Experimental Mechanics (2024) 64:275–290
https://doi.org/10.1007/s11340-023-01020-2

RESEARCH PAPER

Tikhonov Regularization for the Fully Coupled Integral Method of Incremental Hole-Drilling

T. C. Smit[1] · R.G. Reid[1]

Received: 15 August 2023 / Accepted: 15 November 2023 / Published online: 24 January 2024

Abstract

Background The unit pulse integral method is used extensively with the incremental hole-drilling residual stress measurement technique. The ASTM E837 standard, which applies only to isotropic materials, recommends the use of Tikhonov regularization to reduce instability when many depth increments are used. In its current formulation, Tikhonov regularization requires the decoupling of stress, as is possible for isotropic materials. The fully coupled integral method is needed for residual stress determination in layered composite laminates and is currently employed without Tikhonov regularization. This causes greater sensitivity to measurement errors and consequently large stress uncertainties. An approximate method of applying Tikhonov regularization exists for biaxial composites, but is not applicable to more complex laminates.

Objective Extend Tikhonov regularization to the fully coupled integral method to improve residual stress determination in composite laminates.

Methods This work investigates the use of the approximate and fully coupled regularization approaches in an angle ply composite laminate of $[+45/\text{-}45/0/90]_s$ construction. Experimental validation in a $[0/+45/90/\text{-}45]_s$ laminate is also presented where the regularized fully coupled integral method is compared to the series expansion method that includes all in-plane stress and strain directions simultaneously in a least-squares solution.

Results The regularized integral method produces comparable results to those of series expansion while requiring twelve times less FE computation to calculate the compliances. The optimal degree of regularization is also more convenient to determine than the optimal combination of series order required by series expansion.

Conclusions The new method is easily applied and should find wide application in the measurement of residual stresses in composite laminates.

Keywords Residual stress · Incremental hole-drilling · Regularization · Composite materials

Introduction

Composite laminates, such as fibre reinforced plastic (FRP) laminates, undergo complex temperature and pressure cycles during production and residual stresses arise due to differences in the coefficients of thermal expansion and mechanical properties in each ply, tool-part interaction and cure shrinkage of the resin [1–3]. The magnitude and distribution of the residual stresses depend on the material properties of each ply, the laminate configuration and the cure-cycle used. Residual stress distributions in composite laminates are discontinuous at the interfaces between plies of different orientation [4, 5]. The residual stresses that develop in FRP components can introduce transverse cracks, fibre waviness, micro-buckling and delamination [5–7]. These adverse phenomena reduce the overall strength, stiffness and fatigue life of the component [5, 6].

Various residual stress measurement techniques exist including non-destructive diffraction techniques and semi or fully destructive relaxation techniques [8]. Every measurement technique has certain advantages and disadvantages in terms of material applicability, implementation difficulty and cost, maximum measurement depth, depth resolution, measurement accuracy, etc. [9]. Additionally, some techniques are more suited to near surface measurements while others experience large potential error near the surface,

✉ R.G. Reid
robert.reid@wits.ac.za

[1] School of Mechanical, Industrial and Aeronautical Engineering, University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa

therefore, multiple techniques are often used to supplement each other to more accurately describe the residual stress distribution in a specimen. However, non-destructive measurement techniques such as X-ray and neutron diffraction have highly restricted application in FRP laminates since they are non-crystaline [10, 11]. The use of relaxation methods is therefore usually required. Such methods include incremental hole-drilling (IHD) [12], incremental slitting [13] and layer removal [14], for example.

IHD is one of the most commonly used techniques to measure residual stresses due to its general availability, practical implementation, low cost and ability to yield reliable and accurate results [9, 10, 15]. IHD involves incrementally drilling a small hole in a specimen while measuring the released deformations (usually measured as strain using a special IHD rosette) around the hole on the top surface of the specimen. The released strains on the surface are related to the residual stresses that existed in the material prior to drilling through an integral equation. Influence functions, also often referred to as compliance or 'calibration' coefficients, can be obtained through finite element (FE) modelling of the equivalent direct problem for a chosen stress distribution basis function such as piecewise-constant functions or power series functions [16]. Relating measured deformations to the residual stresses requires solution of an ill-posed inverse problem [17]. The inverse problem may also be ill-conditioned since deformations are measured at different locations from where the stressed material is removed and there are technological limitations which prevent sufficiently accurate strain measurements with negligible noise or induced error [17].

Several computational methods are available for IHD, such as power series expansion [16] and the unit pulse integral method [18]. The integral method is used in the standardised IHD test procedure for isotropic materials, ASTM E837-20a [19], and it is also the most commonly used method in composite laminates. The residual stress distribution is represented by a piecewise-constant basis function. Unit pulses of uniform stress are applied individually to every incremental depth as the depth of the hole increases in a forward FE solution of the direct problem. This allows the generation of calibration coefficients which represent the strain that would be measured if a unit pulse of uniform stress was released at each depth increment [18]. Calibration coefficients are determined for each stress depth increment for every hole depth increment and depend on parameters such as material properties, laminate configuration, strain gauge rosette geometry, hole diameter and hole location [20]. Since unit pulses are only applied up the current hole depth, the resulting calibration matrix is lower triangular [8] and the inverse solution is unique since the number of unknown stresses is the same as the number of depth increments [21]. The method is consequently susceptible to computational error sensitivity and error amplification [22, 23]. The greater the number of depth increments, the greater this error sensitivity becomes [17]. Therefore, some form of 'regularization' is required to improve the stability of the solution in the presence of noise. Available 'regularization' methods include coarser discretisation of the solution by limiting the number of depth increments [18], optimising the depth increment distribution [24] and Tikhonov regularization [25], amongst others. Any form of 'regularization' reduces variance in the stress solution by assuming physical knowledge of the solution but introduces some bias. The bias distorts the calculated solution from the true solution and can usually not be quantified. A variance vs. bias trade-off must consequently be considered carefully as shown in Fig. 1 [17]. Solution bias can only be reduced at the expense of increased variance, and vice versa. This behaviour is explained in detail in a recent publication by Beghini et al. [17].

IHD is a discretised technique by nature and so some bias due to discretisation is unavoidable. Fine discretisation (many small depth increments) reduces the bias but introduces unacceptable levels of variance [17]. It is common practice to reduce the degrees of freedom of the solution through coarse discretisation (fewer and larger depth increments) to improve the stability of the solution at the expense of resolution [18]. This approach is limited if the expected variation of the residual stress distribution is high. In such cases, distortion of the stress solution will occur due to averaging effects across the sequence of uniform stresses if the depth increments become too large. This is because the calculated stress in each depth increment is considered constant. However, this is currently the approach adopted by most practitioners when applying the integral method in composite laminates to reduce scattering and instability in the results with an accompanying loss of resolution [26]. Zuccarello [24] proposed optimising the increment distribution

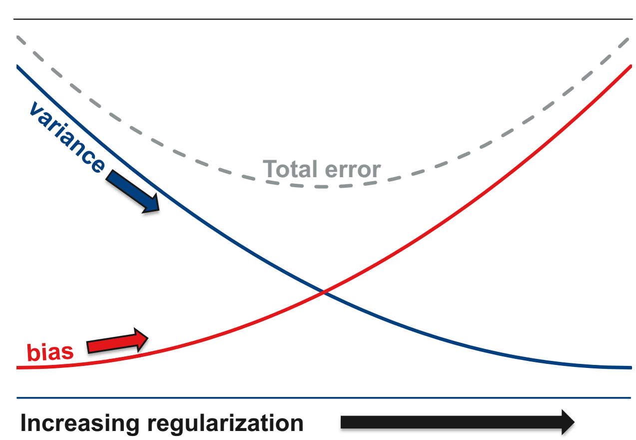

Fig. 1 Illustration of variance vs. bias trade-off

for the integral method in isotropic materials to reduce error sensitivity by ensuring calibration coefficients on the diagonal remain equal in magnitude by gradually increasing the size of each depth increment as the hole depth increases. This approach works well but resolution remains an issue. It is usually not possible to apply this approach in layered composite materials since the depth increment distribution must align with the interfaces between different materials at particular depths to ensure that the discontinuities in stress at the interfaces between material types can be captured correctly. Application of Tikhonov regularization to reduce the sensitivity of IHD stress solutions to noise in the measured data is more practical to use in this situation and is accordingly preferred.

When performing IHD in isotropic materials, second-order Tikhonov regularization [25] is implemented that adds continuity in the stress solution to the objective function by penalizing the norm of the second derivative of the solution. This favours smooth solutions over those with high local curvature, smoothing the calculated residual stresses. The original ill-posed problem is no longer solved and instead a different, slightly biased but well-posed, problem is solved whose refinement is controlled by the regularization parameter α [17]. A discrepancy between the measured strains and those corresponding to the regularized solution is thereby allowed to remove strain noise artefacts. When Tikhonov regularization is applied, the inverse problem no longer suffers from increasing instability with increasing number of depth increments but actually benefits from the larger data set [17]. Tikhonov regularization combined with small depth increments is able to accurately measure residual stress distributions with steep gradients near the surface, such as in material treated by shot peening [27] and laser shock peening [28]. The use of Tikhonov regularization for the integral method of IHD is widely accepted and is recommended in the ASTM standard. This approach does not, however, extend to layered composite materials. In these materials the relationship between the residual stress distributions and released strains does not have a trigonometric form [26] and so it is not possible to decouple equi-biaxial and shear components to which Tikhonov regularization can be applied independently. The fully coupled integral method [26], which considers all three stress and strain directions simultaneously, with nine calibration coefficients representing each stress application depth for each depth increment must accordingly be employed.

Significant stress variation can occur within a single ply of a FRP laminate, for example, due to tool/part interactions at the surface. Currently, no integral computational method exists for IHD that can simultaneously measure all in-plane residual stress components of complex angle-ply laminates where steep stress gradients occur within a single ply; is also tolerant to noisy strain data; and has high depth resolution. Without Tikhonov regularization, the fully coupled integral method requires a limited number of depth increments to reduce instability and variance in the solution. Smit and Reid [29] extended the use of Tikhonov regularization to determine residual stresses in biaxial composite laminates through the use of simplifying approximations. This numerical procedure is effective in smoothing the residual stress profiles and reducing variance in the stress solution but, when used in more complex laminates, the simplifying approximations can potentially result in some numerical errors depending on the laminate configuration and the relative magnitude of the off-diagonal terms in the calibration matrix. Separate series expansion [30] is an alternative computational method that resolves the limitations of IHD in layered composite materials as it is inherently tolerant to noise in the experimental data or to single erroneous measurements due to least-squares curve fitting in the inverse solution. Separate series expansion has been shown to work well in complex laminates [30] but this method requires significantly more FE calculation than the integral method to determine the required calibration and stress matrices [29]. In addition, the selection of the optimal combination of series order in the different ply orientations is not always trivial.

Tikhonov regularization greatly reduces instability and variance when measuring residual stress distributions in isotropic materials. This approach does not, however, extend to complex composite materials where integral methods without Tikhonov regularization are currently still employed which are sensitive to measurement errors that result in large stress uncertainties. This work develops a means to apply Tikhonov regularization to the fully coupled integral method for residual stress measurements in composite laminates of any complexity. The use of the approximate regularization and the fully coupled regularization approaches are investigated in an angle ply composite laminate of $[+45/-45/0/90]_s$ construction using known stress distributions obtained through FE simulations. The regularized fully coupled integral method is then experimentally demonstrated on a $[0/+45/90/-45]_s$ laminate where results are compared to those of the series expansion method that includes all in-plane stress and strain directions simultaneously in a least-squares solution.

Integral Computational Methods for IHD

The following integral computational methods are applicable to instances where stress and strain components cannot be decoupled and therefore fall outside the ASTM standard. This is a common occurrence when performing IHD in composite laminates, but may also arise in isotropic materials under certain conditions such as the presence of hole eccentricity. Since the laminated composite case is more common it will be the focus of this work, but the presented methods can also be applied to IHD in isotropic materials.

Fully Coupled Integral Method

The fully coupled integral method is shown in equation (1) in vector-matrix form. The matrix coefficients of the first two depth increments are given by equation (2) [26], for example.

$$\boldsymbol{\epsilon}_{meas} = \mathbf{C}\, \boldsymbol{\sigma}_{res} \tag{1}$$

where $\boldsymbol{\sigma}_{res}$ is the calculated residual stresses, $\boldsymbol{\epsilon}_{meas}$ is the measured strains and $\mathbf{C}$ is the calibration matrix.

because the thermo-mechanical response of these laminates is the same in both directions. These factors mean that the majority of the strain released in the x and y directions at every depth increment arises from the relaxation of residual stress in those directions, respectively. The off-diagonal coefficients in

$$\left\{\begin{array}{c} \epsilon_{1_x} \\ \epsilon_{1_y} \\ \epsilon_{1_{45^\circ}} \\ \hline \epsilon_{2_x} \\ \epsilon_{2_y} \\ \epsilon_{2_{45^\circ}} \\ \hline \vdots \\ \epsilon_{i_g} \end{array}\right\} = \left[\begin{array}{ccc|ccc|c} C_{1111} & C_{1112} & C_{1113} & 0 & 0 & 0 & \dots \\ C_{1211} & C_{1212} & C_{1213} & 0 & 0 & 0 & \dots \\ C_{1311} & C_{1312} & C_{1313} & 0 & 0 & 0 & \dots \\ \hline C_{2111} & C_{2112} & C_{2113} & C_{2121} & C_{2122} & C_{2123} & \dots \\ C_{2211} & C_{2212} & C_{2213} & C_{2221} & C_{2222} & C_{2223} & \dots \\ C_{2311} & C_{2312} & C_{2313} & C_{2321} & C_{2322} & C_{2323} & \dots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ C_{ig11} & C_{ig12} & C_{ig13} & C_{ig21} & C_{ig22} & C_{ig23} & C_{igjs} \end{array}\right] \left\{\begin{array}{c} \sigma_{1_x} \\ \sigma_{1_y} \\ \tau_{1_{xy}} \\ \hline \sigma_{2_x} \\ \sigma_{2_y} \\ \tau_{2_{xy}} \\ \hline \vdots \\ \sigma_{i_s} \end{array}\right\} \tag{2}$$

where C_{igjs} is the measured strain at the location of strain gauge g when a unit stress is applied in the s direction to the j^{th} depth increment when the hole is i increments deep.

Residual stresses are calculated from the inverse solution of equation (1). Since the calculated stresses are directly linked to the measured strains there is a unique correspondence between measured strain and stress that is consequently very susceptible to measurement uncertainty. Coarse discretization, i.e. limiting the number of calculation steps, can be used to reduce scattering and instability in the results. However, Tikhonov regularization is a superior form of regularization and is preferred.

Approximate Tikhonov Regularization of the Fully Coupled Integral Method

Tikhonov regularization can be applied separately to each measurement direction of the the fully coupled integral method through simplifying approximations to significantly reduce sensitivity to measurement errors. This method was developed for biaxial composites where the strain release in a particular direction predominantly arises from the relaxation of residual stress in that direction. The comprehensive method on the use of Approximate Tikhonov Regularization with IHD in composite materials is described by Smit and Reid [29] and is, therefore, only briefly summarised here.

In the case of biaxial laminates the diagonal coefficients of each 'block' in equation (2) are dominant due to Poisson's effects. Furthermore, the residual stresses in the x and y directions are similar in magnitude with the shear stresses close to zero each 'block' in equation (2) can therefore be discarded during regularization since their contribution to the released strain is negligible. Considering the stress-strain relationships in each in-plane direction, equation (2) can be decoupled so that the stress-strain relationship is written in terms of each direction only, as presented in Fig. 2. Each residual stress component is therefore treated separately, as shown for the x component in equations (3)-(4). Second-order Tikhonov regularization can then be implemented separately for each stress component using equation (5), in the usual fashion as for metallic structures.

$$\left\{\begin{array}{c} \epsilon_{1_x} \\ \epsilon_{2_x} \\ \vdots \end{array}\right\} = \left[\begin{array}{ccc} C_{1111} & 0 & \dots \\ C_{2111} & C_{2121} & \dots \\ \vdots & \vdots & \ddots \end{array}\right] \left\{\begin{array}{c} \sigma_{1_x} \\ \sigma_{2_x} \\ \vdots \end{array}\right\} \tag{3}$$

$$\boldsymbol{\epsilon}_x = \mathbf{C}_x\, \boldsymbol{\sigma}_x \tag{4}$$

$$\left(\mathbf{C}_x^T\, \mathbf{C}_x + \alpha_x\, \mathbf{L}^T\, \mathbf{L}\right)\boldsymbol{\sigma}_x = \mathbf{C}_x^T\, \boldsymbol{\epsilon}_x \tag{5}$$

where α_x is the regularization parameter that controls the degree of regularization applied and $\mathbf{L}$ is the second-order Tikhonov operator with (-1 2 -1) sequence along the main diagonal.

For composite laminates, rows of $\mathbf{L}$ relating the two sides of a material interface are set to zero to prevent regularization across the interface and allow discontinuities in stress to exist at the interfaces between different materials [31].

It is important to note that following separate application of Tikhonov regularization to each stress component, the calculated strains corresponding to the regularized solution in each measurement direction must be combined into a full

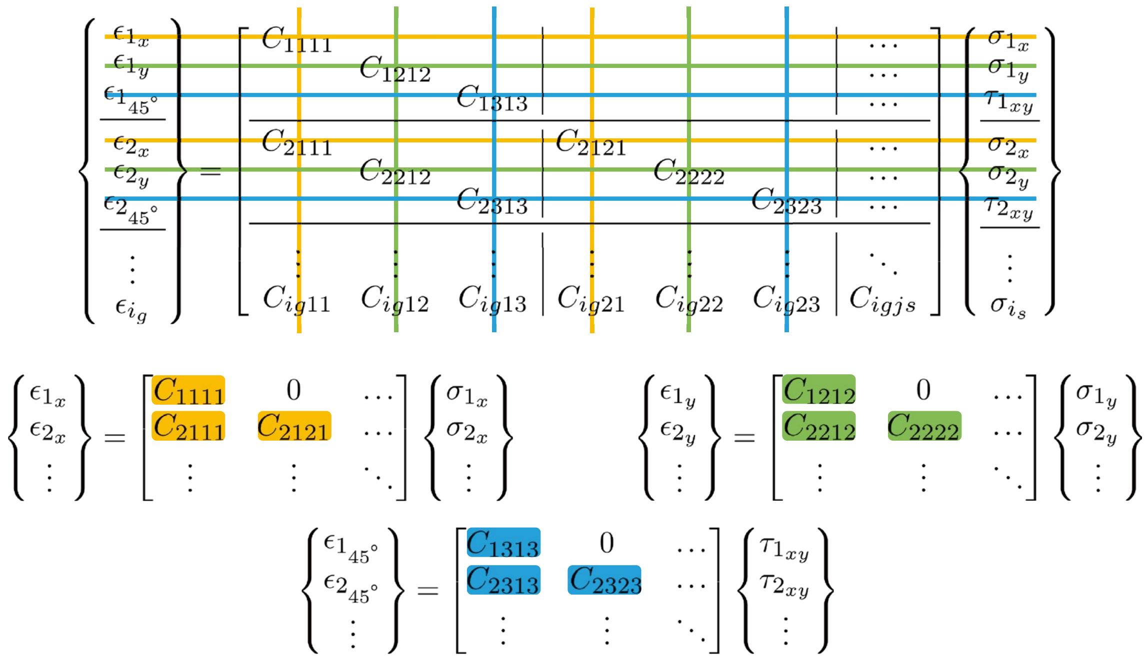

Fig. 2 Approximate decoupling of the fully coupled integral method for the purpose of regularization

strain vector, $\boldsymbol{\epsilon}^{calc}$, in the form of equation (2). The regularized strain vector $\boldsymbol{\epsilon}^{calc}$ is then used with the fully coupled calibration matrix, $\mathbf{C}$, to calculate the regularized residual stress distribution using equation (1). The optimal degree of regularization can be determined initially using the Morozov Discrepancy Principle [32] as will be shown in the subsequent section. The regularization parameters may require adjustment if the stress solution shows signs of excessive variance or local curvature or signs of distortion due to bias.

Tikhonov Regularization of the Fully Coupled Integral Method

All components of residual stress contribute to the measured strain at each strain gauge grid for every depth increment. Depending on the stress distribution relative to the in-plane stiffnesses and Poisson's ratios of the material, the contribution to the released strain in a particular direction due to stresses in other directions may be significant. The approximate decoupling approach of the preceding section which allows separate regularization in each direction was set up in this form to maintain similarity to the approach of the ASTM standard. The method can, however, be combined into a single coupled system in the form of equation (2) to simultaneously penalise the local curvature in all three stress components using an appropriate second-order Tikhonov regularization operator with the (-1 2 -1) sequence along the main diagonal of each measurement direction [33]:

$$L = \left[\begin{array}{ccc|ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline -1 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & \cdots \\ 0 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & -1 & \cdots \\ \hline 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right] \tag{6}$$

While this approach is an improvement on the approximate method because the off-diagonal contributions to strain measurements are accounted for in a single solution, regularization of each stress component remains dependent only its contribution to the strain gauge grid associated with the diagonal of equation (2). Since the off-diagonal contributions of each stress component to the measured strains at the two other corresponding strain gauge grids are not directly included in the regularization simultaneously, a more thorough approach is proposed.

Tikhonov regularization can be applied directly to the fully coupled integral method by constructing a $\mathbf{L}$ matrix that incorporates contributions to each strain measurement direction from all stress components. This allows the amount of data that is incorporated to be tripled, albeit that the strain contributions from the two grids associated with the off-diagonal directions have lower magnitude and sensitivity. If the magnitude of the contribution to off-diagonal strains is small, then incorporation of the additional data will have a negligible effect on the overall regularization and the additional lower quality data will not negatively influence the regularization. However, if the contribution to off-diagonal strains is larger, their inclusion meaningfully contributes to the regularized solution. The variance of a given regularization scheme is reduced when the highest number of measurements are taken [17], therefore the inclusion of the off-diagonal contributions of each stress component to the strain measured at the two associated strain gauge grids benefits the regularized solution. The relative contribution of each stress component to the measured strain in the two off-diagonal directions is always smaller than that of the diagonal direction and so the contribution of these off-diagonal strains to the regularization must be weighted. The ratios of the off-diagonal calibration coefficients to the primary diagonal coefficient of each strain measurement direction for every depth increment are used to appropriately weight the off-diagonal (-1 2 -1) terms of **L** such that the secondary strain components arising from each stress component are not only accounted for, but also used in the regularization of each stress component in a single solution. The form of the accompanying **L** matrix for equation (2) is shown in equations (7)-(11).

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

$$D = \begin{bmatrix} -1 & -\frac{C_{2112}}{C_{2111}} & -\frac{C_{2113}}{C_{2111}} \\ -\frac{C_{2211}}{C_{2212}} & -1 & -\frac{C_{2213}}{C_{2212}} \\ -\frac{C_{2311}}{C_{2313}} & -\frac{C_{2312}}{C_{2313}} & -1 \end{bmatrix} \tag{8}$$

$$E = \begin{bmatrix} 2 & 2\frac{C_{2122}}{C_{2121}} & 2\frac{C_{2123}}{C_{2121}} \\ 2\frac{C_{2221}}{C_{2222}} & 2 & 2\frac{C_{2223}}{C_{2222}} \\ 2\frac{C_{2321}}{C_{2323}} & 2\frac{C_{2322}}{C_{2323}} & 2 \end{bmatrix} \tag{9}$$

The off-diagonal terms corresponding to the final -1 term of the (-1 2 -1) sequence in each row are determined such that the sum of each off-diagonal sequence is equal to zero, ensuring that no net force or moment is added into the solution. Alternatively, the off-diagonal terms of **L** can be weighted using the average of the ratios in equations (8) and (9) to preserve the (-1 2 -1) form, for example, the $-\frac{C_{2112}}{C_{2111}}$ entry in equation (8) would become $-(\frac{C_{2112}}{C_{2111}} + \frac{C_{2122}}{C_{2121}})/2$ and equations (9) and (10) would become $E = -2D$ and $F = D$, respectively. This slight variation of the method has not been employed in the remainder of this work since it produces insignificant benefits in the composite laminates considered herein.

$$F = -[D + E] \tag{10}$$

The pattern illustrated in equations (8)-(10) for the 2nd row can be expanded along the main diagonal, ensuring that the relevant terms of the calibration matrix of equation (2) are used for each depth increment:

$$L = \left[\begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline D & E & F & 0 & \cdots & \ddots \\ \hline 0 & D & E & F & \cdots & \ddots \\ \hline 0 & 0 & D & E & \cdots & \ddots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \tag{11}$$

Finally, each row of **L** can be scaled such that the absolute row-wise sum of the matrix coefficients are all equal to a constant value. For the sake of simplicity this constant can be 4 as would be found for the standard (-1 2 -1) sequence. This ensures that the degree of regularization applied remains constant across all depth increments throughout the measurement depth. Otherwise, an artificial increase or decrease in regularization with depth occurs. As in the approximate method, the rows of **L** bordering any interface must be set to zero to ensure that no regularization is applied across the interfaces between ply orientations or material types.

While a single regularization parameter will usually suffice, regularization parameters can also be controlled individually through the use of a diagonal matrix containing the α_x, α_y and α_{45° parameters corresponding to each in-plane direction. Strains are measured by the same equipment and therefore should have similar standard errors. Nonetheless variations in optimal regularization parameters in each direction usually exist. The use of a diagonal regularization parameter matrix allows simultaneous optimization of each regularization parameter by allowing different degrees of regularization to be applied in each direction.

$$\mathbf{A} = \begin{bmatrix} \sqrt{\alpha_x} & & & & & & \\ & \sqrt{\alpha_y} & & & & & \\ & & \sqrt{\alpha_{45^\circ}} & & & & \\ & & & \sqrt{\alpha_x} & & & \\ & & & & \sqrt{\alpha_y} & & \\ & & & & & \sqrt{\alpha_{45^\circ}} & \\ & & & & & & \ddots \end{bmatrix} \quad (12)$$

The regularized fully coupled stress distribution can be determined from the inverse solution of:

$$\left(\mathbf{C}^T\ \mathbf{C} + \mathbf{L}^T\ \mathbf{A}^T\ \mathbf{A}\ \ \mathbf{L}\right)\boldsymbol{\sigma}_{res} = \mathbf{C}^T\ \boldsymbol{\epsilon}_{meas} \quad (13)$$

The degree of regularization applied can have an appreciable effect on the quality of the calculated stress distribution. The degree of regularization to apply is again a variance vs. bias trade-off that is complicated by the fact that the bias is not accessible to the user [17]. Many strategies exist to determine the optimal degree of regularization, however, the Morozov discrepancy principle has been frequently used and with success in IHD and other relaxation methods so it will be used here as well. The Morozov discrepancy principle can be used to iteratively determine the optimal degree of regularization to apply based on the standard strain error to ensure that maximum noise artefacts are removed from the measured strains without introducing a dominant bias which distorts the residual stress distribution. The standard strain error in each direction can be estimated using the average local misfit norm [34] which assumes that smooth strain variations exist in each ply and that deviations from the smooth variation are a result of measurement noise [35]. Due to the possibility of pronounced changes in the released strain at the interface between plies, the local misfit norm is calculated separately in each ply to exclude the misfit across interfaces [29]. This prevents overestimating the standard error due to slope discontinuities and, consequently, applying excessive regularization that would distort the stress distribution. The Morozov discrepancy principle specifies that an optimal degree of regularization is achieved when the chi-squared statistic, χ^2, equals the number of depth increments, i.e. when the squared discrepancy between the calculated and measured strains is equal to the standard error over the measurement depth. Each ply is treated separately since the estimated standard error, e, tends to vary slightly from ply to ply. The χ^2 statistics can be calculated using equation (14), shown only for strain in the x-direction but similarly for strain in the y and 45° strain directions.

$$\chi_x^2 = \sum_{j=1}^{n}\left(\sum_{i=h_{f_j}}^{h_{e_j}}\left(\frac{\epsilon_{i_x}^{calc} - \epsilon_{i_x}^{meas}}{e_{j_x}}\right)^2\right) \quad (14)$$

where n is the number of plies, h_{f_j} is the first depth increment in ply j, h_{e_j} is the depth increment at the end of ply j (i.e. at the interface with the subsequent ply), e_{j_x} is the standard strain error in the x-direction in ply j, ϵ_x^{meas} is the measured strain data and ϵ_x^{calc} is the calculated strain corresponding to the regularized solution of equation (13):

$$\boldsymbol{\epsilon}^{calc} = \mathbf{C}\ \boldsymbol{\sigma}_{res} \quad (15)$$

The optimal combination of regularization parameters can be found iteratively by starting with sufficiently small regularization parameters and determining the ratio of the χ^2 statistic vs. the total number of depth increments, h, in each measurement direction after solution of equations (13) and (15). The ratios obtained from the current iteration, k, can be used to scale the regularization parameters for the next iteration. Shown only for the x-direction but similarly and simultaneously for the y and 45° strain directions:

$$\alpha_{x_{k+1}} = \left(\frac{h}{\chi_{x_k}^2}\right)^m \times \alpha_{x_k} \quad (16)$$

where a power term m is included simply to allow greater control over the rate of convergence through a simple gradient ascent, but $m = 1$ can be used for the general case and reduced if finer iteration steps are required.

All three regularization parameters are optimised simultaneously and since their respective convergence to the Morozov discrepancy principle is based on their ratio of χ^2 of the current iteration to the total number of depth increments, smaller or larger adjustments are made depending on their relative 'distance' from the optimal value. This allows convergence to the Morozov discrepancy principle in each strain direction through the use of the diagonal regularization parameter matrix. The optimal values for the diagonal regularization parameters for a given standard strain error are unique and a minimum is found for the least-squares solution of Tikhonov regularization with the second derivative penalty term. However, despite the use of the Morozov discrepancy principle it may be necessary to adjust the degree of regularization applied if noise artefacts and local curvature in the stress solution remain or if distortion of the stress solution occurs since the standard strain error is not exactly known and only estimated. Nonetheless, the Morozov discrepancy principle can be used to determine an initial estimate of the optimal degree of regularization to apply. This initial estimate might, however, make it difficult to know whether the solution is somewhat under-or over-regularized. An effective approach when performing Monte Carlo simulation to estimate uncertainty, is to use regularization parameters of say 50%

of that indicated by the Morozov discrepancy principle and gradually increase them while observing the behaviour and roughness of the stress distributions calculated from perturbed strain data. As the degree of regularization is gradually increased, the mean solution should remain relatively unaffected overall except for fine smoothing of the stress through removal of local curvature. This is accompanied by a gradual reduction in variance until a degree of regularization is reached where the first signs of the induced bias becoming the dominant source of error in the solution is observed through deviation of the stress solution. This can be achieved by inspecting the mean and variance of the stress distributions visually or by statistical means as the degree of regularization is gradually increased.

Demonstration of Methods Using Known Stress

To confirm the accuracy of the proposed method of fully coupled regularization, the method is assessed against a known stress distribution in a complex laminate. For comparison, the method is assessed against the approximate method of regularization that is known to work well in biaxial laminates, but has not been assessed elsewhere. Sinusoidal eigenstrain distributions are applied to a FE model to generate a known stress distribution through the thickness of a laminate, whereafter IHD is simulated to obtain a known released strain variation. The same FE model as that of the forward solution is used to avoid any variability arising from different FE models. A laminate configuration of $[+45/-45/0/90]_s$ was selected for the verification tests. This laminate configuration is arbitrary but allows the accuracy of the proposed method to be assessed in a reasonably complex laminate where the off-diagonal compliance terms are large with respect to the diagonal terms (at least 30% the magnitude). A Gaussian distributed noise with standard deviation of 1 µm/m is added to the known strains in a Monte Carlo simulation [36] using 1000 trials. The approximate and fully coupled regularization methods with 75 depth increments up to 1 mm depth (five plies deep) are used to determine the residual stress distribution for each of the 1000 noisy strain datasets. Only 1000 trials are required to obtain repeatable results since only one uncertainty source is considered. Small depth increments are used for the regularization methods to limit the introduction of bias due to discretisation [17]. This increases the variance of the solution significantly which must be reduced by Tikhonov regularization, allowing the performance of the approximate and fully coupled methods to be assessed more independently of the bias introduced by discretisation.

The standard method with coarser discretisation of twenty depth increments up to the same depth is also included as a baseline comparison. Following the Monte Carlo simulations, the mean stress and variance at every information depth is obtained from the 1000 trials used in the Monte Carlo simulation for each method. The variance is determined from the standard deviation in calculated stress at each depth across all 1000 trials, presented as ±2 standard deviations. Stresses are presented at the mid-depth location rather than piecewise constant functions to improve figure readability.

Modest Stress Variation Throughout the Laminate

Firstly, a modest stress variation throughout the laminate, with a released strain presented in Fig. 3, is considered in Figs. 4, 5 and 6. At optimal levels of regularization, both regularized methods are able to accurately determine the applied stress throughout the drilling depth with 3.75 times greater depth resolution than the standard approach while also reducing the stress variance by a factor of around 3. It was clear, however, that the approximate regularization method is more affected by bias and consequently a slightly lower degree of regularization is used for this method. This results in slightly larger variance compared to the fully regularized method even at a degree of regularization where the bias is still noticeable towards the maximum depth.

Strains corresponding to the regularized solutions versus the known strain variation are not presented since the known strain is free of noise or error and the mean calculated strains of both regularization methods fit the known strains indistinguishably well.

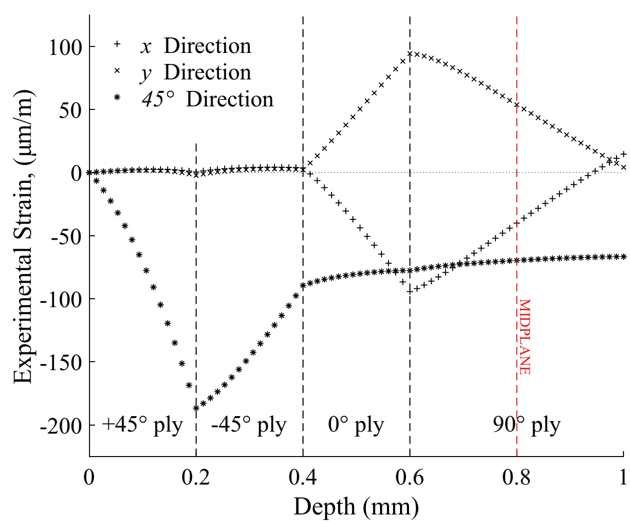

Fig. 3 Known strain variation in the *x*, *y* and *45°* gauge directions in a $[+45/-45/0/90]_s$ laminate

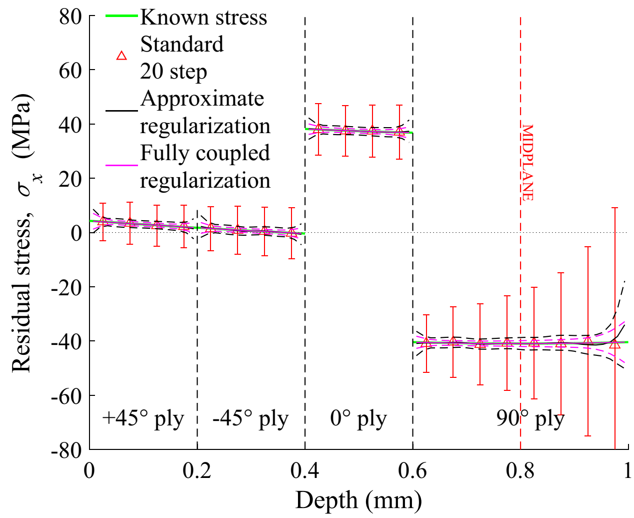

Fig. 4 Comparison of regularized integral solutions of σ_x for a modest stress variation in a [+45/-45/0/90]$_s$ laminate

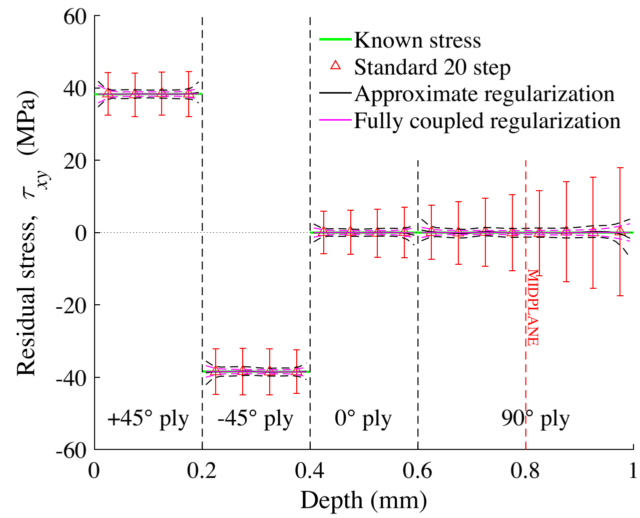

Fig. 6 Comparison of regularized integral solutions of τ_{xy} for a modest stress variation in a [+45/-45/0/90]$_s$ laminate

Near-Surface Stresses Due to Tool/Part Interaction

The manufacturing process of composite laminates can induce significant tool/part stresses near the surface of the laminate during curing and release. These stresses can have a considerable gradient near the surface which dissipates over the thickness of the surface ply, leaving the remaining plies with a relatively constant stress throughout their thickness. A known stress distribution was generated where there is significant stress variation in the first ply to simulate tool/part interaction and little to no variation in the remaining plies. This also serves as a test for the regularization methods to determine if they are able to correctly regularize the solution when there is a combination of steep stress gradients near the surface followed by constant stresses as presented in Fig. 7 for the x-direction only since the other directions show the same behaviour. Again, both regularization methods show very good correlation with the known stress variation and perform between 3 and 4 times better than the standard method in terms of both the depth resolution and the stress variance.

Capturing Significant Stress Variation Within each Ply

To further verify the performance of the regularization methods, a known stress distribution with significant stress variation through the thickness of each individual ply is generated in Fig. 8. Although simple theoretical models indicate

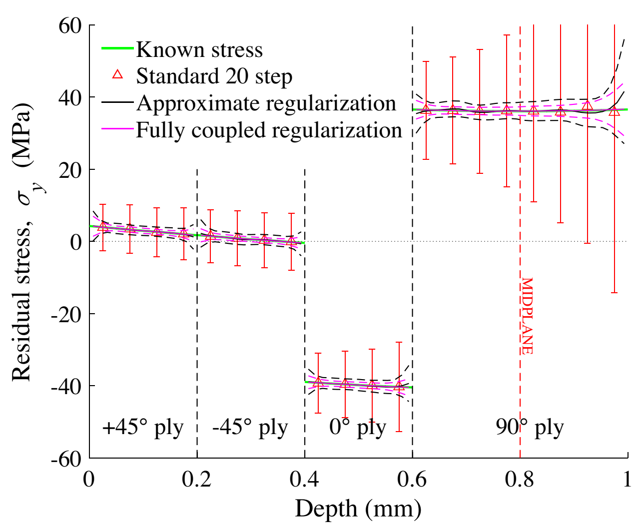

Fig. 5 Comparison of regularized integral solutions of σ_y for a modest stress variation in a [+45/-45/0/90]$_s$ laminate

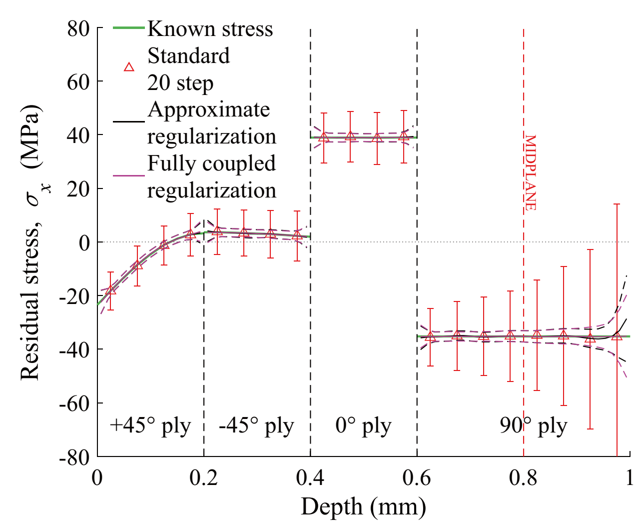

Fig. 7 Comparison of regularized integral solutions of σ_x for significant stress variation in the first ply in a [+45/-45/0/90]$_s$ laminate

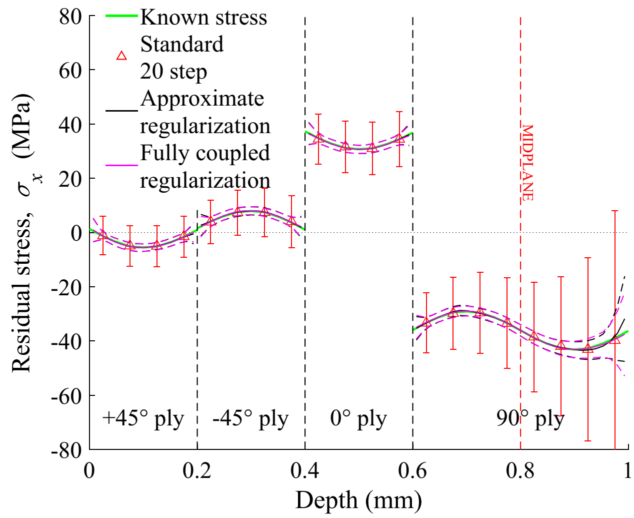

Fig. 8 Comparison of regularized integral solutions of σ_x for significant stress variation within each ply in a $[+45/-45/0/90]_s$ laminate

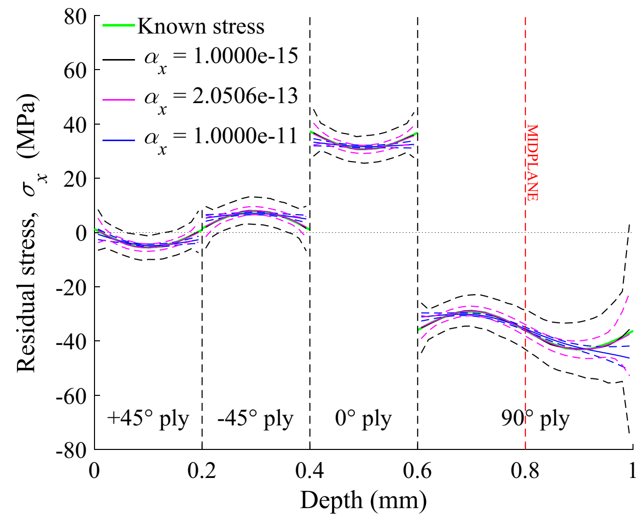

Fig. 9 Effect of under- and over-regularization vs. the optimal degree of regularization of the fully coupled method

such variation to be unlikely, it is however, commonly found in practice [29, 30]. Both methods still perform extremely well throughout the thickness.

The approximate regularization approach is found to produce acceptably accurate residual stress measurements in this particular laminate which compare well to those of the fully regularized method. However, even at optimal regularization levels it can be seen in Figs. 4, 5, 6, 7 and 8 that the stress results of this method start to deviate from the known stress at maximum depth. While this is mostly insignificant in these cases, it does illustrate the error introduced by not performing regularization on the coupled system and applying regularization to the strain in each gauge direction using only the calibration coefficients of the primary stress associated with that direction. A different stress is therefore initially solved and smoothed by the regularization in equation (5) than the actual stress state existing in the material. This smooth, but incorrect, stress state is then used to obtain regularized strains in each direction that are subsequently used to determine the regularized stress solution using the fully coupled method of equations (1) and (2). Despite this error, the approximate method works well in most cases, but the error should be recognised and care should be taken when using this approach. Despite the better than anticipated performance of the approximate regularization approach in a complex laminate, the fully regularized method is superior and more accurate overall over the entire depth of measurement and will be the only method considered hereafter.

Effect of Under-and Over-Regularization of the Fully Coupled Method

Determining the optimal degree of regularization to apply is crucial to the quality of the measurement of the residual stress distribution. Insufficient regularization does not remove all the noise artefacts from the strain data, resulting in large stress variance or even instabilities in the stress solution. Under-regularization obtains the true solution but has large variance. Over-regularization reduces variance but removes true strains from the data and introduces a bias into the over-smoothed stress solution that deviates from the true stress solution. Optimal regularization balances these effects to achieve a good compromise between variance and bias to obtain the best estimate of the stress distribution with minimal variance. The effect of under- and over-regularization of the fully coupled method is presented in Fig. 9 for the *x*-direction. The behaviour in *y* and shear directions is similar. The optimal α_x, α_y and α_{45° parameters were found to be 2.05×10^{-13}, 1.93×10^{-13} and 5.11×10^{-13} respectively. While the α_x and α_y parameters are similar, α_{45° differs significantly since the shear stresses are constant in each ply, but the use of the diagonal parameter matrix allows the optimal α_{45° to be found.

Effect of Increasing Noise on the Fully Coupled Method

The effect of increasing measurement uncertainty on the fully regularized method is presented in Fig. 10 for Gaussian distributed noise with standard deviations of 1 μm/m, 3 μm/m and 5 μm/m. In practice, measurement noise should be minimised by meticulous experimental technique and not reach levels of 5 μm/m which is greater than 10% of the average magnitude of the released strains across all directions and depths in this case. However, despite this level of noise, the fully regularized method is able to accurately determine the residual stress distribution, with a stress variance that enables the measurement to be meaningful.

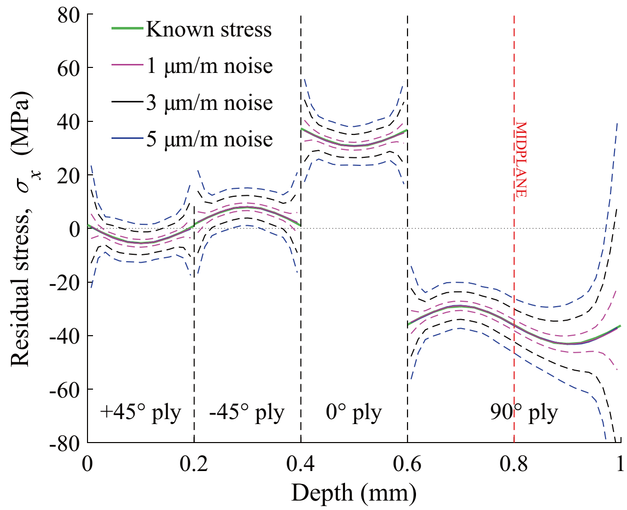

Fig. 10 Effect of increasing noise on the fully coupled regularization method

Experimental

Following successful demonstration of the accuracy of the fully coupled regularization method using known stress and strain from FE simulations, it is important that the method be compared in a real world application to another method beyond the standard fully coupled integral method with limited depth increments used thus far. The series expansion method considers all stress and strain component contributions simultaneously in a least-squares solution and so provides a robust comparison for the results of the fully coupled regularization method. Layered composite materials have discontinuous stress distributions at the interfaces between plies which are difficult to describe by a least-squares fit of a single series, even at high orders. This prevents application of the standard series expansion method in composite materials. These materials can be investigated with IHD through the use of separate, lower order, series expansion in each ply to achieve a good fit to the experimental strain data. The comprehensive method on the use of separate power series expansion with IHD in composite materials is described by Smit and Reid [30] and is, therefore, only briefly summarised here. Stress variations induced by applied power series eigenstrain functions in the forward solution are used to create a stress matrix which allows discontinuities in stress to be captured on each side of a ply interface. The separate series expansion method has been successfully compared against neutron diffraction in a fibre metal laminate of [0/90/steel]$_s$ construction [37].

A laminate configuration of [0/45/90/-45]$_s$ was arbitrarily selected for the experimental case. This configuration is, however, substantially different from the previous laminate where a known stress was used. This is to ensure and demonstrate that the method and optimisation process applies to any configuration and is not inadvertently tuned to perform well where the stress is known and subsequently applied to an experimental case of the same laminate configuration.

Specimen

The [0/45/90/-45]$_s$ composite plate was manufactured from E-glass/epoxy using prepreg material with $V_f \approx 60\%$ and ply thickness of ≈ 200 µm. The cured plate had no noticeable curvature or voids and a thickness of 1.6 mm. Further details around the manufacturing process and orthotropic lamina properties can be found in previous works [29, 30].

IHD

The test specimen was laid flat on a surface and drilled without any other constraints being applied to avoid introducing mounting stresses. To avoid any movement during drilling, a specimen with dimensions of 75 mm × 170 mm was used such that it had sufficient inertia. An HBM foil strain gauge rosette of type 1.5/350M RY61 with 6 strain gauge grids was used. Strain gauges with a resistance of 350 Ω and a gauge excitation voltage of 1.5 V were selected to reduce heating effects on the low-conductivity material whilst achieving the desired sensitivity. Further details regarding the experimental procedure can be found in previous works [29, 37].

Sixty constant depth increments were used up to the midplane, and an additional fifteen increments past the midplane (up to the end of the layer beyond the midplane). The experimentally measured strain variations with depth at each strain gauge location are presented in Fig. 11. The use of small

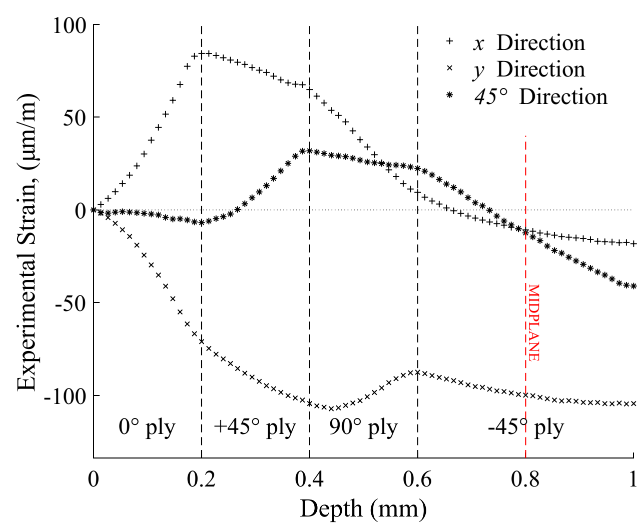

Fig. 11 Measured strain variation in the x, y and $45°$ gauge directions obtained from IHD in a [0/45/90/-45]$_s$ laminate

depth increments not only reduced the introduction of heat into the specimen during drilling but also provided sufficient data for the series expansion method [21] for a fair comparison with the regularized fully coupled integral method which also benefits from larger data sets [17]. The small depth increments also provided a good data set for interpolation of measurements at the depths of the more coarsely discretised 20 step integral method. IHD was conducted beyond the midplane to provide additional data points for series expansion as these additional data points help constrain the series near their end points where greater instability and consequently greater variance in stress can occur [38]. This approach therefore reduced the overall uncertainty in the measured stress distribution at the midplane [29, 38].

Computational

MSC Nastran was used to determine the calibration coefficients required for both IHD computational methods. Calibration coefficients for the integral and series expansion methods were determined from the same FE model to negate variability in the results. The FRP laminate was modelled using HEX20 type 3D elements with 32 elements through the thickness, 4 elements through each ply [29]. The hole diameter measured after IHD was used in the FE model. Due to the material asymmetries, a full model was used with boundary conditions to prevent rigid body motion. Further details regarding the application of the basis functions of each method to the FE model and the calculation of the required calibration coefficients from the FE forward solutions are outlined in previous works [29, 37].

The calibration matrices for the regularized fully coupled integral and series expansion methods must be populated with data corresponding to the 75 experimental depth increments used. Therefore, the calibration coefficients for the 75 experimental depth increments are interpolated from the twenty depth increments used in the FE calculation. The coefficients follow smooth variations with depth, with discontinuities (in slope) at the interfaces between different ply orientations. A separate spline was accordingly fitted to the FE calibration coefficients within each ply orientation to enable these interpolated coefficients to be found.

Propagation of Uncertainties

Only the dominant experimental and computational uncertainty sources [29, 39] were considered in the Monte Carlo simulation of the IHD experiment in the $[0/45/90/-45]_s$ laminate, these are provided in Table 1. Uncertainties due to material properties were not included for simplicity since they affect the integral and series expansion computational methods similarly [29].

Zero depth detection is essential for IHD residual stress measurement and is one of the largest sources of error in the stress solution. The approach used in this work is as described by Smit and Reid [29], where the drilling process is started slightly above the specimen. The first few depth increments do not measure any meaningful strain while the end mill is in the air or within the thickness of the strain gauge. Once the end mill has penetrated the specimen, a strain response is observed in at least one of the gauge directions due to the presence of residual stresses near the surface. However, it is not known if this occurs just as the end mill makes contact with the specimen at the end of an increment or if the end mill was a full depth increment into the specimen. To be conservative, the zero depth uncertainty is set to the size of a full depth increment. After the IHD experiment, the strain variations were investigated and shifted along the x axis to find the most likely position of the surface. The discontinuities in slope at the interfaces between ply orientations were also used to aid the determination of the zero depth position.

Within each Monte Carlo trial, the measured strains are all scaled using the same random variable for the indicated strain uncertainty as they are considered fully correlated [41]. Thereafter, the depth of each strain measurement is adjusted for uncertainty in incremental depth, and all measurements are adjusted for the uncertainty in zero depth position. Spline interpolation is used to determine the strain data, referenced to the zero depth of that trial, at the relevant depths for the inverse solution of each method considered. Some additional bias is introduced by the use of spline interpolation, but this is negligible in comparison with the uncertainty associated with the depth position of each increment. Finally, uncertainty due to strain measurement noise is included for each interpolated strain datum.

Table 1 Sources of uncertainty and their assigned probability density functions

x_i	Description	$p(x_i)$ [40]	Type [40]	Nominal value, uncertainty
h_0	Zero depth	Rectangular	B	0 µm, 13.33 µm
h_i	Incremental depths	Rectangular	B	13.33 µm, 0.50 µm
ϵ_{meas}	Indicated experimental strain	Normal	B	Fig. 11, 1.54%
ϵ_{noise}	Experimental noise	Normal	A	Fig. 11, 0.61 µm/m
FE	Finite element calculations	Normal	B	0, 2%

The regularized integral method and series expansion both contain an inherent bias since only an approximation of the original inverse problem is solved. The calculated strains corresponding to the residual stress solutions do not exactly match the experimental strain data, a so called 'model error' [21]. The induced bias cannot be determined directly but its effect must be estimated and included in the uncertainty analysis. Prime and Hill [21] proposed that this uncertainty be estimated as the standard deviation of the misfit between the experimentally measured strains and those calculated by the least-squares fit of series expansion. This is done separately within each ply orientation for the x, y and 45° directions [30]. The same approach was used for the calculated strains produced by the regularized fully coupled integral method so that both methods are treated similarly for comparison purposes. A similar approach for Tikhonov regularization has been proposed by Olson et al. [42, 43].

FE calculations are not able to fully represent all the deformations that are possible in reality. The calibration coefficients were all varied by the same random variable within each Monte Carlo trial since the matrices are considered to be fully correlated [41].

The residual stress field was determined for each Monte Carlo trial using equation (1) for the standard fully coupled integral method with 20 steps and equation (13) for the regularized method. The mean stress and variance at every information depth is determined as for the earlier known stress cases. Ten thousand trials are used in this Monte Carlo simulation since multiple uncertainty sources are considered and a sufficient number of trials must be used to allow interaction between the different noisy inputs to estimate their combined effect on the stress distribution variance.

Results and Discussion

The residual stress distributions determined using the regularized fully coupled integral method and the standard integral method (discretised to 20 steps) are overlaid with those determined using series expansion in Figs. 12, 13 and 14. The standard integral method is discretised to 20 steps to yield stable stress results as a baseline comparison. A coarser discretisation would not allow the rapid stress variations to be properly captured due to averaging effects while a finer discretisation would lead to meaninglessly large variance and somewhat erratic behaviour. In contrast to the standard method, the regularized method benefits from an increased number of depth increments and produces stress results with both good resolution and considerably lower uncertainty over the measurement depth. The general trend of the integral

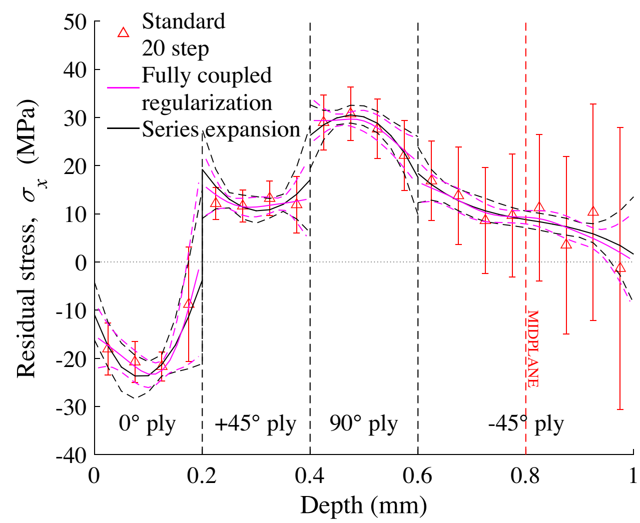

Fig. 12 σ_x distributions and associated uncertainties of each method in a $[0/45/90/\text{-}45]_s$ laminate

methods match each other well. The residual stress distributions of the regularized fully coupled integral method compare favourably with those obtained by separate series expansion. Although the correlation is not perfect (which is expected considering they are based on two entirely different approaches) they are clearly representing the same general trends.

The regularized and least-squares fits to the measured strain data in the x-direction are presented in Fig.15 for the regularized integral method and series expansion, respectively. The curve fits from both methods correctly capture the discontinuity in slope between plies of different orientation. The curve fits to the measured strain data are nearly indistinguishable from each other except for minor discrepancies near the interfaces between plies. This demonstrates that the fully regularized integral method is working well compared to series expansion which directly incorporates all stress and strain components simultaneously in its least-squares curve fitting.

The optimal α_x, α_y and $\alpha_{45°}$ regularization parameters were determined as 1.88×10^{-12}, 1.69×10^{-12} and 5.53×10^{-12}, respectively. These regularization parameters are much smaller than those recommended in the ASTM E837 standard for isotropic materials (10^{-6} – 10^{-4}) simply because the calibration matrices in the ASTM are based on a normalised Young's modulus of 1 MPa and therefore have much higher values with the actual material properties being multiplied in during the inverse solution.

Regularization of the fully coupled method is predictable and stable in its behaviour in the sense that over-regularization simply flattens the stress solution while reducing the variance, indicating the bias becoming the dominant source of error. This effect is, however, noticeable when the stress

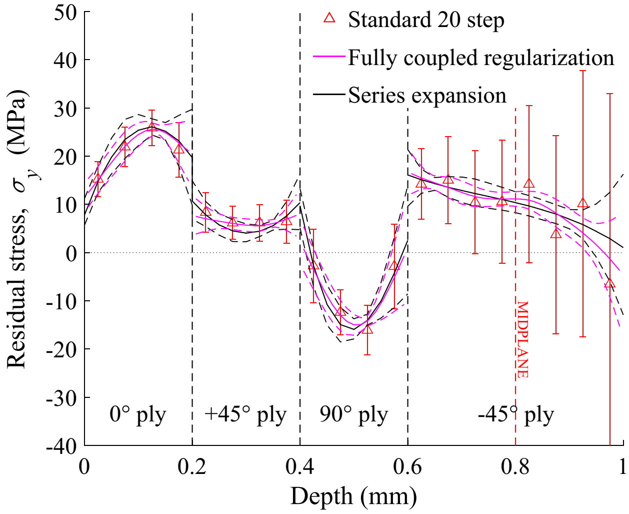

Fig. 13 σ_y distributions and associated uncertainties of each method in a $[0/45/90/\text{-}45]_s$ laminate

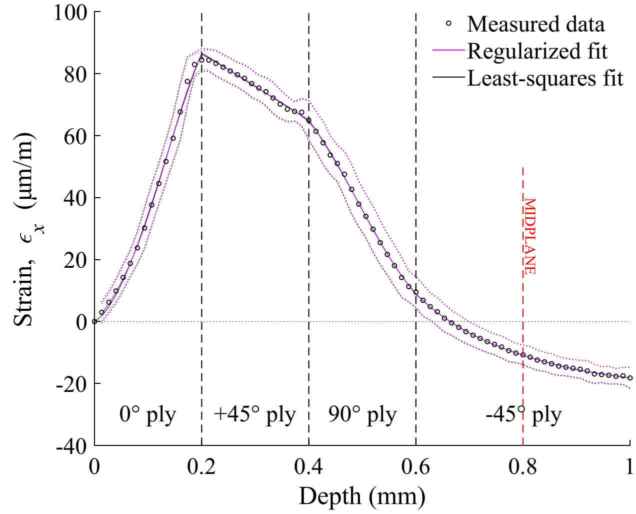

Fig. 15 Regularized and least-squares fits to the measured strain data in the x-direction from IHD in a $[0/45/90/\text{-}45]_s$ laminate

distributions and accompanying variance are plotted with increasing regularization. It then becomes comparatively easy to determine the degree of regularization at which the bias starts to become dominant. As is evident in Fig. 9, increasing the regularization decreases the variance without significant changes to the solution apart from removing low magnitude oscillations. At some point the shape of the solution starts to deviate and it is just before this point that optimal regularization is found, where the variance is minimised without the bias significantly influencing the underlying solution. While the Morozov discrepancy principle is used as an initial guide to determine the first estimate of the optimal regularization parameters, the stress solution and variance for a range of increasing regularization parameters are considered in determining the optimal degree of regularization.

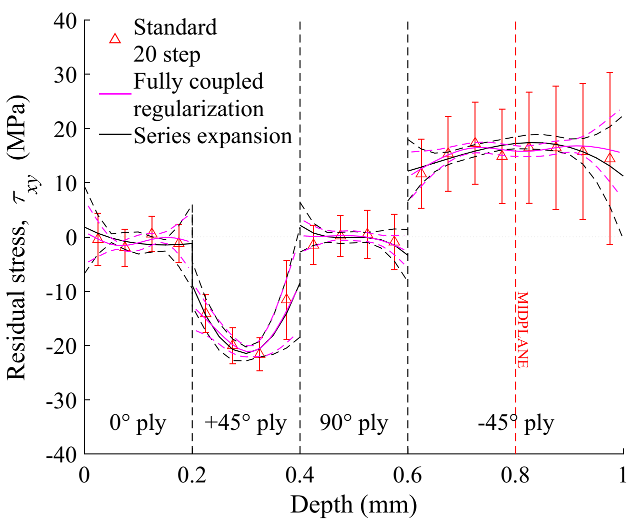

Fig. 14 τ_{xy} distributions and associated uncertainties of each method in a $[0/45/90/\text{-}45]_s$ laminate

In this experimental case it was clear that the shear stress required a higher degree of regularization, especially near the surface. The α_{45° regularization parameter was consequently progressively increased to gradually reduce variance and local curvature without introducing significant bias in the shear stress solution, or in the x and y directions. The effect of increasing the α_{45° regularization parameter on the x and y stress distributions is accounted for in the fully coupled regularization which considers all components simultaneously. If this had been attempted with only a single regularization parameter for all stress components, the x and y stress distributions would have become over-regularized or the shear stress would have been under-regularized. The use of a diagonal matrix for the regularization parameters avoids a compromise between under- and over-regularization in different directions. In this work, the regularization parameters have been kept constant with depth throughout all plies but there is potential scope for further optimization of the regularization parameters by adjusting equation (12) to allow for changes in the degree of regularization applied to each ply orientation, similar to the approach used in the separate series expansion method where the series order can vary from ply to ply.

While a table comparing the stress uncertainty induced by each input uncertainty source would usually be included, it is omitted since the relative contribution of bias towards the solution uncertainty for each method is unknown. This renders such a comparison meaningless [17]. Consequently it is debatable whether the regularized fully coupled integral method or the separate series expansion method produces superior stress results in this particular experiment because the true solution is not known, but it is clear that the results are comparable throughout the entire measurement depth. The fully coupled regularized integral method requires 12 times fewer FE computations, however, to calculate the compliances and since it does not require experimental drilling up to a particular depth, it is more practical in general application. The optimal degree of regularization is also more convenient to determine than the optimal combination of series order in each ply orientation required by series expansion.

Conclusions

The approximate regularization method performs well in all known stress cases considered, but deviates somewhat from the correct solution at maximum depth and may be susceptible to errors in other cases. Tikhonov regularization of the fully coupled integral method of IHD matched the known stress solution extremely accurately and outperformed the approximate solution in the composite laminate considered in this work. Both regularized approaches significantly improve upon the standard approach. The regularized fully coupled integral solution closely matches the series expansion solution in the experimental example presented. Since the series expansion solution employs a least-squares fit of all stress and strain components simultaneously, this close match provides reassuring evidence of the accuracy of the regularized fully coupled integral method presented in this work. The standard fully coupled integral method, the approximate regularization method and the regularized fully coupled method all require the fully coupled calibration matrix and so, once this calibration matrix is obtained, the additional effort required to implement fully coupled regularization is negligible. Fully coupled regularization should consequently be used in general as there is little downside to this method. The regularized integral method requires significantly fewer FE computations compared to separate series expansion. Furthermore, optimal values only need to be found for three regularization parameters instead of an optimal order combination from a large number of possible order combinations that grows rapidly with laminate complexity in the case of separate series expansion. Optimal regularization parameters can be found simultaneously for the fully coupled method. The solution is a true and unique minimum of the objective function for a chosen admissible squared discrepancy between the calculated and measured strains. The new method is easily applied and should find wide application in the measurement of residual stresses in composite laminates and in isotropic materials where the ASTM standard cannot currently be applied.

Funding Open access funding provided by University of the Witwatersrand.

Declarations

Informed Consent Not applicable.

Conflict of Interest The authors declare that they have no conflict of interest.

Statements on Human and Animal Rights Not applicable.

References

1. Barnes JA, Byerly GE (1994) The formation of residual stresses in laminated thermoplastic composites. Compos Sci Technol 51(4):479–494. ISSN 02663538. https://doi.org/10.1016/0266-3538(94)90081-7
2. Regester RF (1969) Behavior of Fiber Reinforced Plastic Materials in Chemical Service. Corrosion 25(4):157–167. ISSN 0010-9312. https://doi.org/10.5006/0010-9312-25.4.157
3. Garstka T, Ersoy N, Potter KD, Wisnom MR (2007) In situ measurements of through-the-thickness strains during processing of AS4/8552 composite. Compos A Appl Sci Manuf 38(12):2517–2526. ISSN 1359-835X. https://doi.org/10.1016/j.compositesa.2007.07.018
4. Shokrieh MM, Akbari S, Daneshvar A (2013) A comparison between the slitting method and the classical lamination theory in determination of macro-residual stresses in laminated composites. Compos Struct 96:708–715. ISSN 02638223. https://doi.org/10.1016/j.compstruct.2012.10.001
5. Parlevliet PP, Bersee HEN, Beukers A (2007) Residual stresses in thermoplastic composites - a study of the literature. Part III: Effects of thermal residual stresses. Compos A Appl Sci Manuf 38(6):1581–1596. ISSN 1359835X. https://doi.org/10.1016/j.compositesa.2006.12.005
6. Safarabadi M, Shokrieh MM (2014) Residual stresses in composite materials, Chapter 8, pages 197–232. Woodhead Publishing Limited. ISBN 9780857092700. https://doi.org/10.1533/9780857098597.2.197
7. Awaja F, Zhang S, Tripathi M, Nikiforov A, Pugno N (2016) Cracks, microcracks and fracture in polymer structures: Formation, detection, autonomic repair. Prog Mater Sci 83:536–573. ISSN 0079-6425. https://doi.org/10.1016/j.pmatsci.2016.07.007
8. Schajer GS (Ed) (2013) Practical Residual Stress Measurement Methods. John Wiley & Sons, Ltd. ISBN 9781118402832. https://doi.org/10.1002/9781118402832
9. Kandil F, Lord D, Fry A (2001) A review of residual stress measurement methods: A guide to technique selection. Technical report, National Physical Laboratory Materials Centre, Teddington, UK
10. Schajer GS (2010) Hole-Drilling Residual Stress Measurements at 75: Origins, Advances, Opportunities. Exp Mech 50(2):245–253. ISSN 1741-2765. https://doi.org/10.1007/s11340-009-9285-y
11. Benedikt B, Predecki PK, Kumosa L, Rupnowski P, Kumosa M (2002) Measurement of residual stresses in fibre reinforced composites based on X-ray diffraction. Adv X-ray Anal 45(c):218–224

12. Niku-Lari A, Lu J, Flavenot JF (1985) Measurement of residual-stress distribution by the incremental hole-drilling method. J Mech Work Technol 11(2):167–188. ISSN 03783804. https://doi.org/10.1016/0378-3804(85)90023-3
13. Prime MB (1999) Residual Stress Measurement by Successive Extension of a Slot: The Crack Compliance Method. Appl Mech Rev 52(2):75. ISSN 00036900. https://doi.org/10.1115/1.3098926
14. Treuting RG, Read WT (1951) A mechanical determination of biaxial residual stress in sheet materials. J Appl Phys 22(2):130–134. ISSN 00218979. https://doi.org/10.1063/1.1699913
15. Scafidi M, Valentini E, Zuccarello B (2011) Error and uncertainty analysis of the residual stresses computed by using the hole drilling method. Strain 47(4):301–312. ISSN 00392103. https://doi.org/10.1111/j.1475-1305.2009.00688.x
16. Schajer GS (1981) Application of Finite Element Calculations to Residual Stress Measurements. J Eng Mater Technol 103(2):157. ISSN 00944289. https://doi.org/10.1115/1.3224988
17. Beghini M, Grossi T, Prime MB, Santus C (2023) Ill-Posedness and the Bias-Variance Tradeoff in Residual Stress Measurement Inverse Solutions. Exp Mech 63(3):495–516. https://doi.org/10.1007/s11340-022-00928-5
18. Schajer GS (1988) Measurement of non-uniform residual stresses using the hole-drilling method. Part II. Practical application of the integral method. J Eng Mater Technol Trans ASME 110(4):344–349. ISSN 00944289. https://doi.org/10.1115/1.3226059
19. ASTM E837-20 (2020) Standard Test Method for Determining Residual Stresses by the Hole-Drilling Strain-Gage Method. ASTM International, West Conshohocken, PA. www.astm.org
20. Schajer GS, Yang L (1994) Residual-stress measurement in orthotropic materials using the hole-drilling method. Exp Mech 34(4):324–333. ISSN 00144851. https://doi.org/10.1007/BF02325147
21. Prime MB, Hill MR (2006) Uncertainty, Model Error, and Order Selection for Series-Expanded, Residual-Stress Inverse Solutions. J Eng Mater Technol 128(2):175. ISSN 00944289. https://doi.org/10.1115/1.2172278
22. Oettel R (2000) The Determination of Uncertainties in Residual Stress Measurement (Using the hole drilling technique). Code of Practice 15, Issue 1, EU Project No. SMT4-CT97-2165 (1):18. ISSN 00392103. https://doi.org/10.1007/bf02327502
23. Schajer GS, Altus E (1996) Stress calculation error analysis for incremental hole-drilling residual stress measurements. J Eng Mater Technol Trans ASME 118(1):120–126. ISSN 0094-4289. https://doi.org/10.1115/1.2805924
24. Zuccarello B (1999) Optimal calculation steps for the evaluation of residual stress by the incremental hole-drilling method. Exp Mech 39(2):117–124. ISSN 0014-4851. https://doi.org/10.1007/BF02331114
25. Tikhonov AN, Goncharsky AV, Stepanov VV, Yagola AG (1995) Numerical Methods for the Solution of Ill-Posed Problems. Kluwer, Dordrecht
26. Akbari S, Taheri-Behrooz F, Shokrieh MM (2014) Characterization of residual stresses in a thin-walled filament wound carbon/epoxy ring using incremental hole drilling method. Compos Sci Technol 94:8–15. ISSN 02663538. https://doi.org/10.1016/j.compscitech.2014.01.008
27. Valentini E, Beghini M, Bertini L, Santus C, Benedetti M (2011) Procedure to perform a validated incremental hole drilling measurement: Application to shot peening residual stresses. Strain 47(s1):e605–e618. https://doi.org/10.1111/j.1475-1305.2009.00664.x
28. Nobre JP, Polese C, van Staden SN (2020) Incremental hole drilling residual stress measurement in thin aluminum alloy plates subjected to laser shock peening. Exp Mech 60(1):553–564. https://doi.org/10.1007/s11340-020-00586-5
29. Smit TC, Reid RG (2020) Tikhonov regularization with Incremental Hole-Drilling and the Integral Method in Cross-Ply Composite Laminates. Exp Mech 60(8):1135–1148. ISSN 17412765. https://doi.org/10.1007/s11340-020-00629-x
30. Smit TC, Reid RG (2018) Residual Stress Measurement in Composite Laminates Using Incremental Hole-Drilling with Power Series. Exp Mech 58(8):1221–1235. ISSN 17412765. https://doi.org/10.1007/s11340-018-0403-6
31. Prime MB, Crane DL (2014) Slitting method measurement of residual stress profiles, including stress discontinuities, in layered specimens. Conf Proc Soc Exp Mech Series 8:93–102. https://doi.org/10.1007/978-3-319-00876-9_12
32. Morozov VA (1984) Methods for solving incorrectly posed problems. Springer, New York. https://doi.org/10.1007/978-1-4612-5280-1
33. Smit TC (2021) Extension of the Incremental Hole-Drilling Technique for Residual Stress Measurement. PhD thesis, University of theWitwatersrand, Johannesburg
34. Schajer GS, Prime MB (2006) Use of Inverse Solutions for Residual Stress Measurements. J Eng Mater Technol 128(3):375. ISSN 00944289. https://doi.org/10.1115/1.2204952
35. Akbari S, Taheri-Behrooz F, Shokrieh MM (2013) Slitting measurement of residual hoop stresses through the wall-thickness of a filament wound composite ring. Exp Mech 53:1509–1518. https://doi.org/10.1007/s11340-013-9768-8
36. BIPM, IEC, IFCC, ILAC, IUPAC, IUPAP, ISO, OIML (2008a) Evaluation of measurement data – Supplement 1 to the “Guide to the expression of uncertainty in measurement” – Propagation of distributions using a Monte Carlo method. JCGM 101:2008
37. Smit TC, Nobre JP, Reid RG, Wu T, Niendorf T, Marais D, Venter AM (2022) Assessment and Validation of Incremental Hole-Drilling Calculation Methods for Residual Stress Determination in Fiber-Metal Laminates. Exp Mech 62(8):1289–1304. https://doi.org/10.1007/s11340-022-00848-4
38. Reid RG, Smit TC (2023) Evaluation of the Maximum Allowable Hole Depth for IHD of GFRP Laminates. In Rachael C. Tighe, John Considine, Sharlotte L.B. Kramer, and Tom Berfield, editors, Thermomechanics & Infrared Imaging, Inverse Problem Methodologies and Mechanics of Additive & Advanced Manufactured Materials 6:1–8, Cham. Springer International Publishing. ISBN 978-3-031-17475-9
39. Smit TC, Reid RG (2020) Use of Power Series Expansion for Residual Stress Determination by the Incremental Hole-Drilling Technique. Exp Mech 60(9):1301–1314. ISSN 17412765. https://doi.org/10.1007/s11340-020-00642-0
40. BIPM, IEC, IFCC, ILAC, IUPAC, IUPAP, ISO, OIML (2008b) Evaluation of measurement data–Guide for the expression of uncertainty in measurement. JCGM 100:2008
41. Peral D, de Vicente J, Porro JA, Ocaña JL (2017) Uncertainty analysis for non-uniform residual stresses determined by the hole drilling strain gauge method. Meas J Int Meas Conf 97:51–63. ISSN 02632241. https://doi.org/10.1016/j.measurement.2016.11.010
42. Olson MD, DeWald AT, Hill MR (2020) An Uncertainty Estimator for Slitting Method Residual Stress Measurements Including the Influence of Regularization. Exp Mech 60(1):65–79. https://doi.org/10.1007/s11340-019-00535-x
43. Olson MD, DeWald AT, Hill MR (2021) Precision of Hole-Drilling Residual Stress Depth Profile Measurements and an Updated Uncertainty Estimator. Exp Mech 61(3):549–564. https://doi.org/10.1007/s11340-020-00679-1