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Minimization of Total Pumping Cost from an Aquifer to a Water Tank, Via a Pipe Network

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Abstract
In this paper, we have investigated minimization of total cost to pump a given flow rate from any number (n) of wells up to a water tank, under steady-state flow conditions. Regarding groundwater flow, we have considered infinite or semi-infinite aquifers, to which the method of images applies. Additional regional groundwater flow can be taken into account, too. The pipe network connecting the wells to the tank can include junctions at the locations of the wells only. Moreover, all pumps have equal efficiency. We have derived a new analytical formula, which holds at the critical points of the total cost function. Based on this formula, we derived a system of n equations and n unknowns, to calculate the well flow rate combinations which correspond to the critical points of the total cost function. The n-1 equations are 2nd degree polynomials, while the remaining one is linear, expressing the constraint that the sum of well flow rates must be equal the required total flow rate. The solution of the system can be achieved using commercial solvers. Moreover, we have concluded that there is one feasible solution that minimizes the total cost. Finally, we present a tabulation process to facilitate the use of solvers and we provide and discuss two illustrative examples.

Keywords Hydraulic friction losses · Groundwater flow · Optimization · Pipe network · Pumping cost · Water resources

1 Introduction

Water is directly and indirectly vital to humanity. From drinking water, sanitation and agriculture to building infrastructures, manufacturing and religion, water is shaping our everyday life. Moreover, not only historically "the availability of water … has been considered an essential part of a civilized way of life in different periods" (Vuorinen et al. 2007), but still water scarcity is one of the main problems of humanity, as its lack is considered to accelerate

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diseases' spread and to cause millions of deaths every year (Montgomery and Elimelech 2007). Climate change seems to aggravate water scarcity on a global scale.

In this framework, optimal management of water resources, including both the supply and the demand side of the water balance (Rijsberman 2006) is a high priority issue. For this reason, the respective bibliography is quite large. A substantial part of it is dedicated to groundwater resources. The respective problems can be broadly classified in two groups: a) maximization of pumping rate under physical or financial constraints and b) minimization of pumping cost of the total flow rate from a system of wells (probably combined with other cost items). This group includes also aquifer restoration problems, where groundwater pumping cost is the main cost item (Matott et al. 2006; Kontos 2013), and problems of controlling high groundwater levels (Bayer et al. 2009).

Groundwater management optimization problems have many forms; for this reason, many optimization methods (and combinations of them) have been used to address them, ranging from versions of linear and non-linear programming (Theodossiou 2004; Bostan et al. 2016; Mani et al. 2016; Seo et al. 2018) to evolutionary techniques, which have gained popularity in the last decades (Nicklow et al. 2010; Tsai 2010; Ghadimi and Ketabchi 2019) and combinations of them with other techniques (Karterakis et al. 2007; Khadem and Afshar 2015; Alizadeh et al. 2017; Moutsopoulos et al. 2017).

In most cases, and for both classes of problems, groundwater flow simulation is part of the optimization procedure and may determine the difficulty of the respective optimization problem (Singh and Panda 2013). In certain cases, surrogate models are used to alleviate the computational burden, in particular when coastal or karst aquifers are involved (Sreekanth and Datta 2014; Ketabchi and Ataie-Ashtiani 2015; Christelis and Mantoglou 2019).

Analytical solutions for pumping cost minimization problems, are rather few, to our knowledge. Katsifarakis (2008) studied the cost K to pump a given total flow rate Q_T from any number and layout of wells, up to a predefined constant level. He assumed steady-state flow in confined infinite and semi-infinite aquifers (using the method of images for the latter) and he ignored friction losses in the well pipes. He proved that K is minimized, when hydraulic head levels at all wells are equal to each other, as long as flow is due to the examined system of wells only. Finally, he presented an analytical calculation procedure of the optimal distribution of Q_T to the individual wells, through solution of a linear system of equations.

Using similar assumptions and the method of images, Katsifarakis and Tselepidou (2009) extended the aforementioned work to aquifers with two zones of different transmissivities and found similar results. Then they took into account regional flow, independent of the operation of the wells. In this more general case, they proved that pumping cost is minimized, when final differences between hydraulic head values at the locations of the wells, resulting from superposition of the regional flow and the operation of the well system, are equal to the half of those, which are due to the regional flow only. Finally, they presented an analytical calculation procedure of the optimal distribution of Q_T to the individual wells.

Ahlfeld and Laverty (2011), using a matrix formulation, have come up with similar results, even for laterally confined flow fields, assuming that the groundwater flow equation is linear, the boundary conditions are not head-dependent and the response matrix of drawdown to pumping is symmetric. Its coefficients have to be calculated numerically. Moreover, symmetry of the numerical model coefficient matrix is a prerequisite. Their proof holds for transient flows with constant well flow rates, under the same assumptions. Later on, the authors have tested their formulation on a hypothetical problem, which was based on an aquifer with complex hydrogeology and large drawdowns in California, USA (Ahlfeld and Laverty 2015).

Katsifarakis et al. (2018) have extended the results of Katsifarakis and Tselepidou (2009) to transient pumping from a system of wells. The transient flow is superposed with a regional steady-state flow, resulting in different initial hydraulic head values at the locations of the wells. The authors proved that, at any time, the instant pumping cost is minimum, when the observed at that instant differences between hydraulic heads at the wells are equal to the half of the initial ones. Moreover, as well flow rates usually remain constant over the pumping period, they have outlined an approximate calculation of the optimal constant flow rate distribution.

Nikoletos (2020) has extended results of Katsifarakis et al. (2018) to stepwise or intermittent pumping from a system of wells and to alternate pumping from groups of wells.

In this paper we present a new analytical solution for pumping cost minimization. We take into account: a) Hydraulic head level drawdown and initial head levels at the wells, as in existing solutions and b) Friction losses in the pipe network, which connects the pumping wells to a central water tank. This problem is more complicated, since the cost function is a 3rd degree polynomial and there are many local optima, as discussed in the following sections. To our knowledge, no analytical solution has been presented up to now, taking into account both groundwater and pipe flow.

2 Mathematical Formulation of the Problem

2.1 The Objective Function

The problem is stated as follows: Minimize the cost K_{tot} to pump a given total flow rate q_{tot} from any number and layout of wells up to a tank, under steady-state flow conditions. Take into account hydraulic head level drawdown due to pumping and initial head levels at the wells, together with friction losses at the pipe network that connects the wells with the tank.

For convenience, K_{tot} can be written as:

$$K_{tot} = b(K_{dr} + K_{el} + K_{trans}) \tag{2.1}$$

where b depends on energy cost.

K_{dr} accounts for hydraulic head level drawdown due to the operation of the well system and is expressed as:

$$K_{dr} = \sum_{j=1}^{n} q_j \, s_j \tag{2.2}$$

where n is the number of wells, s_j the hydraulic head level drawdown at well j and q_j its flow rate.

K_{el} accounts for the difference δ_j between water level at the tank (in practice an average level could be used) and initial hydraulic head level at each well ($\delta_j = z_{tank} - z_j$). It is given as:

$$K_{el} = \sum_{j=1}^{n} q_j \, \delta_j \tag{2.3}$$

Finally, K_{trans} accounts for hydraulic head losses at the pipe network and is given as:

$$K_{trans} = \sum_{i=1}^{n} Q_i \, hf_i \tag{2.4}$$

where Q_i is flow rate through pipe i and hf_i the respective head loss.

If we consider that b in Eq. 2.1 is constant, namely that pump efficiencies are constant and equal to each other, the objective function $K1_{tot}$ of the minimization problem reduces to:

$$K1_{tot} = K_{dr} + K_{el} + K_{trans} \tag{2.5}$$

2.2 The Groundwater Flow

In order to calculate K_{dr} we consider steady-state flow in infinite confined aquifers or semi-infinite ones to which the method of images applies (Katsifarakis 2008). Then, s_j in Eq. (2.2) is given as:

$$s_j = -\frac{1}{2\pi T}\sum_{k=1}^{n} q_k \ln\left(\frac{r_{k,j}}{R}\right) \tag{2.6.a}$$

$$s_j = -\frac{1}{2\pi T}\sum_{k=1}^{n} q_k \ln\left(\frac{r_{kj} r'_{kj}}{R^2}\right) \tag{2.6.b}$$

$$s_j = -\frac{1}{2\pi T}\sum_{k=1}^{n} q_k \ln\left(\frac{r_{kj}}{r'_{kj}}\right) \tag{2.6.c}$$

In Eqs (2.6a,b,c) T represents aquifer's transmissivity, R the radius of influence, r_{kj} the distance between wells j and k, and $r_{kj}{}^{\ddot{e}}$ the distance between well j and the image of well k. Eq. (2.6a) holds for infinite aquifers, while (2.6b) and (2.6c) for semi-infinite ones, bounded by an impermeable or a constant head boundary, respectively.

2.3 The Pipe Network

To calculate K_{trans} we use the Darcy–Weisbach formula for each pipe; according to it, hf_i in Eq. (2.4) is given as:

$$hf_i = \frac{8 L_i f_i}{D_i^5 g \pi^2} Q_i^2 \tag{2.7}$$

where D_i is the diameter of pipe i, L_i its length and f_i the friction coefficient. Then, we can define a constant term ξ_i for each pipe, given as:

$$\xi_i = \frac{8\ f_i\ L_i}{D_i^5 g\ \pi^2} \tag{2.8}$$

and hf_i can be written as:

$$hf_i = \xi_i * Q_i^2 \tag{2.9}$$

Moreover, combining Eqs (2.9) and (2.4), we get:

$$K_{trans} = \sum_{i=1}^{n} \xi_i\ Q_i^3 \tag{2.10}$$

Regarding the structure of the pipe network, we assume that each pipe connects a well with another or directly with the water tank, namely we allow pipe connections at the locations of the wells only. In the following analysis pipes are named after the corresponding well, namely the pipe starting for well W_a is termed P_a.

Flow rate Q_a of pipe P_a "carries" the flow rate q_a of well W_a, plus the flow rates of all the "upstream" wells, which are connected directly or indirectly to W_a through the pipe network. This way each Q_i is the sum of certain well flow rates q_j. To express this relationship, we introduce a matrix C, with elements "$c_{i,,j}$". If P_i carries water pumped from well W_j, then $c_{i,j} = 1$, otherwise $c_{i,j} = 0$. Then, Q values can be calculated from the q ones, using the following matrix:

$$\begin{pmatrix} Q_1 \\ \vdots \\ \vdots \\ Q_n \end{pmatrix} = \begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & c_{2,n} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ \vdots \\ q_n \end{pmatrix} \tag{2.11}$$

Next, we introduce the notion of the "path" S_j of well flow rate q_j: S_j includes all pipes P_i through which water pumped from well W_j flows to the tank. There is only one path for each q_j, but a pipe P_i may belong to more than one paths.

Using the previous definitions, we can see that: 1) The elements $c_{i,i}$ that belong to the diagonal of matrix C, are larger than zero, because Q_a is at least equal to q_a. 2) The matrix is not symmetric. 3) If $c_{i,j} = 0, \ \forall i, j \,\&\, i \neq j$, then every well is directly connected to the water tank and obviously $Q_i = q_j$. 4) $c_{i,j} = 1 \Longleftrightarrow P_i \in S_j$.

2.4 Decision Variables and Constraints

Since Q_i can be expressed as sums of q_j values, we have chosen the n well flow rates q_j as decision variables. They are not independent from each other, though, since they are subject to the following constraint:

$$\sum_{j=1}^{n} q_j = q_{tot} \tag{2.12}$$

We can assume, without loss of generality, that for j ϵ [1, n-1] the respective q_j are independent variables, while the flow rate of well n (q_n) depends upon all the rest, namely:

$$q_n = q_{tot} - \sum_{j=1}^{n-1} q_j \tag{2.13}$$

It follows that:

$$\frac{\partial q_j}{\partial q_j} = 1, \quad \frac{\partial q_n}{\partial q_j} = -1 \quad \forall j \in [1, n-1] \tag{2.14.a}$$

$$\frac{\partial q_i}{\partial q_j} = 0 \quad \forall i, j \in [1, n-1] \,\&\, \ i \neq j \tag{2.14.b}$$

Additionally, for a solution to be feasible, the following constraints should hold:

$$q_j \geq 0 \quad \forall j \in [1, n] \tag{2.15}$$

It results, from constraints (2.12) and (2.15) that:

$$q_j \leq q_{tot} \quad \forall j \in [1, n] \tag{2.16}$$

It results from Eqs. (2.15) and (2.16) that the feasible region is convex.

Moreover, it results from constraint (2.15) that for feasible solutions all Q_i are non-negative.

3 Solution of the Cost Minimization Problem

3.1 Critical Points of the Objective Function

To solve the cost minimization problem, the critical points of the objective function $K1_{tot}$ should be found. Its derivatives with respect to q_m, for m = 1 to n-1, should be calculated first. Since $K1_{tot}$ is the sum of K_{dr}, K_{el} and K_{trans}, the respective calculations can be performed separately for each term.

According to Katsifarakis and Tselepidou (2009), for s_j given by any of Eqs. (2.6a,b,c) the derivative of the sum K_{dr} + K_{el} with respect to q_m is given as:

$$\frac{\partial(K_{dr} + K_{el})}{\partial q_M} = 2(s_m - s_n) + (\delta_m - \delta_n) \quad \forall m \in [1, n-1] \tag{3.1}$$

The derivative of K_{trans} with respect to q_m for any m ϵ [1, n-1], can be expressed as:

$$\frac{\partial K_{trans}}{\partial q_m} = \sum_{i=1}^{i=n} \frac{\partial(\xi_i Q_i^3)}{\partial q_m} = 3 \sum_{i=1}^{i=n} \xi_i Q_i^2 \frac{\partial Q_i}{\partial q_m} = 3 \sum_{i=1}^{i=n} hf_i \frac{\partial Q_i}{\partial q_j} \tag{3.2}$$

The derivatives of the pipe flow rates Q_i with respect to any well flow rate q_m can be calculated using Eqs (2.14a,b). They can take one of the following values: 1, 0 or − 1. Non-zero terms (1 or − 1) may arise along the "paths" of q_m and q_n only. It results that:

$$\frac{\partial K_{trans}}{\partial q_m} = 3(Hf_{sm} - Hf_{sn}) \quad \forall m \in [1, n-1] \tag{3.3}$$

where Hf_{sm} and Hf_{sn} denote the sum of head losses along the paths of q_m and of q_n, respectively. These paths may include common pipes.

Using Eqs. (3.1) and (3.3) we get:

$$\frac{\partial K1_{tot}}{\partial q_m} = 2(s_m - s_n) + 3(Hf_{sm} - Hf_{sn}) + (\delta_m - \delta_n) \quad \forall m \in [1, .., n-1] \tag{3.4}$$

Then, the coordinates of the critical points, namely the respective combinations of q_m values, can be found by solving a system of n equations and n unknowns. The first n-1 equations have the following form:

$$2(s_m - s_n) + 3(Hf_{sm} - Hf_{sn}) + (\delta_m - \delta_n) = 0 \tag{3.5}$$

Equation (2.12), expressing the main constraint of the problem, completes the equation system. Equation (2.12) is linear, while all the rest are 2nd degree polynomials. Then, according to the theorem of Bezout, the system has at most $2^{(n-1)}$ solutions.

3.2 Minimum Feasible Value

To find the type of the critical points, the second derivatives of $K1_{tot}$ should be investigated. Starting from Eq. (3.4) and since all δ_i are constant, we get, for any m ϵ [1, n-1]:

$$\frac{\partial^2 K1_{tot}}{\partial q_m^2} = 2\frac{\partial(s_m - s_n)}{\partial q_m} + 3\frac{\partial(Hf_{sm} - Hf_{sn})}{\partial q_m} \tag{3.6}$$

According to Katsifarakis (2008) the first term of the right-hand side of (3.6) is a positive constant, for s_m given by any of Eqs. (2.6a,b,c). Regarding the second term, and starting from Eq. (3.2), we get:

$$\frac{\partial^2 K_{trans}}{\partial q_m^2} = 6\sum_{i=1}^{i=n} \xi_i Q_i \left(\frac{\partial Q_i}{\partial q_m}\right)^2 \tag{3.7}$$

It follows that this term is non-negative, provided that $Q_i \geq 0$. This condition holds throughout the feasible region. The same result can be reached using a physical argument, in connection with Eq. (3.3): If q_m increases (at the expense of q_n), then the head loss Hf_{sm} increases or remains constant, while Hf_{sn} decreases or remains constant. They cannot remain constant at the same time, though. Hence, their difference increases with q_m and the respective derivative is positive. A similar argument holds for the mixed second derivative of K_{trans} with respect to q_m, q_j for any m, j ϵ [1, n-1]. In this case, though, Hf_{sm} and Hf_{sn} can remain constant at the same time.

Up to now, we have proved that in the feasible region

$$\frac{\partial^2 K1_{tot}}{\partial q_m^2} \geq 0 \quad \forall m \in [1, .., n-1] \tag{3.8.a}$$

$$\frac{\partial^2 K1_{tot}}{\partial q_m \partial q_j} \geq 0 \quad \forall m, j \in [1, .., n-1] \& m \neq j \tag{3.8.b}$$

It results from Eqs. (3.8a,b) that no local maxima exist in the feasible region. Moreover, if a local minimum exists, it is unique, since existence of more local minima would require existence of local maxima "between" them, as well, since the feasible region is convex.

To investigate whether a minimum exists in the feasible region, we turn to the "extreme" cases. As a first step, we take into account K_{dr} only, namely we assume that: a) head losses in the pipe network are negligible and b) all δ_j are equal to each other, namely we can ignore K_{el}. In this case (Katsifarakis 2008), K_{dr} is concave with one local minimum only, which is always located inside the feasible region.

The other extreme case is to consider K_{trans} only, namely to assume that drawdown at the water source is negligible (e.g. pumping from lakes with constant level). The minimum value at the feasible region may correspond to inflection point on its boundary, which is described as follows:

For any well j not directly connected to the tank, $Q_j = q_j = 0$.
If wells j and m are directly connected to the tank, then:

$$\frac{Q_j}{Q_m} = \frac{q_j}{q_m} = \left(\frac{\xi_m}{\xi_j}\right)^{1/2} \tag{3.9}$$

Moreover, q_j obey the constraint (2.12), namely their sum equals q_{tot}.

If all wells are directly connected to the tank, then all $q_j = Q_j > 0$ and the critical point is a local minimum. In any case, K_{trans} is concave in the feasible region. It follows from the preceding analysis that in the feasible region the sum of K_{dr} and K_{trans} is concave, with one minimum (or an inflection point in the aforementioned limiting case).

When there is additional flow, resulting in different δ_j values, K_{el} should be taken into account. When examined together with K_{dr}, it may sometimes shift the minimum outside the feasible region, namely to a negative q_j value (Katsifarakis and Tselepidou 2009). In such cases the respective well is excluded, and calculations are repeated with the remaining wells. Such a case may arise when minimizing $K1_{tot}$, too.

4 Solution of the Equation System

The system of the n equations and n unknowns which is described in section 3.1, cannot be solved analytically. It can be easily solved, though, using commercial solvers of equation systems. To facilitate their use, the following matrices are introduced, to express the terms of the equation system:

$$\begin{pmatrix} 2(s_1 - s_n) \\ 2(s_2 - s_n) \\ \vdots \\ 2(s_{n-1} - s_n) \\ q_{total} \end{pmatrix} = 2\{A\}\{q\} = 2\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{(n-1),1} & \cdots & a_{(n-1),n} \\ \frac{1}{2} & \cdots & \frac{1}{2} \end{pmatrix}\begin{pmatrix} q_1 \\ \vdots \\ \vdots \\ q_n \end{pmatrix} \tag{4.1}$$

The last row corresponds to the constraint of Eq. (2.12), whereas the terms of the other n-1 rows depend on the form of Eq. (2.6). Specifically, for infinite aquifers, namely from Eq. (2.6a), and taking into consideration the assumption that two wells do not affect each other if their distance exceeds R, we have:

$$\mathrm{if}\, r_{ji} < R \,\&\, r_{jn} > R : \alpha_{i,j} = -\frac{1}{2\pi T}\ln\left(\frac{r_{ji}}{R}\right) \tag{4.2.a}$$

$$\mathrm{if}\, r_{ji} > R \,\&\, r_{jn} < R : \alpha_{i,j} = -\frac{1}{2\pi T}\ln\left(\frac{R}{r_{jn}}\right) \tag{4.2.b}$$

$$\mathrm{if}\ r_{ji}, r_{jn} < R : \quad \alpha_{i,j} = -\frac{1}{2\pi T}\ln\left(\frac{r_{ji}}{r_{jn}}\right) \tag{4.2.c}$$

$$\mathrm{if}\ r_{ji}, r_{jn} > R : \quad \alpha_{i,j} = 0 \tag{4.2.d}$$

The derivative of K_{trans} can be expressed with the aid of matrix C, defined in Eq. (2.11). Specifically, the transpose matrix C^T is needed, as it is necessary to represent pipe paths using columns and wells using rows. This way we get:

$$3\begin{pmatrix} Hf_{s1}-Hf_{sn} \\ Hf_{s2}-Hf_{sn} \\ \vdots \\ Hf_{sn}-Hf_{sn} \end{pmatrix} = 3\left(\begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & c_{2,n} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{pmatrix}^{T} - \begin{pmatrix} c_{1,n} & \cdots & c_{1,n} \\ c_{2,n} & \cdots & c_{2,n} \\ \vdots & \ddots & \vdots \\ c_{n,n} & \cdots & c_{n,n} \end{pmatrix}^{T}\right)\begin{pmatrix} \xi_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi_n \end{pmatrix}\begin{pmatrix} Q_1^2 \\ \vdots \\ Q_n^2 \end{pmatrix} \quad (4.3)$$

Finally, constant terms like δ_j and q_{tot} are moved to the right-hand side of the matrix equation. This way we derive Eqs. (4.4a,b), or in extended form, Eqs (4.5a,b). Equation (4.4a) represents the system of equations that needs to be solved to minimize the total cost, while (4.4b) needs to be used to connect the pipe flow rates (Q) with the well flow rates (q). We anticipate to get up to $2^{(n-1)}$ solutions, one of them in the feasible region.

$$2\{A\}\{q\}-3\{C_{i-n}^{T}\}\{\xi\}\{Q^2\} = \{\Delta\delta, q_t\} \quad (4.4a)$$

$$\{Q\} = \{C\}\{q\} \quad (4.4b)$$

$$2\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{(n-1),1} & \cdots & a_{(n-1),n} \\ \frac{1}{2} & \cdots & \frac{1}{2} \end{pmatrix}\begin{pmatrix} q_1 \\ \vdots \\ \vdots \\ q_n \end{pmatrix} + 3\begin{pmatrix} c_{1,1}-c_{1,n} & \cdots & c_{1,n}-c_{1,n} \\ c_{2,1}-c_{2,n} & \cdots & c_{2,n}-c_{2,n} \\ \vdots & \ddots & \vdots \\ c_{n,1}-c_{n,n} & \cdots & c_{n,n}-c_{n,n} \end{pmatrix}^{T}\begin{pmatrix} \xi_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \xi_n \end{pmatrix}\begin{pmatrix} Q_1^2 \\ \vdots \\ Q_n^2 \end{pmatrix} = \begin{pmatrix} \Delta\delta_{1,n} \\ \vdots \\ \Delta\delta_{(n-1),n} \\ q_{total} \end{pmatrix} \quad (4.5a)$$

$$\begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix} = \begin{pmatrix} c_{1,1} & \cdots & c_{1,n} \\ c_{2,1} & \cdots & c_{2,n} \\ \vdots & \ddots & \vdots \\ c_{n,1} & \cdots & c_{n,n} \end{pmatrix}\begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \quad (4.5b)$$

5 Application Examples

The procedure of calculating the optimal distribution of q_{tot} to n wells connected to a central tank is illustrated through the following examples.

5.1 Example 1

In this example a simple well layout is examined (Fig. 1): 3 wells are pumping from an infinite aquifer, with $T = 0.025$ m^2/s. The hydraulic head surface is initially horizontal, namely all δ_i are equal. Moreover, $q_{tot} = 0.1$ m^3/s, $R = 3000$ m, $r_0 = 0.2$ m for all wells, $L_i = 500$ m $D_1 = 0.5$ m, $D_2 = D_3 = 0.3$ m and $f_i = 0.03$ for all pipes. The initial choice of f_i can be checked ex post, based on the obtained pipe flow rates. In case of significant discrepancies, the calculation procedure can be repeated, using updated f_i values.

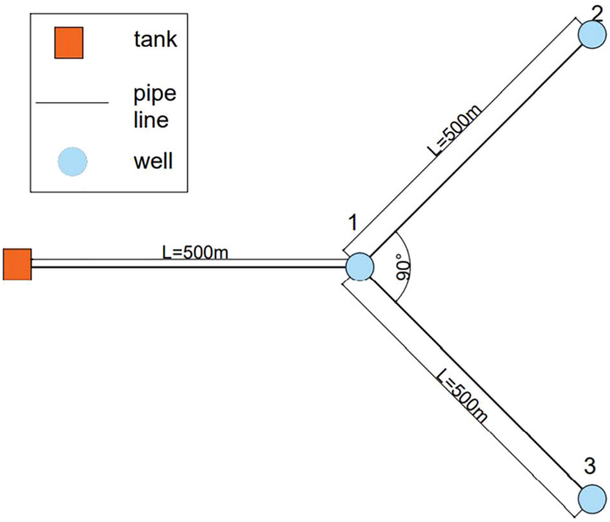

Fig. 1 Layout of the pipe network connecting the 3 wells to the tank (Example 1)

It results that: $\xi_1 = 39.70$, $\xi_2 = \xi_3 = 510.55$ while

$$A = \begin{pmatrix} 49.81 & 2.21 & -49.81 \\ 0 & 52.02 & -52.02 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Regarding matrix C, pipe 1 is included in every flow path S_1, S_2, S_3. For this reason, all the elements of the first row will be equal to 1. Following the same line of thought $P_2 \in S_2$ and $P_3 \in S_3$. This way we have $S_1 = \{P_1\}$, $S_2 = \{P_1, P_2\}$, $S_3 = \{P_1, P_3\}$. Another way of approaching the same matrix is observing that: $Q_1 = q_1 + q_2 + q_3$, $Q_2 = q_2$, $Q_3 = q_3$. Therefore:

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C_{i-n}^T = \begin{pmatrix} 1-1 & 1-1 & 1-1 \\ 0-0 & 1-0 & 0-0 \\ 0-1 & 0-1 & 1-1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Finally this set of data is inserted in Wolfram Mathematica as shown in Fig. 2, to solve the system of Eq. (4.5a) so that $2\Delta s + 3\Delta hf = 0$ for the first (n-1) wells and $q_{tot} = q_1 + q_2 + q_3$:

The q_i values for the only feasible solution ($q > 0$) are the following: $q_1 = 0.041$ m^3/s, $q_2 =$ 0.029 m^3/s, $q_3 = 0.029$ m^3/s. The flow rate q_1 is substantially larger than the other two, due to the cost term K_{trans}, while $q_2 = q_3$ due to symmetry. If we ignore friction losses in the pipe network, we get: $q_1 = 0.032$ m^3/s, $q_2 = q_3 = 0.034$ m^3/s.

5.2 Example 2

In this example, 6 wells, which pump from an infinite aquifer with $T = 0.0025$ m^2/s, are connected to a tank through a pipe network, shown in Fig. 3. The coordinates of the wells are

```
q = {q1, q2, q3}
Q = {Q1, Q2, Q3}
X = {Q1^2, Q2^2, Q3^2}

j = {{39.70, 0, 0}, {0, 510.55, 0}, {0, 0, 510.55}}
a = {{49.81, 2.21, -49.81}, {0, 52.02, -52.02}, {1/2, 1/2, 1/2}}
c = {{1, 1, 1}, {0, 1, 0}, {0, 0, 1}}
cc = {{0, 0, -1}, {0, 1, -1}, {0, 0, 0}}
d = {{0}, {0}, {qtotal}}
b = cc.j
qtotal = 0.1

Solve[Rationalize[2*a.q + 3*b.X == d && Q == c.q] && q1 > 0 && q2 > 0 && q3 > 0, {q1, q2, q3}, {Q1, Q2, Q3}, Reals]
NSolve[Rationalize[2*a.q + 3*b.X == d && Q == c.q] && q1 > 0 && q2 > 0 && q3 > 0, {q1, q2, q3}, {Q1, Q2, Q3}, Reals]
```

Fig. 2 Inserting the matrices in Wolfram Mathematica. Command "Solve" is used to obtain the results

presented in Table 1 and they are the same with those used by Katsifarakis (2008), for comparison purposes. A tank is added, with coordinates $(x_0, y_0) = (0,0)$. The total flow rate that should be pumped is $q_{tot} = 0.5$ m^3/s. Finally, diameter of all wells is $r_o = 0.2$ m and the radius of influence of the system of wells is chosen as $R = 3000$ m. Moreover, we consider that two "extra" wells we$_1$ and we$_2$, with coordinates $(x_{e1}, y_{e1}) = (500,500)$ and $(x_{e2}, y_{e2}) = (800,300)$ respectively, pump independently from the same aquifer, resulting in different initial hydraulic head levels (and different δ_i values) at the 6 wells of the examined system. Regarding the pipe network, the friction coefficient in the pipe network is taken equal to 0.03.

To tackle this problem, R Studio is used. Specifically, packages "sp", "rgeos" and "gdistance" (van Etten 2017) are used, so that anyone using the code can project the wells onto a real map, get the distances and project the pipes. Packages "tmap" (Tennekes 2018) and "tmaptools" are used to plot Fig. 3. Package "nleqslv" (Dennis and Schnabel 1996) is used to solve the equations and finally "ggplot2" (Gómez-Rubio 2017) is used to produce the diagram of Fig. 4.

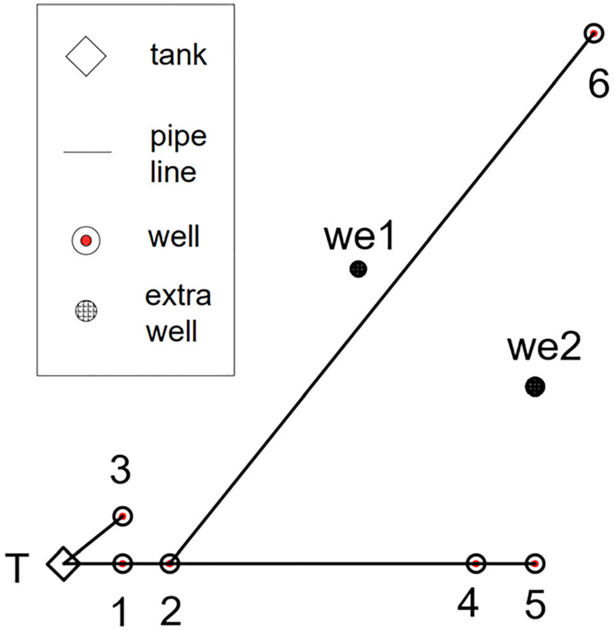

Fig. 3 Layout of the wells and the tank (Example 2)

Table 1 Well coordinates (Katsifarakis 2008)

well	w_1	w_2	w_3	w_4	w_5	w_6
x_i	100	180	100	700	800	900
y_i	0	0	80	0	0	900

Our aim is to investigate the effect of: a) the relative magnitude of s_i and hf_i and b) differences in δ_i values, on the optimal q_i distribution. We use the diameter of pipes as a parameter; to facilitate comparisons, we assume that all network pipes have the same diameter.

In the following, three cases (A, B, C) are discussed. The respective optimal q_i distributions appear in Table 2, together with the two limiting cases LC1 and LC2. In LC1, friction losses in the pipe network are neglected, as in Katsifarakis (2008). In LC2, we take into account friction losses in the pipe network only, as discussed in Section 3 of this paper.

In case A, a large pipe diameter is used ($D = 2$ m) for all pipes, rendering the contribution of friction losses very small, compared to the hydraulic head level drawdowns s_i at the wells. Moreover, the flow rates of we_1 and we_2 are set to 0, namely all δ_i are equal to each other and can be neglected. The results for the pumping rates are shown in Table 2. They are practically identical to those of case LC1 and lead to equal s_i values at the wells.

In cases B and C the pipe diameter is $D = 0.4$ m. In case B the flow rates of the extra wells we_1 and we_2 are zero, while in case C $q_{e1} = 0.1$ m^3/s and $q_{e2} = 0.4$ m^3/s, resulting in different δ_i values. Results appear again in Table 2.

Comparing results of Case B with those of case A, we see that differences are smaller than 2.5%, with the exception of the flow rate of w_3, which is close to the tank and is directly connected to it.

Comparing results of case C with those of case B, we see that the flow rates of the wells w_4 and w_5 are substantially smaller, for the following reason: These wells are closer than the others to the "extra" well we_2; consequently, δ_4 and δ_5 are larger than the others in case C.

Finally, to illustrate the effect of the relevant magnitude of s_i and hf_i, we have produced Fig. 4. It displays the change of optimal flow rates of wells w_1 and w_6 (the closest and the more distant from the tank) with respect to the diameter of the pipe network.

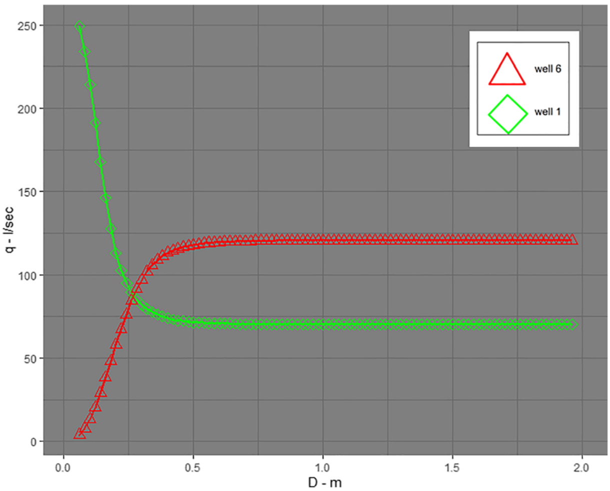

Fig. 4 Optimal pumping rates of wells 1 and 6, with respect to pipe diameter D

Table 2 Optimal well flow rates for all cases (in l/s)

	q_1	q_2	q_3	q_4	q_5	q_6
Case A	70.21	69.92	73.46	80.32	85.48	120.60
Case B	71.61	68.75	80.15	78.23	83.35	117.91
Case C	78.96	73.07	85.45	68.26	72.81	121.46
Case LC1	70.21	69.92	73.46	80.32	85.49	120.60
Case LC2	265.48	0	234.52	0	0	0

6 Conclusions

In this paper, we have investigated the minimization of the total cost to pump a given flow rate q_{tot} from any number n of wells up to a central water tank, taking into account friction losses in the pipe network. Regarding the groundwater flow, we have considered steady state flow in infinite or semi-infinite aquifers, to which the method of images applies. Additional regional groundwater flow can be taken into account, too. Regarding the pipe network that carries water from the wells to the central tank, it can include junctions at the locations of the wells only.

Splitting the objective function $K1_{tot}$, which is proportional to the total pumping cost, in 3 terms, we have derived a new analytical expression (Eq. 3.5) that holds at the critical points of the cost function. Qualitatively, it states that hydraulic head level drawdowns s_i in the aquifer should be smaller at pumping wells that require higher hydraulic head Hf_{si} to overcome friction losses in the pipe network.

Then, based on the aforementioned formula, we have produced a system of n equations and n unknowns, to calculate the sets of well flow rates that correspond to the critical points of the total cost function. The n-1 equations are 2nd degree polynomials, described by Eq. (3.5). The remaining equation is linear, expressing that the sum of well flow rates must be equal to q_{tot}. The solution of the system can be achieved by means of commercial solvers.

While the maximum number of solutions is $2^{(n-1)}$, we have concluded that there is only one feasible solution, that minimizes $K1_{tot}$. This solution can be quite different from the analytical solution, obtained by neglecting the head losses in the pipe network, as indicated in the illustrative examples. Generally, that analytical solution tends to underestimate the flow rates of wells, which are directly connected to the tank.

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Data Availability Not applicable.

Compliance with Ethical Standards

Conflict of Interest None.

Code Availability Code is available by the authors.

List of Symbols

$a_{i,j}$, Coefficient used to calculate the difference of interference of well j with wells i and n (for i = 1 to n-1);

A, Matrix of a_{ij};

b, Cost coefficient;

$c_{i,j}$, Parameter indicating whether pipe P_i carries water pumped from well W_j;

C, Matrix of c_{ij};

D_i, Diameter of pipe i 8;

f_i, Friction coefficient along pipe i;

g, Gravity constant;

hf_i, Head loss along pipe i;

Hf_{sm}, Sum of head losses along the path of q_m;

$K1_{tot}$, objective function of the minimization problem;

K_{dr}, Cost factor, accounting for groundwater hydraulic head level drawdown;

K_{el}, Cost factor, accounting for the difference δ_j between water level at the tank and initial hydraulic head level at each well;

K_{tot}, Total pumping cost;

K_{trans}, Cost factor, accounting for hydraulic head losses at the pipe network;

L_i, Length of pipe i;

n, Number of wells;

Q_i, Flow rate through pipe i;

q_j, Flow rate of well j;

q_{tot}, Total required flow rate;

R, Radius of influence of the wells;

r_{kj}, Distance between wells j and k;

r'_{kj}, Distance between well j and the image of well k;

S_j, "Path" of well flow rate q_j; it includes all pipes P_i through which water pumped from well W_j flows to the tank;

T, Aquifer's transmissivity;

δ_j, Difference between water level at the tank and initial hydraulic head level at each well;

ξ_i, Head loss coefficient of pipe i

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