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# A Family of Stable Multipath Dual Congestion Control Algorithms

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**Abstract** We consider the problem of multipath congestion control in the Internet. The aim is to take advantage of multiple paths diversity to achieve efficient bandwidth allocation and improve network efficiency. But there exist some potential difficulties when one directly uses the well-known network utility maximization model to design stable multipath congestion control algorithms for the alternative paths. In this paper, we propose a generalized multipath utility maximization model to consider the problem of joint routing and rate control, which can be reduced to specific models with different parameter settings. And then we develop a family of multipath dual congestion control algorithms which are stable in the absence of delays. We also derive decentralized and scalable sufficient conditions for a particular scheme when propagation delays exist in networks. The simulation results show that the proposed multipath dual congestion control algorithms with appropriate parameter settings can achieve stable resource shares while maintaining fairness among the involved users.

**Keywords** flow control, resource allocation, duality, stability, multipath congestion control

## 1 Introduction

Today, there exist many applications and services that can be realized by flexible multipath routing. If multiple paths exist, applications that have different performance requirements can use different routings, and if multiple paths exist, traffic can switch quickly to an alternative path when a link or router fails[1]. In that scheme, packets belonging to the same origin-destination pair are transmitted along several routes between them instead of a single path. In this paper, we consider networks where multiple paths are available to each user between its source and destination, and the user can direct its flow along these paths using source routing.

There are many questions in multipath routing[2-4]. One question is: given the availability of multiple routes between each origin-destination pair, how does one design stable congestion control algorithms that exploit the multipath routing capability? We are interested in this question in the paper.

The Transmission Control Protocol and the Internet Protocol, known as TCP/IP, are widespread for guiding traffic flows in the Internet. In a packet-switch network, a route is computed and selected to send packets from a source to a destination, and the sending rate is determined by TCP. Traditionally, a single path routing scheme is deployed, where the shortest path is chosen by IP routing in terms of hop count or distance, and the flow rate is varied according to congestion level along that path. Ideally, both routes and flow rates should be guided to guarantee the efficiency and fairness in link bandwidth utilization. There thus has long been a

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desire to direct routes selection and rates variation according to congestion level. However, studies, e.g., [5], have shown that making paths selection consistent with congestion level may result in network occlusions and routing instability. Despite that IP routing is highly scalable, the static or single path routing scheme fails to react to instantaneous network congestion. In contrast, multipath routing can more easily reach equilibrium: instead of drastic switches of large bulks of traffic, it can gradually adapt the traffic mix between different routes. Mathematically, when we consider the optimization of a convex congestion cost to serve a matrix of end-to-end demands over single path routes, the problem is non-convex, but its relaxation to multipath amounts to convex multi-commodity optimization.

Many researchers try to find a multipath congestion control protocol implemented in a decentralized way by source and routers. Furthermore the protocol should make the system stay at a stable equilibrium point which satisfies some basic requirements, such as the high utilization of resources, small queues, and a degree of control over resource allocation. All of these are required to be scalable, i.e., holding for an arbitrary network, with possibly high capacity and delay.

A multipath proposal is present in Kelly's original work[6] which defines a rate variable for each end-to-end path from the source to the destination, and the sources control all these variables to optimize overall utility. There are some technical difficulties in the proof in [6] due to the possible existence of multiple equilibria. A major difficulty for the multipath congestion control is that the natural optimization problem to consider is concave but not strictly concave, which brings a huge challenge to design stable multipath congestion control algorithms for the possible existence of multiple equilibriums. In [7], Voice has investigated the possibility of stable multipath dual congestion control algorithms. In the scheme proposed in [7], the presence of the parameter $p$ in stability conditions demonstrates a trade-off between the stability and the degree of approximation. Also, in order to approximate the optimal solution, the parameter $p$ needs to be very large (ideally $\infty$). It means that in order to get an ideal solution, there is a large risk of numerical instability for the large approximation parameter in [7].

To solve the above problems in [7], we propose a new model for multipath TCP congestion control. In our model, we use two parameters to reduce the computational complexity and numerical instability. Our algorithm only needs fewer product and addition operations to approximate the optimal solution. In contrast, [7] needs the $p$-th power operation to get the optimization solution. Our algorithm is simpler to be implemented than the one in [7].

This paper proposes a generalized multipath utility maximization model, which is strictly concave and ensures equilibria satisfying desirable static properties. Then we derive a family of multipath dual congestion control algorithms. We show that the proposed algorithms are stable in the absence of propagation delays, based on which we derive decentralized and scalable sufficient conditions for a particular scheme when propagation delays exist in the networks.

The main contributions of our work can be stated as follows.

1) We propose a generalized multipath utility maximization model, which can be reduced to specific models with different parameter settings. A family of multipath dual congestion control algorithms is derived from the proposed model that can fully utilize resources, while maintaining the fairness among different users at the state of equilibrium.

2) We also derive the decentralized and scalable sufficient conditions for a particular scheme when propagation delays exist in the networks.

3) We implement rate-based simulations using Matlab. To validate the efficiency of the proposed algorithms, a comparison is made between the proposed algorithm and the ones in [7] and [8].

The remainder of this paper is organized as follows. Related work is briefly reviewed in Section 2. We present the proposed multipath utility maximization model in Section 3. The stability in the absence of delays and the stability in the presence of delays are shown in Section 4 and Section 5, respectively. Following those are the simulation results in Section 6. We conclude the paper in Section 7.

## 2 Related Work

In recent years, theoreticians have developed a framework that allows a congestion control algorithm such as Jacobson's TCP to be interpreted as a distributed mechanism solving a global optimization problem[6,9]. The framework is based on fluid-flow models, and the form of the optimization problem makes the equilibrium resource allocation policy of the algorithm explicit, which can often be restated in terms of a fairness criterion. The dynamics of the fluid-flow models allows the machinery of control theory to be used to study stability, and to develop rate control algorithms that can scale to arbitrary capacities.

These algorithms can be classified into two major groups, i.e., primal algorithms and dual algorithms. In general, the equilibrium point of the algorithm solves the original problem, an approximation problem or the relaxed one (where the capacity constraint is replaced by penalties) respectively[6]. Some related studies are summarized in Table 1.

**Table 1**. Equilibrium and Dynamic Properties for Congestion Control Algorithms

| Stability | Primal Single Path | Dual Single Path | Primal Multipath | Dual Multipath |
|---|---|---|---|---|
| Global $\natural$ | [6]$^\ddagger$ | [6]$^\ddagger$, [11]$^\dagger$, [12]$^\dagger$, [13]$^\dagger$ | [6]$^£$ | [6]$^£$, [7]$^\ddagger$, [16]$^\ddagger$ |
| Local $\sharp$ | [6]$^\ddagger$, [10]$^\dagger$ | [6]$^\ddagger$, [14]$^\dagger$, [15]$^\dagger$ | [3]$^£$, [8]$^£$ | [7]$^\ddagger$ |

Note: $\dagger, \ddagger$ and £ denote the equilibrium point solving the original problem, the approximation problem and the relaxed problem respectively. $\natural$ and $\sharp$ denote the stability in the absence and the presence of propagation delays respectively.

For the single-path case, Vinnicombe[10] derived decentralized and scalable stability conditions for a fluid approximation of a class of Internet-like communications networks operating a modified form of TCP-like congestion control. Dual algorithms are classified into two groups, namely, delay-based dual algorithm and fair dual algorithms[14]. The delay-based dual algorithms allow a natural interpretation of the link price as either a real or virtual queueing delay[6,11-13]. It was, however, difficult to reconcile fairness with stability. Kelly[14] designed a class of fair dual algorithm, which can achieve weighted-fairness, and have straightforward delay and stochastic stability properties.

In the multipath case, there may exist multiple equilibria because the natural optimization problem to consider is concave but not strictly concave. Meanwhile, when one attempts to use a dual approach, the dual function may not be differentiable at every point[17]. This led to a discontinuous multipath dual algorithm, proposed in [6], which has undesirable stability properties[18].

To circumvent these difficulties, Lin and Shroff[16] proposed some ideas from proximal point algorithms. Han *et al.*[3] modified the utility function to ensure a unique equilibrium point and generalized the algorithm for the case of single path[10] to multipath. Kelly and Voice[8] improved the results obtained by Han *et al.*[3], and proposed an algorithm with a sufficient condition for local stability that is decentralized in the stronger sense that the gain parameter for each route is restricted by the round-trip time of this route. However, the majority of above research focuses on the extensions of the primal algorithms proposed by Kelly *et al.*[6], a class of single path primal congestion controls. The primal algorithms exhibit a trade-off between the rate of convergence and the bandwidth utilization at equilibrium since the desired equilibrium point only solves the relaxed problem.

Paganini and Mallada[19] proposed a node-based multipath routing algorithm, which allows routers to split their traffic in a controlled way between the outgoing links. They formulated global optimization criteria, combining those used in the congestion control and traffic engineering, and proposed decentralized controllers at sources and routers to reach these optimal points, based on congestion price feedback. Node-based congestion control algorithms need routers to help to choose route[20-21]. The source-based multipath congestion control and the node-based multipath congestion control have their own advantages and have different application scenes. This paper mainly discusses the source-based multipath routing.

## 3 Generalized Multipath Utility Maximization Model

We suppose that the network comprises an interconnection of a set of sources① $\mathcal{S}$, with a set of links $\mathcal{J}$. Each source $s \in \mathcal{S}$ identifies a unique origin-destination pair. Associated with each source is a collection of routes, each route being a list of links. If a source $s$ transmits along a route $r$, then we write $r \in s$. For a route $r$, we let $s(r)$ be the (unique) source such that $r \in s(r)$. We let $\mathcal{R}$ denote a subset of paths chosen by the network operators or the routing protocol.

In the following, we use notations $S, J$ and $R$ to denote the cardinalities of sets $\mathcal{S}, \mathcal{J}$ and $\mathcal{R}$ respectively. We use the notation $a = (b)_c^+$ to denote that $a = b$ if

①In this paper, the term "source" also refers to "user", "session", and "connection".

$c > 0$ and $a = \max(0, b)$ if $c = 0$. Given any $p > 1$, we use notation $\|\boldsymbol{\lambda}_s\|_{1-p}$ to denote $(\sum_{r\in s}\lambda_r^{1-p})^{\frac{1}{1-p}}$ where $\lambda_r$ is the price of route $r$. For any natural number $n$, the set $\mathbb{R}^n$ consists of all $n$-tuples of real numbers ($\mathbb{R}$). If $y = f(t)$, then $\dot{y}$ denotes the first derivative of $y$ with respect to $t$.

### 3.1 Novel Multipath Utility Maximization Model

In the model, a route $r$ has been associated with a flow rate $x_r(t) \geqslant 0$, which represents a dynamic fluid approximation to the rate at which the source $s(r)$ is sending packets along route $r$ at time $t$.

For each route $r$ and link $j \in r$, let $T_{rj}$ denote the propagation delay from $s(r)$ to $j$, i.e., the length of time it takes for a packet to travel from source $s(r)$ to link $j$ along route $r$. Let $T_{jr}$ denote the propagation delay from $j$ to $s(r)$, i.e., the time it takes for congestion control feedback to reach $s(r)$ from link $j$ along route $r$. In the protocols under consideration, a packet must reach its destination before an acknowledgement packet (ACK), containing congestion feedback (via some form of explicit congestion notification), is returned to its source. Further, we assume queueing delays are negligible, which is a standard assumption on analyzing the local stability of the congestion algorithm[10]. It is argued that in the limiting regime, capacities increase and queueing delays and queue emptying time become small in relation to propagation delays. Thus for all $j \in r$, $T_{rj} + T_{jr} = T_r$, the round trip time for route $r$.

Let $y_s$ be the aggregate flow rate with source $s \in \mathcal{S}$. A utility function $U_s(y_s)$ is associated with source $s$, which is an increasing, strictly concave and continuously differentiable function of $y_s$ over the range $y_s > 0$. And $U_s(y_s) \to \infty$ if $y_s \to \infty$. As an example, suppose that

$$U_s(y_s) = \begin{cases} w_s \dfrac{y_s^{1-\alpha}}{1-\alpha}, & \text{if} \quad \alpha \neq 1, \\ w_s \log y_s, & \text{if} \quad \alpha = 1, \end{cases} \tag{1}$$

for $w_s > 0, \alpha > 0$, so that the resource proportions obtained by different sources are weighted $\alpha$-fair[22]. If $w_s = 1, s \in \mathcal{S}$, the cases $\alpha \to 0, \alpha = 1$ and $\alpha \to \infty$ correspond to an allocation which achieves maximum throughput, is proportionally fair or is max-min fair[22] respectively. TCP fairness, in the case where each source has just a single route, corresponds to the choice $\alpha = 2$ with $w_s$, the reciprocal of the square of the (single) round trip time for source $s$[9]. We use $U_s'$ to denote the derivative of $U_s$ with respect to $y_s$ and define the demand function $D_s(\lambda_s) = (U_s')^{-1}(\lambda_s)$, a continuous, strictly decreasing function. Then, the demand function derived from the class of utility functions defined in (1) is

$$D_s(\lambda_s) = \left(\frac{w_s}{\lambda_s}\right)^{1/\alpha}, \tag{2}$$

where $w_s$ and $\alpha$ are the parameters in (1).

For the convenience of analysis, we introduce a routing matrix to succinctly express the relationships between routes and links. Let $A_{jr} = 1$ if $j \in r$, so that link $j$ lies on route $r$, and set $A_{jr} = 0$ otherwise. This defines a 0-1 matrix $\boldsymbol{A} = (A_{jr}, j \in \mathcal{J}, r \in \mathcal{R})$. The aggregate rate for source $s$ is $y_s = \sum_{r\in s} x_r$. Since we wish the total network utility to be high, it is desirable for a congestion control algorithm to asymptotically solve the classical Kelly's formulation[6]:

$$\underset{\boldsymbol{x}\geqslant\boldsymbol{0}}{\text{maximize}} \quad \sum_{s\in\mathcal{S}} U_s(y_s), \tag{3}$$

$$\text{s.t.} \quad y_s = \sum_{r\in s} x_r, \ \forall s \in \mathcal{S}, \tag{4}$$

$$\boldsymbol{Ax} \leqslant \boldsymbol{c}, \tag{5}$$

where $\boldsymbol{x} = (x_r, r \in \mathcal{R})$ is the rate vector and $\boldsymbol{c} = (c_j, j \in \mathcal{J})$ is the capacity vector with $c_j$ being the capacity of link $j$. Note that even if $U_s$ is strictly concave, the whole objective function is not, due to the linear relationship (4).

To solve the non-strict concave of the objective function in (3) with $\boldsymbol{x}$, only the $\theta$ fraction of the $\frac{1}{q}$-th power of aggregate rate $y_s$ is substituted by $\sum_{r\in s} x_r^{\frac{1}{q}}$ in (4), and then we get the generalized model for the multipath utility maximization problem (3)~(5), i.e.,

$$\underset{\boldsymbol{x}\geqslant\boldsymbol{0}}{\text{maximize}} \quad \sum_{s\in\mathcal{S}} U_s(u_s^q), \tag{6}$$

$$\text{s.t.} \quad u_s \leqslant \theta \sum_{r\in s} x_r^{\frac{1}{q}} + (1-\theta) y_s^{\frac{1}{q}}, \tag{7}$$

$$y_s = \sum_{r\in s} x_r, \ \forall s \in \mathcal{S}, \tag{8}$$

$$\boldsymbol{Ax} \leqslant \boldsymbol{c}, \tag{9}$$

where $\theta \in [0, 1]$ and $q > 1$ are parameters.

**Remark 1.** *Formulations* (6)~(9) *are equivalent to minimizing* $\sum_{s\in\mathcal{S}} U_s((\theta \sum_{r\in s} x_r^{\frac{1}{q}} + (1-\theta) y_s^{\frac{1}{q}})^q)$ *subject to* (8) *and* (9) *for the fact that* $U_s(u_s^q)$ *is increasing with* $u_s$. *To ensure that the objective function is strictly concave with* $\boldsymbol{x}$*, we make assumption H: for each* $s \in \mathcal{S}, U_s(u_s^q)$ *is a strictly concave function of* $u_s$. *This is true for the weighted* $\alpha$*-fairness utility function* (1) *if* *the parameter $p$ is chosen such as $\alpha p > 1$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$*②.

Given $q$, (6)∼(9) reduce to the one proposed by Voice in [7] with $\theta = 1$ and to the classical Kelly's formulation (3)∼(5) with $\theta = 0$. Based on (6)∼(9), a family of multipath dual congestion control algorithms derived can fully utilize resources, while maintaining the fairness among different users at the state of equilibrium. When $\theta$ approximates to 0, the proposed algorithm with a fixed $q$ reduces to the dual congestion control with respective to the previous work in [6] and [11], which provides a direct insight into the reason for the stability of the dual congestion control. When $\theta$ approximates to 1, the proposed algorithm with a fixed $q$ reduces to the one proposed by Voice[7]. The former uses addition while the latter uses the $p$-th power operation, which needs that $p$ approximates $\infty$ to maximize the original utility function $\sum_{s\in\mathcal{S}} U_s(y_s)$. Compared with the one in [7], the algorithms here use parameters $\theta$ and $p$ to control the approximation error to the total utility.

### 3.2 Approximation Error

We can bound how far the solution to (6)∼(9) is from maximizing the aggregate user utility.

**Lemma 1** (Approximation Error). *Let $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ be any optimal solution to (3)∼(5), and $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u})$ be the optimal solution to (6)∼(9). We define $|s|$ be the number of route serving for source $s$. Then we have*

$$\sum_{s\in\mathcal{S}} U_s(y_s^*) \geqslant \sum_{s\in\mathcal{S}} U_s(y_s) \tag{10}$$

*and*

$$\sum_{s\in\mathcal{S}} U_s(e_{\theta p} y_s) \geqslant \sum_{s\in\mathcal{S}} U_s(y_s^*), \tag{11}$$

*where $y_s^*$ is the aggregate rate of source $s$ and the factor $e_{\theta p} = \big(\theta |s|^{\frac{1}{p}} + (1-\theta)\big)^q$.*

*Proof.* It is obvious that $(\boldsymbol{x}, \boldsymbol{y})$ is feasible for (3)∼(5). Thus the inequality (10) is followed by the optimality of $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ to (3)∼(5).

For $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have $\sum_{r\in s} x_r^{\frac{1}{q}} \leqslant |s|^{\frac{1}{p}} \big(\sum_{r\in s} x_r\big)^{\frac{1}{q}}$ with the famous Hölder inequality[23]. Plugging it into (7), we have

$$u_s^q \leqslant e_{\theta p} y_s \tag{12}$$

by increasing function $(\cdot)^q$ and (8). Now let $u_s^*$ be $\theta \sum_{r\in s} (x_r^*)^{\frac{1}{q}} + (1-\theta)(y_s^*)^{\frac{1}{q}}$. It is obvious that $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{u}^*)$ is feasible for (6)∼(9). By the optimality of $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u})$ to (6)∼(9), we have

$$U_s(u_s^q) \geqslant U_s((u_s^*)^q). \tag{13}$$

Since $(\cdot)^{\frac{1}{q}}$ is a subadditive function and $(\cdot)^q$ is increasing, we have

$$(u_s^*)^q \geqslant y_s^* \tag{14}$$

for (4). Combining inequalities (12), (13), (14) and the fact that $U_s(\cdot)$ is increasing, we get the desired result (11). □

It can be verified that $e_{\theta p} = \big(1 + \theta(|s|^{\frac{1}{p}} - 1)\big)^q$. The factor $e_{\theta p}$ is increasing with $\theta \in [0, 1]$ for a fixed $p$. And the facts that $e_{0p} = 1$ and $e_{1p} = |s|^{\frac{1}{p-1}}$ hold. Thus $e_{\theta p} \in [1, |s|^{\frac{1}{p-1}}]$ for a given $p$. Then Lemma 1 in [7] is a special case of Lemma 1 here with $\theta = 1$. To get the original utility maximum $\sum_{s\in\mathcal{S}} U_s(y_s^*)$, the parameter $p$ in the algorithm in [7] needs approximating to $\infty$. In the algorithm proposed in this paper, for a fixed $p$, when the parameter $\theta$ approximates to 0, one also gets the original utility maximum $\sum_{s\in\mathcal{S}} U_s(y_s^*)$.

The approximation factor curves $e_{\theta p}$ with parameters $p$ and $\theta = 1$ are shown in Fig.1(a), i.e., the one proposed by Voice[7]. The approximation factor curves $e_{\theta p}$ with parameter $p = 5, 10$ and 20 are shown in Figs.1(b), 1(c) and 1(d) respectively. In the same figure, four curves represent $e_{\theta p}$ corresponding to the cases that there are 2, 3, 4 and 5 paths serving some source $s$. Take Figs.1(a) and 1(b) as an example, if we make the approximation factor be 1.2 and 1.025, for $\theta = 1$ the parameter $p$ must take 10 and 50 respectively, while for $p = 5$, the parameter $\theta$ takes 0.1 and 0.01 respectively.

### 3.3 Dual Problem of Generalized Model

Given dual variables $\boldsymbol{\mu} \in \mathbb{R}^J$ and $\boldsymbol{\nu} \in \mathbb{R}^S$, let $\boldsymbol{\lambda} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu}$. Here $\mu_j$ can be interpreted as the congestion price of link $j$. Then $\lambda_r = \sum_{j\in r} \mu_j$ is the price of route $r$. The Lagrangian of (6)∼(9) is $\sum\limits_{s} \big[U_s(u_s^q) - \sum\limits_{r\in s} (\lambda_r - \nu_s) x_r - \nu_s y_s\big] + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{\mu}$. For any $s \in \mathcal{S}$, we

$$\begin{aligned} \underset{u_s, y_s, \boldsymbol{x}_s}{\text{maximize}} \quad & U_s(u_s^q) - \sum_{r\in s} (\lambda_r - \nu_s) x_r - \nu_s y_s, \\ \text{s.t.} \quad & u_s \leqslant \theta \sum_{r\in s} x_r^{\frac{1}{q}} + (1-\theta) y_s^{\frac{1}{q}}, \\ & \boldsymbol{x}_s \geqslant \boldsymbol{0}, \end{aligned} \tag{15}$$

②Parameter $p$ is determined by parameter $q$, i.e., $p = \frac{q}{q-1}$ and $p$ approximates to $\infty$ as $q$ approximates to 1.

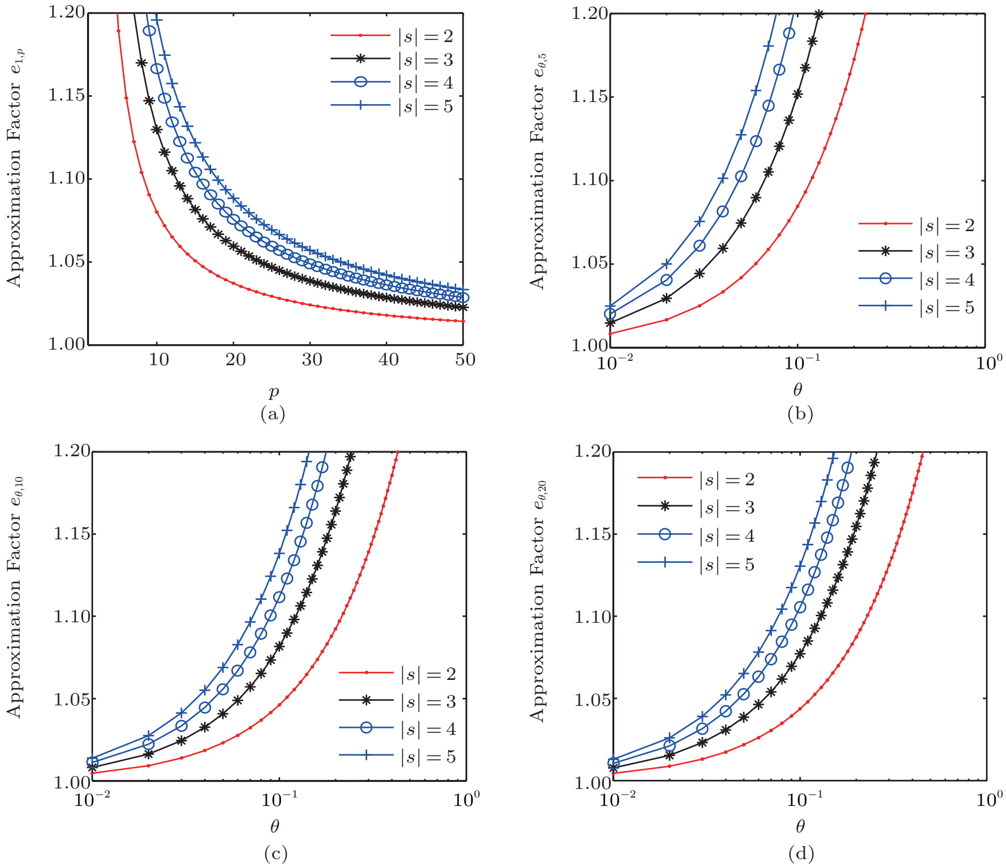


Fig. 1. Approximation factor curves with parameters $p$ and $\theta$ where the number of route serving some source $s$ is 2, 3, 4, and 5 respectively. (a) $\theta = 1$. (b) $p = 5$. (c) $p = 10$. (d) $p = 20$.

where $\boldsymbol{x}_s = (x_r, r \in s)$ is the rate vector for source $s$. We denote the optimal value of (15) by $W_s(\boldsymbol{\lambda}_s, \nu_s)$ where $\boldsymbol{\lambda}_s = (\lambda_r, r \in s)$. It can be verified that $W_s(\boldsymbol{\lambda}_s, \nu_s)$ is finite only if $\lambda_r - \nu_s > 0$ for all $r \in s$ and $\nu_s > 0$. Otherwise, $W_s(\boldsymbol{\lambda}_s, \nu_s) = +\infty$. In the following derivation, we assume $\lambda_r > \nu_s > 0$ for all $r \in s$ and $s \in \mathcal{S}$. Finally, the Lagrangian dual problem of (6)~(9) is to

$$\underset{\boldsymbol{\mu}\geqslant \boldsymbol{0},\boldsymbol{\nu}}{\text{minimize}}\, W(\boldsymbol{\mu}, \boldsymbol{\nu}) := \sum_{s\in\mathcal{S}} W_s(\boldsymbol{\lambda}_s, \nu_s) + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{\mu}. \quad (16)$$

The control laws that we will propose can be viewed as decentralized algorithms to solve problem (16) and the primal problem (6)~(9) simultaneously.

## 4 Family of Stable Multipath Dual Congestion Control Algorithms

In this section, we will derive a family of multipath dual congestion control algorithms from the generalized multipath utility maximization model. The stability with the absence of delay will be stated later.

### 4.1 Multipath Dual Congestion Control Algorithms

To design the dual algorithm, we need the close form of the solution of (15) for the given $\boldsymbol{\lambda}_s, \nu_s$.

If $(u_s, y_s, \boldsymbol{x}_s)$ is a minimizer of (15), then there exists Lagrange multiplier $\eta_s$ such that $(u_s, y_s, \boldsymbol{x}_s)$ and $\eta_s$ satisfy the following system:

$$u_s = \theta \sum_{r\in s} x_r^{\frac{1}{q}} + (1-\theta) y_s^{\frac{1}{q}}, \quad (17)$$

$$q u_s^{q-1} U_s'(\hat{y}_s) - \eta_s = 0, \quad (18)$$

$$\theta \eta_s x_r^{-\frac{1}{p}} - q(\lambda_r - \nu_s) = 0, \quad r \in s, \quad (19)$$

$$(1-\theta)\eta_s y_s^{-\frac{1}{p}} - q\nu_s = 0, \quad (20)$$

where

$$\hat{y}_s = u_s^q. \quad (21)$$

Plugging (18) and (21) into (19) and (20) respectively, we get

$$x_r = \hat{y}_s \Big(\frac{\theta U_s'(\hat{y}_s)}{\lambda_r - \nu_s}\Big)^p \quad (22)$$

for all $r \in s$ and

$$y_s = \hat{y}_s\Big(\frac{(1-\theta)U_s'(\hat{y}_s)}{\nu_s}\Big)^p. \tag{23}$$

Substituting (22) and (23) into (21), we obtain

$$U_s'(\hat{y}_s) = \Bigg(\theta\sum_{r\in s}\Big(\frac{\theta}{\lambda_r-\nu_s}\Big)^{p-1} + (1-\theta)\Big(\frac{1-\theta}{\nu_s}\Big)^{p-1}\Bigg)^{\frac{1}{1-p}}.$$

Combining it with the definition of the demand function, we get

$$\hat{y}_s = D_s\Big(\big(\theta^p\|\boldsymbol{\lambda}_s-\nu_s\mathbf{1}\|_{1-p}^{1-p}+(1-\theta)^p\nu_s^{1-p}\big)^{\frac{1}{1-p}}\Big), \tag{24}$$

where $\mathbf{1}$ is an $|s|$-dimensional vector with all 1.

**Lemma 2.** *Let $\boldsymbol{\mu} \geqslant \mathbf{0}$ and $\boldsymbol{\lambda} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu}$. Then the objective $W(\boldsymbol{\mu},\boldsymbol{\nu})$ of (16) is convex and differentiable with the partial derivative $\frac{\partial W}{\partial \mu_j} = c_j - z_j$, $\forall j \in \mathcal{J}$, and $\frac{\partial W}{\partial \nu_s} = \sum_{r\in s} x_r - y_s$, where $z_j = \sum_{j\in r} x_r$, $x_r$ and $y_s$ are defined by (22)∼(24).*

*Proof.* Firstly, the dual function $W(\boldsymbol{\mu},\boldsymbol{\nu})$ is always convex (refer to [17], Proposition 5.1.2). Secondly, by the assumption H, the solution to (15) is unique; hence, the optimal value function $W_s(\boldsymbol{\lambda}_s,\nu_s)$ is differential (refer to [17], Proposition 6.1.1) with

$$\frac{\partial W}{\partial \mu_j} = c_j - z_j,\ \forall j \in \mathcal{J},$$
$$\frac{\partial W}{\partial \nu_s} = \sum_{r\in s} x_r - y_s,\ \forall s \in \mathcal{S},$$

where $z_j = \sum_{j\in r} x_r$ and $(\boldsymbol{x},\boldsymbol{y})$ solve problem (15). By the derivation of (22), (23), and (24), we get the desired results. □

We are now in a position to state a family of multipath dual algorithms. For all links $j$,

$$\dot{\mu}_j(t) = \kappa_j(\mu_j(t))\big(z_j(t)-c_j\big)^+_{\mu_j(t)} \tag{25}$$

for some positive function $\kappa_j$, and for all sources $s$,

$$\dot{\nu}_s(t) = \kappa_s(\nu_s(t))\big(y_s(t) - \sum_{r\in s} x_r(t)\big) \tag{26}$$

for some positive function $\kappa_s$, where

$$z_j(t) = \sum_{j\in r} x_r(t), \tag{27}$$

$$x_r(t) = \hat{y}_{s(r)}(t)\Big(\frac{\theta U_{s(r)}'(\hat{y}_{s(r)}(t))}{\lambda_r(t)-\nu_{s(r)}(t)}\Big)^p, \tag{28}$$

$$y_s(t) = \hat{y}_s(t)\Big(\frac{(1-\theta)U_s'(\hat{y}_s(t))}{\nu_s(t)}\Big)^p, \tag{29}$$

$$\lambda_r(t) = \sum_{j\in r}\mu_j(t), \tag{30}$$

and

$$\hat{y}_s = D_s\Big(\big(\theta^p\|\boldsymbol{\lambda}_s(t)-\nu_s(t)\mathbf{1}\|_{1-p}^{1-p} + (1-\theta)^p\nu_s(t)^{1-p}\big)^{\frac{1}{1-p}}\Big). \tag{31}$$

**Theorem 1.** *Let $(\boldsymbol{\mu},\boldsymbol{\nu})$ be an equilibrium point of the system (25)∼(26), and let $(\boldsymbol{x},\boldsymbol{y},\hat{\boldsymbol{y}})$ be defined through (27)∼(31). Then $(\boldsymbol{\mu},\boldsymbol{\nu})$ solves the dual problem (16). Let $u_s = \hat{y}_s^{\frac{1}{q}}$ for all $s \in \mathcal{S}$. Then $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u})$ is the unique solution of the primal problem (6)∼(9).*

*Proof.* From (25) and (30), it holds that $\boldsymbol{\mu} \geqslant \mathbf{0}$ and $\boldsymbol{\lambda} = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\mu}$. From (28), (29), and (31), the equalities (22), (23), and (24) hold. Furthermore, from (25) and the positivity of function $\kappa_j$, we obtain the fact that $(z_j-c_j)^+_{\mu_j} = 0$, which is equivalent to

$$z_j - c_j = 0,\quad \text{if } \mu_j > 0, \tag{32}$$
$$z_j - c_j \leqslant 0,\quad \text{if } \mu_j = 0. \tag{33}$$

From (26) and the positivity of function $\kappa_s$, we obtain the fact

$$y_s - \sum_{r\in s} x_r = 0. \tag{34}$$

By Lemma 2, $W(\boldsymbol{\mu},\boldsymbol{\nu})$ is convex and differential. Then we obtain that

$$W(\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\nu}}) \geqslant W(\boldsymbol{\mu},\boldsymbol{\nu}) + \sum_{j\in\mathcal{J}}\frac{\partial W}{\partial \mu_j}(\hat{\mu}-\mu_j) + \sum_{s\in\mathcal{S}}\frac{\partial W}{\partial \nu_s}(\hat{\nu_s}-\nu_s),$$

for all $\hat{\boldsymbol{\mu}} \in \mathbb{R}^J$ and $\hat{\boldsymbol{\nu}} \in \mathbb{R}^S$ because of the property of the convex and differential function (refer to [17], Proposition B.3). Again with Lemma 2, submitting (32), (33) and (34) into the inequality, then

$$W(\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\nu}}) \geqslant W(\boldsymbol{\mu},\boldsymbol{\nu}) + \sum_{j\in\mathcal{J},\mu_j=0}(c_j-z_j)\hat{\mu}_j, \forall\hat{\boldsymbol{\mu}}, \forall\hat{\boldsymbol{\nu}}.$$

Again using (33), we know $W(\boldsymbol{\mu},\boldsymbol{\nu})$ is the minimum of $W(\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\nu}})$ subject to $\hat{\boldsymbol{\mu}} \geqslant \mathbf{0}$, i.e., $(\boldsymbol{\mu},\boldsymbol{\nu})$ solves the dual problem (16).

Moreover, with (28), (29), (31) and the definition of $u_s$, it can be checked that $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u})$ satisfies (7) with equality. From (32), (33) and (34), $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u})$ satisfies (8) and (9). Thus $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u})$ is feasible for (6)∼(9). In addition, (32), (33) and (34) implicate the complementarity holding. Therefore $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u})$ is the solution of

(6)∼(9) (refer to [23], p.217, Theorem 1). Furthermore, since the objective function of (6)∼(9) is strictly concave with $\boldsymbol{x}$ (see Remark 1), the maximizer of (6)∼(9) is unique. Then we obtain the desired result. □

### 4.2 Global Stability

Assume that matrix $\boldsymbol{A}$ has full row rank to ensure a unique equilibrium point, which is the general assumption when analyzing the dual congestion control algorithms, such as [7], [12] and [14]. Note, the assumption is needed only for the links that would be a bottleneck; thus our assumption is quite generic. This condition is sufficient to deduce that the system (25)∼(31) has a unique equilibrium point. Following is the first main result of this paper with a detailed proof. Here the proof is completely different from the one in [7].

**Theorem 2.** *Given the system defined by* (25)∼(31)*, the unique equilibrium point* $(\boldsymbol{\mu}^*, \boldsymbol{\nu}^*)$ *is globally asymptotically stable.*

*Proof.* The proof is based on Lasalle's invariance principle applied to a suitable Lyapunov function. Now we introduce the candidate Lyapunov function $V(\boldsymbol{\mu}, \boldsymbol{\nu}) = W(\boldsymbol{\mu}, \boldsymbol{\nu})$. For any state vector $(\boldsymbol{\mu}(t), \boldsymbol{\nu}(t)) \neq (\boldsymbol{\mu}^*, \boldsymbol{\nu}^*)$ of system (25)∼(31), it can be seen that $(\boldsymbol{\mu}(t), \boldsymbol{\nu}(t))$ is feasible for the dual problem (16). For $(\boldsymbol{\mu}^*, \boldsymbol{\nu}^*)$ is the unique solution of (16), we have $W(\boldsymbol{\mu}, \boldsymbol{\nu}) > W(\boldsymbol{\mu}^*, \boldsymbol{\nu}^*)$.

We now take the derivative of $V(\boldsymbol{\mu}, \boldsymbol{\nu})$ along trajectories of our system:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V &= \sum\nolimits_{j\in\mathcal{J}} \frac{\partial W}{\partial \mu_j}\dot{\mu}_j + \sum\nolimits_{s\in\mathcal{S}} \frac{\partial W}{\partial \nu_s}\dot{\nu}_s \\ &= \sum\nolimits_{j\in\mathcal{J}}(c_j - z_j)\dot{\mu}_j + \sum\nolimits_{s\in\mathcal{S}}\left(\sum\nolimits_{r\in s} x_r - y_s\right)\dot{\nu}_s \\ &= \sum\nolimits_{j\in\mathcal{J}} v_j - \sum\nolimits_{s\in\mathcal{S}} \kappa_s(\nu_s)\left(y_s - \sum\nolimits_{r\in s} x_r\right)^2,\end{aligned}$$

where we have denoted $v_j := (c_j - z_j)\dot{\mu}_j$. Note that the second equality follows Lemma 2. We will now show that $v_j \leqslant 0$ for each $j$. For this, we must apply the dynamic equations (25) and (26), and distinguish between the two cases:

1) $\mu_j > 0$. Here $v_j = -\kappa_j(\mu_j)(z_j - c_j)^2$.

2) $\mu_j = 0$. Here $v_j = \kappa_j(\mu_j)(c_j - z_j)\max(0, z_j - c_j)$. There are two cases:

$$\begin{aligned}&v_j = 0, \quad \text{for } z_j < c_j, \\ &v_j = -\kappa_j(\mu_j)(z_j - c_j)^2, \quad \text{for } z_j \geqslant c_j.\end{aligned}$$

We thus confirm that $v_j \leqslant 0$ for every $j$, and thus $\dot{V} \leqslant 0$. Invoking Lyapunov's stability theorem, we conclude that the trajectory $(\boldsymbol{\mu}(t), \boldsymbol{\nu}(t))$ must remain bounded over time, and that the equilibrium point $(\boldsymbol{\mu}^*, \boldsymbol{\nu}^*)$ is stable in the sense of Lyapunov, namely, trajectories starting close to it will remain inside a neighborhood.

To establish the stronger claim of asymptotic stability, we must show that trajectories will converge to equilibrium as time goes to infinity. We do this by means of LaSalle's invariance principle (see, e.g., [24]). To apply it, we must study the set of states $(\boldsymbol{\mu}, \boldsymbol{\nu})$ where the derivative $\dot{V}$ is 0, i.e.,

$$I = \{(\boldsymbol{\mu}, \boldsymbol{\nu}) : y_s = \sum_{r\in s} x_r, \forall s \in \mathcal{S}, v_j = 0, \forall j \in \mathcal{J}\}.$$

Reviewing the cases above, we find that $v_j = 0$ can only happen when either

1) $z_j = c_j$, or

2) $z_j < c_j$ and $\mu_j = 0$.

Thus any point $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in I$ is the solution of (16). By the assumption that matrix $\boldsymbol{A}$ has full row rank, the solution to (16) is unique. Thus $I$ does not contain any point, except the unique equilibrium point $(\boldsymbol{\mu}^*, \boldsymbol{\nu}^*)$. Here $(\boldsymbol{\mu}^*, \boldsymbol{\nu}^*)$ is the unique solution to (16).

Now, the union of complete trajectories contained entirely in the set $I$ contains no trajectory of the system except the trivial trajectory $\boldsymbol{\mu}(t) \equiv \boldsymbol{\mu}^*, \boldsymbol{\nu}(t) \equiv \boldsymbol{\nu}^*$, and then $(\boldsymbol{\mu}^*, \boldsymbol{\nu}^*)$ is globally asymptotically stable. □

## 5 Delay Stability

As transmission delay universally exists in network environment, in this section, we will present a particular scheme, where the proposed algorithms can achieve the stability in the presence of propagation delays.

### 5.1 Choice of Scheme

When we include propagation delays, we get the following algorithms, for which we can provide scalable, decentralized stability conditions.

For all links $j$,

$$\dot{\mu}_j(t) = \kappa_j\mu_j(t)\Big(\sum_{j\in r} x_r(t - T_{rj}) - c_j\Big)^+_{\mu_j(t)} \tag{35}$$

for some positive constant $\kappa_j$, and for all sources $s$,

$$\dot{\nu}_s(t) = \kappa_s\nu_s(t)\big(y_s(t) - \sum_{r\in s} x_r(t - T_r)\big), \tag{36}$$

$$\dot{\hat{y}}_s(t) = \rho_s\left(\theta\sum_{r\in s} x_r(t - T_r)^{\frac{1}{q}} + (1-\theta)y_s(t)^{\frac{1}{q}} - \hat{y}_s(t)^{\frac{1}{q}}\right) \tag{37}$$

for some positive constants $\kappa_s$ and $\rho_s$, where

$$x_r(t) = \hat{y}_{s(r)}(t)\left(\frac{\theta U'_{s(r)}(\hat{y}_{s(r)}(t))}{\lambda_r(t) - \nu_{s(r)}(t)}\right)^p, \quad (38)$$

$$y_s(t) = \hat{y}_s(t)\left(\frac{(1-\theta)U'_s(\hat{y}_s(t))}{\nu_s(t)}\right)^p, \quad (39)$$

and

$$\lambda_r(t) = \textstyle\sum_{j\in r}\mu_j(t - T_{jr}). \quad (40)$$

Compared with (25)∼(31), we set $\kappa_j\mu_j(t)/p$ and $\kappa_s\nu_s(t)/p$ as the dynamic gain factor for link $j$ and source $s$ respectively. Moreover, we relax the algebraic equation (31) to differential equation (37) for the existence of delay.

Some properties of the equilibrium point are presented, which are useful in the proof of the main result.

**Lemma 3.** *We have* $a_s := -\frac{U''_s(\hat{y}_s)}{U'_s(\hat{y}_s)} - \frac{1}{p\hat{y}_s} > 0$, *where* $\hat{y}_s = u_s^q$.

*Proof.* We have $b_s := U''_s(u_s^q) < 0$ since assumption H holds. It can be checked that

$$\begin{aligned} b_s &= q(q-1)u_s^{q-2}U'_s(\hat{y}_s) + q^2u_s^{2(q-1)}U''_s(\hat{y}_s) \\ &= -q^2u_s^{2(q-1)}U'_s(\hat{y}_s)a_s. \end{aligned}$$

Combining it with $U'_s(\hat{y}_s)$ positivity by $U_s(\hat{y}_s)$ increasing with $\hat{y}_s$, we get the desired result. □

**Remark 2.** *We need the positivity of* $a_s$ *to guarantee the local stability, e.g., the condition* (48) *in Theorem* 3.

**Lemma 4.** *Let* $(\boldsymbol{\mu}, \boldsymbol{\nu}, \hat{\boldsymbol{y}})$ *be an equilibrium point of system* (35)∼(37) *and* $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda})$ *defined by* (38)∼(40). *Then for each* $r \in \mathcal{R}$, *we have that*

$$\frac{x_r}{\hat{y}_{s(r)}^{\frac{1}{p}}U'_{s(r)}(\hat{y}_{s(r)})} = \frac{\theta x_r^{\frac{1}{q}}}{\lambda_r - \nu_{s(r)}} \quad (41)$$

and

$$1 + \frac{\nu_{s(r)}}{\lambda_r - \nu_{s(r)}} + \sum_{j\in r}\frac{\mu_j}{\lambda_r - \nu_{s(r)}} \leqslant \frac{2}{\theta}. \quad (42)$$

*Proof.* Firstly, we have $x_r = \hat{y}_{s(r)}\left(\frac{\theta U'_{s(r)}(\hat{y}_{s(r)})}{\lambda_r - \nu_{s(r)}}\right)^p$ by (38). Then raising both sides of this equation to the $1/p$ power, we get

$$x_r^{\frac{1}{p}} = \hat{y}_{s(r)}^{\frac{1}{p}}\frac{\theta U'_{s(r)}(\hat{y}_{s(r)})}{\lambda_r - \nu_{s(r)}}. \quad (43)$$

Multiplying (43) with $x_r^{\frac{1}{q}}$ and combining the fact $1/p + 1/q = 1$, we have (41).

Similarly with (43), we have

$$y_{s(r)}^{\frac{1}{p}} = \hat{y}_{s(r)}^{\frac{1}{p}}\frac{(1-\theta)U'_{s(r)}(\hat{y}_{s(r)})}{\nu_{s(r)}} \quad (44)$$

by (39). Combining (43) with (44), we get

$$\frac{\nu_{s(r)}}{\lambda_r - \nu_{s(r)}} = \left(\frac{1}{\theta} - 1\right)\left(\frac{x_r}{y_{s(r)}}\right)^{\frac{1}{p}}.$$

Combining it with $x_r \leqslant y_{s(r)}$, we have $\frac{\nu_{s(r)}}{\lambda_r - \nu_{s(r)}} \leqslant \frac{1}{\theta} - 1$. Then

$$\begin{aligned} & 1 + \frac{\nu_{s(r)}}{\lambda_r - \nu_{s(r)}} + \textstyle\sum_{j\in r}\frac{\mu_j}{\lambda_r - \nu_{s(r)}} \\ &= 1 + \frac{\nu_{s(r)}}{\lambda_r - \nu_{s(r)}} + \frac{\lambda_r}{\lambda_r - \nu_{s(r)}} \\ &= 2 + \frac{2\nu_{s(r)}}{\lambda_r - \nu_{s(r)}} \\ &\leqslant \frac{2}{\theta}. \end{aligned}$$

□

### 5.2 Main Result

We now turn to our main concern, the local stability of the system (35)∼(40).

We define link $j$ to be almost saturated if $\mu_j = 0$ and conditions

$$\sum_{j\in r} x_r = c_j, \quad j \in \mathcal{J} \quad (45)$$

hold. We thus rule out the possible degeneracy that both terms in the product (35) might vanish.

**Theorem 3.** *Let* $(\boldsymbol{\mu}, \boldsymbol{\nu}, \hat{\boldsymbol{y}})$ *be an equilibrium point of system* (35)∼(40) *with no almost saturated links, and assume that for all* $j \in \mathcal{J}$,

$$\kappa_j\frac{p}{\theta}\sum_{j\in r}x_rT_r < \frac{\pi}{4}, \quad (46)$$

*and for all* $s \in \mathcal{S}$,

$$\kappa_s\frac{p}{\theta}\sum_{r\in s}x_rT_r < \frac{\pi}{4} \quad (47)$$

*and*

$$\rho_s\frac{pa_s}{q}\sum_{r\in s}x_r^{\frac{1}{q}}T_r < \frac{\pi}{4}. \quad (48)$$

*Then there exists a neighborhood* $\mathcal{N}$ *of* $\boldsymbol{\mu}$ *such that for any initial trajectory* $((\boldsymbol{\mu}(t), \boldsymbol{\nu}(t), \hat{\boldsymbol{y}}(t)), t \in (-T_{\max}, 0))$ *with* $\boldsymbol{\mu}(t)$ *lying within the neighborhood* $\mathcal{N}$, $(\boldsymbol{\mu}(t), \boldsymbol{\nu}(t))$ *converges as* $t \to \infty$ *to the solution* $(\boldsymbol{\mu}, \boldsymbol{\nu})$ *of* (16) *and* $(\boldsymbol{x}(t), \boldsymbol{y}(t), \hat{\boldsymbol{y}}(t))$ *converges as* $t \to \infty$ *to the solution* $(\boldsymbol{x}, \boldsymbol{y}, \hat{\boldsymbol{y}})$ *of* (6)∼(9), *where* $T_{\max} = \max(T_r, r \in \mathcal{R})$.

*Proof.* Initially assume that $\mu_j > 0$ for $j \in \mathcal{J}$, and thus (45) holds for each $j \in \mathcal{J}$. Later we shall see this assumption can be made without loss of generality.

Let $x_r(t) = x_r + u_r(t), y_s(t) = y_s + v_s(t)$, $\hat{y}_s(t) = \hat{y}_s + \hat{v}_s(t), \lambda_r(t) = \lambda_r + v_r(t), \mu_j(t) = \mu_j + w_j(t)$, and $\nu_s(t) = \nu_s + w_s(t)$. Then, linearizing the system (35)∼(40) about $\boldsymbol{\mu}, \boldsymbol{\nu}$ and $\hat{\boldsymbol{y}}$, and using the relation (41), we obtain the following equations

$$\dot{w}_j(t) = \kappa_j\mu_j \sum_{j\in r} u_r(t - T_{rj}), \quad (49)$$

$$\dot{w}_s(t) = \kappa_s\nu_s(v_s(t) - \sum_{r\in s} u_r(t - T_r)), \quad (50)$$

$$\dot{\hat{v}}_s(t) = \frac{\rho_s}{q}(\theta \sum_{r\in s} x_r^{-\frac{1}{p}} u_r(t - T_r) + (1-\theta)y_s^{-\frac{1}{p}} v_s(t) - \hat{y}_s^{-\frac{1}{p}} \hat{v}_s(t)), \quad (51)$$

where

$$u_r(t) = -px_r\left(a_s\hat{v}_s(t) + \frac{\sum_{j\in r} w_j(t - T_{jr})}{\lambda_r - \nu_s} - \frac{w_s(t)}{\lambda_r - \nu_s}\right),$$

$$v_s(t) = -py_s\left(a_s\hat{v}_s(t) + \frac{w_s(t)}{\nu_s}\right),$$

and $a_s = -\frac{U_s''}{U_s'} - \frac{1}{p\hat{y}_s} > 0$ by Lemma 3.

Let us overload notation and write $u_r(\omega), v_s(\omega)$, $\hat{v}_s(\omega)$ and $v_r(\omega), w_j(\omega), w_s(\omega)$ as the Laplace transforms of $u_r(t), v_s(t), \hat{v}_s(t)$ and $v_r(t), w_j(t), w_s(t)$, respectively. We may deduce from (49)∼(51),

$$\begin{aligned}
\omega w_j(\omega) &= \kappa_j\mu_j \textstyle\sum_{j\in r} e^{-\omega T_{rj}} u_r(\omega),\\
\omega w_s(\omega) &= \kappa_s\nu_s(v_s(\omega) - \textstyle\sum_{r\in s} e^{-\omega T_r} u_r(\omega)),\\
\omega\hat{v}_s(\omega) &= \frac{\rho_s}{q}\left(\theta \textstyle\sum_{r\in s} x_r^{-\frac{1}{p}} e^{-\omega T_r} u_r(\omega) + (1-\theta)y_s^{-\frac{1}{p}} v_s(\omega) - \hat{y}_s^{-\frac{1}{p}}\hat{v}_s(\omega)\right),\\
u_r(\omega) &= -px_r\left(a_s\hat{v}_s(\omega) + \textstyle\sum_{j\in r} e^{-\omega T_{jr}} \frac{w_j(\omega)}{\lambda_r - \nu_s} - \frac{w_s(\omega)}{\lambda_r - \nu_s}\right),\\
v_s(\omega) &= -py_s\left(a_s\hat{v}_s(\omega) + \frac{w_s(\omega)}{\nu_s}\right).
\end{aligned} \quad (52)$$

By (52), we have

$$(\omega + \sigma_s)\hat{v}_s(\omega) = \frac{\rho_s}{q}\left(\textstyle\sum_{r\in s} \theta x_r^{-\frac{1}{p}} e^{-\omega T_r} u_r(\omega) + (1-\theta)y_s^{-\frac{1}{p}} v_s(\omega)\right),$$

where $\sigma_s = \rho_s\hat{y}_s^{-\frac{1}{p}}/q$.

We calculate that

$$\begin{pmatrix}\hat{\boldsymbol{v}}(\omega)\\ \boldsymbol{w}(\omega)\end{pmatrix} = -\boldsymbol{G}(\omega)\begin{pmatrix}\hat{\boldsymbol{v}}(\omega)\\ \boldsymbol{w}(\omega)\end{pmatrix}.$$

The matrix $\boldsymbol{G}(\omega)$ is called the return ratio for $(\hat{\boldsymbol{v}}, \boldsymbol{w})$ and

$$G_{s's'}(\omega) = \frac{p}{q}\frac{\rho_{s'}a_{s'}}{\omega + \sigma_{s'}}(\textstyle\sum_{r\in s'} \theta x_r^{\frac{1}{q}} e^{-\omega T_r} + (1-\theta)y_{s'}^{\frac{1}{q}}),$$

$$G_{s'(S+s)}(\omega) = \frac{p}{q}\frac{\rho_{s'}}{\omega + \sigma_{s'}}\left(-\textstyle\sum_{r\in s} \frac{\theta x_r^{\frac{1}{q}}}{\lambda_r - \nu_s} e^{-\omega T_r} + (1-\theta)\frac{y_s^{\frac{1}{q}}}{\nu_s}\right), s = s',$$

$$G_{s'(2S+j)}(\omega) = \frac{p}{q}\frac{\rho_{s'}}{\omega + \sigma_{s'}}\textstyle\sum_{r\in s', j\in r} \frac{\theta x_r^{\frac{1}{q}}}{\lambda_r - \nu_{s'}} \times e^{-\omega(T_r + T_{jr})},$$

$$G_{(S+s)s'}(\omega) = \frac{p\kappa_s a_s\nu_s}{\omega}(y_s - \textstyle\sum_{r\in s} x_r e^{-\omega T_r}), s = s',$$

$$G_{(S+s)(S+s)}(\omega) = \frac{p\kappa_s\nu_s}{\omega}\left(\frac{y_s}{\nu_s} + \textstyle\sum_{r\in s} \frac{x_r}{\lambda_r - \nu_s} \times e^{-\omega T_r}\right),$$

$$G_{(S+s)(2S+j)}(\omega) = -\frac{p\kappa_s\nu_s}{\omega}\textstyle\sum_{r\in s, j\in r} \frac{x_r}{\lambda_r - \nu_s} \times e^{-\omega(T_r + T_{jr})},$$

$$G_{js'}(\omega) = \frac{p\kappa_j\mu_j a_{s'}}{\omega}\textstyle\sum_{j\in r, r\in s'} x_r e^{-\omega T_{rj}},$$

$$G_{j(S+s)}(\omega) = -\frac{p\kappa_j\mu_j}{\omega}\textstyle\sum_{j\in r, r\in s} \frac{x_r}{\lambda_r - \nu_s} e^{-\omega T_{rj}},$$

$$G_{j(2S+j')}(\omega) = \frac{p\kappa_j\mu_j}{\omega}\textstyle\sum_{j\in r, j'\in r} \frac{x_r}{\lambda_r - \nu_s} e^{-\omega(T_{rj} + T_{j'r})},$$

and all other entries are 0.

Let $\bar{\boldsymbol{G}}$ be a $(2S + J) \times (2S + J)$ matrix with

$$\bar{G}_{s's'}(\omega) = \frac{p}{q}\frac{\rho_{s'}a_{s'}}{\omega + \sigma_{s'}}(1-\theta)y_{s'}^{\frac{1}{q}},$$

$$\bar{G}_{s'(S+s)}(\omega) = \frac{p}{q}\frac{\rho_{s'}}{\omega + \sigma_{s'}}(1-\theta)\frac{y_s^{\frac{1}{q}}}{\nu_s}, \quad s = s',$$

$$\bar{G}_{(S+s)s'}(\omega) = \frac{p\kappa_s\nu_s a_s}{\omega}y_s, \quad s = s',$$

$$\bar{G}_{(S+s)(S+s)}(\omega) = \frac{p\kappa_s}{\omega}y_s,$$

and all other entries are 0. It can be verified that

$$\boldsymbol{G}(\omega) = \boldsymbol{P}\boldsymbol{Y}(\omega)\boldsymbol{R}(-\omega)^{\mathrm{T}}\boldsymbol{X}(\omega)\boldsymbol{R}(\omega)\boldsymbol{P}^{-1} + \bar{\boldsymbol{G}}(\omega),$$

where $\boldsymbol{X}(\omega)$ is an $R \times R$ diagonal matrix with entries $X_{rr}(\omega) = e^{-\omega T_r}/(\omega T_r)$, $\boldsymbol{Y}(\omega)$ is a $(2S + J) \times (2S + J)$ diagonal matrix with entries $Y_{s's'}(\omega) = \frac{\omega}{\omega + \sigma_{s'}}, Y_{ss}(\omega) = 1, Y_{jj}(\omega) = 1$, $\boldsymbol{P}$ is a $(2S + J) \times (2S + J)$ diagonal matrix with entries $P_{s's'} = \left(\frac{p\rho_{s'}}{qa_{s'}\hat{y}_s^{\frac{1}{p}}U_s'}\right)^{\frac{1}{2}}, P_{ss} =$

$(p\kappa_s\nu_s)^{\frac{1}{2}}, P_{jj} = (p\kappa_j\mu_j)^{\frac{1}{2}}$, and $\boldsymbol{R}(\omega)$ is an $R \times (2S+J)$ matrix where

$$R_{rs'}(\omega) = \left(\theta x_r^{\frac{1}{q}} T_r\right)^{\frac{1}{2}} \left(\frac{p}{q}\rho_{s'} a_{s'}\right)^{\frac{1}{2}}, r \in s',$$

$$R_{rs}(\omega) = -\left(\frac{x_r T_r}{\lambda_r - \nu_s}\right)^{\frac{1}{2}} (p\kappa_s\nu_s)^{\frac{1}{2}}, r \in s,$$

$$R_{rj}(\omega) = \left(\frac{x_r T_r}{\lambda_r - \nu_s}\right)^{\frac{1}{2}} (p\kappa_j\mu_j)^{\frac{1}{2}} e^{-\omega T_{jr}}, j \in r,$$

and all other entries are 0. Since the open loop system (35)∼(37) is stable, we just need to show that the eigenvalues of the return ratio $\boldsymbol{G}(\omega)$, for $\omega = i\theta$, do not encircle the point $-1$ from the generalized Nyquist stability criterion. Now, these eigenvalues are identical to those of

$$\boldsymbol{Y}(\omega)\boldsymbol{R}(-\omega)^{\mathrm{T}}\boldsymbol{X}(\omega)\boldsymbol{R}(\omega) + \boldsymbol{P}^{-1}\bar{\boldsymbol{G}}(\omega)\boldsymbol{P}.$$

If $\lambda$ is an eigenvalue of $\boldsymbol{G}(i\theta)$, then we can find a unit vector $\boldsymbol{z}$ such that

$$\lambda\boldsymbol{z} = \boldsymbol{Y}(i\theta)\boldsymbol{R}(i\theta)^*\boldsymbol{X}(i\theta)\boldsymbol{R}(i\theta)\boldsymbol{z} + \boldsymbol{P}^{-1}\bar{\boldsymbol{G}}(i\theta)\boldsymbol{P}\boldsymbol{z},$$

where $*$ represents the matrix conjugate transpose. Thus

$$\begin{aligned}\lambda\boldsymbol{z}^*\boldsymbol{Y}(i\theta)^{-1}\boldsymbol{z} \;&=\; \boldsymbol{z}^*\boldsymbol{R}(i\theta)^*\boldsymbol{X}(i\theta)\boldsymbol{R}(i\theta)\boldsymbol{z}+\\ &\quad\boldsymbol{z}^*\boldsymbol{Y}(i\theta)^{-1}\boldsymbol{P}^{-1}\bar{\boldsymbol{G}}(i\theta)\boldsymbol{P}\boldsymbol{z}.\end{aligned}$$

If $\lambda$ is real, since the real parts of $\boldsymbol{z}^*\boldsymbol{Y}(i\theta)^{-1}\boldsymbol{z}$ and $\boldsymbol{z}^*\boldsymbol{Y}(i\theta)^{-1}\boldsymbol{P}^{-1}\bar{\boldsymbol{G}}(i\theta)\boldsymbol{P}\boldsymbol{z}$ are 1 and 0 respectively, we have

$$\lambda = \mathrm{Re}(\boldsymbol{z}^*\boldsymbol{R}(i\theta)^*\boldsymbol{X}(i\theta)\boldsymbol{R}(i\theta)\boldsymbol{z}).$$

Let $\boldsymbol{d} = \boldsymbol{R}(i\theta)\boldsymbol{z}$. Then, since $\boldsymbol{X}$ is diagonal,

$$\lambda = \sum_r |d_r|^2\mathrm{Re}(X_{rr}(i\theta)) = \sum_r |d_r|^2\mathrm{Re}\left(\frac{e^{-i\theta T_r}}{i\theta T_r}\right).$$

Since $\mathrm{Re}\left(\frac{e^{-i\theta T_r}}{i\theta T_r}\right) \geqslant -\frac{2}{\pi}$ for all $\theta$[10], $\lambda \geqslant (-2/\pi)K$, where $K = \|\boldsymbol{R}(i\theta)\boldsymbol{z}\|^2$.

Next, we bound $K$. Let $\boldsymbol{Q}$ be the $(2S+J)\times(2S+J)$ diagonal matrix taking values $Q_{s's'} = \sqrt{\frac{q\hat{y}_s^{\frac{1}{p}}U_s'}{p\rho_s a_s}}$, $Q_{ss} = \sqrt{\frac{\nu_s}{p\kappa_s}}$ and $Q_{jj} = \sqrt{\frac{\mu_j}{p\kappa_j}}$. Let $\rho(\cdot)$ denote the spectral radius, and $\|\cdot\|_\infty$ the maximum row sum matrix norm. Then

$$\begin{aligned}K \;&=\; \boldsymbol{z}^*\boldsymbol{R}(i\theta)^*\boldsymbol{R}(i\theta)\boldsymbol{z}\\ &\leqslant\; \rho(\boldsymbol{R}(i\theta)^*\boldsymbol{R}(i\theta))\\ &=\; \rho(\boldsymbol{Q}^{-1}\boldsymbol{R}(i\theta)^*\boldsymbol{R}(i\theta)\boldsymbol{Q})\\ &\leqslant\; \|\boldsymbol{Q}^{-1}\boldsymbol{R}(i\theta)^*\boldsymbol{R}(i\theta)\boldsymbol{Q}\|_\infty\\ &<\; \tfrac{\pi}{2},\end{aligned}$$

and the last inequality follows from (42) (proved in Lemma 4) and the given conditions (46), (47), and (48).

Therefore we have that $\lambda > -1$ for any real eigenvalue $\lambda$. Thus, when the loci of the eigenvalues of $\boldsymbol{G}(i\theta)$ for $-\infty < \theta < \infty$ cross the real axis, they do so to the right of $-1$. Hence the loci of the eigenvalues of $\boldsymbol{G}(i\theta)$ cannot encircle $-1$, the generalized Nyquist stability criterion is satisfied and the system (35)∼(40) is stable, in the sense that $v_s(t) \to 0, \hat{v}_s(t) \to 0, w_j(t) \to 0$ exponentially, for all $s, j$, as $t \to \infty$. There remains the possibility that $\boldsymbol{\mu}(t)$ might hit a boundary of the positive orthant, and invalidate the linearization (49), (50), and (51). To rule out this possibility, note that there exists an open neighborhood of $\boldsymbol{\mu}$, say $\mathcal{N}$, such that $\boldsymbol{\mu}(t) > 0, t \in (-T_{\max}, 0)$, and the linearization is valid. Thus $\mathcal{N}$ is as required.

Finally, we relax the assumption that $\mu_j > 0$ for all $j$. Since $(\boldsymbol{\mu}, \boldsymbol{\nu}, \hat{\boldsymbol{y}})$ is an equilibrium point of system (35)∼(40) with no almost saturated links, $\mu_j = 0$ implies $\dot{\mu}_j(t) < 0$. Thus there is a neighborhood of $\boldsymbol{\mu}$, say $\mathcal{M}$, such that, on $\mathcal{M}$, the linearization of (35)∼(40) coincides with the case where we discard all $j$ such that $\mu_j = 0$. Therefore, as above, we may choose an open neighborhood $\mathcal{N} \subset \mathcal{M}$ such that for any initial trajectory $((\boldsymbol{\mu}(t), \boldsymbol{\nu}(t), \hat{\boldsymbol{y}}(t)), t \in (-T_{\max}, 0))$ with $\boldsymbol{\mu}(t)$ lying within the neighborhood $\mathcal{N}$, $(\boldsymbol{\mu}(t), \boldsymbol{\nu}(t))$ converges as $t \to \infty$ to the solution $(\boldsymbol{\mu}, \boldsymbol{\nu})$ to the optimization problem (16) and $(\boldsymbol{x}(t), \boldsymbol{y}(t), \hat{\boldsymbol{y}}(t))$ converges as $t \to \infty$ to the solution $(\boldsymbol{x}, \boldsymbol{y}, \hat{\boldsymbol{y}})$ to the optimization problem (6)∼(9). □

Corresponding to weighted $\alpha$-fair utility function, defined by (1), the $a_s$ defined in Lemma 3 becomes $\frac{\alpha p-1}{p\hat{y}_s}$. Then the local stability condition (48) reduces to

$$\rho_s \frac{\alpha p - 1}{q} \frac{\sum_{r\in s} x_r^{\frac{1}{q}} T_r}{\hat{y}_s} < \frac{\pi}{4}.$$

Since $\hat{y}_s \geqslant y_s = \sum_{r:r\in s} x_r$ (similar to (14)), the conditions (46), (47) and

$$\rho_s \frac{\alpha p - 1}{q} \frac{\sum_{r\in s} x_r^{\frac{1}{q}} T_r}{y_s} < \frac{\pi}{4} \tag{53}$$

are sufficient to derive the local stability for this special case.

For given parameter $p$, these conditions are attractive because they are local and decentralized. Let the maximum available rate for source $s$ be $M_s$. They lead to a highly scalable parameter choice scheme: each source and each link choose their gain parameters to be $\kappa_s = \frac{\theta\kappa}{M_s\bar{T}_s}$ and $\rho_s = \frac{p\kappa}{(\alpha p-1)\bar{T}_s}$, and $\kappa_j = \frac{\theta\kappa}{c_j\bar{T}_j}$ respectively for some $\kappa \in (0, \frac{\pi}{4})$, where $\bar{T}_s = \frac{\sum_{r\in s} x_r T_r}{y_s}$ is the average round trip time of packets transmitted by source $s$ and $\bar{T}_j = \frac{\sum_{j\in r} x_r T_r}{\sum_{j\in r} x_r}$ is the average round trip time of packets passing through link $j$. As a desirable feature, the gain parameters in the proposed algorithms can be derived from local information only. The independence of state information in networks leads the conditions for delay stability to be scalable and decentralized.

**Remark 3.** *If $\theta = 1$, the conditions* (46) *and* (53) *reduce to the sufficient ones determined by Voice*[7] *except that they use an estimation of the average round trip time of packets transmitted by source s. The delay stability condition for single path fair dual algorithm given in* [14] *has minor difference from these conditions, which were derived by linearizing the system about the flow rate $x_r(t)$.*

## 6 Some Experiments on a Toy Network

We will illustrate the proposed model and algorithms by simulation experiments. First we illustrate the properties of the proposed model in terms of the theoretical equilibrium point. We solve the optimization problem (6)∼(9) with fmincon.m in Matlab and compare the results with the ones resulting from the models in [7] and [8]. Then we study the dynamic performance of the proposed algorithms with and without the presence of propagation delay. The implementation of the algorithms in a real network has to be performed in discrete time. This is accomplished by obtaining a difference equation using the forward rule approximation. We implement the rate-based algorithm with Matlab.

### 6.1 Simulation Setup

The network topology used in the simulation is given in Fig.2, where all the links share the same bandwidth, i.e., 200 Mb/s. The link index and delay are shown in the arc, i.e., the delay for link 1 is 15 ms and the delay for the others is 10 ms. Delays are the same in the reverse link.

We consider the case of three origin-destination pairs, where the routes serving these origin-destination pairs are represented by link index and shown in Table 2. The route number is shown in the parentheses in the third column, i.e., route 1 serves for source 1, route 2 and route 3 serve for source 2, and route 4, route 5 and rout 6 serve for source 3.

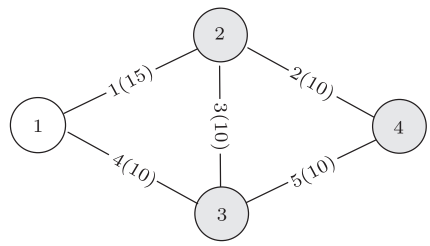


Fig.2. Toy network used in simulations.

**Table 2.** Routes for Different Origin-Destination Pairs

| $s$ | Origin-Destination Pair | Routes Serving for $s$ |
|---|---|---|
| 1 | (1, 2) | 1(1) |
| 2 | (1, 3) | 4(2); 1-3(3) |
| 3 | (1, 4) | 1-2(4); 1-3-5(5); 4-5(6) |

In the sequel, the following aggregate utility function is assumed,

$$\log y_1 + \log y_2 + \log y_3.$$

The resource shares obtained by different sources based on (3)∼(5) with this utility function are proportionally fair. The link marking function in the model in [8] is taken to be $p_j(z_j) = (\frac{z_j}{c_j})^\beta$, where $c_j$ is the capacity of link $j$ and $\beta$ is a parameter describing the link responsiveness. This marking function has been extensively used in the research work, such as [3, 8, 12].

The integration step (sampling time) is set to 5 ms in the algorithm implementation with and without the presence of propagation delay.

### 6.2 Equilibrium Point

To illustrate the properties of the proposed model in terms of the theoretical equilibrium point, we solve the optimization problem (6)∼(9) and the ones in [7] and [8] for different parameter settings with fmincon.m in Matlab. To indicate how the parameters affect the performance of the resource allocation schemes, seven schemes are considered and described in Table 3.

**Table 3.** Resource Allocation Schemes Resulting from Different Models with Different Parameter Settings

| Scheme | Model | Parameters Setting |
|---|---|---|
| Desired | (3)∼(5) | |
| V-a | (6)∼(9) | $\theta = 1, p = 5$ |
| V-b | (6)∼(9) | $\theta = 1, p = 10$ |
| LLXS-a | (6)∼(9) | $\theta = 0.5, p = 5$ |
| LLXS-b | (6)∼(9) | $\theta = 0.1, p = 5$ |
| KV-a | (6) and (8) in [8] | $\beta = 10$ |
| KV-b | (6) and (8) in [8] | $\beta = 100$ |

The equilibrium points corresponding to each scheme are listed in Table 4. The following facts can be observed from the data in Table 4. In the fore five schemes, the aggregate rates are all 200 Mb/s and the resource is used fully. In the last two ones, the aggregate rates are less than 200 Mb/s and increase with parameter $\beta$.

Then seven groups of resource shares corresponding to each scheme are plotted in Fig.3, which provide a reference point when we study the dynamic performance of the proposed algorithms in Subsections 6.3 and 6.4. For one group resource shares, the less the difference among the source rates is, the more fair the resource allocation scheme is. And the larger the source rate is, the more effective the resource allocation scheme is.

In Fig.3, the resource shares obtained by three sources in the Desired scheme, labeled as 1, are proportionally fair. It is a desired resource allocation scheme in terms of efficiency and fairness which the system is achieving. The resource shares obtained by three sources in the LLXS-b scheme, labeled as 5, is the closest one to the ones in the Desired scheme among all the listed groups.

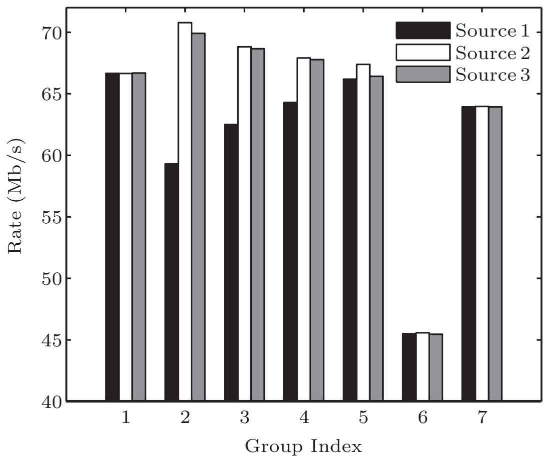


Fig.3. Seven groups of resource shares. Each group corresponds to three sources. All the rate values are computed according to the flow rates in Table 4.

## 6.3 Stability and Convergence Rate

Our global stability result (Theorem 2) imposes no constraint on gains $\kappa_j$ or $\kappa_s$. However, the choice of the gains may still influence the transient performance of the congestion control algorithm defined by (25) and (26). In this subsection, we will study via simulation the effect of the gains on system performance. From our simulations, we will see that the values can be chosen for the given approximation parameter $\theta$ judiciously to make the network converge quickly.

At time 0, the link price $\mu_j = 0.1$ for all $j$ and $\nu_s = 0.05$ for all $s$. We run the simulations for the three groups of parameter settings as the subfigure captions in Fig.4. Here, we run the simulation for 10 s. The simulation results are shown in Fig.4. Here we observe that the algorithm will converge to the theoretical source rate in scheme V-b and scheme LLXS-a (shown in Fig.3 and labeled as 3 and 4 respectively) respectively. In each case, the source rates in the two schemes are almost the same.

Consider the scheme LLXS-a. From the dynamic evolution shown in Fig.4(b), we notice that the rates converge quite quickly in the case $\kappa_j = \kappa_s = 0.1$, while the rates converge slowly in the case $\kappa_j = \kappa_s = 0.01$ as shown in Fig.4(c). Therefore, the better choice of gains for this scheme seems to be 0.1. For this value of $\kappa_j$ and $\kappa_s$, the rate allocated to each source converges fast to the theoretical value. This suggests that, even though the network may be stable for many values of $\kappa_j$ and $\kappa_s$, these parameters have to be chosen carefully to provide a trade-off between transient performance and the rate at which the equilibrium is reached. We do not have a prescription for the choice of the gains in this paper, to which more attention will be paid in the further study.

## 6.4 Trade-Off Between Fairness and Convergence Rate

Our local stability result (Theorem 3) imposes constraints on gains $\kappa_j$, $\kappa_s$ and the approximation param-

**Table 4**. Equilibrium Point (Flow Rate (Mb/s)) for Different Resource Allocation Schemes Listed in Table 3

| Flow | Desired | V-a | V-b | LLXS-a | LLXS-b | KV-a | KV-b |
|---|---|---|---|---|---|---|---|
| 1 | 66.6651 | 59.3029 | 62.5125 | 64.3052 | 66.1873 | 45.5069 | 63.9420 |
| 2 | 52.3744 | 54.3997 | 53.6667 | 53.3341 | 52.7061 | 41.5134 | 56.5802 |
| 3 | 14.2775 | 16.3849 | 15.1535 | 14.4781 | 14.6816 | 4.0625 | 7.3939 |
| 4 | 9.5287 | 12.1561 | 11.1670 | 10.5583 | 9.5656 | 10.0782 | 14.8445 |
| 5 | 47.6256 | 45.6003 | 46.3333 | 46.6659 | 47.2939 | 26.7483 | 39.3463 |
| 6 | 9.5287 | 12.1561 | 11.1670 | 10.5583 | 9.5656 | 8.6279 | 9.7460 |

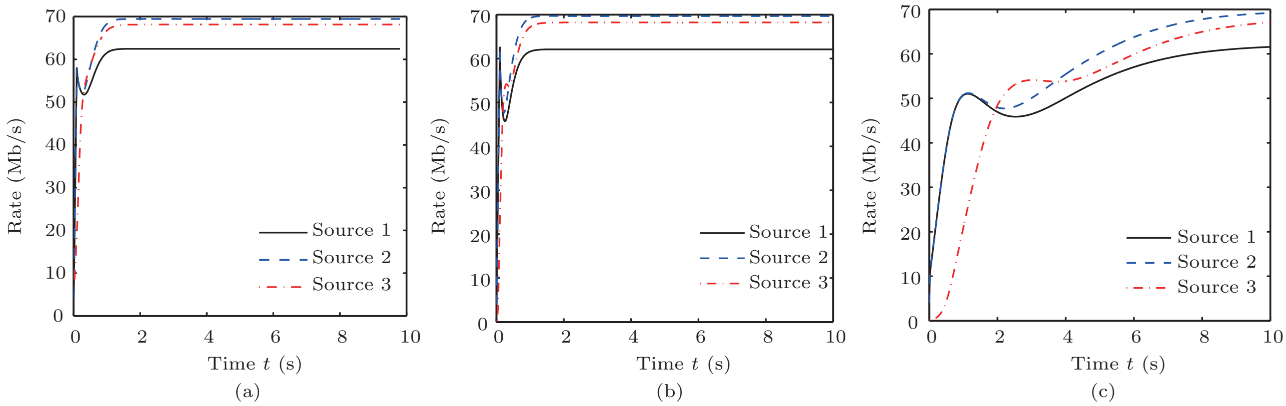


Fig.4. Source rates without propagation delay for different parameter settings. (a) Scheme V-b with $\kappa_j = \kappa_s = 0.1$. (b) Scheme LLXS-a with $\kappa_j = \kappa_s = 0.1$. (c) Scheme LLXS-a with $\kappa_j = \kappa_s = 0.01$.

eters $p$ and $\theta$. From the approximation factor curve in Fig.1, we know the solution of problem (6)$\sim$(9) approximates to the desired well for large $p$ and small $\theta$. However, based on the local stability conditions, i.e., (46), (47) and (48), we notice that, to guarantee the stability, gains $\kappa_j, \kappa_s$ must remain small, which means the slow convergence rate. We will study via simulation the trade-off between the fairness (the approximation degree to the Desired scheme) and the convergence rate. From our simulations, we will see that the algorithm proposed in this paper will stably converge to a more fairness equilibrium point with the same gains than the algorithm in [7].

For the existing prorogation delay, the initial conditions (before time 0) of the link price are $\mu_j = 1$ for all $j$ and $\nu_s = 0.5$ for all $s$. The initial conditions (before 0) of the flow rate are $x_r = 20$ Mb/s for all $r$.

To compare the dynamic evolution of the algorithm with the one without propagation delay shown in Fig.4, we run the simulations for three groups of parameter settings. Here, we run the simulation for 10 s with the same gains $\kappa_j = 0.03$ and $\kappa_s = \rho_s = 0.1$. The simulation results are shown in Fig.5.

Compared with Fig.4(a), the source rates shown in Fig.5(a) diverge at 4 seconds because of the propagation delay. When parameter $p$ is decreased to 5, the source rates shown in Fig.5(b) converge to the theoretical resource shares as $x$-label 2 group in Fig.3. In this case, the fairness among the sources is worse than the one shown in Fig.5(c). To some extent, this observation shows the algorithm proposed in [7] is sensitive to small parameter $p$ when delay exists.

The source rates shown in Fig.5(c) converge to the same resource shares as those in Figs.4(b) and 4(c). Since there exists delay, the convergence in Fig.5(c) needs longer time than that in Fig.4(b); the conver-

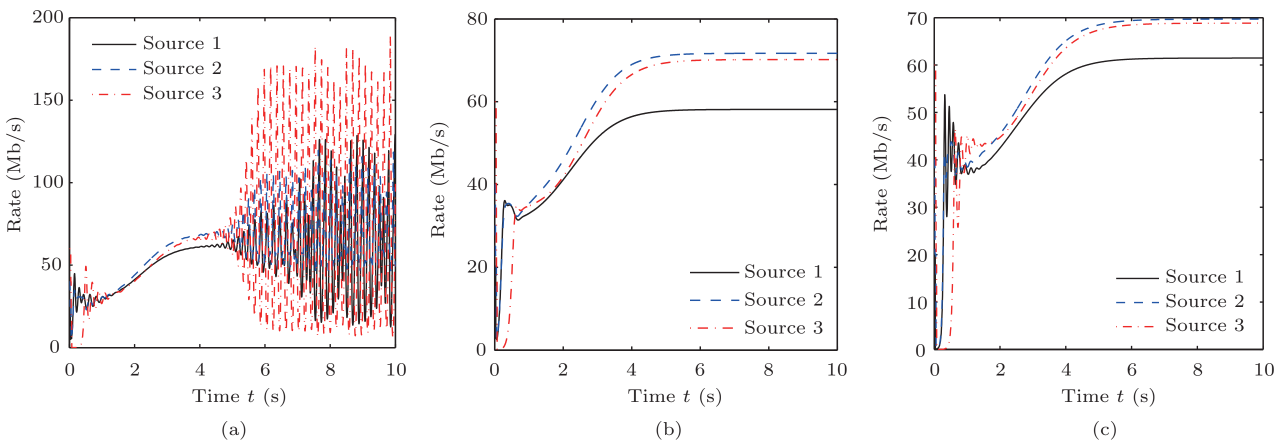


Fig.5. Source rates with propagation delay for different parameter settings. (a) Scheme V-b with $\kappa_j = 0.03, \kappa_s = \rho_s = 0.1$. (b) Scheme V-a with $\kappa_j = 0.03, \kappa_s = \rho_s = 0.1$. (c) Scheme LLXS-a with $\kappa_j = 0.03, \kappa_s = \rho_s = 0.1$.

gence in Fig.5(c) arises slight oscillation within the first second compared with the one in Fig.4(c).

## 7 Conclusions

This paper considered the well known problem of joint multipath routing and flow control, and brought new insight into congestion control algorithms. Specifically, we proposed a generalized multipath utility maximization model, and then derived a family of mutipath dual congestion control algorithms. Based on the results of this paper, one can understand the unstability of the natural mutipath dual congestion control algorithm which is a special case in the proposed family. The one proposed in [7], a special case in this family, is at risk to choose a sufficient large $p$ to approximate the solution of the original problem. The simulation results show that the proposed multipath dual algorithms with appropriate parameter settings can achieve more stable resource shares while maintaining the fairness among the involved users than the one proposed in [7].

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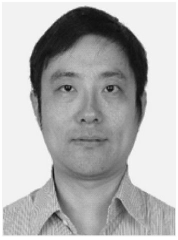

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