPUs with 16 processors running at 3.3 GHz, 1 disk with 3 TB and 64 GB of physical memory. The operating system is Linux ubuntu 16.04 LTS. In the process of executing the algorithms, we monitor the resource status of the server with Ganglia[36]. We analyze the algorithm's network cost by monitoring the state of resources usage. It mainly calculates the consumption of computing resources during the execution of algorithms, such as CPU and memory.

We compare our embedding algorithm comb with the natural embedding[40] and the random embedding. The natural embedding (natural for short) is a bijection $f : \{1, ..., n\} \rightarrow \{1, ..., n\}$ such that $f(x) = (x + 1)$, $x < n$, and $f(n) = 1$. The random embedding (random for short) is a random bijection $f : \{1, ..., n\} \rightarrow \{1, ..., n\}$.

Fig.6 illustrates the network cost of three embedding schemes. When the number of nodes is less than 32, the cost of the three algorithms is relatively close. As the number of nodes increases, the random's cost becomes larger than those of the other two algorithms. Due to the random mapping of nodes, the congestion and the dilation of some links become quite large. This will increase the communication cost.

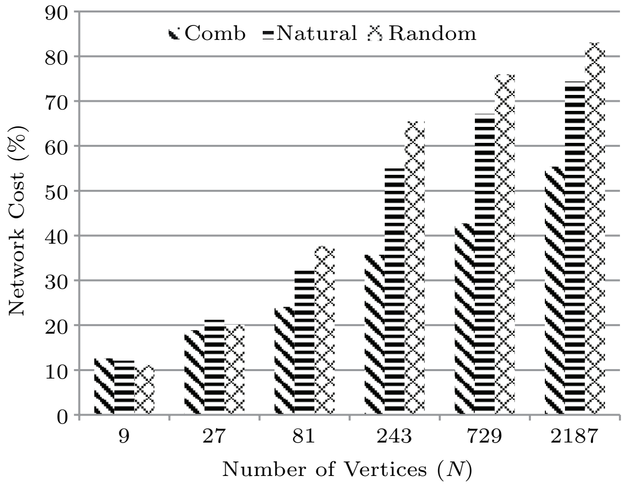

Fig.6. Network cost of three embedding schemes.

As shown in Fig.7, the comb embedding induces the lower wirelength compared with the other two embeddings. Obviously, random is the worst embedding with the maximum wirelength required. As the number of nodes increases, comb embedding has better performance than natural embedding.

6 Conclusions

We proposed optimal embedding of 3-ary n-cube into linear arrays and grids. We first proved that a Q_n^3 can be embedded into linear arrays and grids with minimum wirelength. We then showed that a Q_n^3 can be embedded into a square grid with minimal dilation and congestion. Finally, we proposed an algorithm for embedding Q_n^3 into a grid with balanced communication. The main contribution of this work is that our Q_n^3 embedding into square grid is the first embedding with multiple optimized targets.

The k-ary n-cube is an underlying network model of both theoretical and practical importance, of which the cubes of lower k are particularly important in practice. Therefore it is a worthwhile undertaking to investigate the embedding of Q_n^3 into simpler platforms, optimizing single/multiple objectives. The results of this paper provide more attributes of Q_n^3 to take into account when considering it as a candidate for interconnection network.

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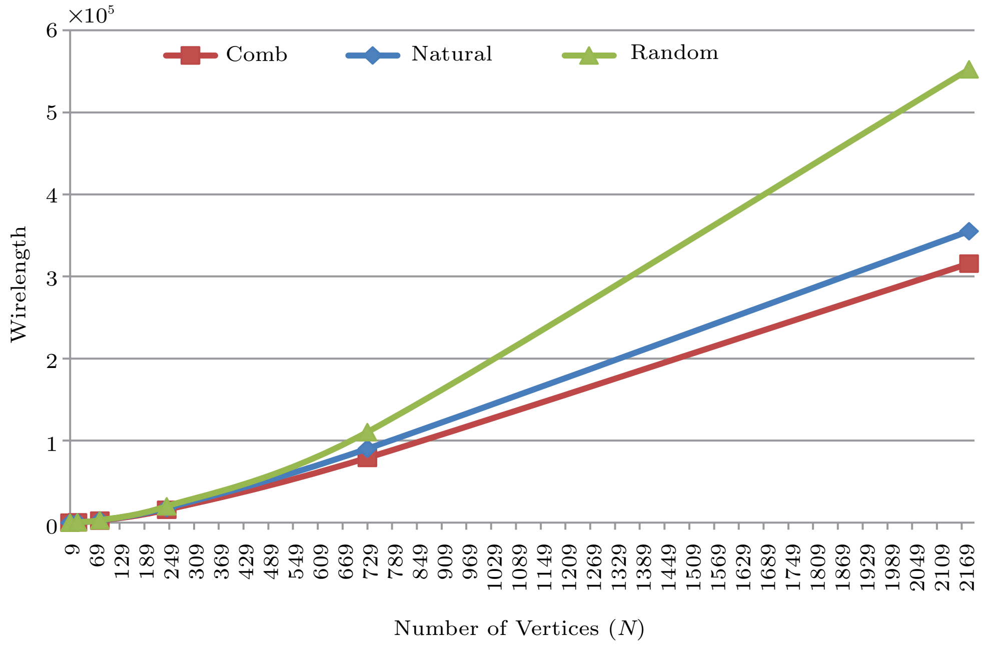

Fig.7. Comparison of three embedding schemes.

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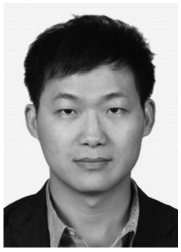

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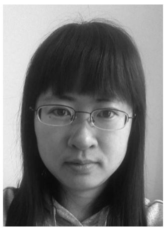

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