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On Convolution and Product Theorems for FRFT

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Abstract The fractional Fourier transform (FRFT), which is considered as a generalization of the Fourier transform (FT), has emerged as a very efficient mathematical tool in signal processing for signals which are having time-dependent frequency component. Many properties of this transform are already known, but the generalization of convolution theorem of Fourier transform for FRFT is still not having a widely accepted closed form expression. In the recent past, different authors have tried to formulate convolution theorem for FRFT, but none have received acclamation because their definition do not generalize very appropriately the classical result for the FT. A modified convolution theorem for FRFT is proposed in this article which is compared with the existing ones and found to be a better and befitting proposition.

Keywords FRFT · Time-frequency plane · Convolution

1 Introduction

The fractional Fourier transform (FRFT) was introduced way back in 1920s, but remained largely unknown until the work of Namias in 1980 [1]. The fractional Fourier transform can be viewed as the chirp-basis expansion directly from its definition, but essentially it can be interpreted as a rotation in the time-frequency plane, i.e. the unified time-frequency transform. With the order increasing from 0 to 1, the fractional Fourier transform can show the characteristics of the signal changing from the time domain to the frequency domain i.e., FRFT is a generalization of Fourier transform (FT). The FRFT has an advantage over other

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Integral Transforms (IT), being used in the application areas like—signal processing and optics [2–6].

Many properties of FRFT are derived, developed or established in the recent past [2,3,7–10], which includes multiplication property, differentiation property, shifting property, modulation property, and convolution theorem etc. Since the convolution theorem plays an important role in digital signal processing so it is extensively investigated for the refinement of a well accepted closed-form expression.

In mathematics, the convolution theorem states that under suitable conditions the Fourier transform of a convolution of two signals is the point wise product of respective Fourier transforms of both the signals. In other words, convolution in one domain (e.g., time domain) equals point-wise multiplication in the other domain (e.g., frequency domain). The usefulness of convolution theorem can be best explained by its application in filtering. Since filtering can be performed both way i.e., time domain filtering and frequency domain filtering. Simultaneously, if the computational complexity is a basis parameter then it can be shown that under different input conditions one type of filtering has advantage over other and vice-versa [4].

For non-stationary signals and noise, the time and frequency domain filtering both fails because the signal and noise may have their respective Wigner distribution overlapping to each other in time and frequency domains. In this case, fractional Fourier based filtering can provide a better solution, where for a rotated domain in time-frequency plane corresponding to an optimum value of angle parameter, the Wigner distribution of signal and noise may be separated and filtering of the signal from the noise can be performed by designing a filter in FRFT domain with this optimum angle parameter value [2].

For designing such a FRFT domain filter, the convolution theorem for the FRFT should be known. In literature many definitions of convolution theorem for FRFT is being proposed such as by Almeida [7], Zayed [8], Deng et al. [9] and Wei et al. [10].

In this article, modified expressions for the convolution and product theorems are proposed which are different from those given in [7–10]. The modified definitions preserve all the properties of classical convolution and product theorems. An attempt is also made to establish a comparative study, in terms of variable dependability, FT conversion and computational error. It has been established that the proposed convolution and product theorems are the better proposition and can be considered as generalization of classical convolution and product theorems.

2 FRFT and Convolution Theorem

The FRFT of a signal $x(t)$ with angle parameter '$\boldsymbol{\alpha}$', represented by $X_\alpha(u)$ is defined for entire time-frequency plane as

$$X_\alpha(u) = \begin{cases} \sqrt{\frac{1-j\cot\alpha}{2\pi}} \int_{-\infty}^{\infty} x(t) e^{\frac{j}{2}\{(t^2+u^2)\cot\alpha - 2tu\csc\alpha\}} dt, & \text{if } \alpha \neq k\pi \\ x(u), & \text{if } \alpha = 2k\pi \\ x(-u), & \text{if } \alpha = (2k+1)\pi \end{cases} \tag{1}$$

Classically, the convolution theorem of the FT for the signals $x(t)$ and $y(t)$ with associated Fourier transforms, $X(\omega)$ and $Y(\omega)$, respectively is given by

$$x(t) \otimes y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \quad \leftrightarrow \quad \sqrt{2\pi}X(\omega)Y(\omega) \tag{2}$$

where '⊗' denotes the linear convolution operation.

This theorem states that convolution of two signals in time domain results in simple multiplication of their Fourier transforms in frequency domain. This convolution theorem satisfies all the property that a convolution theorem of an integral transform should satisfy—Commutative property, Associative property, and Distributive property. The convolution and product theorems for FRFT have been suggested by many researchers in the past. The chronological development in this dimension is given in the following paragraph.

In 1997, Almeida [7] gave a definition by considering two functions, $x, y \in L^1(R) \cap W$. A convolution theorem for these two functions in FRFT domain is given by

$$\begin{aligned} z(t) &= (x \otimes y)(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \leftrightarrow \\ Z_\alpha(u) &= |\sec\alpha| e^{-j\left(\frac{u^2}{2}\right)\tan\alpha} \int_{-\infty}^{\infty} X_\alpha(v)y[(u-v)\sec\alpha]e^{j\left(\frac{v^2}{2}\right)\tan\alpha}dv \end{aligned} \tag{3}$$

The FRFT of a convolution (i.e., $(x \otimes y)(t)$) can be obtained by multiplying a chirp to the FRFT of one of the signals and convolving it with a scaled version of the other signal, subsequently multiplying again by another chirp and a scale factor, as evident from (3). And product theorem is given by

$$\begin{aligned} z(t) = x(t)y(t) \leftrightarrow Z_\alpha(u) &= \frac{|\csc\alpha|}{\sqrt{2\pi}} e^{j\left(\frac{u^2}{2}\right)\cot\alpha} \\ &\times \int_{-\infty}^{\infty} X_\alpha(v)Y[(u-v)\csc\alpha]e^{-j\left(\frac{v^2}{2}\right)\cot\alpha}dv \end{aligned} \tag{4}$$

where $Y(u)$ is Fourier transform of $y(t)$.

After one year, Zayed [8] has documented a different method to calculate FRFT of the convolution of two signals. In this method, the chirp and scale factor are multiplied to the convolution of chirp multiplied signals as given in (5). Subsequently, the FRFT of the obtained convolution is derived which is entirely in the FRFT domain.

$$\begin{aligned} z(t) = (x * y)(t) &= \sqrt{\frac{1 - j\cot\alpha}{2\pi}} e^{-\frac{j}{2}t^2\cot\alpha} \\ &\times \int_{-\infty}^{\infty} x(\tau)e^{\frac{j}{2}\tau^2\cot\alpha} \quad y(t-\tau)e^{\frac{j}{2}(t-\tau)^2\cot\alpha}\, d\tau \\ &\leftrightarrow \quad Z_\alpha(u) = e^{-\frac{j}{2}u^2\cot\alpha} X_\alpha(u)Y_\alpha(u) \end{aligned} \tag{5}$$

And product theorem given by Zayed [8] was

$$z(t) = x(t)y(t)e^{-\frac{j}{2}t^2 \cot\alpha} \leftrightarrow Z_\alpha(u) = \sqrt{\frac{1 + j\cot\alpha}{2\pi}} e^{-\frac{j}{2}u^2 \cot\alpha} \times \int_{-\infty}^{\infty} X_\alpha(v) e^{-\frac{j}{2}v^2 \cot\alpha} Y_\alpha(u - v) e^{-\frac{j}{2}(u-v)^2 \cot\alpha} dv \tag{6}$$

After Zayed [8], it took almost 8 years to propose a different definition of convolution theorem in FRFT domain by Deng et al. [9] in 2006. Actually the definition was given for linear canonical transform (LCT) and it can easily be shown that definition of convolution theorem for FRFT is just a particular case of what is given for LCT. Here both convolution and product theorems are shown for FRFT.

The convolution theorem given by Deng et al. [9] is

$$z(t) = (x * y)(t) = \sqrt{\frac{1 - j\cot\alpha}{2\pi}} e^{-\frac{j}{2}t^2 \cot\alpha} \int_{-\infty}^{\infty} x(\tau) e^{\frac{j}{2}\tau^2 \cot\alpha} y(t - \tau) e^{\frac{j}{2}(t-\tau)^2 \cot\alpha} d\tau \leftrightarrow \quad Z_\alpha(u) = e^{-\frac{j}{2}u^2 \cot\alpha} X_\alpha(u) Y_\alpha(u) \tag{7}$$

And product theorem is given by

$$z(t) = x(t)y(t) \quad \leftrightarrow \quad Z_\alpha(u) = \frac{|\csc\alpha|}{\sqrt{2\pi}} e^{j\left(\frac{u^2}{2}\right)\cot\alpha} \times \int_{-\infty}^{\infty} X_\alpha(v) Y[(u - v)\csc\alpha] e^{-j\left(\frac{v^2}{2}\right)\cot\alpha} dv \tag{8}$$

Later on in year 2009, a different definition of convolution theorem in LCT domain is given by Wei et al. [10]. Prior to defining convolution theorem, author defines a τ-generalized translation of signal $y(t)$ denoted by $y(t\theta\tau)$ where

$$y(t\theta\tau) = \int_{-\infty}^{\infty} Y_\alpha(u) K_\alpha(u, \tau) K^*(u, t) du \tag{9}$$

where K and K^* represent FRFT and IFRFT kernel respectively. Based on this generalized function convolution theorem is defined as

$$(f \circ g)(t) = \int_{-\infty}^{\infty} x(\tau) y(t\theta\tau) d\tau \quad \leftrightarrow \quad X_\alpha(u) Y_\alpha(u) \tag{10}$$

In this way the convolution and product theorems were proposed and developed. During the examination of these propositions the following observations are made

(a) The definition given by Almeida [7] has failed to convert into its Fourier transform (FT) equivalent, when $\boldsymbol{\alpha} = \pi/2$.
(b) The definitions given by Zayed [8] and Deng et al. [9] have required three chirp multiplication to evaluate the defined convolution integral.
(c) In the definition given by Wei et al. [10], the generalized convolution operation defined in time domain is not only dependent on time variable but it also depends on transform domain variable 'u' in which it has to be transformed.

This necessitates that a new convolution theorem should be developed in a manner

- which is capable to convert the convolution identity into its FT equivalent, at $\boldsymbol{\alpha} = \pi/2$,
- in which the convolution integral should be dependent on time variable only and its transform will depend on transform domain variable, and
- where the computation of convolution identity requires less number of chirp multiplication.

3 Modified Identities for FRFT

The unavailability of a well defined convolution and product theorems, as explained in the previous section, has motivated to propose another method to define these identities for FRFT which is a modified version of the identities proposed by Zayed [8]. These modified identities not only satisfy variable dependability and FT conversion (for $\boldsymbol{\alpha} = \pi/2$) but also satisfy all the properties that these classical identities in FT domain satisfies.

3.1 Convolution Theorem

The circular convolution has been defined separately for Discrete Fourier Transform (DFT) in place of linear convolution, whereas the circular and linear convolutions are related and one can be determined if other is known. Similarly, a weighted convolution is proposed for FRFT, which can be made linear with $\boldsymbol{\alpha} = \pi/2$.

Definition 1 For any two functions $x(t)$ and $y(t)$, the weighted convolution operation is defined as (here Θ symbol is used to represent proposed modified convolution operation).

$$z(t) = (x\Theta y)(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)e^{j\tau(\tau-t)\cot\alpha}d\tau \tag{11}$$

Theorem 1 *Let $z(t)$is modified convolution of two functions $x(t)$ and $y(t)$, and $X_\alpha(u)$, $Y_\alpha(u)$ and $Z_\alpha(u)$ are defined as FRFT of $x(t)$, $y(t)$ and $z(t)$ respectively. Then*

$$z(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)e^{j\tau(\tau-t)\cot\alpha}\mathrm{d}\tau$$
$$\leftrightarrow \quad Z_\alpha(u) = \sqrt{\frac{2\pi}{1-j\cot\alpha}}e^{-\frac{j}{2}u^2\cot\alpha}X_\alpha(u)Y_\alpha(u) \tag{12}$$

(a) *Proof* From the definition of FRFT

$$Z_\alpha(u) = \sqrt{\frac{1-j\cot\alpha}{2\pi}}\int_{-\infty}^{\infty} z(t)e^{\frac{j}{2}\{(t^2+u^2)\cot\alpha-2tu\csc\alpha\}}dt$$
$$= \left(\sqrt{\frac{1-j\cot\alpha}{2\pi}}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)$$
$$\times\, e^{\frac{j}{2}\{(t^2+u^2)\cot\alpha-2tu\csc\alpha+2\tau(\tau-t)\cot\alpha\}}d\tau dt \tag{13}$$

By putting the change of variable, $t - \tau = m$,

$$Z_\alpha(u) = \left(\sqrt{\frac{1 - j\cot\alpha}{2\pi}}\right) \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau)\, e^{\frac{j}{2}\{(\tau^2+u^2)\cot\alpha - 2\tau u \csc\alpha\}} d\tau \right\} \times y(m)\, e^{\frac{j}{2}\{m^2\cot\alpha - 2mu\csc\alpha\}} dm \tag{14}$$

Multiply and divide by $\sqrt{\frac{1-j\cot\alpha}{2\pi}} e^{\frac{j}{2}u^2\cot\alpha}$,

$$\begin{aligned} Z_\alpha(u) &= \sqrt{\frac{2\pi}{1 - j\cot\alpha}} e^{-\frac{j}{2}u^2\cot\alpha} \left\{\sqrt{\frac{1 - j\cot\alpha}{2\pi}}\right\}^2 \\ &\quad \times \left\{ \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} x(\tau)\, e^{\frac{j}{2}\{(\tau^2+u^2)\cot\alpha - 2\tau u\csc\alpha\}} d\tau \right\} y(m)\, e^{\frac{j}{2}\{(m^2+u^2)\cot\alpha - 2mu\csc\alpha\}} dm \right\} \\ &= \sqrt{\frac{2\pi}{1 - j\cot\alpha}} e^{-\frac{j}{2}u^2\cot\alpha} X_\alpha(u) Y_\alpha(u) \end{aligned} \tag{15}$$

Finally, convolution theorem transform pair is given as

$$\int_{-\infty}^{\infty} x(\tau)\, y(t - \tau)\, e^{j\tau(\tau - t)\cot\alpha} d\tau \leftrightarrow \sqrt{\frac{2\pi}{1 - j\cot\alpha}} e^{-\frac{j}{2}u^2\cot\alpha} X_\alpha(u) Y_\alpha(u) \tag{16}$$

(b) *Special Case:* By putting $\boldsymbol{\alpha} = \pi/2$ in the transform pair, the linear convolution for Fourier transform can be obtained as

$$\int_{-\infty}^{\infty} x(\tau) y t - \tau) d\tau \quad \leftrightarrow \quad \sqrt{2\pi}\, X_{\pi/2}(u) Y_{\pi/2}(u) \tag{17}$$

where $X_{\pi/2}(u)$ and $Y_{\pi/2}(u)$ are Fourier transform of $x(t)$ and $y(t)$ respectively.

(c) *Properties satisfied by convolution theorem:*

(i) *Commutative:*

$$\begin{aligned} (x\Theta y)(t) &\leftrightarrow \sqrt{\frac{2\pi}{1 - j\cot\alpha}} e^{-\frac{j}{2}u^2\cot\alpha} X_\alpha(u) Y_\alpha(u) \\ (y\Theta x)(t) &\leftrightarrow \sqrt{\frac{2\pi}{1 - j\cot\alpha}} e^{-\frac{j}{2}u^2\cot\alpha} X_\alpha(u) Y_\alpha(u) \\ (x\Theta y)(t) &= (y\Theta x)(t) \end{aligned} \tag{18}$$

(ii) *Associative:*

$$
\begin{aligned}
((x\Theta y)\Theta f)(t) &\leftrightarrow \left(\frac{2\pi}{1-j\cot\alpha}\right)e^{-ju^2\cot\alpha}X_\alpha(u)Y_\alpha(u)F_\alpha(u)\\
(x\Theta(y\Theta f))(t) &\leftrightarrow \left(\frac{2\pi}{1-j\cot\alpha}\right)e^{-ju^2\cot\alpha}X_\alpha(u)Y_\alpha(u)F_\alpha(u)\\
((x\Theta y)\Theta f)(t) &= (x\Theta(y\Theta f))(t)
\end{aligned}
\tag{19}
$$

(iii) *Distributive:*

$$
\begin{aligned}
(x\Theta(y+f))(t) &\leftrightarrow \sqrt{\frac{2\pi}{1-j\cot\alpha}}e^{-\frac{j}{2}u^2\cot\alpha}X_\alpha(u)(Y_\alpha(u)+F_\alpha(u))\\
(x\Theta y+x\Theta f)(t) &\leftrightarrow \sqrt{\frac{2\pi}{1-j\cot\alpha}}e^{-\frac{j}{2}u^2\cot\alpha}(X_\alpha(u)Y_\alpha(u)+X_\alpha(u)F_\alpha(u))\\
(x\Theta(y+f))(t) &= (x\Theta y+x\Theta f)(t)
\end{aligned}
\tag{20}
$$

3.2 Product Theorem

Definition 2 For any two functions $x(t)$ and $y(t)$, the modified product operation is defined as

$$
z(t) = x(t)y(t)e^{\frac{j}{2}t^2\cot\alpha} \tag{21}
$$

Theorem 2 *Let $z(t)$ is weighted product of two functions $x(t)$ and $y(t)$, and $X_\alpha(u)$, $Y_\alpha(u)$ and $Z_\alpha(u)$ are defined as FRFT of $x(t)$, $y(t)$ and $z(t)$ respectively. Then*

$$
\begin{aligned}
z(t) = x(t)y(t)e^{\frac{j}{2}t^2\cot\alpha} \leftrightarrow \quad Z_\alpha(u) &= \sqrt{\frac{1+j\cot\alpha}{2\pi}}\\
&\times\int_{-\infty}^{\infty} X_\alpha(v)Y_\alpha(u-v)e^{\frac{j}{2}v(u-v)\cot\alpha}dv
\end{aligned}
\tag{22}
$$

(a) *Proof* This is same as of proposed convolution theorem, therefore it is omitted.
(b) *Properties satisfied by product theorem:* Same as that of convolution theorem.

4 Comparative Analysis of Convolution Theorems

A comparative analysis of available definitions of convolution function on the following parameters is presented in this section.

4.1 Variable Dependability

The convolution defined in one domain and its transformed counterpart in transformed domain should have mathematical expressions in terms of respective domain variables only. For example, when the time domain convolution of rectangular window function $\{(r\otimes r)(t)\}$ is Fourier transformed, it is having a closed form expression in frequency domain, $\text{sinc}^2(\pi f/2)$, dependent upon frequency as variable only. This parameter is assumed in order to assure that a quantity defined in one domain when transformed will result in an equivalent quantity in transformed domain. In the Table 1, Yes is included for the method, which

Table 1 Computational complexity for all the methods for convolution theorem

Parameter	Wei et al. [10]		Deng et al. [9]		Zayed [8]		Almeida [7]		Proposed	
Variable dependability	No		Yes		Yes		Yes		Yes	
FT conversion at $\alpha = \pi/2$	Yes		Yes		Yes		No		Yes	
Hardware complexity	LHS	RHS	LHS	RHS	LHS	RHS	LHS	RHS	LHS	RHS
No. of chirp functions	7	6	3	7	3	7	–	5	2	7

LHS represents the defined convolution process by different methods and RHS represents their transforms

transform a convolution function defined in one domain variable into equivalent function of transform domain variable and No is for the method in which either convolution function is dependent on both variable or transformed equivalent quantity is function of both variable.

4.2 Equivalent FT Conversion

The proposed convolution theorem should be converted into classical convolution theorem for Fourier transform with angle parameter, $\alpha = \pi/2$, it is due to the basic property of FRFT as expression of FRFT converts into expression of FT at angle parameter, $\alpha = \pi/2$. In the Table 1, Yes is entered for the method where the relation is converted into classical convolution theorem of Fourier transform at $\alpha = \pi/2$ and No is mentioned otherwise. This parameter has given a prime importance because FRFT is assumed as the generalization of Fourier Transform (FT). Therefore, any property defined for FRFT should be converted to the analogous property for FT. In the existing definitions, the identity given by Almeida [7] is not satisfying this parameter; hence the simulation has not been performed for this method.

4.3 Simulation Results

The mathematical relation describing the convolution operation in fractional domain by Zayed [8] and Wei et al. [10] is compared with the proposed identity for convolution by simulating both the expressions on the platform of Wolfram Mathematica® software (version-7.0) on a system having configuration Pentium-4, Intel(R) CPU-1.8 GHz processor having 1 GB RAM.

A rectangular window function '$r(t)$' of unity amplitude and unit duration ($0 \leq t \leq 1.0$) is convolved linearly with itself $\{(r \otimes r)(t)\}$, gives a triangular (or Bartlett) window function

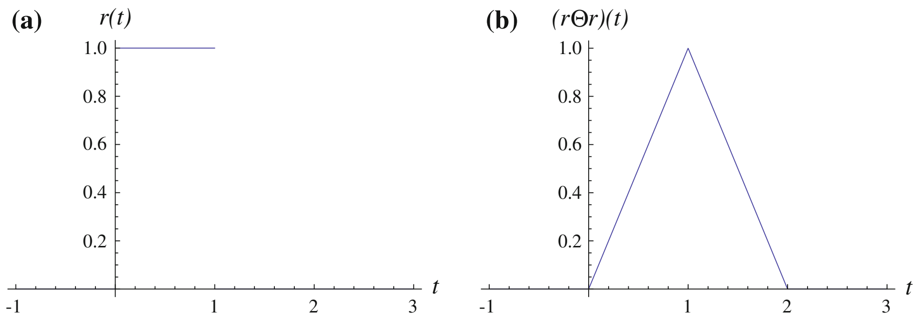

Fig. 1 **a** Rectangular function, '$r(t)$', and **b** Convolved signal '$\{(r \otimes r)(t)\}$' i.e., Bartlet window

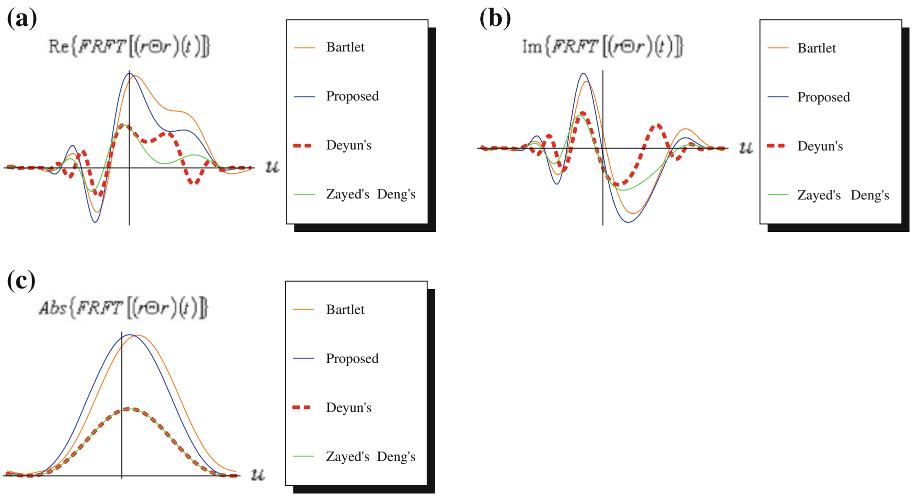

Fig. 2 FRFT of $(r\Theta r)(t)$for $\alpha = \pi/4$ by Zayed, Deyun and Proposed method along with FRFT of Bartlett window **a** Real value of, **b** Imaginary component of, and **c** Absolute component of FRFT

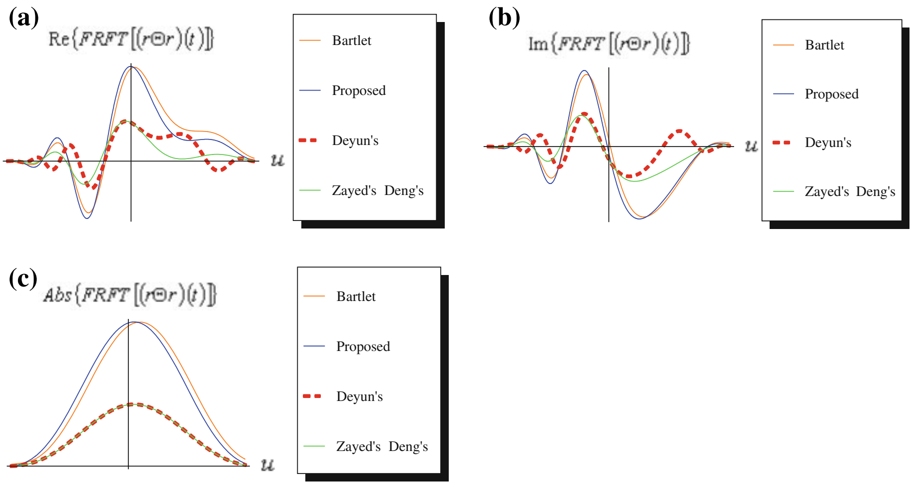

Fig. 3 FRFT of $(r\Theta r)(t)$ for $\alpha = \pi/3$ by Zayed, Deyun and Proposed method along with FRFT of Bartlett window **a** Real value of, **b** Imaginary component of, and **c** Absolute component of FRFT

of duration double to it, in the time domain as shown in Fig. 1. As, the methods given by Zayed [8] and Deng et al. [9] are having similar closed form expressions, therefore, the definition given by Zayed is only simulated for the analysis purpose. The source codes for the convolution theorem given by Zayed [8], Wei et al. [10] and of the proposed one are tested by choosing the two functions as '$r(t)$' in respective convolution integral. The results of the FRFT determined, at an angle $\alpha = \pi/4$, by the Zayed's method, Deyun's method and also by the proposed convolution identity are shown in the Fig. 2 and for angle $\alpha = \pi/3$ are shown in Fig. 3. Simultaneously, the FRFT of the triangular window function is also evaluated for the same angle to make a comparison. It has been shown, that the FRFT for $\alpha = \pi/4$ and $\alpha = \pi/3$ of the triangular window function resembles maximally, i.e. the real (Re), imaginary (Im) and absolute (Abs) components to the FRFT of the convolved output

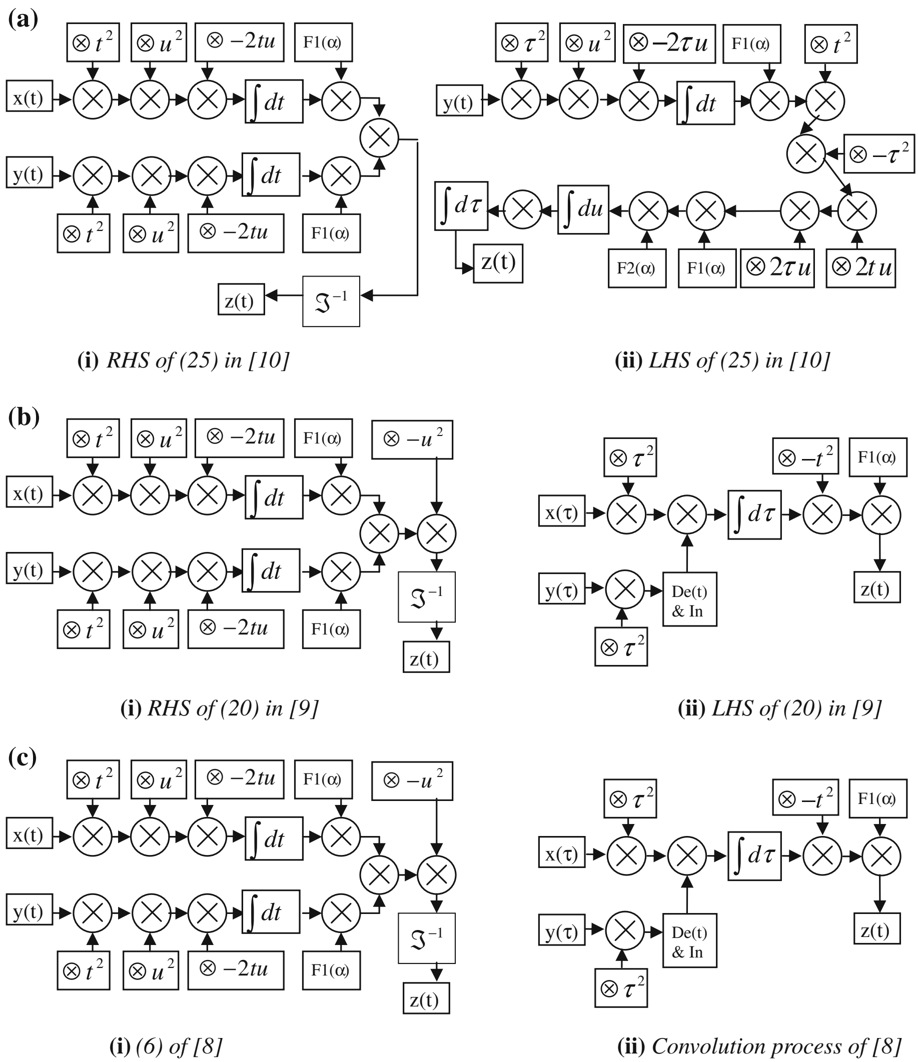

Fig. 4 **a** Block diagram representation of Wei et al. [10] method. **b** Block diagram representation of Deng et al. [9] method. **c** Block diagram representation of Zayed [8] method. **d** Block diagram representation of Almeida [7] method. **e** Block diagram representation of proposed method. **f** Blocks used in the block diagram representation

with the proposed theorem. It is also visible from the Fig. 2a, b in case of angle $\alpha = \pi/4$ and in the Fig. 3a, b in case of angle $\alpha = \pi/3$ that the Real and Imaginary components are more oscillatory in the case of Deyun's method than the similar components determined by the proposed method.

4.4 Computational Error

A simple block diagram implementation of the mathematical relations given by various authors for establishing the convolution theorem is also used for the comparison purpose. The relationships (both the LHS and RHS) given by Wei et al. [10], Deng et al. [9], Zayed [8], Almeida [7] and proposed one are shown with the help of their block diagram realizations in Fig. 4a–e, respectively. The nomenclature of various building blocks is given in Fig. 4f.

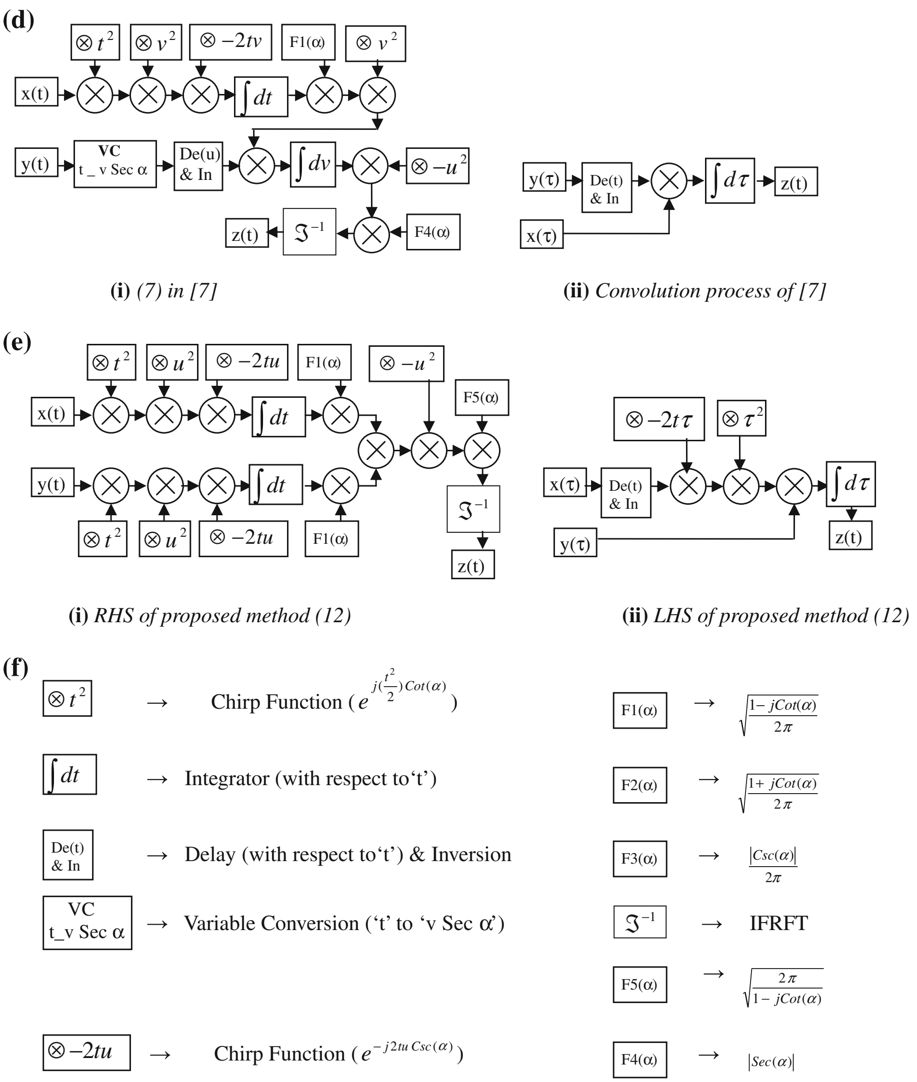

Fig. 4 continued

5 Conclusion

A modified expression for the convolution integral for FRFT has been introduced. This can be treated as convolution theorem for FRFT and enhances the support for FRFT for its consideration as an integral transform. The proposed definition satisfies all the properties of classical convolution theorem for Fourier transform, i.e. the commutative property, associative property and distributive property. As shown in the Table 1, the definition given by Wei et al. [10] has failed to satisfy the variable dependability condition, while all other obeys this aspect of classical convolution theorem. Similarly, the definition given by Almeida [7] is not converted to the classical convolution theorem with $\alpha = \pi/2$, while others satisfy this parameter. So analyzing all the five method of defining the convolution theorem on the first and second parameters, it can be clearly noticed that the definition given by Zayed [8] and proposed one are only methods (since, the method given by Deng et al. [9] is same as given by Zayed [8]) having satisfied both the parameters, variable dependability and equivalent Fourier transform conversion.

As can be seen from the simulation results of Figs. 2 and 3, the proposed weighted convolution theorem is giving results better than the convolution theorem given by Zayed [8] and Wei et al. [10]. The results determined by the proposed theorem are closer in shape and of matching values to the FRFT of a triangular window function. The results determined by the convolution expression of Wei et al. [10] have more oscillations in both the Real and Imaginary components, as it is visible from the Figs. 2 and 3 for different value of angle α. These oscillations are significant and present due to the chirp signal included in the calculation of convolution integral by Wei et al. [10], which also contains variable of the transformed domain. In the context of computational complexity, the comparison has been established in terms of the number of chirp multiplications performed in realizing the different convolution theorems. Finally, comparing the definition given by Zayed [8] and proposed one on the basis of computational complexity, it is found that number of chirp multiplications is lesser in the case of proposed method. Therefore, proposed modified definition is found to be a better proposition to other four definitions given by Almeida [7], Zayed [8], Deng et al. [9] and Wei et al. [10].

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R. Saxena born at Gwalior in Madhya Pradesh in 1961, obtained B.E. (Electronics & Telecommunication Engineering) in the year 1982 from Jabalpur University, Jabalpur. Subsequently, Dr. Saxena joined the Reliance Industries, Ahmedabad, as Graduate Trainee. In 1984, Dr. Saxena joined Madhav Institute of Technology & Science, Gwalior as Lecturer in Electronics Engineering. He obtained his M.E. (Digital Techniques & Data Processing) from Jiwaji University, Gwalior in 1990. The Ph.D. degree was conferred on him in 1996–1997 in Electronics & Computer Engineering from IIT, Roorkee (erstwhile UOR, Roorkee). Currently Dr. Saxena is head and professor in ECE department at JUET, Guna.