Environmental Science and Pollution Research (2018) 25:34720–34729
https://doi.org/10.1007/s11356-018-3433-9

RESEARCH ARTICLE

Semi-analytical solutions for two-dimensional convection–diffusion–reactive equations based on homotopy analysis method

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Received: 18 July 2018 / Accepted: 9 October 2018 / Published online: 15 October 2018

Abstract
Convention applied to describe contaminant transport in landfills and groundwater systems is typically characterized by simplified geometries and boundary conditions. As a result, they neglect the more general boundary conditions encountered in the real world, including convection and diffusion of contaminants (e.g., landfill leachate) associated with fluid transportation in the lateral direction. Here, we present semi-analytical solutions that can be used to describe and estimate the contaminants' fate in two-dimensional space. This is achieved by applying the homotopy analysis method (HAM) to create a different order deformation equation series, the sum of which is the solution of the two-dimensional target problem. To ensure the accuracy of the semi-analytical solution, elements of the equation series have been defined and adapted to satisfy the partial differential equation of the discussed problem. Similarly, the convergence of the HAM solution has been achieved by adopting proper convergent control parameters, ensuring the convergence of each element of the deformation equation series. This guarantees that the sum of the equation series is convergent. HAM has been applied to three cases with more general and smooth initial conditions. Good agreement between HAM solutions and numerical solutions from the literature demonstrates the capacity of HAM.

Keywords Homology analysis method · Convection–diffusion–reactive · Two-dimensional · Semi-analytical solutions

Introduction

Convection–diffusion–reactive transport models have been widely used to describe, evaluate, and remedy contaminant transport in landfills, groundwater, dredged sludge heap sites, and waste disposal areas (Chen et al. 2005; Yu et al. 2018). Governing equations for the description of contaminant transport should be expressed in three dimensions if they are to be applied in realistic industrial settings. However, many cases in three-dimensional convection–diffusion analysis can be reduced to two-dimensional problems owing to the symmetry of the spatial domain. Further reduction of two-dimensional problems to one-dimensional cases has been limited to problems characterized by simple geometry and boundary conditions (Douglas and Peaceman 1955). Thus, a two-dimensional variable would lead to more general scenarios for contaminant transport and transformation analysis.

Analytical solutions for two-dimensional contaminant convection and diffusion problems involve complicated transformation techniques (e.g., Laplace transformations or Fourier transformations) in both the temporal and spatial domains (Chen et al. 2009; Li and Cleall 2010; Li and Cleall 2011; Oñate et al. 2017; Van Genuchten 1985). This fact limits the analytical solutions to a subset of problems with more general boundary conditions and smooth initial conditions. This shortcoming can be overcome today by developing computationally intensive approximate solutions for partial differential equations of convection–diffusion–reactive in two-dimensional space.

Accurate numerical solutions for partial differential equations describing the convection–transport process of contaminants are particularly difficult to obtain and have posed a great challenge for the environmental science and computational fluid dynamics community (Liu et al. 1999;

Responsible editor: Marcus Schulz

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Phongthanapanich and Dechaumphai 2009; Tian 2011; Zhang et al. 2010). Nevertheless, numerous numerical approaches for contaminant transport equations have been developed in the past few decades. The main strategies for solving the convection–diffusion equations are the finite difference method (FDM) and the finite element method (FEM); these strategies are associated with two main approaches, the implicit and explicit approaches. Because the implicit approach is computationally expensive and may not convergence for strongly nonlinear equations, the explicit approach is preferred in industrial applications. The explicit approach also benefits from simplicity and opportunities for parallelization (Phongthanapanich and Dechaumphai 2009). Unfortunately, the explicit approach has been limited by the Courant–Friedrichs–Lewy (CFL) stability criterion, meaning that solutions with acceptable accuracy can only be obtained using small incremental time steps (Abbo and Sloan 1996; Ma 2014; Ma and Zhao 2018; Ma et al. 2016). In some convection-dominated diffusion problems or convection problems, spurious oscillations of numerical solutions will occur without special treatment (Codina 1993; Joshi and Jaiman 2017; Knopp et al. 2002; Phongthanapanich and Dechaumphai 2009). In recent decades, numerous approaches to suppress the spurious oscillations of numerical solutions for convection problems or convection-dominated problems have been proposed. Notable contributions include combined FEM–FDM with the method of characteristics (Douglas and Russell 1982), Petrov–Galerkin methods with discontinuous weighting functions (Hughes and Brooks 1982), the discontinuity–capturing crosswind–dissipation approach (Codina 1993), higher order compact (HOC) schemes (Kalita et al. 2002), the FEM with shock-capturing technique (Knopp et al. 2002), techniques that combine finite volume and FEM (Phongthanapanich and Dechaumphai 2009), the two-grid characteristic finite volume element method (Chen et al. 2013), and the positivity preserving variational method (Joshi and Jaiman 2017).

Non-FEM–FDM approaches are also feasible for solving convection–diffusion problems. For example, Liu et al. (1999) applied the lattice Boltzmann method (LBM) to solve two-dimensional convection–dispersion equations. Methods for applying LBM models to multi-dimensional convection–diffusion problems have subsequently been proposed (Hu et al. 2016; Li et al. 2015; Wang et al. 2018; Wang et al. 2017).

However, the analytical and some numerical models above are limited to more general boundary and initial conditions, a characteristic which restricts their capacity and feasibility in most industrial applications. Although various modifications (or treatments) by mathematical techniques have been developed to remove these limitations, these modifications normally involve more complicated and difficult mathematical theories and impose further limitations for engineering applicability. However, solution procedures suggested by these methods require a professional mathematical and computational background; engineers may need professional training before they are able to employ them. Although LBM and other numerical tools have become much more powerful with the ongoing rapid development of computational technology, semi-analytical solutions based on mathematic tools are still valuable given that semi-analytical approaches may provide benchmarks for numerical solutions.

Therefore, the main objective of this paper is to remove the limitations imposed by current numerical techniques and analytical approaches by applying the homotopy analysis method (HAM) to solve two-dimensional convection–diffusion–reactive equations. HAM, proposed by Liao (1992), is based on the concept of homotopy of topology; it treats a nonlinear problem as the sum of infinite linear sub-problems with different-order deformation equations. To ensure the accuracy of HAM solutions, elements of the created deformation equation series must be defined properly so that they satisfy the partial differential equation of the problem. Similarly, the convergence of a HAM solution can only be achieved by adopting proper convergent control parameters and ensuring the convergence of each element of the deformation equation series, which would guarantee convergence of the sum. Thus, HAM is a semi-analytical method for treating highly nonlinear problems without resorting to complex mathematical theories (Brociek et al. 2014; Liao 2004; Liao 2012). In this paper, HAM has been applied to describe the convection–diffusion–reactive process of a contaminant in two-dimensional space, with more general and smooth initial conditions being considered.

Validation of HAM for a two-dimensional convection–diffusion transport model

Governing equations and boundary conditions

A two-dimensional convection–diffusion transport model investigated by Liu et al. (1999), which is characterized by general boundary and smooth initial conditions, is solved for the validation of HAM. The governing differential equation for the description of contaminant convection–diffusion without decay effects can be expressed as follows:

$$\frac{\partial c}{\partial t}+u\left(\frac{\partial c}{\partial x}+\frac{\partial c}{\partial y}\right)=D\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}\right) \tag{1}$$

where $c = c(x, y, t)$ (mg L^{-1}) is the contaminant concentration at time t (a); u is the fluid flow rate in the soil system (m a^{-1}); D (m^2 a^{-1}) is the diffusion coefficient; and x and y (m) are the coordinates in two-dimensional space.

The initial conditions presented in Eq. (2) show a pre-existing plume of contaminant within a two-dimensional domain at time $t = 0$; the conditions are expressed as the product of an exponential function and two trigonometric functions. Four borders of the investigated domain are contaminant-free for the consideration of inlet and outlet boundaries that exposed to air and fresh water source. These initial conditions can be applied to describe the variation of contaminant distribution in soil systems where the ground water flow rate and diffusion may vary seasonally. It is readily apparent that the convection–diffusion equations with more general initial conditions normally cannot be solved by most traditional transform methods or numerical approaches without special technical treatment,

$$\begin{aligned} & c(x,y,0) = ae^{\frac{u}{2D}(x+y)}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) \\ & c(x,0,t) = c(x,l_y,t) = c(0,y,t) = c(l_x,y,t) = 0 \end{aligned} \tag{2}$$

where a is the initial contaminant concentration and l_x and l_y are the size of the investigated domain in the x and y direction.

Solutions by HAM

According to the principle of HAM, the initial approximations are adopted based on the initial conditions:

$$c(x,y,t) = ae^{\frac{u}{2D}(x+y)}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) \tag{3}$$

Then, a linear operator embedding in HAM can be defined by introducing a variable q:

$$L[\phi(x,y,t;q)] = \frac{\partial\phi(x,y,t;q)}{\partial t}. \tag{4}$$

ϕ has been defined as the solution of the prescribed problem, $c(x, y, t)$, with

$$L[C] = 0 \tag{5}$$

in which C is the integration constant.

Based on the principle of HAM and the target nonlinear problem, a nonlinear operator is defined:

$$N[\phi(x,y,t;q)] = \frac{\partial\phi}{\partial t} - D\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + u\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right) \tag{6}$$

Since the HAM solution is the sum of a different-order deformation equation series (Liao 1992), with the variable q, the zero-order equation can be created as

$$(1-q)L[\phi - c_0(x,y,t)] = q\hbar N[\phi(x,y,t;q)] \tag{7}$$

$q = 0$ would yield

$$\phi(x,y,t;0) = c_0(x,y,t) \tag{8}$$

and $q = 1$ leads to

$$\phi(x,y,t;1) = c(x,y,t). \tag{9}$$

Therefore, as q increases from 0 to 1, $\phi(x, y, t; q)$ approaches the exact analytical solution $c(x, y, t)$ from an initial guess $c_0(x, y, t)$. At $q = 0$, $\phi(x, y, t; q)$ can be expressed as a Taylor series:

$$\phi(x,y,t;q) = \phi(x,y,t;0) + \sum_{m=1}^{+\infty} c_m(x,y,t)q^m, \tag{10}$$

where

$$c_m(x,y,t) = \left.\frac{\partial^m\phi(x,y,t;q)}{m!\partial q^m}\right|_{q=0}. \tag{11}$$

As the equation series must be convergent at $q = 1$, we have

$$c(x,y,t) = c_0(x,y,t) + \sum_{m=1}^{+\infty} c_m(x,y,t) \tag{12}$$

According to the principle of HAM (Liao 1995; Liao 2005; Liao 1992; Liao 2004), Eq. (12) is the solution for the problem described by Eq. (1). In line with the derivation of zero-order deformation equations, the mth-order deformation equation is subsequently created by differentiating Eq. (7) m times with respect to q, setting $q = 0$ and dividing them by $m!$:

$$L[c_m(x,y,t) - \chi_m c_{m-1}(x,y,t)] = \hbar R_m\left(\overrightarrow{c_{m-1}(x,y,t)}\right), \tag{13}$$

in which

$$\begin{aligned} & R_m\left(\overrightarrow{c_{m-1}}\right) = \frac{\partial c_{m-1}}{\partial t} + u\left(\frac{\partial c_{m-1}}{\partial x} + \frac{\partial c_{m-1}}{\partial y}\right) - D\left(\frac{\partial^2 c_{m-1}}{\partial x^2} + \frac{\partial^2 c_{m-1}}{\partial y^2}\right) \\ & \chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases} \end{aligned} \tag{14}$$

and $\hbar$ is the auxiliary parameter that controls the convergence of series solutions.

Equation (7) can be solved as

$$c_m(x,y,t) = \chi_m c_{m-1}(x,y,t) + \hbar\int_0^t R_m\left(\overrightarrow{c_{m-1}(x,y,\tau)}\right)d\tau \tag{15}$$

or expressed as follows:

$$c_m(x,y,t) = \chi_m c_{m-1} + \hbar\int_0^t\left(\frac{\partial c_{m-1}}{\partial t} + u\left(\frac{\partial c_{m-1}}{\partial x} + \frac{\partial c_{m-1}}{\partial y}\right) - D\left(\frac{\partial^2 c_{m-1}}{\partial x^2} + \frac{\partial^2 c_{m-1}}{\partial y^2}\right)\right)d\tau \tag{16}$$

Thus, series solutions can be obtained as follows:

$$c_0(x,y,t) = ae^{\frac{u}{2D}(x+y)}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right), \quad (17a)$$

$$c_1(x,y,t) = \hbar\left(\frac{atu^2}{D} + 2D\pi^2 at - \frac{atu^2}{2D}\right)e^{\frac{u}{2D}(x+y)}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right). \quad (17b)$$

In HAM, the convergence region for the series solutions of the target nonlinear problems is achieved by adjusting the auxiliary parameter $\hbar$ (Liao 1995; Liao 1992; Liao 2004). The plot of $c_{tt}(0, 0, 1)$ versus $\hbar$ curve for this case is depicted in Fig. 1, which suggests that, in this case, the series solutions would converge if the value of $\hbar$ ranges from −2.4 to 0.6.

Note that the selected value of convergence control parameter $\hbar$ would determine the convergence area of the created equation series and its convergence rate (Liao 1995; Liao 2010; Liao 2004). Thus, we can adopt the minimum of the residual error square technique proposed by Liao (2010) in this study to obtain the optimal value of $\hbar$. The squared residual error of the governing differential equation is defined as follows:

$$E_n(\hbar) = \iint_{\Omega}\left(N\left[\sum_{m=0}^{n} c_m(x,y,t)\right]\right)^2 dxdy. \quad (18)$$

In this case, the optimal value of $\hbar$ is −0.5.

Numerical results

A two-dimensional convection–diffusion problem without decay effects investigated by Liu et al. (1999) is here reconsidered for model validation, with model parameters as follows: $D = 0.01$ m^2 a^{-1}, $u = 0.001$ m a^{-1}, $l_x = l_y = 1$ m, and

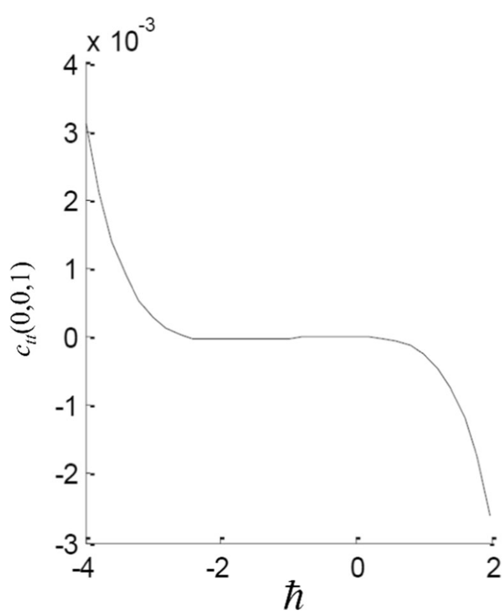

Fig. 1 $\hbar$ curve of $c_{tt}(0, 0, 1)$

$a = 1$ mg L^{-1}. The semi-analytical solutions reproduced by HAM are compared with LBM solutions (Liu et al. 1999) at $t = 1.2$ a in Fig. 2. Good agreement between semi-analytical solutions produced by HAM and LBM solutions produced by Liu et al. (1999) demonstrates that HAM is capable of solving the problem of two-dimensional convection–diffusion contaminant transport. For a better demonstration of the proposed HAM solution, contaminant concentration profiles along two perpendicular sections ($x = 0.5$ m and $y = 0.5$ m) at three different time intervals ($t = 0.4$, 0.8 and 1.2) are depicted in Fig. 3. As can be seen from Figs. 2 and 3, the distribution of contaminants decreases from the central zone to its four borders due to the convection and diffusion process associated with the transport of the contaminant. The HAM solutions agree well with LBM solutions for all cases under consideration, which further demonstrates the capacity of HAM.

HAM solutions for a reactive transport model with a variable decay coefficient

Governing equations and boundary conditions

In this section, a two-dimensional contaminant reactive transport problem is investigated under the influence of a time-dependent decay coefficient. In practice, this type of problem is seen in cases where contaminants, such as nuclear radioactive waste or nitric oxide, among others, decay over time during the convection–diffusion process in ground water or soil systems (Chen 2014; Yu et al. 2018). Here, we adapt general boundary conditions and the pre-existing plume of contaminant described in "Validation of HAM for a two-dimensional convection–diffusion transport model." The governing differential equation for convection–diffusion with a time-dependent decay coefficient is expressed as

$$\frac{\partial c}{\partial t} + u\left(\frac{\partial c}{\partial x} + \frac{\partial c}{\partial y}\right) = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right) - \lambda e^{-kt}c, \quad (19)$$

where λ (a^{-1}) and k (a^{-1}) are the decay coefficients.

Solutions by HAM

Based on the initial conditions, the initial approximations can take the same form as those presented in Eq. (3). The same linear operator adopted in Eq. (4) can also be applied to define the nonlinear operator for the partial differential equations:

$$N[\phi(x,y,t;q)] = \frac{\partial\phi}{\partial t} - D\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + u\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right) + \lambda e^{-kt}\phi \quad (20)$$

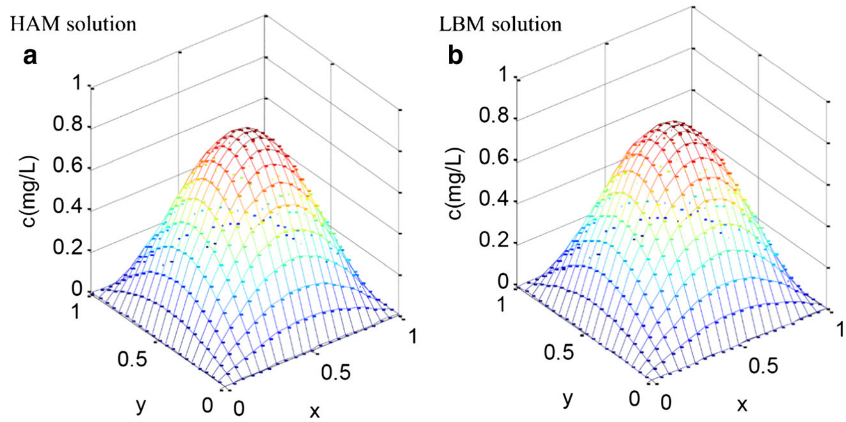

Fig. 2 Comparison between **a** HAM solution and **b** LBM solution by Liu et al. (1999)

Following the same procedure demonstrated in "Validation of HAM for a two-dimensional convection–diffusion transport model," a different-order deformation equation series can be achieved via HAM:

$$c_0(x,y,t) = ae^{\frac{u}{2D}(x+y)}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) \tag{21a}$$

$$\begin{aligned} c_1(x,y,t) &= \hbar\left(\frac{a\lambda\left(1-e^{-kt}\right)}{k}-\frac{atu^2}{D}+2D\pi^2at-\frac{atu^2}{2D}\right)e^{\frac{u}{2D}(x+y)}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) \\ &+\hbar\pi atue^{\frac{u}{2D}(x+y)}\cos\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right)-\hbar\pi atue^{\frac{u}{2D}(x+y)}\sin\left(\frac{\pi x}{l_x}\right)\cos\left(\frac{\pi y}{l_y}\right) \\ &\vdots \end{aligned} \tag{21b}$$

Adopting the minimum of the residual error square technique, we obtain $\hbar = 0.6$ for this case.

Simulation results and discussion

The model parameters adopted in this case are $D = 0.01\ \mathrm{m^2\ a^{-1}}$, $u = 0.001\ \mathrm{m\ a^{-1}}$, $l_x = l_y = 1$ m, $a = 1\ \mathrm{mg\ L^{-1}}$, $\lambda = 1\ \mathrm{a^{-1}}$, and $k = 0.1\ \mathrm{a^{-1}}$. The resulting contaminant concentration profiles along two perpendicular sections ($x = 0.5$ m and $y = 0.5$ m) at three different time intervals ($t = 0.5$, 1.0, and 1.5 a) produced by both HAM and LBM are compared in Fig. 4. The HAM solutions match well with the LBM solutions (Liu et al. 1999) for all cases considered.

For a better evaluation of model performance, we investigate the effects of the decay coefficient on contaminant concentrations. Figure 5 compares profiles of contaminant concentrations along two perpendicular cross sections ($y = 0.5$ m and $x = 0.5$ m, at time $t = 1$ a) with three different decay coefficients ($k = 0.1\ \mathrm{a^{-1}}$, $0.2\ \mathrm{a^{-1}}$, and $0.3\ \mathrm{a^{-1}}$). Contaminant concentrations decrease gradually as the decay coefficient decreases, except the four inlet/outlet boundaries. In addition, comparing Fig. 5a, b, we can see that the effects of the decay coefficient are the same in two directions. This finding indicates that large values of k would lead to a slower dissipation or transport of contaminants.

HAM solutions for contaminant transport model with variable coefficients of convection and diffusion

Governing equations and boundary conditions

A two-dimensional contaminant transport model with variable coefficients of both convection and diffusion is considered in this section. Variations in both convection and diffusion coefficients have been reported in many contaminant transport problems owing to seasonally varying groundwater flow rates and changing diffusion rates (Chen et al. 2005; Chen 2014; Feng and Zheng 2015; Yu et al. 2018). This section describes a two-dimensional problem of contaminant convection–diffusion that is fairly realistic. For a more general and realistic consideration, we employ the contaminant-free boundary conditions and a pre-existing plume of contaminant investigated in "Validation of HAM for a two-dimensional convection–diffusion transport model." The governing differential equation with sinusoidal variation in convection and diffusion coefficients is expressed as follows:

$$\frac{\partial c}{\partial t}+u(1-\sin(mt))\left(\frac{\partial c}{\partial x}+\frac{\partial c}{\partial y}\right) = D(1-\sin(mt))\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}\right) \tag{22}$$

where m ($\mathrm{a^{-1}}$) is the flow-resistant coefficient.

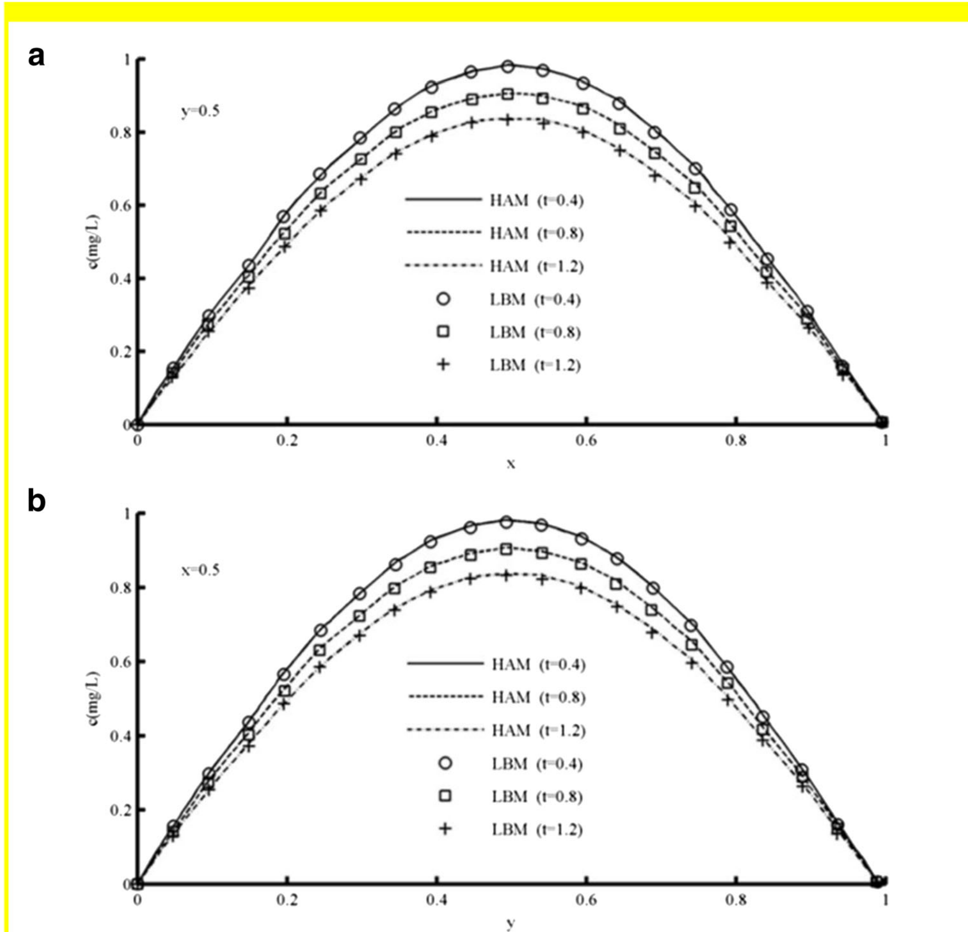

Fig. 3 Comparison between HAM solutions and LBM solutions by Liu et al. (1999) for different time intervals for **a** section $y = 0.5$ m and **b** section $x = 0.5$ m

Solutions using HAM

In line with the procedures of HAM solution demonstrated in "Validation of HAM for a two-dimensional convection–diffusion transport model" and considering the initial conditions, the initial approximations can take the form of Eq. (3); the linear operator by Eq. (4) is applied to define the nonlinear operator for the partial differential equations:

$$N[\phi(x,y,t;q)] = \frac{\partial \phi}{\partial t} - D(1-\sin(m\,t))\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + u(1-\sin(m\,t))\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\right) \tag{23}$$

Following the same procedure of HAM solution demonstrated in "Validation of HAM for a two-dimensional convection–diffusion transport model," Eqs. (7)–(17), the HAM deformation equation series can be obtained as

$$c_0(x,y,t) = ae^{\frac{u}{2D}(x+y)}\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) \tag{24a}$$

$$\begin{aligned} c_1(x,y,t) = \hbar e^{\frac{u}{2D}(x+y)}\left(\frac{atu^2}{2D} + 2D\pi^2 at + \frac{\left(au^2 + 4aD^2\pi^2\right)(1-\cos(mt))}{2mD}\right)\sin\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) \\ + \frac{\hbar\pi aue^{\frac{u}{2D}(x+y)}}{m}(1-\cos(mt))\cos\left(\frac{\pi x}{l_x}\right)\sin\left(\frac{\pi y}{l_y}\right) \\ \vdots \end{aligned} \tag{24b}$$

Applying the minimum of the residual error square technique (Liao 2010; Liao 2012), we obtain $\hbar = -0.5$ for this case.

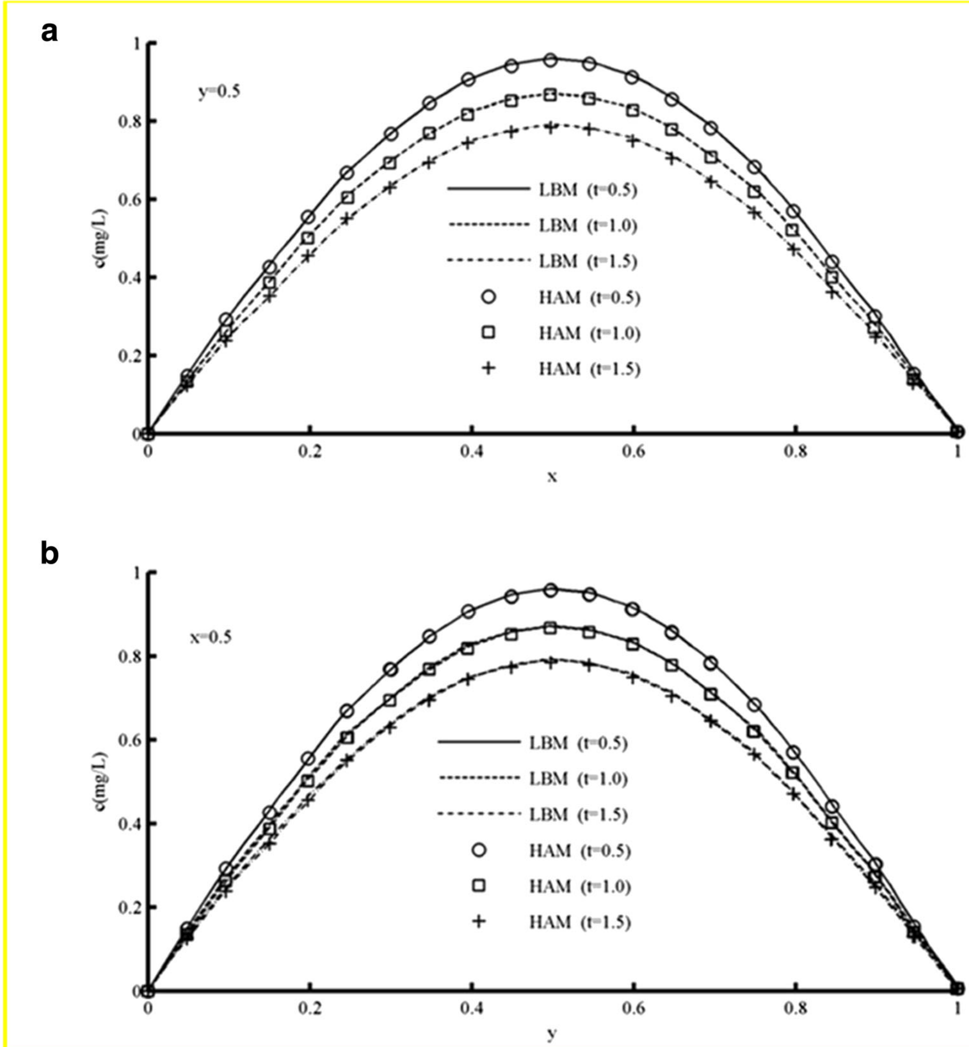

Fig. 4 Comparison between HAM solutions and LBM solutions by Liu et al. (1999) under different time intervals for **a** section $y = 0.5$ m and **b** section $x = 0.5$ m

Simulation results and discussion

In this case, the contaminant transport model is simulated with the following parameters: $D = 0.01$ m^2 a^{-1}, $u = 0.001$ m a^{-1}, $l_x = l_y = 1$ m, $a = 1$ mg L^{-1}, and $m = 0.1$ a^{-1}. Simulation results obtained using HAM are compared with those obtained from Liu et al. (1999) using LBM in Fig. 6, where contaminant concentration profiles along two perpendicular sections ($x = 0.5$ m and $y = 0.5$ m) at two time intervals ($t = 1$ a, and $t = 2$ a) are depicted. As observed, there is

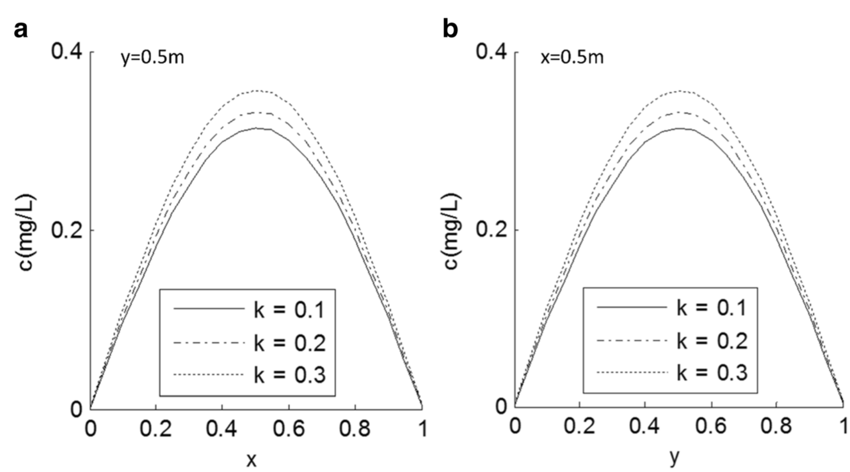

Fig. 5 Contaminant concentration profiles with various k (a^{-1}) values in two cross sections. **a** Section $y = 0.5$ m. **b** Section $x = 0.5$ m

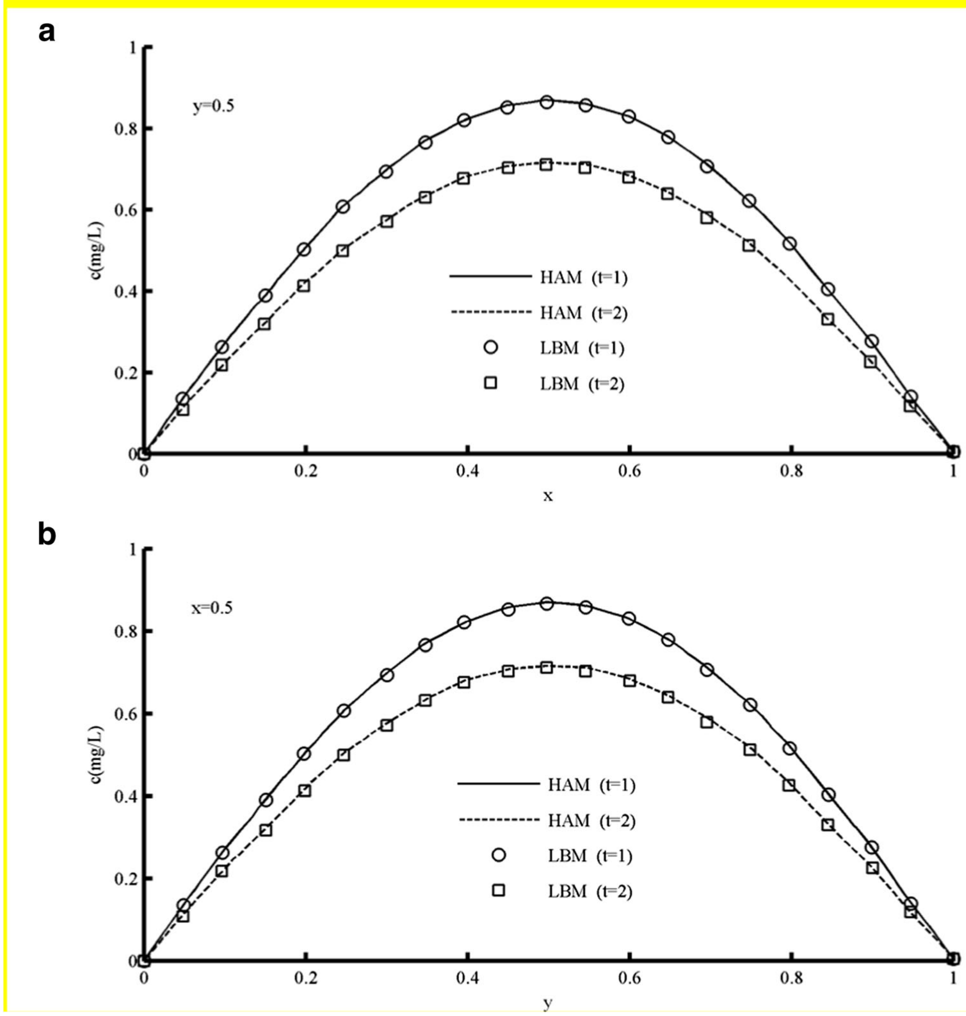

Fig. 6 Comparison between the HAM solutions and the LBM solutions obtained by Liu et al. (1999) under different time intervals for **a** section $y = 0.5$ m and **b** section $x = 0.5$ m

good agreement between the HAM solutions and the LBM solutions (Liu et al. 1999) for all cases considered. This illustrates that HAM can solve contaminant transport problems with sinusoidal variation in convection and diffusion coefficients.

Figure 7 shows comparison profiles of contaminant concentrations along two perpendicular cross sections ($y = 0.5$ m and $x = 0.5$ m, at time $t = 2$ a) with four different flow-resistant coefficients ($m = 0.01$, 0.1, 1, and 10 a^{-1}). As can be seen, contaminant concentrations in the two cross sections increase

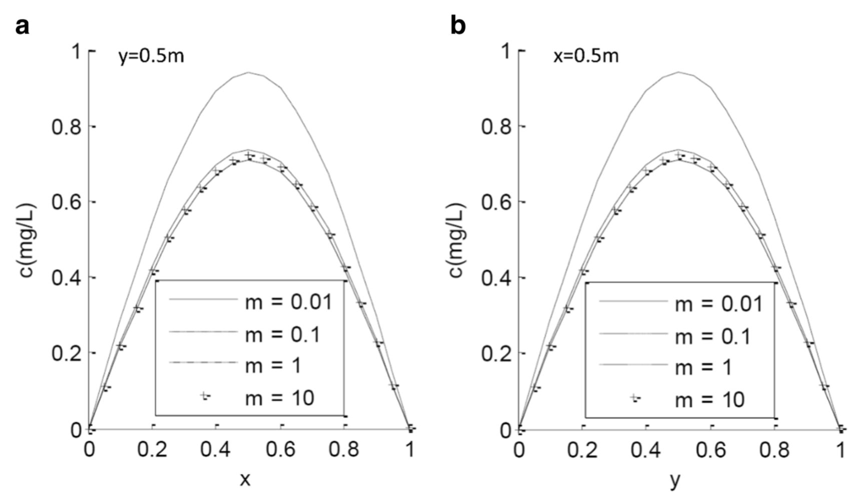

Fig. 7 Contaminant concentration profiles for various m (a^{-1}) for two cross sections. **a** Section $y = 0.5$ m. **b** Section $x = 0.5$ m

with the increasing in m (from 0.01 to 1 a^{-1}); however, they decrease when $m = 10\ a^{-1}$, with concentration profiles between those of $m = 0.01$ and $m = 0.1\ a^{-1}$. Thus, an increasing flow-resistant coefficient would to lead to an initial slowing down effect on contaminant transport and to faster contaminant dissipation effects in later stages. Note that the effects of changing the flow-resistant coefficient are the same in two directions, as seen in by comparing Fig. 7a with Fig. 7b.

Conclusions

This study presents semi-analytical solutions for a two-dimensional convection–diffusion–reactive model obtained using the homotopy analysis method (HAM). This is achieved by creating different-order deformation equation series, the sum of which leads to solutions for the target problems. The convergence of the HAM solution was achieved by selecting proper convergent control parameters based on the minimum residual error square. Thus, rather than involving complicated mathematical and computational theories/techniques, HAM provides a simple yet effective way to resolve two-dimensional convection–diffusion–reactive problems.

HAM was first validated by solving a two-dimensional convection–diffusion problem without decay effects. The results were in good agreement with LBM solutions from the literature, confirming the applicability of HAM to two-dimensional contaminant transport problems ("Validation of HAM for a two-dimensional convection–diffusion transport model" section). Then, HAM was applied to two cases with more general and smooth initial conditions. Two cases were investigated here: one case with a variable decay coefficient ("HAM solutions for a reactive transport model with a variable decay coefficient" section) and another with a sinusoidal variation in the convection and diffusion coefficients. Good agreement between HAM solutions and LBM solutions revealed that HAM can solve two-dimensional contaminant transport models with more general boundary and realistic initial conditions. We also carried out a model parameter sensitivity study for two transport problems, demonstrating the importance of flow-resistant and decay coefficients for the analysis or estimation of contaminant concentration.

Funding information This work was supported by the National Natural Science Foundation of China (Nos. 41372264, 51608192, 51508416), the Zhejiang Provincial Natural Science Foundation (No. LY13E080013), and the Zhejiang Provincial Governmental Public Industry Research Special Funds (Nos. 2015C33220, 2017C33220).

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