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Novel Linear Decodable QO-STBC for Four Transmit Antennas with Transmit Antenna Shuffling

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Abstract A conventional quasi orthogonal space time block code (QO-STBC) scheme can achieve full rate, but at the cost of decoding complexity. This limitation of the conventional QO-STBC scheme is mainly due to interference terms in the detection matrix. In this article, a novel QO-STBC scheme is proposed which eliminates the interference terms. The proposed method achieves improved diversity as compared to the conventional QO-STBC scheme, also providing a considerable reduction in decoding complexity. A transmit antenna shuffling scheme for the proposed code is also illustrated. It is shown that by adaptively mapping space time sequences of the proposed code to appropriate transmit antennas depending on channel condition, proposed scheme can improve its transmit diversity with limited feedback information. Lastly, simulation results show that the symbol error rate performance is improved considerably.

Keywords Space time block code · QO-STBC · Transmit diversity · Zero-forcing decoding · Antenna shuffling and simple linear decoding

1 Introduction

Multiple input multiple output (MIMO) system has been studied extensively in over past few years as a method of combating impairments in wireless fading channel. In order to approach the capacity of MIMO systems, space time coding has received the significant amount of attention. In 1998, Alamouti [1] proposed the transmit diversity scheme which is usually regarded as the first space time coding with two transmit antennas and has been used in 3G cellular communications. This scheme is a complex orthogonal design with full rate and full diversity and allows for a simple linear maximum likelihood decoding. These kinds of

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space time coding are known as orthogonal space time block codes (O-STBC). However, it is demonstrated that the complex orthogonal full rate design, offering full diversity, was limited to the case of two transmit antennas [2]. When three or four transmit antennas were considered, the maximum symbol transmission rate of the complex O-STBCs with linear processing was 3/4. In order to achieve the advantages of O-STBC schemes with properties close to such optimal codes providing full rate, the so called QO-STBCs were proposed [3]. These space time block codes (STBC) were developed from quasi-orthogonal designs, where orthogonality is relaxed to provide higher rate. A QO-STBC scheme can achieve full rate, but it also adds interference terms resulting from neighboring signals during signal detection. These results in increase in decoding complexity and a decrease in performance gain, with respect to the O-STBC schemes. Two maximum likelihood detectors are used in parallel to decode pairs of transmitted symbols in QO-STBC, which results in higher complexity decoding at the receiver. With increase in modulation level, receiver computes the decision metric over larger number of symbols in the constellation [6], which subsequently increases decoding complexity and hence increases transmission delay.

In this paper, based on the quasi-orthogonal code structures in the Jafarkhani scheme, an efficient QO-STBC scheme for four transmit antennas is proposed, which can achieve both the full rate and simple linear decoding. Using symmetric nature of detection matrix of conventional QO-STBC scheme, detection matrix is changed accordingly in order to eliminate interference terms, and derive an encoding matrix corresponding to the interference free detection matrix. Later on, a transmit antenna shuffling scheme (TAS) is used for the proposed code. The optimum antenna shuffling pattern can be selected to improve the transmit diversity with limited feedback information during the whole signal transmission.

2 Conventional QO-STBC Scheme

First QO-STBC code was proposed by Jafarkhani [3] for four transmit antenna and one receive antenna, which is given by S (4×4):

$$S = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2^* & s_1^* & -s_4^* & s_3^* \\ -s_3^* & -s_4^* & s_1^* & s_2^* \\ s_4 & -s_3 & -s_2 & s_1 \end{bmatrix} \tag{1}$$

Assuming a flat fading channel over four time slots and the channel gain between the transmit antennas and receiver antenna is denoted by:

$$\overline{H} = [h_1 \quad h_2 \quad h_3 \quad h_4]^T \tag{2}$$

where T stands for transpose operator

Received signal is given by:

$$R = S\overline{H} + N \tag{3}$$

where, $N = [n_1\, n_2\, n_3\, n_4]^T$ and n_i is the complex white Gaussian noise added in the ith time slot.

Equivalent virtual channel matrix (EVCM) for this code, which is formed by after applying a complex conjugate operation to the second and the third elements of the received signal, is given by:

$$\mathrm{H} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \\ h_2^* & -h_1^* & h_4^* & -h_3^* \\ h_3^* & h_4^* & -h_1^* & -h_2^* \\ h_4 & -h_3 & -h_2 & h_1 \end{bmatrix} \tag{4}$$

Now, received signal is expressed as:

$$\mathrm{R_n} = \mathrm{HC_n} + \overline{\mathrm{N}} \tag{5}$$

where, $\mathrm{C_n} = [\mathrm{s_1\ s_2\ s_3\ s_4}]^{\mathrm{T}}, \overline{\mathrm{N}} = [\mathrm{n_1\ n_2^*\ n_3^*\ n_4}]^{\mathrm{T}}$

In case of the orthogonal scheme, the received signals are decoded using a detection matrix D defined as $H^H H$ where H^H is Hermitian of H. For an O-STBC scheme, the detection matrix is always a diagonal matrix and this enables simple linear decoding .However for QO-STBC scheme simple linear decoding cannot be applied because detection matrix is not diagonal. For example, the detection matrix for the aforementioned four transmit antenna QO-STBC scheme is expressed by:

$$\mathrm{D} = \mathrm{H^H H} = \begin{bmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -b & a & 0 \\ b & 0 & 0 & a \end{bmatrix} \tag{6}$$

where, $\mathrm{a} = \sum_{i=1}^{4} |h_i|^2$ represents channel gain for the four transmit antennas, $\mathrm{b} = 2\mathrm{Re}\left(h_1 h_4^* - h_2 h_3^*\right)$ represents interference terms.

For decoding this, a more complex decoding method to detect the estimate $\hat{\mathrm{C}}$ has been introduced (zero forcing decoding), as given by:

$$\hat{\mathrm{C}} = \left(\mathrm{H^H H}\right)^{-1} \mathrm{H^H R_n} \tag{7}$$

3 Proposed QO-STBC Scheme

Due to the presence of interference terms in detection matrix D, more complex decoding is required. In this section, a method to eliminate the interference terms is presented so that simple linear decoding can be applied. As detection matrix D is symmetric, which means $\mathrm{D^T} = \mathrm{D}$ and so, by using property of symmetric matrix, D can be expressed as

$$\mathrm{D} = \mathrm{QD_n Q^T} \tag{8}$$

where, Q is orthogonal matrix and D_n is diagonal matrix whose diagonal elements are eigen values of D. Equation (8) can also be written as

$$\mathrm{Q^T D Q} = \mathrm{D_n} \tag{9}$$

By Eq. (9) it is cleared that by pre and post multiplying the detection matrix D with $\mathrm{Q^T}$ and Q, diagonal matrix $\mathrm{D_n}$ is obtained which is interference free. In Taha Scheme [6], they used the same concept of symmetry of detection matrix but the value of Q was taken as general unitary matrix for constructing their code. But in proposed scheme singular value decomposition is used to derive value of Q (Appendix). So Q is given by:

$$\mathrm{Q} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \end{bmatrix} \tag{10}$$

New interference free detection matrix D_n is given by

$$\mathrm{D_n} = \begin{bmatrix} a+b & 0 & 0 & 0 \\ 0 & a+b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{bmatrix} \tag{11}$$

Then, new channel matrix is derived by D_n as:

$$\begin{aligned} \mathrm{D_n} &= \mathrm{Q^T D\,Q = Q^H D\,Q = Q^H H^H H\,Q} \\ &= \mathrm{(H\,Q)^H\,(H\,Q)} \end{aligned} \tag{12}$$

So, new channel matrix is defined as $\mathrm{H_n = H\,Q}$, which can be expressed as following matrix:

$$\mathrm{H_n} = \begin{bmatrix} (h_1+h_2-h_3+h_4)/2 & (h_1-h_2+h_3+h_4)/2 & (h_1+h_2+h_3-h_4)/2 & (-h_1+h_2+h_2+h_4)/2 \\ (h_2^*-h_1^*-h_4^*-h_3^*)/2 & (h_2^*+h_1^*+h_4^*-h_3^*)/2 & (h_2^*-h_1^*+h_4^*+h_3^*)/2 & (-h_2^*-h_1^*+h_4^*-h_3^*)/2 \\ (h_3^*+h_4^*+h_1^*-h_2^*)/2 & (h_3^*-h_4^*-h_1^*-h_2^*)/2 & (h_3^*+h_4^*-h_1^*+h_2^*)/2 & (-h_3^*+h_4^*-h_1^*-h_2^*)/2 \\ (h_4-h_3+h_2+h_1)/2 & (h_4+h_3-h_2+h_1)/2 & (h_4-h_3-h_2-h_1)/2 & (-h_4-h_3-h_2+h_1)/2 \end{bmatrix} \tag{13}$$

Now, new encoding matrix corresponding to $\mathrm{H_n}$, which is expressed as:

$$\mathrm{S_n} = \begin{bmatrix} (s_1+s_2+s_3-s_4)/2 & (s_1-s_2+s_3+s_4)/2 & (-s_1+s_2+s_3+s_4)/2 & (s_1+s_2-s_3+s_4)/2 \\ (-s_1^*+s_2^*-s_3^*-s_4^*)/2 & (s_1^*+s_2^*+s_3^*-s_4^*)/2 & (-s_1^*-s_2^*+s_3^*-s_4^*)/2 & (-s_1^*+s_2^*+s_3^*+s_4^*)/2 \\ (s_1^*-s_2^*-s_3^*-s_4^*)/2 & (-s_1^*-s_2^*+s_3^*-s_4^*)/2 & (s_1^*+s_2^*+s_3^*-s_4^*)/2 & (s_1^*-s_2^*+s_3^*+s_4^*)/2 \\ (s_1+s_2-s_3+s_4)/2 & (s_1-s_2-s_3-s_4)/2 & (-s_1+s_2-s_3-s_4)/2 & (s_1+s_2+s_3-s_4)/2 \end{bmatrix} \tag{14}$$

The new encoding matrix $\mathrm{S_n}$ is Quasi-orthogonal, but its channel matrix $\mathrm{H_n}$ is orthogonal matrix, so simple linear decoding can be used to reduce decoding complexity at the receiver. The decoding matrix is given by:

$$\hat{\mathrm{C}} = \mathrm{H_n^H H_n C_n + H_n^H N} \tag{15}$$

where $\mathrm{C_n = [s_1\ s_2\ s_3\ s_4]^T}$, $\overline{\mathrm{N}} = \mathrm{[n_1\ n_2^*\ n_3^*\ n_4]^T}$

4 TAS Scheme for Proposed QO-STBC

From Fig. 3, it is apparent that the performance of the proposed QO-STBC scheme is slightly better than Taha scheme [6]. So, to increase the performance of the proposed QO-STBC scheme, transmit antenna shuffling is employed in proposed QO-STBC for four transmit antennas so that optimum antenna shuffling pattern can be selected to improve the transmit diversity. Detection matrix for proposed QO-STBC scheme for four transmit antenna and one receive antenna is given by Eq. (11), where $(a+b)$, $(a+b)$, $(a-b)$ and $(a-b)$ are total channel gains for four transmit antenna respectively. For error free decoding, these channel gains must be more positive. But they are strongly dependent on channel coefficients. As channel gain is described by two factors a and b, where a is always positive quantity but b can be positive large or negative large quantity depending on the channel condition, which in turn decrease the positivity of half of the channel gains. So, for all channel gains, to have consistency in its positivity, value of b should be minimized. It is known that by using shuffling in transmit

Table 1 Six different shuffling having different value of *b*

Index	Six different shuffling	Value of *b*
1	[1A,2,3,4]	2real $(h_1h_4^* - h_2h_3^*)$
2	[1A,2,4,3]	2real $(h_1h_3^* - h_2h_4^*)$
3	[1A,4,3,2]	2real $(h_1h_2^* - h_3h_4^*)$
4	[1B,2,3,4]	−2real $(h_1h_4^* + h_2h_3^*)$
5	[1B,2,4,3]	−2real $(h_1h_3^* + h_2h_4^*)$
6	[1B,4,3,2]	−2real $(h_1h_2^* + h_3h_4^*)$

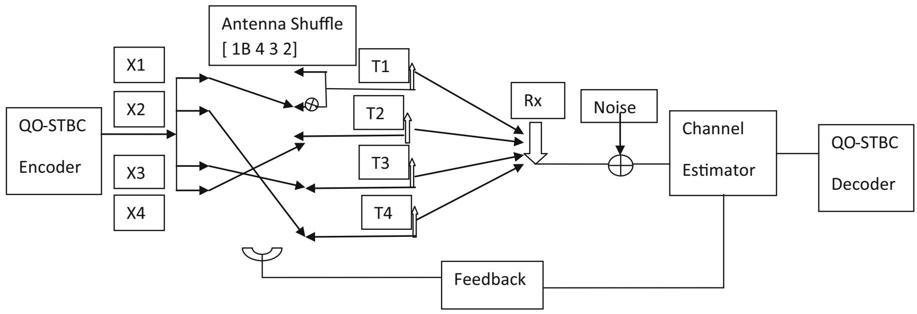

Fig. 1 Proposed closed loop system

antenna, six different shuffled antenna codes is possible for which detection matrices are same but having different values of channel gain because of different value of *b* which are given in Table 1. So, at the transmitter end that shuffled antenna code is used for which $|b|$ has minimum value. Here, six different type of shuffling is used, so three bit feedback is used to tell transmitter about which shuffling undergoes during transmission having minimum value of $|b|$. This in turns increase the performance of proposed QO-STBC scheme. The block diagram of the proposed QO-STBC scheme using transmit antenna shuffling scheme (TAS) [4] scheme with four transmit antennas and one receive antenna is given in Fig. 1. Here, an antenna shuffling structure [1B 4 3 2] between the QO-STBC encoder and four transmit antennas is implemented.

To achieve different values of *b*, the possible number of antenna shuffling patterns for the proposed QO-STBC code is six, which are shown in Table 1.

From Table 1, pattern [1A,2,4,3] means that the four rows of the QOSTBC will be transmitted from antenna 1,2,4 and 3 respectively and pattern [1B,3,4,2] representing that the four rows of the QOSTBC are transmitted from antenna 1(with 180° phase shift before transmission), 3, 4 and 2 respectively.

4.1 Adaptive Antenna Shuffling Algorithm

It is assumed that the CSI information is perfectly known at the receiver and both transmitter & receiver has knowledge of different possible shuffling of the transmit antennas at different channel conditions. Each antenna shuffling is associated with a particular value of channel dependent parameter e.g. *b*. The association of the channel dependent parameter and antenna shuffling is given in Table 1. The steps of the algorithm are:

- Calculate the values of the channel dependent parameter *b* at current channel condition using the channel coefficients. The values of *b* are calculated according to the groupings of channel coefficients as given in Table 1.

- Select the index of the grouping providing min $|b|$ from all calculated $|b|$ values for transmission to the transmitter as feedback.
- The transmitter selects the antenna shuffling associated with the received index.
- The transmitter shuffles the antennas according to the selected antenna shuffling based on minimum value of $|b|$.
- At receiver, multiply the received signal by the hermitian of channel matrix of transmitted shuffled antenna code so that simple linear decoding is possible.

Since, six different type of transmit antenna shuffling is employed, therefore a three bit feedback (which selects the optimum antenna shuffling) is used to achieve minimum value of $|b|$. This in turns increase the performance of the proposed QO-STBC code.

5 Comparison of Conventional QO-STBC Scheme and Proposed QO-STBC Scheme when Both Employing TAS

Estimated transmitted signal for conventional QO-STBC scheme is given by:

$$\begin{bmatrix} \tilde{s}_1 \\ \tilde{s}_2 \\ \tilde{s}_3 \\ \tilde{s}_4 \end{bmatrix} = \begin{bmatrix} as_1 + bs_4 \\ as_2 - bs_3 \\ as_3 - bs_2 \\ as_4 + bs_1 \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \\ \tilde{n}_4 \end{bmatrix}$$

Estimated transmitted signal for proposed QO-STBC scheme is given by:

$$\begin{bmatrix} \tilde{s}_1 \\ \tilde{s}_2 \\ \tilde{s}_3 \\ \tilde{s}_4 \end{bmatrix} = \begin{bmatrix} (a + b)s_1 \\ (a + b)s_2 \\ (a - b)s_3 \\ (a - b)s_4 \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \\ \tilde{n}_4 \end{bmatrix}$$

where $\tilde{s}_i$ is the estimated symbol; s_i is the transmitted symbol and $\tilde{n}_i$ is the noise component

By using the transmit antenna shuffling scheme, minimum value of the parameter b is obtained in both the cases, which is the main reason behind the increase in performance of both the schemes. So, the value of parameter b should be equal to zero. But in the actual wireless environment, value of b (which is channel dependant parameter) is not always equal or nearly equal to zero and effect of which is different in both proposed QO-STBC and conventional QO-STBC. In conventional QO-STBC, if this condition occurs, let's say for $(as_1 + bs_4)$, value of $(as_1 + bs_4)$ is not nearly equal to as_1, which means that simple linear decoding is not possible and degradation of the performance takes place. But on the other hand, in the proposed QO-STBC, let's say for $(a + b)s_1$, increase or decrease in value of b, is added to s_1 itself so that simple linear decoding can be used easily with no degradation of performance.

6 Simulation Results

Using four transmit antennas, the performance of the proposed QO-STBC scheme is evaluated over a Rayleigh fading channel (assuming that channel is quasi-static and receiver has perfect knowledge about the channel). Quadrature Phase-Shift Keying (QPSK) is chosen for the modulation format leading to information rate of 2 bits/sec/Hz.

Figure 2 shows symbol error rate performance of the following schemes: conventional QO-STBC scheme [3], Park's scheme [5], proposed QO-STBC scheme without TAS and

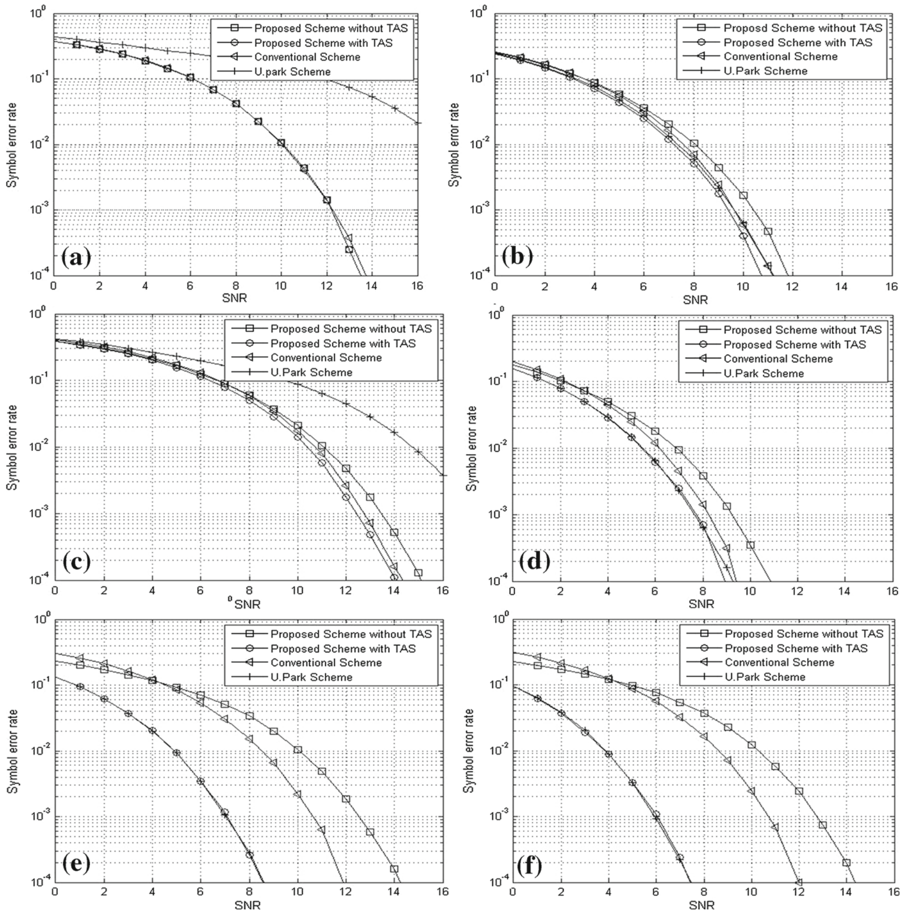

Fig. 2 **a**, **b**, **c**, **d**, **e** and **f** shows performance of different shuffled proposed codes under optimum channel conditions respectively. **a** Using shuffled antenna code corresponding to min |b| when $b = 2\text{real}\ (h_1h_4^* - h_2h_3^*)$. **b** Using shuffled antenna code corresponding to min |b| when $b = 2\text{real}\ (h_1h_3^* - h_2h_4^*)$. **c** Using shuffled antenna code corresponding to min |b| when $b = 2\text{real}(h_1h_2^* - h_3h_4^*)$. **d** Using shuffled antenna code corresponding to min |b| when $b = -2\text{real}\ (h_1h_4^* + h_2h_3^*)$. **e** Using shuffled antenna code corresponding to min |b| when $b = -2\text{real}\ (h_1h_3^* + h_2h_4^*)$. **f** Using shuffled antenna code corresponding to min |b| when $b = -2\text{real}\ (h_1h_2^* + h_3h_4^*)$

proposed QO-STBC scheme with TAS. In these figures, taking the noise constant for all simulations, the performance of six types of shuffled antenna code is compared with other schemes in their optimum channel conditions. The shuffling of the antenna is based on the feedback information given by the receiver. The feedback information has index of minimum $|b|$ value from all calculated $|b|$ values at the receiver, where b is channel dependent parameter. By analyzing figures from (a) to (f) in Fig. 2, it is apparent that proposed QO-STBC scheme with transmit antenna shuffling (TAS) achieves better performance than all the other schemes in changing channel condition.

Figure 3 shows that proposed QO-STBC scheme performs slightly better than Taha scheme [6] but after using transmit antenna shuffling the proposed QO-STBC scheme performs far better than Taha scheme. It is clear from the figures that proposed QO-STBC scheme with TAS

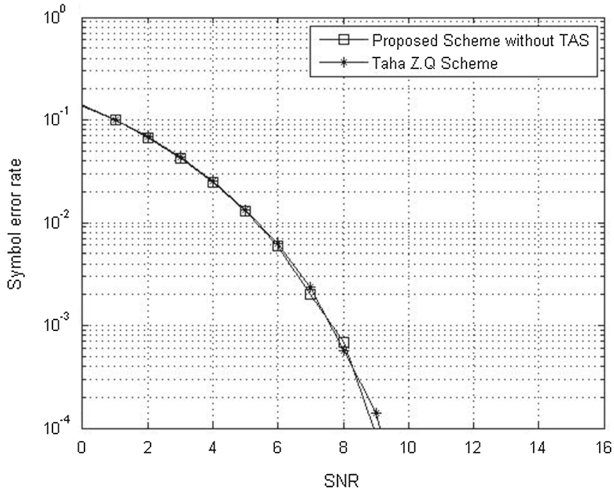

Fig. 3 Symbol error rate versus SNR

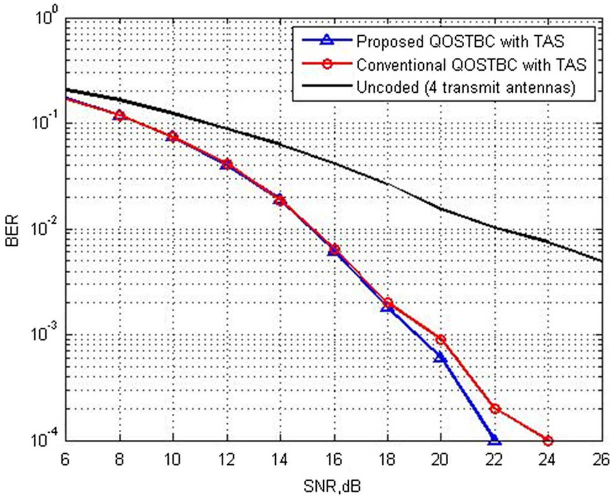

Fig. 4 BER performance of proposed QO-STBC versus conventional QO-STBC when TAS is used

achieved better performance than any other scheme discussed above with limited feedback information.

Figure 4 shows the comparison of bit error rate between the conventional QO-STBC scheme and proposed QO-STBC scheme when both are employing transmit antenna shuffling (TAS).

7 Conclusion

In this work, a new QO-STBC is proposed whose decoding complexity is considerably lower than the decoding complexity of maximum likelihood decoder and zero forcing decoder. QO-

STBC is developed by using the symmetry property of the detection matrix of Jafarkhani code. The new encoding matrix S_n is Quasi-Orthogonal but its equivalent virtual channel matrix H_n is an Orthogonal matrix, so that simple linear decoding is possible for the code which in turn reduces the transmission delay. Hence, the decoding complexity of proposed code is much lower than conventional QO-STBC without losing BER performance. It is known that transmission strategy can be improved when CSI is known to the transmitter. For this closed loop QO-STBC is employed using transmit antenna shuffling (TAS) for both proposed QO-STBC and conventional QO-STBC. It is clear that at bit error probability of 10^{-4}, the proposed QO-STBC with TAS provides about 2dB power gain over the conventional QO-STBC. So, by using transmit antenna shuffling technique, proposed QO-STBC scheme has minimum decoding complexity and can be achieve better BER performance than any other QO-STBC schemes.

Appendix

Singular value decomposition is based on linear algebra which says that a rectangular matrix D can be broken down into the product of three matrices - orthogonal matrix U, a diagonal matrix S and the transpose of an orthogonal matrix W. The theorem is usually presented like this:

$$\mathrm{D} = \mathrm{USW}^{\mathrm{T}} \tag{16}$$

where $\mathrm{UU}^{\mathrm{T}} = \mathrm{I}$, $\mathrm{WW}^{\mathrm{T}} = \mathrm{I}$; the columns of U are orthogonal eigenvectors of DD^{T}, the columns of W are orthogonal eigenvectors of $\mathrm{D}^{\mathrm{T}}\mathrm{D}$, and S is a diagonal matrix containing the square roots of eigen values from U or W in descending order. Start with the matrix

$$\mathrm{D} = \begin{bmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -b & a & 0 \\ b & 0 & 0 & a \end{bmatrix} \tag{17}$$

In order to find U, start with DD^{T}

$$\begin{aligned} \mathrm{DD}^{\mathrm{T}} &= \begin{bmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -b & a & 0 \\ b & 0 & 0 & a \end{bmatrix} \begin{bmatrix} a & 0 & 0 & b \\ 0 & a & -b & 0 \\ 0 & -b & a & 0 \\ b & 0 & 0 & a \end{bmatrix} \\ &= \begin{bmatrix} a^2+b^2 & 0 & 0 & 2ab \\ 0 & a^2+b^2 & -2ab & 0 \\ 0 & -2ab & a^2+b^2 & 0 \\ 2ab & 0 & 0 & a^2+b^2 \end{bmatrix} \end{aligned} \tag{18}$$

Now, to find the eigen values and corresponding eigenvectors of DD^{T}. It is known that eigenvectors are defined by the equation $\mathrm{A}\vec{v} = \lambda\vec{v}$, applying this to DD^{T} gives us:

$$\begin{bmatrix} a^2+b^2 & 0 & 0 & 2ab \\ 0 & a^2+b^2 & -2ab & 0 \\ 0 & -2ab & a^2+b^2 & 0 \\ 2ab & 0 & 0 & a^2+b^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{19}$$

Rewrite above matrix as the set of equations

$$(a^2 + b^2 - \lambda)x_1 + (2ab)\ x_4 = 0 \tag{20}$$

$$(a^2 + b^2 - \lambda)x_2 + (\text{-}2ab)\ x_3 = 0 \tag{21}$$

$$(\text{-2ab})\ x_2 + (a^2 + b^2 - \lambda)\ x_3 = 0 \tag{22}$$

$$(\text{2ab})\ x_1 + (a^2 + b^2 - \lambda)\ x_4 = 0 \tag{23}$$

which are solved by setting determinant of coefficient matrix to zero,

$$\begin{bmatrix} a^2+b^2-\lambda & 0 & 0 & 2ab \\ 0 & a^2+b^2-\lambda & -2ab & 0 \\ 0 & -2ab & a^2+b^2-\lambda & 0 \\ 2ab & 0 & 0 & a^2+b^2-\lambda \end{bmatrix} = 0 \tag{24}$$

which works out as

$$\Rightarrow (a^2+b^2-\lambda)[(a^2+b^2-\lambda)^3 - 4(a^2+b^2-\lambda)a^2b^2$$

$$-2ab[\text{2ab}(a^2+b^2-\lambda)^2 - 2ab(4a^2b^2)] = 0 \tag{25}$$

$$\Rightarrow (a^2+b^2-\lambda)^4 - 8(a^2+b^2-\lambda)^2a^2b^2 + 16a^4b^4 = 0 \tag{26}$$

This gives eigen values $\lambda = (a+b)^2$; $\lambda = (a+b)^2$; $\lambda = (a-b)^2$ and $\lambda = (a-b)^2$. Plugging λ back to the original equations gives eigenvectors.

For $\lambda = (a+b)^2$,

$$(a^2+b^2-(a+b)^2)x_1 + (2ab)x_4 = 0 \quad \text{or} \quad (-2ab)x_1 + (2ab)x_4 = 0 \tag{27}$$

$$(a^2+b^2-(a+b)^2)x_2 + (-2ab)x_3 = 0 \quad \text{or} \quad (-2ab)x_2 + (-2ab)x_3 = 0 \tag{28}$$

$$(-2ab)x_2 + (a^2+b^2-(a+b)^2)x_3 = 0 \quad \text{or} \quad (-2ab)x_2 + (-2ab)x_3 = 0 \tag{29}$$

$$(2ab)x_1 + (a^2+b^2-(a+b)^2)x_4 = 0 \quad \text{or} \quad (2ab)x_1 + (-2ab)x_4 = 0 \tag{30}$$

By solving these equation $x_1 = x_4$ and $x_2 = -x_3$ is obtained. Thus, corresponding to eigen value $\lambda = (a+b)^2$, eigenvectors are [1 1 −1 1] and [1 −1 1 1].

For $\lambda = (a-b)^2$,

$$(a^2+b^2-(a-b)^2)x_1 + (2ab)x_4 = 0 \quad \text{or} \quad (2ab)x_1 + (2ab)x_4 = 0 \tag{31}$$

$$(a^2+b^2-(a-b)^2)x_2 + (-2ab)x_3 = 0 \quad \text{or} \quad (2ab)x_2 + (-2ab)x_3 = 0 \tag{32}$$

$$(-2ab)x_2 + (a^2+b^2-(a-b)^2)x_3 = 0 \quad \text{or} \quad (-2ab)x_2 + (2ab)x_3 = 0 \tag{33}$$

$$(2ab)x_1 + (a^2+b^2-(a-b)^2)x_4 = 0 \quad \text{or} \quad (2ab)x_1 + (2ab)x_4 = 0 \tag{34}$$

By solving these equation $x_1 = -x_4$ and $x_2 = x_3$ is obtained. Thus, corresponding to eigen value $\lambda = (a-b)^2$, eigenvectors are [1 1 1 −1] and [−1 1 1 1].

These eigenvectors become column vectors in a matrix ordered by the size of the corresponding eigen value. In the matrix below, the eigenvectors for $\lambda = (a+b)^2$ are in column one and two, and, the eigenvectors for $\lambda = (a-b)^2$ are in column three and four.

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \tag{35}$$

Finally, by applying Gram-Schmidt orthonormalization process to the column vector, above matrix is converted into an orthogonal matrix. Begin by normalizing

For

$$\overrightarrow{v_1} = [1\,1\,{-1}\,1]$$
$$\overrightarrow{u_1} = \overrightarrow{v_1}/|\overrightarrow{v_1}| = [1\,1\ -1\,1]/\sqrt{1^2+1^2+-1^2+1^2} = [1\,1\ -1\,1]/2 = [0.5\,0.5\ -0.5\,0.5] \tag{36}$$

For

$$\overrightarrow{v_2} = [1\ -1\,1\,1]$$
$$\overrightarrow{w_2} = \overrightarrow{v_2} - \overrightarrow{u_1}.\overrightarrow{v_2} * \overrightarrow{u_1}$$
$$\overrightarrow{u_2} = \overrightarrow{w_2}/|\overrightarrow{w_2}| = [1\,{-1}\,1\,1]/\sqrt{1^2+-1^2+1^2+1^2} = [1\ -1\,1\,1]/2 = [0.5\ -0.5\,0.5\,0.5] \tag{37}$$

For

$$\overrightarrow{v_3} = [1\,1\,1\ -1]$$
$$\overrightarrow{w_3} = \overrightarrow{v_3} - \overrightarrow{u_1}.\overrightarrow{v_3} * \overrightarrow{u_1} - \overrightarrow{u_2}.\overrightarrow{v_3} * \overrightarrow{u_2}$$
$$\overrightarrow{u_3} = \overrightarrow{w_3}/|\overrightarrow{w_3}| = [1\,1\,1\ -1]/\sqrt{1^2+1^2+1^2+-1^2} = [1\,1\,1\ -1]/2 = [0.5\,0.5\,0.5\ -0.5] \tag{38}$$

For

$$\overrightarrow{v_4} = [-1\,1\,1\,1]$$
$$\overrightarrow{w_4} = \overrightarrow{v_4} - \overrightarrow{u_1}.\overrightarrow{v_4} * \overrightarrow{u_1} - \overrightarrow{u_2}.\overrightarrow{v_4} * \overrightarrow{u_2} - \overrightarrow{u_3}.\overrightarrow{v_4} * \overrightarrow{u_3}$$
$$\overrightarrow{u_4} = \overrightarrow{w_4}/|\overrightarrow{w_4}| = [-1\,1\,1\,1]/\sqrt{-1^2+1^2+1^2+1^2} = [-1\,1\,1\,1]/2 = [-0.5\,0.5\,0.5\,0.5] \tag{39}$$

All these $\vec{u}$ ‘s result in an orthogonal matrix U which is given by

$$\begin{bmatrix} 0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \end{bmatrix} \tag{40}$$

In order to find W, start with D^TD

As D is the symmetric matrix, so $D^TD = DD^T$. This means that value of W will be equal to the value of U which was derived earlier by DD^T. So, W^T is given by

$$\begin{bmatrix} 0.5 & 0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{41}$$

For finding S, take square roots of the non-zero eigen values and populate the diagonal with them, putting largest first. So, S is given by

$$\begin{bmatrix} a+b & 0 & 0 & 0 \\ 0 & a+b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{bmatrix} \tag{42}$$

So, $D = USW^T$ is given by

$$D = \begin{bmatrix} 0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} a+b & 0 & 0 & 0 \\ 0 & a+b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{43}$$

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