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Optimal Path Embedding in the Exchanged Crossed Cube

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Abstract The $(s+t+1)$-dimensional exchanged crossed cube, denoted as $ECQ(s,t)$, combines the strong points of the exchanged hypercube and the crossed cube. It has been proven that $ECQ(s,t)$ has more attractive properties than other variations of the fundamental hypercube in terms of fewer edges, lower cost factor and smaller diameter. In this paper, we study the embedding of paths of distinct lengths between any two different vertices in $ECQ(s,t)$. We prove the result in $ECQ(s,t)$: if $s \geqslant 3, t \geqslant 3$, for any two different vertices, all paths whose lengths are between $\max\{9, \lceil\frac{s+1}{2}\rceil + \lceil\frac{t+1}{2}\rceil + 4\}$ and $2^{s+t+1}-1$ can be embedded between the two vertices with dilation 1. Note that the diameter of $ECQ(s,t)$ is $\lceil\frac{s+1}{2}\rceil+\lceil\frac{t+1}{2}\rceil+2$. The obtained result is optimal in the sense that the dilations of path embeddings are all 1. The result reveals the fact that $ECQ(s,t)$ preserves the path embedding capability to a large extent, while it only has about one half edges of CQ_n.

Keywords interconnection network, exchanged crossed cube, path embedding, parallel computing system

1 Introduction

It is widely known that interconnection networks take key roles in parallel computing systems. An interconnection network can usually be conveniently modeled by a simple connected graph $G = (V, E)$, where V is the vertex set, E is the edge set, and vertices and edges are used to denote processors and communication links between processors, respectively. Throughout this paper, we use these terms, graph and interconnection network, edge and link, and vertex and node interchangeably, and all graphs mean simple undirected graphs.

The n-dimensional crossed cube CQ_n, is one of the most desirable interconnection network structures, proposed by Efe[1]. It is n-regular with 2^n vertices, but has only about one half the diameter[1], wide diameter[2], and fault diameter[2] of the hypercube with the same dimension. The $(s + t + 1)$-dimensional exchanged hypercube $EH(s,t)$, proposed by Loh *et al.*, is also an attractive topology with the lower hardware cost by systematically removing edges from a hypercube[3]. Based on CQ_n and $EH(s,t)$, Li *et al.*[4] proposed a new interconnection network named the exchanged crossed cube $ECQ(s,t)$. The interconnection topology both maintains most of the topological features of $EH(s,t)$ and combines many attractive features of CQ_n. For example, the diameter of $ECQ(s,t)$ is nearly the same with that of CQ_n, but much smaller than that of $EH(s,t)$. And the hardware cost of $ECQ(s,t)$ is nearly as much as that of $EH(s,t)$, but much lower than that of CQ_n. $ECQ(s,t)$ also has expandability, isomorphism, decomposition[4], and strong connectivity[5-6].

A crucial issue in designing and analysing an interconnection network is how to use one network to simulate another popular network. Embeddability of graphs is an important metric to measure the per-

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formance of simulation capability of an interconnection network. Graph embedding is a technique that maps one graph (guest graph) into another graph (host graph)[7]. There are many applications of graph embedding, such as parallel algorithms transplanting, architecture simulation[8-9], and VLSI chip design[10]. In parallel computing, a large process is usually divided into a set of small subprocesses that can execute in parallel with communications among these subprocesses. Hence, the problem of allocating these subprocesses into a parallel computing system can be again modeled as a graph embedding problem[11].

Most of the studies on graph embedding consider paths, cycles, trees, meshes as guest graphs, such as [9, 12-20], because these interconnection networks are widely used in parallel and distributed computing systems. In particular, paths and cycles, are more common and useful in embedding regular topologies such as linear arrays and rings. What is more, they are suitable for designing simple parallel algorithms with low communication cost. These algorithms like data (control) flow architectures can be applied to parallel and distributed computing systems. Many efficient parallel algorithms developed on paths (cycles) can be used to solve graph problems, algebraic problems, some parallel applications and so on. Besides, some special paths play important roles in parallel and distributed computing. A shortest path between any two vertices in an interconnection network is an optimal path in the aspect of communication delay. Efe[1] and Chang *et al.*[2] respectively discussed different embeddings of the shortest paths between any two vertices in crossed cubes. Edge-disjoint paths between two vertices are basic routing in high-speed interconnection networks[21], while node-disjoint paths are significant for fault-tolerant routing[22]. Lai and Hsu presented the nearly shortest path embedding in crossed cubes[23]. A Hamiltonian path can be used in dual-path and multipath multicast routing algorithms to decrease congestion or avoid deadlock in parallel computing systems[24]. Huang *et al.* studied the fault-tolerant Hamiltonian path embedding in crossed cubes[25]. Chang *et al.* proposed the end-to-end longest path problem[26]. Finally, if paths of consecutive lengths can be embedded, the number of simulated processors can be adjusted to match the elasticity of demand. Fan *et al.* gave the results on embedding of paths of consecutive lengths: for any two vertices, all paths whose lengths are greater than or equal to the distance between the two vertices plus 2 can be embedded between the two vertices with dilation 1[14]. And in [15], it was proved that paths of all lengths from $\lceil \frac{n+1}{2} \rceil + 1$ to $2^n - 1$ can be embedded between two arbitrary distinct vertices with dilation 1 in CQ_n. In summary, it is significant to embed paths of all the possible lengths into an interconnection network.

So far, the path embedding problem, which deals with various lengths of the paths, has been discussed in numerous interconnection networks, such as [27-34].

This paper mainly focuses on the path embeddability of the exchanged crossed cube. We will discuss the optimal embedding of paths of various lengths between two arbitrary distinct vertices in $ECQ(s,t)$. The contributions of this paper are as follows.

If $s \geqslant 3, t \geqslant 3$, for any two distinct vertices x and y, and any integer l with $\max\{9, \lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 4\}$, a path of length l can be embedded between x and y with dilation 1 in $ECQ(s,t)$. Note that the diameter of $ECQ(s,t)$ is $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 2$.

The Hamiltonian connectivity is an important property in interconnection networks. The results obtained in this paper show a stronger connectivity for $ECQ(s,t)$[4]. That is, we prove that any two different vertices are connected not only by a Hamiltonian path, but also by many other paths of consecutive lengths in $ECQ(s,t)$. What is more, the obtained results are optimal in the sense that the dilations of path embeddings are all 1.

The rest of this paper is organized as follows. Section 2 presents some related definitions and notations used in the paper. Section 3 discusses embedding paths of lengths from $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 4$ to $2^{s+t+1} - 1$ in $ECQ(s,t)$. Finally, the work is summarized in Section 4.

2 Preliminaries

In graph $G = (V, E)$, an $\langle x, y\rangle$-path of length k from vertex x to vertex y is denoted by $P = \langle x_0,\ x_1, \ldots,\ x_k\rangle$, where $x_0 = x$ and $x_k = y$ are called the two end vertices of path P, and all the vertices $x_0, x_1, \ldots, x_k$ are distinct. The length of a path is the number of edges contained in it. We also denote path P by $\langle x_0, x_1, \ldots, x_i, P_1, x_j, x_{j+1}, \ldots, x_k\rangle$, where P_1 is the subpath $\langle x_i, x_{i+1}, \ldots, x_j\rangle$ and $0 \leqslant i < j \leqslant k$. For simplicity, we also use $\langle x_0, x_1, \ldots, x_i\rangle + P_1 + \langle x_j, x_{j+1}, \ldots, x_k\rangle$ to denote path P. If a vertex x is not in path P, then we use $P + \langle x, x_0\rangle$ to denote the path between x and x_k obtained by adding the edge (x, x_0) into P.

A Hamiltonian path is defined as a path which traverses each vertex of graph G exactly once. If there exists a Hamiltonian path between any two distinct vertices of graph G, then we say that graph G is a Hamiltonian connected graph. The length of a shortest $\langle x,y\rangle$-path in G between x and y is called the distance between x and y, denoted by $dist(G,x,y)$. The diameter of G, denoted by $D(G)$, is the maximum of $dist(G,x,y)$ for any distinct vertices x and y[35]. The neighbor set of vertex x in G is defined to be the set of vertices adjacent to x.

A graph $G_1=(V_1,E_1)$ is a subgraph $G_2=(V_2,E_2)$ (written by $G_1\subseteq G_2$) if $V_1\subseteq V_2$ and $E_1\subseteq E_2$[35]. For two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, we say that G_1 is an edge-induced subgraph of G_2 if $E_1\subseteq E_2$ and V_1 is the set of end vertices of edges E_1. Two graphs G_1 and G_2 are said to be isomorphic if and only if there are bijections $\Theta:V_1\longrightarrow V_2$ and $\Phi:E_1\longrightarrow E_2$ such that Φ maps each edge (u,v) to $(\Theta(u),\Theta(v))$ and vice versa.

Graph embedding can be defined as: given two graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, G_1, which represents the network (guest graph) to be embedded, and G_2, which represents the network (host graph) into which other networks are to be embedded, an embedding is to find an injective mapping $f:V_1\to V_2$[34]. An important performance standard of embedding is dilation. The dilation of embedding f is defined as follows: $dil(G_1,G_2,f)=\max\{dist(G_2,f(x),f(y))|(x,y)\in E_1\}$. The smaller the dilation of an embedding is, the shorter the communication delay that the graph G_2 simulates the graph G_1. We say f is the optimal embedding $G_1\to G_2$ if f has the smallest dilation in all the embeddings from G_1 to G_2. Clearly, $dil(G_1,G_2,f)$ is at least 1. If $dil(G_1,G_2,f)=1$, then G_1 is isomorphic to a subgraph of G_2. This is the most ideal embedding, which is also called isomorphic embedding. Under this circumstance, f is surely optimal in the sense of communication delay. Finding the isomorphic embedding of graphs is NP-hard[15].

In order to demonstrate the process of constructing $ECQ(s,t)$, here we follow [4].

Definition 1[1]. *Two binary strings, $u=u_1u_0$ and $v=v_1v_0$, are pair related, denoted by $u\sim v$, if and only if $(u,v)\in\{(00,00),(10,10),(01,11),(11,01)\}$. If u and v are not pair related, then we write $u\nsim v$.*

Definition 2[1]. *The n-dimensional crossed cube, denoted by CQ_n, is the labeled graph recursively defined as follows. CQ_1 is the complete graph on two vertices whose binary strings are 0 and 1. CQ_n consists of two subcubes CQ^0_{n-1} and CQ^1_{n-1}. The most significant bit of the binary strings of the vertices of CQ^0_{n-1} and CQ^1_{n-1} is 0 and 1, respectively. The vertices $u=u_{n-1}u_{n-2}\cdots u_1u_0$ and $v=v_{n-1}v_{n-2}\cdots v_1v_0$, where $u_{n-1}=0$ and $v_{n-1}=1$, are joined by an edge in CQ_n if and only if*

1) $u_{n-2}=v_{n-2}$ if n is even, and

2) $u_{2i+1}u_{2i}\sim v_{2i+1}v_{2i}$ for $0\leqslant i<\lfloor\frac{n-1}{2}\rfloor$.

Fig.1 demonstrates CQ_3 and CQ_4.

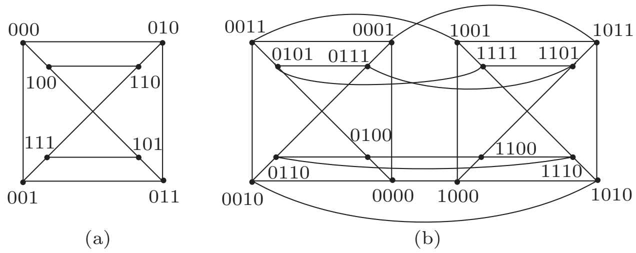

Fig.1. (a) 3-dimensional crossed cube CQ_3. (b) 4-dimensional CQ_4.

Let $x=x_sx_{s-1}\cdots x_0$ be a binary string of length $s+1$. We set $x[i]=x_i$ and $x[j:k]=x_jx_{j-1}\cdots x_k$, where $0\leqslant k\leqslant j\leqslant s$. By the definition of $ECQ(s,t)$[4], we can give its equivalent definition as follows.

Definition 3. *The exchanged crossed cube $ECQ(s,t)=(V,E)$, where $s\geqslant 1$ and $t\geqslant 1$ are positive integers, $V=\{a_{s-1}\dots a_0b_{t-1}\dots b_0c\ |a_i,b_j,c\in\{0,1\}$, $i\in[0,s-1]$, and $j\in[0,t-1]\}$ and $E=E_1\cup E_2\cup E_3$. For $(u,v)\in E$, E_1, E_2 and E_3 are defined as follows.*

1) $(u,v)\in E_1$, if and only if $u[0]\neq v[0]$ and $u\oplus v=1$, where $\oplus$ is the exclusive-OR operator.

2) $(u,v)\in E_2$, if and only if $u[t:1]=v[t:1]$, $u[0]=v[0]=0$, and $(u[s+t:t+1],v[s+t:t+1])\in E(CQ_s)$.

3) $(u,v)\in E_3$, if and only if $u[s+t:t+1]=v[s+t:t+1]$, $u[0]=v[0]=1$, and $(u[t:1],v[t:1])\in E(CQ_t)$.

By the above definition, there exist 2^{s+t+1} vertices and $(s+t+2)2^{s+t-1}$ edges in $ECQ(s,t)$. The subgraphs of $ECQ(s,t)$ induced by the edges E_2 (respectively, E_3) are a set of disjoint CQ_s's (respectively, CQ_t's). What is more, there are 2^t different induced subgraphs $CQ_s[i]$ in $ECQ(s,t)$ for $1\leqslant i\leqslant 2^t$, and there are 2^s different induced subgraphs $CQ_t[i]$ in $ECQ(s,t)$ for $1\leqslant i\leqslant 2^s$. Fig.2 demonstrates $ECQ(2,3)$, where the edges from E_1, E_2, and E_3 are denoted by dashed lines, bold lines, and solid lines, respectively.

Notation 1. *$ECQ(s,t)$ can be decomposed into two subgraphs $ECQ^0_{s,t}$ and $ECQ^1_{s,t}$, where $V(ECQ^\lambda_{s,t})=\{a_{s-1}\cdots a_0\lambda b_{t-2}\cdots b_0c\mid a_i,b_j,c\in\{0,1\}$, $i\in[0,s-1]$, $j\in[0,t-2]\}$, for $\lambda\in\{0,1\}$. Obviously, both $ECQ^0_{s,t}$ and $ECQ^1_{s,t}$ are isomorphic to $ECQ(s,t-1)$. And an edge between $ECQ^0_{s,t}$ and $ECQ^1_{s,t}$ belongs to E_3.*

$ECQ(s,t)$ can be also decomposed into two subgraphs $ECQ_0(s,t)$ and $ECQ_1(s,t)$, where $V(ECQ_\lambda(s,t)) = \{\lambda a_{s-2}\cdots a_0 b_{t-1}\dots b_0 c \mid a_i, b_j, c \in \{0,1\}, i \in [0, s-2], j \in [0, t-1]\}$, for $\lambda \in \{0,1\}$. Obviously, both $ECQ_0(s,t)$ and $ECQ_1(s,t)$ are also isomorphic to $ECQ(s-1,t)$. An edge between $ECQ_0(s,t)$ and $ECQ_1(s,t)$ belongs to E_2.

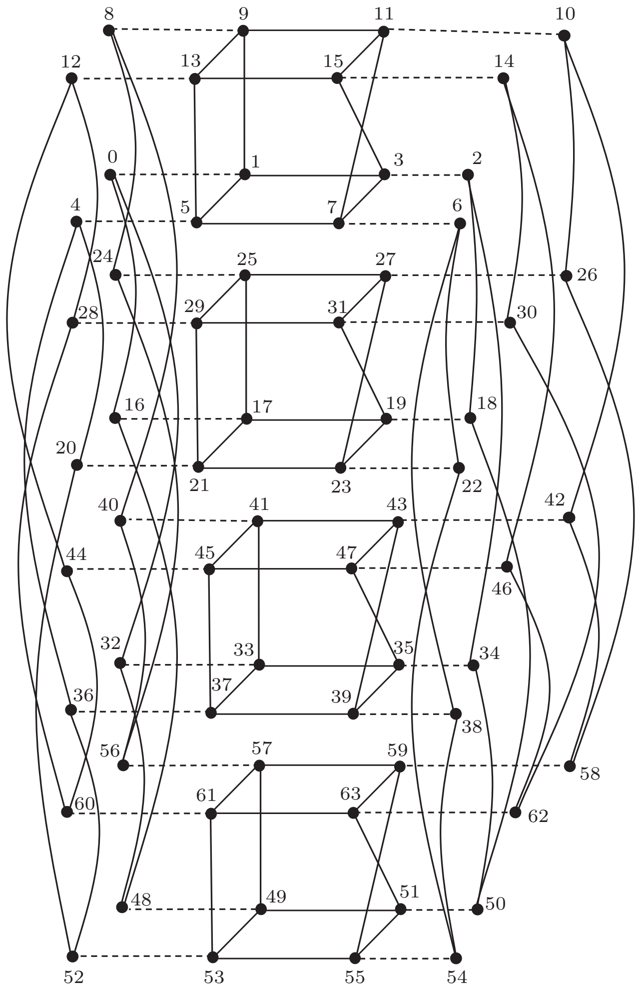

Fig.2. Exchanged crossed cube $ECQ(2,3)$.

The definitions of distance-preserving pair related and pair related distance were proposed by Chang *et al.* in crossed cubes[2]. Here, for convenience, we rephrase them as follows.

Let $u = u_{n-1}u_{n-2}\dots u_0$ and $v = v_{n-1}v_{n-2}\dots v_0$ be two distinct vertices in CQ_n. The i-th double bit of vertex u is defined as a 2-bit string $u_{2i+1}u_{2i}$ for $0 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor - 1$, and as a single bit u_{2i} for $i = \lfloor \frac{n}{2} \rfloor$ when n is odd. Bit l is called the most significant differing bit between u and v if $u_s = v_s$ for all s with $l+1 \leqslant s \leqslant n-1$ and $u_l \neq v_l$. Let $i^* = \lfloor \frac{l}{2} \rfloor$, called the most significant differing double bit. A function ρ on u, v is defined as follows.

$$\rho_j(u,v) = 0 \text{ for all } j \geqslant i^* + 1,$$

$$\rho_{i^*}(u,v) = \begin{cases} 2, & \text{if } u_{2i^*+1}u_{2i^*} = \bar{v}_{2i^*+1}\bar{v}_{2i^*}, \\ 1, & \text{otherwise.} \end{cases}$$

Furthermore, for $j \leqslant i^* - 1$, $\rho_j(u,v)$ can be defined with the notion of distance-preserving pair related (abbreviated as d.p. pair related) as follows.

Definition 4[2]. *$u_{2j+1}u_{2j}$ and $v_{2j+1}v_{2j}$, for $j \leqslant i^* - 1$, are distance-preserving pair related if one of the following conditions holds*:

1) $(u_{2j+1}u_{2j}, v_{2j+1}v_{2j}) \in \{(01,01),(11,11)\}$

and $\sum_{k=j+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_k(u,v)$ is even,

2) $(u_{2j+1}u_{2j}, v_{2j+1}v_{2j}) \in \{(01,11),(11,01)\}$

and $\sum_{k=j+1}^{\lfloor \frac{n-1}{2} \rfloor} \rho_k(u,v)$ is odd, and

3) $(u_{2j+1}u_{2j}, v_{2j+1}v_{2j}) \in \{(00,00),(10,10)\}$.

We write $u_{2j+1}u_{2j} \backsim^{\text{d.p.}} v_{2j+1}v_{2j}$ if $u_{2j+1}u_{2j}$ and $v_{2j+1}v_{2j}$ are d.p. pair related, and $u_{2j+1}u_{2j} \nsim^{\text{d.p.}} v_{2j+1}v_{2j}$ otherwise. Then $\rho_j(u,v)$ for $j \leqslant i^*-1$ is recursively defined as follows.

$$\rho_j(u,v) = \begin{cases} 0, & \text{if } u_{2j+1}u_{2j} \backsim^{\text{d.p.}} v_{2j+1}v_{2j}, \\ 1, & \text{otherwise.} \end{cases}$$

The pair related distance between u and v, denoted by $\rho(u,v)$, is defined as

$$\rho(u,v) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \rho_j(u,v).$$

By the definitions of distance-preserving pair related and pair related distance, Chang *et al.* further provided an $O(n)$ algorithm to get more shortest paths between any different vertices in CQ_n, called the CSH algorithm (for details on the algorithm, refer to [2]).

3 Path Embeddings in $ECQ(s,t)$

In this section, we will show that paths of lengths l with $\max\{9, \lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 4\} \leqslant l \leqslant 2^{s+t+1} - 1$ can be embedded between any two distinct vertices with dilation 1 in $ECQ(s,t)$, where $s \geqslant 3$ and $t \geqslant 3$. In the discussion of this section, we always assume $s \leqslant t$ except in the proof of Lemma 13, where $ECQ(4,3)$ will be used.

To prove the main results, Theorem 1 and Theorem 2, the basic lemmas are first listed as follows.

Lemma 1[2]. *For any integer $n \geqslant 1$ and any $u, v \in V(CQ_n)$, dist$(CQ_n, u, v) = \rho(u, v)$.*

Lemma 2[4]. *The diameter of an $ECQ(s,t)$ is $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 2$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x.*

Lemma 3[4]. *$ECQ(s,t)$ is isomorphic to $ECQ(t,s)$.*

Lemma 4[4]. *$ECQ(s,t)$ can be partitioned into two copies of $ECQ(s-1,t)$ or $ECQ(s,t-1)$.*

Lemma 5[4]. *$ECQ(s,t)$ can be divided into 2^t disjoint subgraphs isomorphic to CQ_s and 2^s disjoint subgraphs isomorphic to CQ_t.*

According to Lemma 1 and the definitions of distance-preserving pair related distance, pair related distance, and $ECQ(s,t)$, we can easily obtain the following results.

Lemma 6. *For any $u, v \in V(ECQ^i_{s,t})$ or $u, v \in V(ECQ_i(s,t))$ and any $i \in \{0,1\}$, dist$(ECQ(s,t), u, v) =$ dist$(ECQ^i_{s,t}, u, v)$ or dist$(ECQ(s,t), u, v) =$ dist$(ECQ_i(s,t), u, v)$.*

Lemma 7[36]. *For any integer $l \in \{2^{s+t+1} - 2,\ 2^{s+t+1} - 1\}$, there exists an $\langle x, y \rangle$-path of length l between any two distinct vertices x and y in $ECQ(s,t)$ where $s \geqslant 2$ and $t \geqslant 3$.*

We can verify Lemma 8 by a computer program and Lemma 7.

Lemma 8. 1) *In $ECQ(3,3)$, for any integer l with $9 \leqslant l \leqslant 127$, there exists an $\langle x, y \rangle$-path of length l between any two distinct vertices x and y. And there exist two distinct vertices $x, y \in V(ECQ(3,3))$ (e.g., $x = 0000000, y = 0000001$), such that there does not exist a path of length 8 between x and y.* 2) *In $ECQ(3,4)$, there exists an $\langle x, y \rangle$-path of length 9 between any two distinct vertices x and y. And there exist two distinct vertices $x, y \in V(ECQ(3,4))$ (e.g., $x = 00000000, y = 00100000$), such that there does not exist a path of length 8 between x and y.*

Lemma 9[14]. *For any integer $n \geqslant 3$, any $x, y \in V(CQ_n)$ with $x \neq y$, and any integer l with dist$(CQ_n, x, y) + 2 \leqslant l \leqslant 2^n - 1$, a path of length l can be embedded between x and y with dilation 1 in CQ_n.*

Lemma 10. *In $ECQ(3,4)$, for any two distinct vertices $x \in V(ECQ^0_{3,4})$ and $y \in V(ECQ^1_{3,4})$ with $(x, y) \notin E_1$, there exists a path P of length l between x and y in $ECQ(3,4)$, where $9 \leqslant l \leqslant 13$.*

Proof. By Lemma 5, $ECQ(3,4)$ can be divided into 16 copies of CQ_3 and 8 copies of CQ_4 (refer to Fig.3, let $s = 3$, $t = 4$). Hence, we can denote $CQ_3[1], CQ_3[2], \ldots, CQ_3[16]$ as 16 copies of CQ_3 that are composed of the edges E_2, and $CQ_4[1], CQ_4[2], \ldots, CQ_4[8]$ as 8 copies of CQ_4 that are composed of the edges E_3. More specifically, we denote $u^1_1, u^2_1, \ldots, u^8_1$ as 8 vertices of $CQ_3[1]$, $u^1_2, u^2_2, \ldots, u^8_2$ as 8 vertices of $CQ_3[2]$, $\ldots$, and $u^1_{16}, u^2_{16}, \ldots, u^8_{16}$ as 8 vertices of $CQ_3[16]$. And we denote $v^1_1, v^2_1, \ldots, v^{16}_1$ as 16 vertices of $CQ_4[1]$, $v^1_2, v^2_2, \ldots, v^{16}_2$ as 16 vertices of $CQ_4[2]$, $\ldots$, and $v^1_8, v^2_8, \ldots$, and v^{16}_8 as 16 vertices of $CQ_4[8]$. Then, we have the following cases.

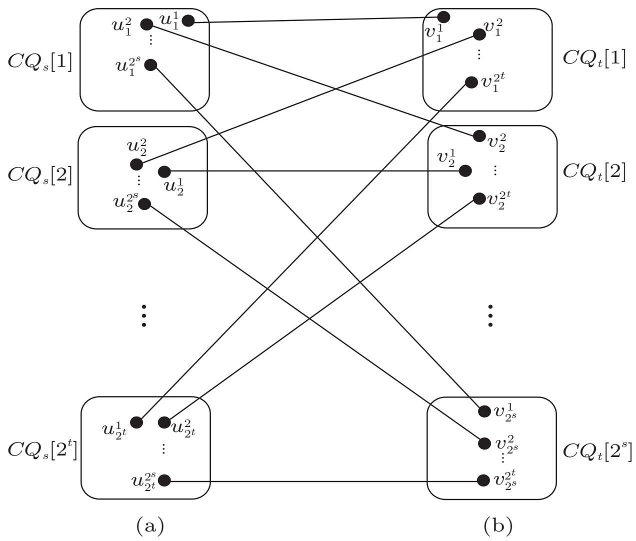

Fig.3. Another representation of $ECQ(s,t)$, (a straight line denotes an edge). (a) 2^t copies of CQ_s. (b) 2^s copies of CQ_t.

Case 1: $x, y \in V(CQ_4[k])$ for some integers $k \in [1, 8]$. Without loss of generality, suppose that $x, y \in V(CQ_4[1])$. Let $x = v^1_1$, $y = v^2_1$. By Lemma 9, there exists a path P_t of length l_t between x and y in $CQ_4[1]$, where $dist(CQ_4[1], x, y) + 2 \leqslant l_t \leqslant 15$. Since $dist(CQ_4[1], x, y) + 2 < 9 \leqslant l \leqslant 13 < 15$, there exists an $\langle x, y \rangle$-path of length l between x and y in $ECQ(3,4)$.

Case 2: $x \in V(CQ_3[k_1])$ and $y \in V(CQ_3[k_2])$ for some integers $k_1, k_2 \in [1, 16]$ with $k_1 \neq k_2$. Without loss of generality, suppose that $x = u^1_1 \in V(CQ_3[1])$ and $y = u^1_2 \in V(CQ_3[2])$. In $ECQ(3,4)$, each vertex of $CQ_3[1]$ has exactly one neighbor in $CQ_4[1], CQ_4[2], \ldots, CQ_4[8]$, respectively. And each vertex of $CQ_4[1]$ has exactly one neighbor in $CQ_3[1], CQ_3[2], \ldots, CQ_3[16]$, respectively. Without loss of generality, suppose that x has exactly one neighbor v^1_1 in $CQ_4[1]$. Select $v^2_1 \in V(CQ_4[1]) - \{v^1_1\}$ and let u^i_2 $(1 \leqslant i \leqslant 8)$ be the neighbor of v^2_1 in $CQ_3[2]$. We have the following cases.

Case 2.1: $i \neq 1$. Without loss of generality, suppose that $i = 2$. That is, $(v^2_1, u^2_2) \in E_1$. By Lemma 9, there exists a path P_t of length l_t between v^2_1 and v^1_1 in $CQ_4[1]$, where $dist(CQ_4[1], v^2_1, v^1_1) + 2 \leqslant l_t \leqslant 15$.

By using the CSH algorithm, we can get a shortest path P_s between u_2^2 and u_2^1 in $CQ_3[2]$, whose length is $dist(CQ_3[2], u_2^2, u_2^1)$. Then, $\langle x, v_1^1\rangle + P_t + \langle v_1^2, u_2^2\rangle + P_s$ is a path of length $l_t + dist(CQ_3[2], u_2^2, u_2^1) + 2$ between x and y in $ECQ(3,4)$. Since $dist(CQ_4[1], v_1^2, v_1^1) + dist(CQ_3[2], u_2^2, u_2^1) + 4 \leqslant 9 \leqslant l \leqslant 13 < 17 + dist(CQ_3[2], u_2^2, u_2^1)$, $P = \langle x, v_1^1\rangle + P_t + \langle v_1^2, u_2^2\rangle + P_s$ is a path of length l between x and y in $ECQ(3,4)$.

Case 2.2: $i = 1$. By Lemma 9, there exists a path P_t of length l_t between v_1^1 and v_1^2 in $CQ_4[1]$, where $dist(CQ_4[1], v_1^1, v_1^2) + 2 \leqslant l_t \leqslant 15$. Then, $\langle x, v_1^1\rangle + P_t + \langle v_1^2, y\rangle$ is a path of length $l_t + 2$ between x and y in $ECQ(3,4)$. Since $dist(CQ_4[1], v_1^1, v_1^2) + 4 < 9 \leqslant l \leqslant 13 < 17$, $P = \langle x, v_l^1\rangle + P_t + \langle v_1^2, y\rangle$ is a path of length l between x and y in $ECQ(3,4)$.

Case 3: $x \in V(CQ_4[k_1])$ and $y \in V(CQ_4[k_2])$ for some integers $k_1, k_2 \in [1,8]$ with $k_1 \neq k_2$. Without loss of generality, suppose that $x = v_1^1 \in V(CQ_4[1])$ and $y = v_2^1 \in V(CQ_4[2])$. In $ECQ(3,4)$, each vertex of $CQ_4[1]$ has exactly one neighbor in $CQ_3[1], CQ_3[2], \ldots, CQ_3[16]$, respectively. And each vertex of $CQ_3[1]$ has exactly one neighbor in $CQ_4[1], CQ_4[2], \ldots, CQ_4[8]$, respectively. Without loss of generality, suppose that x has exactly one neighbor u_1^1 in $CQ_3[1]$. Select $(u_1^2, v_2^2) \in E_1$. Obviously, $v_2^2 \neq y$. By Lemma 9, there exists a path P_t of length l_t between v_2^2 and v_2^1, where $dist(CQ_4[2], v_2^2, v_2^1) + 2 \leqslant l_t \leqslant 15$. By using the CSH algorithm, we can get a shortest path P_s between u_1^2 and u_1^1 in $CQ_3[1]$, whose length is $dist(CQ_3[1], u_1^2, u_1^1)$. Then, $\langle x, u_1^1\rangle + P_s + \langle u_1^2, v_2^2\rangle + P_t$ is a path of length $l_t +$ $dist(CQ_3[1], u_1^2, u_1^1) + 2$ between x and y in $ECQ(3,4)$. Since $dist(CQ_3[1], u_1^2, u_1^1) + dist(CQ_4[2], v_2^2, v_2^1) + 4 \leqslant 9 \leqslant l \leqslant 13 < 17 + dist(CQ_3[1], u_1^2, u_1^1)$, $P = \langle x, u_1^1\rangle + P_s + \langle u_1^2, v_2^2\rangle + P_t$ is a path of length l between x and y in $ECQ(3,4)$.

Case 4: $x \in V(CQ_4)$ and $y \in V(CQ_3)$. Without loss of generality, suppose that $x = v_1^1 \in V(CQ_4[1])$ and $y = u_2^1 \in V(CQ_3[2])$. In $ECQ(3,4)$, each vertex of $CQ_3[1]$ has exactly one neighbor in $CQ_4[1], CQ_4[2], \ldots, CQ_4[8]$, respectively. And each vertex of $CQ_4[1]$ has exactly one neighbor in $CQ_3[1], CQ_3[2], \ldots, CQ_3[16]$, respectively. Select $v_1^2 \in V(CQ_4[1]) - \{v_1^1\}$ and let u_2^i $(1 \leqslant i \leqslant 8)$ be the neighbor of v_1^2 in $CQ_3[2]$. We have the following cases.

Case 4.1: $i \neq 1$. Without loss of generality, let $i = 2$. By Lemma 9, there exists a path P_t of length l_t between v_1^1 and v_1^2, where $dist(CQ_4[1], v_1^1, v_1^2) + 2 \leqslant l_t \leqslant 15$. By using the CSH algorithm, we can get a shortest path P_s between u_2^2 and u_2^1 in $CQ_3[2]$, whose length is $dist(CQ_3[2], u_2^2, u_2^1)$. Then, $P_t + \langle v_1^2, u_2^2\rangle + P_s$ is a path of length $l_t +$ $dist(CQ_3[2], u_2^2, u_2^1) + 1$ between x and y in $ECQ(3,4)$. Since $dist(CQ_4[1], v_1^1, v_1^2) + dist(CQ_3[2], u_2^2, u_2^1) + 3 < 9 \leqslant l \leqslant 13 < 16 + dist(CQ_3[2], u_2^2, u_2^1)$, $P = P_t + \langle v_1^2, u_2^2\rangle + P_s$ is a path of length l between x and y in $ECQ(3,4)$.

Case 4.2: $i = 1$. By Lemma 9, there exists a path P_t of length l_t between v_1^1 and v_1^2 in $CQ_4[1]$, where $dist(CQ_4[1], v_1^1, v_1^2) + 2 \leqslant l_t \leqslant 15$. Then, $P_t + \langle v_1^2, y\rangle$ is a path of length $l_t + 1$ between x and y in $ECQ(3,4)$. Since $dist(CQ_4[1], v_1^1, v_1^2) + 3 < 9 \leqslant l \leqslant 13 < 16$, $P = P_t + \langle v_1^2, y\rangle$ is a path of length l between x and y in $ECQ(3,4)$.

Case 5: $x \in V(CQ_3)$ and $y \in V(CQ_4)$. Using the similar method in case 4, we can prove that there exists a path of length l between x and y in $ECQ(3,4)$. Thus, the proof of case 5 is omitted. □

Lemma 11. *If $s \geqslant 4, t \geqslant 4$, and $s \leqslant t$, for any two distinct vertices $x \in V(ECQ_{s,t}^0)$ and $y \in V(ECQ_{s,t}^1)$ with $(x,y) \notin E_1$, there exists a path P of length l between x and y in $ECQ(s,t)$, where $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 4 \leqslant l \leqslant 2^t - 1$.*

Proof. By Lemma 5, $ECQ(s,t)$ can be divided into 2^t copies of CQ_s and 2^s copies of CQ_t. Hence, we can denote $CQ_s[1], CQ_s[2], \ldots, CQ_s[2^t]$ as 2^t copies of CQ_s who are composed of the edges E_2, and $CQ_t[1], CQ_t[2], \ldots, CQ_t[2^s]$ as 2^s copies of CQ_t who are composed of the edges E_3. More specifically, we denote $u_1^1, u_1^2, \ldots, u_1^{2^s}$ as 2^s vertices of $CQ_s[1]$, $u_2^1, u_2^2, \ldots, u_2^{2^s}$ as 2^s vertices of $CQ_s[2], \ldots$, and $u_{2^t}^1, u_{2^t}^2, \ldots, u_{2^t}^{2^s}$ as 2^s vertices of $CQ_s[2^t]$. And we denote $v_1^1, v_1^2, \ldots, v_1^{2^t}$ as 2^t vertices of $CQ_t[1]$, $v_2^1, v_2^2, \ldots, v_2^{2^t}$ as 2^t vertices of $CQ_t[2], \ldots$, and $v_{2^s}^1, v_{2^s}^2, \ldots, v_{2^s}^{2^t}$ as 2^t vertices of $CQ_t[2^s]$. Then, we may verify the result by five cases below (shown in Fig.3).

Case 1: $x, y \in V(CQ_t[k])$ for some integers $k \in [1, 2^s]$. Without loss of generality, suppose that $x, y \in V(CQ_t[1])$. Let $x = v_1^1, y = v_1^2$. By Lemma 9, there exists a path P_t of length l_t between v_1^2 and v_1^1 in $CQ_t[1]$, where $dist(CQ_t[1], v_1^1, v_1^2) + 4 \leqslant l_t \leqslant 2^t - 1$. Since $dist(CQ_t[1], v_1^1, v_1^2) + 2 < \lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 4$ $(s \geqslant 4, t \geqslant 4)$, P_t is a path of length l between x and y in $ECQ(s,t)$.

Case 2: $x \in V(CQ_s[k_1])$ and $y \in V(CQ_s[k_2])$ for some integers $k_1, k_2 \in [1, 2^t]$ with $k_1 \neq k_2$. Without loss of generality, suppose that $x = u_1^1 \in V(CQ_s[1])$ and $y = u_2^1 \in V(CQ_s[2])$. In $ECQ(s,t)$, each vertex of $CQ_s[1]$ has exactly one neighbor in $CQ_t[1], CQ_t[2], \ldots, CQ_t[2^s]$, respectively. And each vertex of $CQ_t[1]$ has exactly one neighbor in $CQ_s[1], CQ_s[2], \ldots, CQ_s[2^t]$, respectively. Without loss of generality, suppose that x has exactly one neighbor v_1^1 in $CQ_t[1]$. Select $v_1^2 \in V(CQ_t[1]) - \{v_1^1\}$ and let u_2^i $(1 \leqslant i \leqslant 2^s)$ be the neighbor of v_1^2 in $CQ_s[2]$. We consider the following cases.

Case 2.1: $i \neq 1$. Without loss of generality, let $i = 2$. That is, $(v_1^2, u_2^2) \in E_1$. By Lemma 9, there exists a path P_t of length l_t between v_1^2 and v_1^1 in $CQ_t[1]$, where $dist(CQ_t[1], v_1^2, v_1^1)+2 \leqslant l_t \leqslant 2^t-1$. By using the CSH algorithm, we can get a shortest path P_s between u_2^2 and u_2^1 in $CQ_s[2]$, whose length is d_s. Then, $\langle x, v_1^1\rangle + P_t + \langle v_1^2, u_2^2\rangle + P_s$ is a path of length $l_t + d_s + 2$ between x and y in $ECQ(s,t)$. Since $dist(CQ_t[1], v_1^2, v_1^1) + d_s + 4 \leqslant \lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 4 \leqslant l \leqslant 2^t - 1 < 2^t + d_s + 1$ $(4 \leqslant s \leqslant t)$, $P = \langle x, v_1^1\rangle + P_t + \langle v_1^2, u_2^2\rangle + P_s$ is a path of length l between x and y in $ECQ(s,t)$.

Case 2.2: $i = 1$. By Lemma 9, there exists a path P_t of length l_t between v_1^1 and v_1^2 in $CQ_t[1]$, where $dist(CQ_t[1], v_1^1, v_1^2) + 2 \leqslant l_t \leqslant 2^t - 1$. Then, $\langle x, v_1^1\rangle + P_t + \langle v_1^2, y\rangle$ is a path of length $l_t + 2$ between x and y in $ECQ(s,t)$. Since $dist(CQ_t[1], v_1^1, v_1^2) + 4 < \lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 4 \leqslant l \leqslant 2^t - 1 < 2^t + 1$ $(4 \leqslant s \leqslant t)$, $P = \langle x, v_1^1\rangle + P_t + \langle v_1^2, y\rangle$ is a path of length l between x and y in $ECQ(s,t)$.

Case 3: $x \in V(CQ_t[k_1])$ and $y \in V(CQ_t[k_2])$ for some integers $k_1, k_2 \in [1, 2^s]$ with $k_1 \neq k_2$. The method is similar to case 3 of Lemma 10. Thus, the proof of case 3 is omitted.

Case 4: $x \in V(CQ_t)$ and $y \in V(CQ_s)$. Without loss of generality, suppose that $x = v_1^1 \in V(CQ_t[1])$ and $y = u_2^1 \in V(CQ_s[2])$. In $ECQ(s,t)$, each vertex of $CQ_s[1]$ has exactly one neighbor in $CQ_t[1], CQ_t[2], \ldots, CQ_t[2^s]$, respectively. And each vertex of $CQ_t[1]$ has exactly one neighbor in $CQ_s[1], CQ_s[2], \ldots, CQ_s[2^t]$, respectively. Select $v_1^2 \in V(CQ_t[1]) - \{v_1^1\}$ and let u_2^i $(1 \leqslant i \leqslant 2^s)$ be the neighbor of v_1^2 in $CQ_s[2]$. We have the following cases.

Case 4.1: $i \neq 1$. Without loss of generality, suppose that $i = 2$. By Lemma 9, there exists a path P_t of length l_t between v_1^1 and v_1^2 in $CQ_t[1]$, where $dist(CQ_t[1], v_1^1, v_1^2) + 2 \leqslant l_t \leqslant 2^t - 1$. By using the CSH algorithm, we can get a shortest path P_s between u_2^2 and u_2^1 in $CQ_s[2]$, whose length is d_s. Then, $P_t + \langle v_1^2, u_2^2\rangle + P_s$ is a path of length $l_t + d_s + 1$ between x and y in $ECQ(s,t)$. Since $dist(CQ_t[1], v_1^1, v_1^2) + d_s + 3 < \lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 4 \leqslant l \leqslant 2^t - 1 < 2^t + d_s$, $P = P_t + \langle v_1^2, u_2^2\rangle + P_s$ is a path of length l between x and y in $ECQ(s,t)$.

Case 4.2: $i = 1$. By Lemma 9, there exists a path P_t of length l_t between v_1^1 and v_1^2 in $CQ_t[1]$, where $dist(CQ_t[1], v_1^1, v_1^2) + 2 \leqslant l_t \leqslant 2^t - 1$. Then, $P_t + \langle v_1^2, y\rangle$ is a path of length $l_t + 1$ between x and y in $ECQ(s,t)$. Since $dist(CQ_t[1], v_1^1, v_1^2) + 3 < \lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 4 \leqslant l \leqslant 2^t - 1 < 2^t$ $(4 \leqslant s \leqslant t)$, $P = P_t + \langle v_1^2, y\rangle$ is a path of length l between x and y in $ECQ(s,t)$.

Case 5: $x \in V(CQ_s)$ and $y \in V(CQ_t)$. Using the similar method in case 4, we can prove that there exists a path of length l between x and y in $ECQ(s,t)$. Thus, the proof of case 5 is omitted. □

Theorem 1. *If $s \geqslant 3$ and $t \geqslant 3$, for any two distinct vertices $x, y \in V(ECQ(s,t))$, and any integer l with $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 5 \leqslant l \leqslant 2^{s+t+1} - 1$, there exists a path P of length l between x and y in $ECQ(s,t)$.*

Proof. In fact, we only need to prove that there exists a path of length l between x and y in $ECQ(s,t)$. We proceed by induction on $s+t$. As a basis, the conclusion clearly holds for $ECQ(3,3)$ by Lemma 8. Suppose that the conclusion is true for $6 \leqslant s+t = k$. When $s+t = k+1$, since $s \leqslant t$, we have $s \geqslant 3$ and $t \geqslant 4$. We deal with the following cases, respectively.

Case 1: $x, y \in V(ECQ_{s,t}^0)$. For $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 5 \leqslant l \leqslant 2^{s+t+1} - 1$, we have the following cases.

Case 1.1: $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 5 \leqslant l \leqslant 2^{s+t} - 1$. By the induction hypothesis, there exist paths of lengths $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t}{2} \rceil + 5, \lceil \frac{s+1}{2} \rceil + \lceil \frac{t}{2} \rceil + 6, \ldots, 2^{s+t} - 1$ between x and y in $ECQ_{s,t}^0$. Notice that $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t}{2} \rceil + 5 \leqslant \lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 5$, there exist paths of lengths l between x and y in $ECQ_{s,t}^0$ and, thus, in $ECQ(s,t)$.

Case 1.2: $2^{s+t} \leqslant l \leqslant 2^{s+t+1} - 1$. By Lemma 7, there exists a Hamiltonian path $P_0: \langle x = x^{(0)}, x^{(1)}, \ldots, x^{(l_1)}, \ldots, x^{(2^{s+t}-1)} = y\rangle$ between x and y in $ECQ_{s,t}^0$. Select an $\langle x^{(0)}, x^{(l_1)}\rangle$-path of length l_1 on P_0 where $1 \leqslant l_1 \leqslant 2^{s+t} - 2$. We consider the following cases, respectively.

Case 1.2.1: $x^{(l_1)}, y \in V(CQ_t[i])$, where $1 \leqslant i \leqslant 2^s$. We can write $l = l_1 + l_2 + 2$ where $2^{s+t} - 3 \leqslant l_2 \leqslant 2^{s+t} - 1$. Let u be the neighbor of $x^{(l_1)}$ in $ECQ_{s,t}^1$ and v be the neighbor of y in $ECQ_{s,t}^1$. Since $2^{s+t} - 3 > D(ECQ_{s,t}^1) + 3$, by the induction hypothesis, there exists a path P_2 of length l_2 between u and v in $ECQ_{s,t}^1$. Then, $P = \langle x, x^{(1)}, \ldots, x^{(l_1)}, u, P_2, v, y\rangle$ is an $\langle x, y\rangle$-path of length l in $ECQ(s,t)$. Fig.4(a) illustrates the subcase.

Case 1.2.2: $x^{(l_1)}, y \in V(CQ_s[j])$, where $1 \leqslant j \leqslant 2^t$. We can write $l = l_1 + l_2 + 4$ where $2^{s+t} - 5 \leqslant l_2 \leqslant 2^{s+t} - 3$. Choose $(x^{(l_1)}, u_1) \in E_1$ with $u_1 \neq x^{(l_1-1)}$ in $ECQ_{s,t}^0$ and $(y, v_1) \in E_1$ with $v_1 \neq x^{(2^{s+t}-2)}$ in $ECQ_{s,t}^0$. Let u_2 be adjacent to u_1 in $ECQ_{s,t}^1$ and v_2 be the neighbor of v_1 in $ECQ_{s,t}^1$. Since $2^{s+t} - 5 > D(ECQ_{s,t}^1) + 3$, by the induction hypothesis, there exists a path P_2 of length l_2 between u_2 and v_2 in $ECQ_{s,t}^1$. Then, $P = \langle x, x^{(1)}, \ldots, x^{(l_1)}, u_1, u_2, P_2, v_2, v_1, y\rangle$ is an $\langle x, y\rangle$-

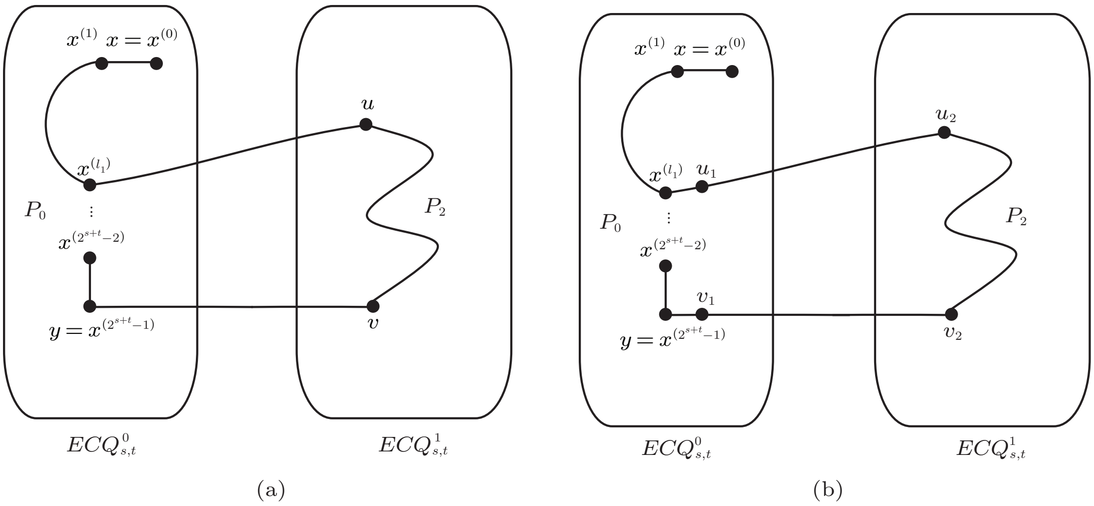

Fig.4. Illustration for the proof of (a) case 1.2.1 and (b) case 1.2.2 (a straight line denotes an edge and a curved line denotes a path between two vertices).

path of length l in $ECQ(s,t)$. Fig.4(b) illustrates the subcase.

Case 1.2.3: $x^{(l_1)} \in V(CQ_t[i])$ and $y \in V(CQ_s[j])$, where $1 \leqslant i \leqslant 2^s$ and $1 \leqslant j \leqslant 2^t$. We can write $l = l_1 + l_2 + 3$ where $2^{s+t} - 4 \leqslant l_2 \leqslant 2^{s+t} - 2$. Choose $(y, v_1) \in E_1$ with $v_1 \neq x^{(2^{s+t}-2)}$ in $ECQ^0_{s,t}$. Let u_1 be adjacent to $x^{(l_1)}$ in $ECQ^1_{s,t}$ and v_2 be the neighbor of v_1 in $ECQ^1_{s,t}$. Since $2^{s+t} - 4 > D(ECQ^1_{s,t}) + 3$, by the induction hypothesis, there exists a path P_2 of length l_2 between u_1 and v_2 in $ECQ^1_{s,t}$. Then, $P = \langle x, x^{(1)}, \ldots, x^{(l_1)}, u_1, P_2, v_2, v_1, y\rangle$ is an $\langle x, y\rangle$-path of length l in $ECQ(s,t)$. Fig.5(a) illustrates the subcase.

Case 1.2.4: $x^{(l_1)} \in V(CQ_s[j])$ and $y \in V(CQ_t[i])$, where $1 \leqslant j \leqslant 2^t$ and $1 \leqslant i \leqslant 2^s$. The discussion is similar to that of case 1.2.3. Thus, the proof of case 1.2.4 is omitted.

Case 2: $x \in V(ECQ^0_{s,t})$ and $y \in V(ECQ^1_{s,t})$. We have the following cases.

Case 2.1: $2^{s+t} \leqslant l \leqslant 2^{s+t+1} - 1$. Select $x' \in V(CQ_t[i])$ in $V(ECQ^0_{s,t}) - \{x\}$, where $1 \leqslant i \leqslant 2^s$. Let y' be a neighbor of x' in $ECQ^1_{s,t}$ and $y' \neq y$. By the induction hypothesis, there exists an $\langle x, x'\rangle$-path P_0 of length l_0 in $ECQ^0_{s,t}$, where $l_0 \in \{D(ECQ^0_{s,t}) + 3,\ 2^{s+t} - 1\}$; there also exists a $\langle y', y\rangle$-path P_1 of length l_1 in $ECQ^1_{s,t}$, where $l_1 \in \{D(ECQ^1_{s,t}) + 3,\ 2^{s+t} - 1\}$. Since $2^{s+t} > D(ECQ^0_{s,t}) + D(ECQ^1_{s,t}) + 7$, then $P = P_0 + \langle x', y'\rangle + P_1$ is a path of length l between x and y in $ECQ(s,t)$.

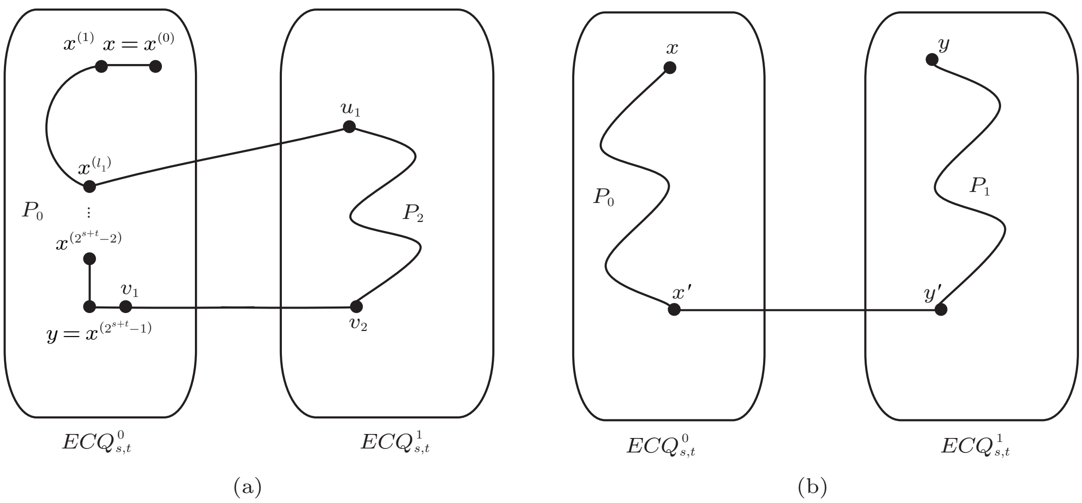

Fig.5. Illustration for the proof of (a) case 1.2.3 and (b) case 2.1 (a straight line denotes an edge and a curved line denotes a path between two vertices).

Fig.5(b) illustrates the subcase.

Case 2.2: $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 5 \leqslant l \leqslant 2^{s+t} - 1$. We deal with the following cases.

Case 2.2.1: $dist(x, y) \geqslant 2$. We consider the following cases.

Case 2.2.1.1: $x \in V(CQ_t[i])$, where $1 \leqslant i \leqslant 2^s$. Let x' be a neighbor of x in $ECQ^1_{s,t}$ and $x' \neq y$. We further deal with two cases according to the odevity of t.

If t is odd, by the induction hypothesis, there exists an $\langle x', y\rangle$-path P_1 of length l_1 in $ECQ^1_{s,t}$, where $l_1 \in \{D(ECQ^1_{s,t}) + 3,\ 2^{s+t} - 1\}$. Then $\langle x, x'\rangle + P_1$ is a path of length $l_1 + 1$. By Lemma 10 and Lemma 11, there exists an $\langle x, y\rangle$-path of length $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 5$ between x and y. Thus, $P = \langle x, x'\rangle + P_1$ is a path of length l between x and y in $ECQ(s, t)$.

Otherwise, by the induction hypothesis, there exists an $\langle x', y\rangle$-path P_1 of length l_1 in $ECQ^1_{s,t}$, where $l_1 \in \{D(ECQ^1_{s,t}) + 3,\ 2^{s+t} - 1\}$. Then, $\langle x, x'\rangle + P_1$ is an $\langle x, y\rangle$-path of length $l_1 + 1$ in $ECQ(s, t)$. Thus, $P = \langle x, x'\rangle + P_1$ is a path of length l between x and y in $ECQ(s, t)$ (see Fig.6(a) for illustration).

Case 2.2.1.2: $x \in V(CQ_s[j])$, where $1 \leqslant j \leqslant 2^t$. Select $w \in V(CQ_t[i])$ and w is the neighbor of x in $ECQ^0_{s,t}$. We deal with the following cases.

For $dist(x, y) \geqslant 3$, let x' be a neighbor of w with $x' \neq y$ in $ECQ^1_{s,t}$. We further deal with two cases according to the odevity of t.

If t is odd, by the induction hypothesis, there exists an $\langle x', y\rangle$-path P_1 of length l_1 in $ECQ^1_{s,t}$, where $D(ECQ^1_{s,t}) + 3 \leqslant l_1 \leqslant 2^{s+t} - 1$. Then $\langle x, w, x'\rangle + P_1$ is a path of length $l_1 + 2$. By Lemma 10 and Lemma 11, there exists an $\langle x, y\rangle$-path of the length between $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 5$ and $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 6$. Thus, $P = \langle x, w, x'\rangle + P_1$ is a path of length l between x and y in $ECQ(s, t)$. Otherwise, by the induction hypothesis, there exists an $\langle x', y\rangle$-path P_1 of length l_1 in $ECQ^1_{s,t}$, where $D(ECQ^1_{s,t}) + 3 \leqslant l_1 \leqslant 2^{s+t} - 1$. Then $\langle x, w, x'\rangle + P_1$ is a path of length $l_1 + 2$ between x and y. By Lemma 10 and Lemma 11, there exists an $\langle x, y\rangle$-path of length $\lceil \frac{s+1}{2} \rceil + \lceil \frac{t+1}{2} \rceil + 5$. Thus, $P = \langle x, w, x'\rangle + P_1$ is a path of length l between x and y in $ECQ(s, t)$ (see Fig.6(b) for illustration).

For $dist(x, y) = 2$, obviously, $x \in V(CQ_s[j])$, $w \in V(CQ_t[i])$, and $y \in V(CQ_t[i])$. We further deal

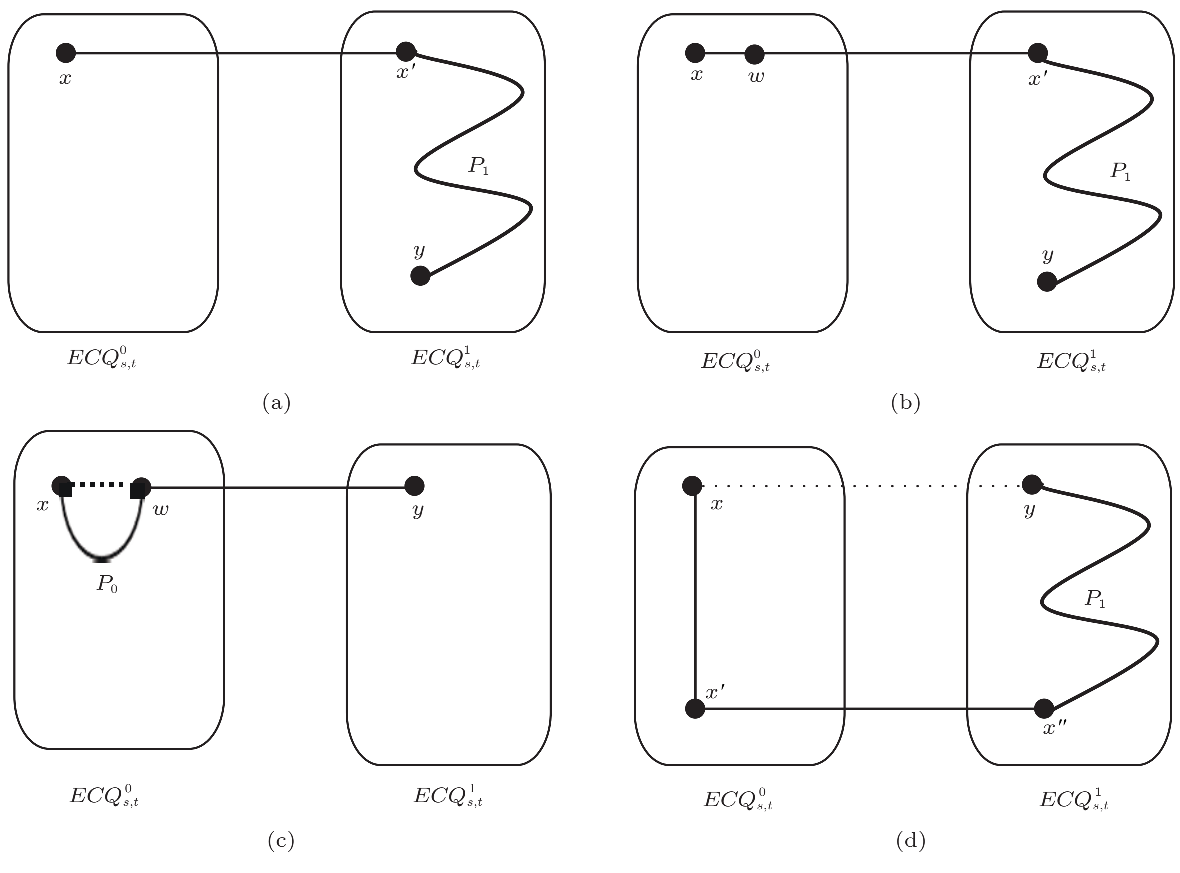

Fig.6. Illustration of case 2.2 (a straight line denotes an edge, a dash line denotes a removed edge, and a curved line denotes a path between two vertices). (a) Case 2.2.1.1. (b) Case 2.2.1.2: $dist(x, y) \geqslant 3$. (c) Case 2.2.1.2: $dist(x, y) = 2$. (d) Case 2.2.2: $dist(x, y) = 1$.

with two cases according to the odevity of t.

If t is odd, by the induction hypothesis, there exists an $\langle x,w\rangle$-path P_0 of length l_0 in $ECQ^0_{s,t}$, where $D(ECQ^0_{s,t})+3\leqslant l_0\leqslant 2^{s+t}-1$. Then $P_0+\langle w,y\rangle$ is a path of length l_0+1. By Lemma 10 and Lemma 11, there exists an $\langle x,y\rangle$-path of length $\lceil\frac{s+1}{2}\rceil+\lceil\frac{t+1}{2}\rceil+5$. Thus, $P=P_0+\langle w,y\rangle$ is a path of length l between x and y in $ECQ(s,t)$. Otherwise, by the induction hypothesis, there exists an $\langle x,w\rangle$-path P_0 of length l_0 in $ECQ^0_{s,t}$, where $D(ECQ^0_{s,t})+3\leqslant l_0\leqslant 2^{s+t}-1$. Then $P_0+\langle w,y\rangle$ is an $\langle x,y\rangle$-path of length l_0+1 in $ECQ(s,t)$. Thus, $P=P_0+\langle w,y\rangle$ is a path of length l between x and y in $ECQ(s,t)$ (see Fig.6(c) for illustration).

Case 2.2.2: $dist(x,y)=1$. Obviously, $x,y\in V(CQ_t[i])$. Let x' be a neighbor of x with $x'\in V(CQ_t[i])$ in $ECQ^0_{s,t}$. And let x'' be a neighbor of x' and $x''\neq y$ in $ECQ^1_{s,t}$. We further deal with two cases according to the odevity of t.

If t is odd, by the induction hypothesis, there exists an $\langle x'',y\rangle$-path P_1 of length l_1 in $ECQ^1_{s,t}$, where $D(ECQ^1_{s,t})+3\leqslant l_1\leqslant 2^{s+t}-1$. Then $\langle x,x',x''\rangle+P_1$ is a path of length l_1+2. By Lemma 10 and Lemma 11, there exists an $\langle x,y\rangle$-path of length between $\lceil\frac{s+1}{2}\rceil+\lceil\frac{t+1}{2}\rceil+5$ and $\lceil\frac{s+1}{2}\rceil+\lceil\frac{t+1}{2}\rceil+6$ in $ECQ(s,t)$. Thus, $P=\langle x,x',x''\rangle+P_1$ is a path of length l between x and y in $ECQ(s,t)$. Otherwise, by the induction hypothesis, there exists an $\langle x'',y\rangle$-path P_1 of length l_1 in $ECQ^1_{s,t}$, where $D(ECQ^1_{s,t})+3\leqslant l_1\leqslant 2^{s+t}-1$. Then $\langle x,x',x''\rangle+P_1$ is a path of length l_1+2. By Lemma 10 and Lemma 11, there exists an $\langle x,y\rangle$-path of length $\lceil\frac{s+1}{2}\rceil+\lceil\frac{t+1}{2}\rceil+5$ in $ECQ(s,t)$. Thus, $P=\langle x,x',x''\rangle+P_1$ is a path of length l between x and y in $ECQ(s,t)$ (see Fig.6(d) for illustration).

Case 3: $x,y\in V(ECQ^1_{s,t})$. The discussion is similar to $x,y\in V(ECQ^0_{s,t})$. Thus, the proof is omitted.

Case 4: $x\in V(ECQ^1_{s,t})$ and $y\in V(ECQ^0_{s,t})$. The discussion is similar to $x\in V(ECQ^0_{s,t})$ and $y\in V(ECQ^1_{s,t})$. Thus, the proof of case 4 is also omitted. □

By Lemma 8 and Theorem 1, we easily get Lemma 12.

Lemma 12. *In $ECQ(3,4)$, there exists an $\langle x,y\rangle$-path of length l between any two distinct vertices x and y in $ECQ(3,4)$, where $9\leqslant l\leqslant 255$.*

Lemma 13. *In $ECQ(4,4)$, for any integer l with $10\leqslant l\leqslant 511$, there exists an $\langle x,y\rangle$-path of length l between any two distinct vertices x and y.*

So far, our discussion has assumed $s\leqslant t$. However, $ECQ(4,3)$ will be used in the following proof for Lemma 13.

Proof. By Theorem 1, there exists an $\langle x,y\rangle$-path of length l between any two distinct vertices x and y with $11\leqslant l\leqslant 511$ in $ECQ(4,4)$. Thus, we only need to prove that there exists a path of length 10 between x and y in $ECQ(4,4)$. We have the following cases.

Case 1: $x,y\in V(ECQ^i_{4,4}), i\in\{0,1\}$. By Lemma 12 and Lemma 3, there exists a path of length 10 between x and y in $ECQ^i_{4,4}$ and, thus, in $ECQ(4,4)$.

Case 2: $x\in V(ECQ^i_{4,4})$, $y\in V(ECQ^{1-i}_{4,4})$, $i\in\{0,1\}$. Obviously, $(x,y)\notin E_1$. By Lemma 11, there exist paths of length 10 between x and y in $ECQ(4,4)$. □

Theorem 2. *If $s\geqslant 3$ and $t\geqslant 4$, for any two distinct vertices $x,y\in V(ECQ(s,t))$, and any integer l, $\lceil\frac{s+1}{2}\rceil+\lceil\frac{t+1}{2}\rceil+4\leqslant l\leqslant 2^{s+t+1}-1$, there exists an $\langle x,y\rangle$-path of length l between x and y in $ECQ(s,t)$.*

Proof. We only need to prove that there exists a path of length l between x and y in $ECQ(s,t)$. We proceed by induction on $s+t$.

As a basis, the conclusion clearly holds for $ECQ(3,4)$ by Lemma 12. Suppose that the conclusion is true for $7\leqslant s+t=k$. Let $s+t=k+1$. Then $s\geqslant 3$ and $t\geqslant 5$, or $s=t=4$. By Lemma 13, the conclusion clearly holds for $ECQ(4,4)$. For $s\geqslant 3$ and $t\geqslant 5$, we partition $ECQ(s,t)$ into two subgraphs: $ECQ^0_{s,t}$ and $ECQ^1_{s,t}$. Using the similar method in Theorem 1, we can prove that there exists an $\langle x,y\rangle$-path of length l in $ECQ(s,t)$. □

4 Conclusions

This paper revealed the fact that $ECQ(s,t)$ preserves the path embedding capability to a large extent, while it only has about one half edges of CQ_n. We proved that paths of all lengths between $\max\{9,\lceil\frac{s+1}{2}\rceil+\lceil\frac{t+1}{2}\rceil+4\}$ and $2^{s+t+1}-1$ can be embedded between any two distinct vertices with dilation 1 in $ECQ(s,t)$, where $s\geqslant 3$ and $t\geqslant 3$. Since failures are inevitable in any interconnection network, it is important to study its fault-tolerant embedding. In particular, studying the embeddability of the path with fault vertices/edges in the exchanged crossed cube is our future work.

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