



Improved Water Allocation under Limited Water Supplies Using Integrated Soil-Moisture Balance Calculations and Nonlinear Programming

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Abstract

Water allocation under limited water supplies is becoming more important as water becomes scarcer. Optimization models are frequently used to provide decision support to enhance water allocation under limited water supplies. Correct modelling of the underlying soil-moisture balance calculations at the field scale, which governs optimal allocation of water is a necessity for decision-making. Research shows that the mathematical programming formulation of soil-moisture balance calculations presented by Ghahraman and Sepaskhah (2004) may malfunction under limited water supplies. A new model formulation is presented in this research that explicitly models deep percolation and evapotranspiration as a function of soil-moisture content. The new formulation also allows for the explicit modelling of inefficiencies resulting from nonuniform irrigation. Modelling inefficiencies are key to the evaluation of the economic profitability of deficit irrigation. Ignoring increasing efficiencies resulting from deficit irrigation may render deficit irrigation unprofitable. The results show that ignoring increasing efficiencies may overestimate the impact of deficit irrigation on maize yields by a maximum of 2.2 tons per hectare.

Keywords Deficit irrigation · Nonlinear programming · Optimization · Soil-moisture balance · Water allocation

1 Introduction

Globally, irrigation agriculture is the largest user of water (Molden 2007). As a result, many research efforts are devoted towards optimal allocation of limited water supplies in agriculture. A key component of any water allocation model is the representation of temporal crop water use. It is fairly easy to derive irrigation schedules that will satisfy crop water demand under full water allocation, provided that information on soil-moisture holding capacity, initial soil-moisture

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content, rainfall and crop water demand is available (Allen et al. 1998). Water allocation becomes more complex when water supply is limited because water stress and resulting crop yield reductions set in when root water content falls below a critical point. The problem is that the onset and duration of water stress can only be determined by tracking the root water content.

Mathematical programming¹ models are preferred by many researchers to study water management problems (Singh 2014). Including soil-moisture balance calculations into optimization models is troublesome because the boundaries placed on soil-moisture balance components result in discontinuous nonsmooth functions. Standard gradient based solvers such as MINOS (Murtagh et al. 1998) are therefore unsuitable to solve those models. Consequently, some form of simplification or reformulation is necessary to incorporate soil-moisture balance calculations into mathematical programming models.

Ghahraman and Sepaskhah (2004) are frequently credited with integrating soil-moisture balance calculations into linear and nonlinear programming models to optimally allocate limited water supplies (Moghaddasi et al. 2010; Montazar et al. 2010; Safa et al. 2012; Kanooni and Monem 2014; Sadati et al. 2014; Singh 2014). The difficulty of including realistic soil-moisture balance calculations into mathematical programming models is accentuated by the fact that deep percolation is modelled as a constant ratio of applied irrigation water irrespective of the soil-moisture status, thus defeating the purpose of including soil-moisture balance calculations, which is to determine the onset of water stress and the duration thereof. Furthermore, the standard reformulation of the MIN function with two inequalities is used to relate root water content to evapotranspiration results in time periods with zero evapotranspiration, even though the water content is above the wilting point. Several researchers have applied the Ghahraman and Sepaskhah (2004) formulation (GSF) to model soil-moisture balances, even though they warned against the shortcomings thereof.

Moghaddasi et al. (2010) used the GSF to compare an optimization approach in which water was allocated during droughts with the traditional approach of equitable water distribution. The authors did, however, include a constraint to preclude zero evapotranspiration within a specific time interval. The optimization approach proved to yield 42% more income compared to equitable water reduction. In another study, Safa et al. (2012) integrated the GSF of agricultural water use optimization with long-term river flow forecasts to enhance water allocation during drought periods. Kanooni and Monem (2014) used the GSF to determine the optimal intra-seasonal water distribution for different seasonal water availabilities using linear programming. The information was used in a subsequent model to optimize water delivery via irrigation canals. Sadati et al. (2014) used a genetic algorithm (GA) to solve their water allocation model with an integrated soil-moisture balance. The solution procedure employed by GAs is more suited to handle the inherent discontinuities of the MIN functions used in hydrological modelling. The model developed by Sadati et al. (2014) correctly placed the appropriate bounds on evapotranspiration calculations. However, deep percolation was modelled as being proportionate to irrigation irrespective of the state of the soil-moisture content.

The inability of previous efforts to model deep percolation as a result of the soil-moisture status clearly demonstrates the lack of mathematical programming procedures available to model water allocation under limited water supply. Haro et al. (2012) argue that the simplifications modelers make to overcome hardware and software limitations lead to optimization models, which may not approximate the reality of the systems modelled. In this paper, it is argued that the assumption that deep percolation is proportionate to irrigation amounts is a misrepresentation of

¹ Mathematical programming refers to the use of linear and non-linear programming in this paper.

reality. Furthermore, the use of the standard reformulation of the MIN function with two inequalities to model evapotranspiration is challenged. The reformulation works well under condition that evapotranspiration is maximized through a term in the objective function (GAMS Development Corporation 2017). Under unlimited water supply conditions, deficit irrigation is not considered, and evapotranspiration is maximized to achieve near potential crop yields. However, under limited water supply, it might be profitable for the crop to sustain evapotranspiration deficits in one time period to avoid deficits in subsequent periods when the crop is more sensitive to crop water stress. Consequently, it will be optimal to minimize evapotranspiration in a specific time period, which will cause the standard reformulation to malfunction. Consequently, improved decision-making may be hampered due to a lack of trust in the modelling results.

The contribution of this paper to the literature on water allocation is that a new model formulation that appropriately models deep percolation and evapotranspiration as a function of the status of soil-moisture content is presented. The improved formulation allows for the explicit modelling of inefficiencies resulting from nonuniform irrigation water applications, which provides a better representation of water allocation under limited water supply. The model is applied to an intraseasonal water allocation problem where both the irrigation area and distribution of water over the growing season are choice variables. In this paper, the nonlinear programming water allocation model used by Ghahraman and Sepaskhah (2004) and an improved formulation of the model is presented. The two model formulations are compared under unlimited and limited water supply conditions to highlight the caveats of the procedures adopted in the GSF under limited water supply. Finally, the implications of using the model for water allocation are discussed.

2 Nonlinear Programming Model Specifications

Both the GSF and the improved formulation are discussed in this section. The most significant difference between the two models is the way in which the soil-moisture balance calculations are included in the optimization model. The convention is used whereby variables are denoted by capital letters and data otherwise when discussing the models.

2.1 Objective Function

According to English et al. (2002), economic efficiency principles should be used to guide irrigation application in contrast to applying irrigation water to achieve maximum crop yield. The following objective function is common to the two model specifications and aims at maximizing the margin above specified costs (MAS) by changing the area irrigated and the irrigation schedule:

$$MAS = (prYa - ac - \sum_r IR_r \times idc - ycYa)A \quad (1)$$

where MAS is the total gross margin calculated for the total irrigated area in Rands, A is the irrigated area in hectares, and the IR_r are the daily irrigation amounts in millimetres of water applied per hectare. Production income per hectare is calculated by multiplying the price of maize (pr) in Rands per ton with the resulting maize yield (Ya) in tons per ha. The MAS calculation allows for cost components to vary with irrigated area, irrigation amounts and the resulting crop yields. ac represents the area dependent costs in Rands per hectare, and idc represents the irrigation dependent costs in Rands per millimetres of water applied. Harvesting cost is dependent on crop yield and is calculated by multiplying yield dependent harvesting costs (yc) in Rands per ton with the resulting

crop yield. Collectively, the terms in parenthesis calculate the margin above specified costs per hectare, which is multiplied with the irrigated area to determine the MAS for the total area.

Calculation of Y_a is complex under limited water supply conditions and requires knowing the soil-moisture depletion in relation to critical soil-moisture depletion levels, which necessitates the inclusion of soil-moisture balance calculations into the model. The procedures used in the GSF to relate soil-moisture balance calculations to crop yield is presented next, followed by the improved formulation.

2.2 Crop Yield Calculation: Ghahraman and Sepaskhah Formulation

Dated crop water production functions are typically used in water resource allocation models to determine the effect of water deficits on crop yield. The multiplicative form presented by Stewart et al. (1977) is a popular method to relate evapotranspiration deficits to crop yield:

$$Y_a = ym \times \prod_g \left[1 - ky_g \left(1 - \frac{\sum_{t \in g} ETa_t}{\sum_{t \in g} etm_t} \right) \right] \quad (2)$$

where Y_a is the actual crop yield in tons per hectare, ym is potential crop yield in tons per hectare under conditions of no water stress, ky_g is an indication of the sensitivity of the crop to water stress in stage g , ETa_t and etm_t are, respectively, the actual and the no water stress potential evapotranspiration rates in mm on day t . The impact of water stress on crop growth is different between the vegetative, flowering, yield formation and ripening crop growth stages (g) as modelled with ky_g . The product indicates that the crop needs water in all growth stages to realize a crop yield, however, water deficits greater than 50% are invalid in any stage (Doorenbos and Kassam 1979). etm_t is an input that is calculated through the multiplication of reference evapotranspiration with the crop coefficient or K_c (Allen et al. 1998).

According to Allan et al. (1998), ETa_t is a function of the soil-moisture status. Total available soil-moisture is defined as the difference between field capacity and permanent wilting point. Theoretically, water is available for production until the permanent wilting point. However, transpiration is reduced before the permanent wilting point is reached because soil water becomes more tightly bound to the soil matrix as the soil gets dryer (Allan et al. 1998). Once a threshold soil-moisture content is reached, potential transpiration is reduced, and the crop suffers from crop water stress. Soil-moisture is related to ETa_t in the model through Constraints 3 and 4 as follows:

$$ETa_t \leq \frac{SM_t}{(1-p_t)(fc-pwp)} etm_t \quad (3)$$

$$ETa_t \leq etm_t \quad (4)$$

where SM_t is calculated with a daily soil-moisture balance and represents the average soil-moisture content in mm on day t in the root zone. fc is the soil-moisture in mm at field capacity, and pwp is the permanent wilting point in mm, while the difference between soil-moisture contents is the total available water in the root zone. p_t is the fraction of total available water that is readily available to the crop. Constraint 3 shows that the maximum value of ETa_t is restricted to a portion of etm_t once SM_t falls below a critical depletion level, p_t . The maximum value of ETa_t is restricted to etm_t with Constraint 4 to ensure that ETa_t is not adjusted above etm_t with Constraint 3 when SM_t is above the threshold soil-moisture $(1 - p_t)(fc)$

–pwp) where water stress commences. The assumption is that if ETA_t is maximized, only one of Constraints 3 or 4 will be dominant in the determination of ETA_t . It is important to note that Constraints 3 and 4 together do not preclude zero evapotranspiration and the correct calculation of ETA_t is critically dependent on the notion that ETA_t is maximized. The SM_t at the beginning of any given day could be represented by Constraint 5:

$$SM_t = SM_{t-1} + rain_{t-1} + IR_{t-1} - ETA_{t-1} - DP_{t-1} + \Delta sm_{t-1} \tag{5}$$

where $rain_{t-1}$ and IR_{t-1} , respectively, represent the rainfall and irrigation amounts in mm that enter the soil profile on the previous day, DP_t is deep percolation in mm, and Δsm_t is the change in the soil-moisture content due to root growth. IR_t is the only decision variable that is under direct control of the irrigator, while ETA_t and DP_t are derived from IR_t through the SM_t calculation. Percolation below the root zone will not influence the quantity of water that is added to the SM_t given the layer below the root zone is at field capacity. Δsm_t could therefore be calculated based on deterministic root growth and thus is a constant. It is important to note that DP_t is not governed by Constraint 5 since DP_t only occurs if the addition to SM_t is more than the capacity. GSF uses the following equation to govern the behaviour of DP_t in their model:

$$DP_t \geq IR_t(1 - e_a) \tag{6}$$

where e_a is the irrigation efficiency. Constraint 6 implies a constant efficiency even though it is well known that deficit irrigation increases the efficiency of water applications (English and Raja 1996) and therefore also reduces the chance of deep percolation. Ghahraman and Sepaskhah (2004) developed procedures to model increasing efficiency with deficit irrigation. However, their procedure only applies to seasonal response functions and is not a function of daily soil-moisture. In this paper, it is argued that the assumption that DP_t is proportional to irrigation applications is unrealistic. In the next section, an improved formulation of soil-moisture balance constraints is proposed that addresses the unrealistic assumption.

2.3 Crop Yield Calculation: Improved Formulation

The formula used to calculate the crop yield with the improved formulation is in essence exactly the same as the GSF. However, the improved formulation uses multiple soil-moisture balances to model nonuniform water applications as a source of inefficiencies. Nonuniform water applications imply that a portion of the irrigated field will be under irrigated and a portion of the field will be over irrigated (Li 1998). Consequently, a crop yield needs to be calculated for each portion of the field using a separate soil-moisture balance. The actual crop yield included in the objective function of the improved model is the average crop yield across the different soil-moisture balances and is calculated as follows:

$$Ya = \sum_w \left\{ ym \times \prod_g \left[1 - ky_g \left(1 - \frac{\sum_{t \in g} ETA_{tw}}{\sum_{t \in g} etm_t} \right) \right] \right\} / card(w) \tag{7}$$

where the subscript w is used to indicate the different soil-moisture balances, and $card(w)$ indicates the number of soil-moisture balances included in the model. The calculation of ETA_t and SM_t were reformulated in the improved model to ensure that the calculation of ETA_t and deep percolation is in complete harmony with the status of the soil-moisture balances included

in the optimization model. An MIN function is used to ensure that the answers that are calculated for ETa_{tw} and SM_{tw} conform to a closed form calculation as follows:

$$ETa_{tw} = MIN \left(\frac{SM_{tw}}{(1-p_t)(fc-pwp)} etm_t, etm_t \right) \tag{8}$$

$$SM_{tw} = MIN(SM_{t-1w} + rain_{t-1} + sf_w IR_{t-1} - ETa_{t-1w} + \Delta sm_{t-1w}, (fc-pwp)) \tag{9}$$

It is important to note that Constraint 9 does not include DP_{tw} in the calculation of SM_{tw} because the level of soil-moisture in the root zone is truncated at $fc-pwp$, which defines the maximum amount of water that could be stored in the root zone. Thus, the excess water is attributed to DP_{tw} . The result is that DP_{tw} is now a function of the status of SM_{tw} in the root zone as influenced by any additions of $rain_t$, IR_t and changes in the soil-moisture due to root growth or losses due to ETa_{tw} . Such a specification allows for the incorporation of the impact of nonuniform water applications as the impact of over irrigation on DP_{tw} is explicitly modelled. Nonuniform irrigation applications were modelled by dividing the irrigated area into three equal portions. Each of the areas received a different amount of water, which is calculated based on the assumption that irrigation applications were uniformly distributed between specified minimum and maximum amounts. Only three soil-moisture balances were included in the programming model to represent nonuniform water applications to reduce the size of the programming matrix. The scaling factor (sf_w) with which IR_t is adjusted for the portion of the field that receives more ($w = upper$) or less ($w = lower$) water than IR_t is calculated as follows while assuming a uniform distribution (Li 1998):

$$sf_w = \begin{cases} 1 + (1-CU) \text{ given } w \text{ is upper} \\ 1 \text{ given } w \text{ is actual} \\ 1 - (1-CU) \text{ given } w \text{ is lower} \end{cases} \tag{10}$$

where CU defines the Christiansen coefficient of uniformity as a fraction. The assumption that irrigation applications are uniformly distributed between a minimum and a maximum ensures that IR_t equates to the average irrigation application over the whole field. Consequently, IR_t enters the objective function as the average irrigation application.

Including Constraints 8 and 9 into a mathematical programming model is problematic because standard solvers are unable to handle the discontinuity embedded in the functions. A smooth approximation of the MIN function is undertaken using the following generalized format (GAMS Development Corporation 2017):

$$z = \left(f + g - \sqrt{(f-g)^2 + (\delta)^2} \right) / 2 \tag{11}$$

where δ represents a small number. Applying the generalization to Constraint 8 for example requires the following substitutions:

$$f = \frac{SM_{tw}}{(1-p_t)(fc-pwp)} etm_t \tag{12}$$

$$g = etm_t \tag{13}$$

Consequently, the model becomes highly nonlinear in the constraint set and more difficult to solve.

2.4 Limits on Variables

Limits on variables are necessary to ensure that the model conforms to the situation being modelled and to increase the validity thereof. The following two equations are common to both optimization model formulations and, respectively, constrain the total and daily irrigation water applications:

$$\sum_t IR_t A \leq alloc \times ce \quad (14)$$

$$IR_t \leq icap \quad (15)$$

where *alloc* is the total water allocation in mm, *ec* is the conveyance efficiency as a fraction, and *icap* is the irrigation system delivery capacity in mm over time period *t*. Constraint 14 limits the total amount of water applied to be less than the water allocation while considering losses due to conveyance. On the other hand, Constraint 15 limits the irrigation application within a specific time period to be less than the irrigation system delivery capacity.

According to Doorenbos and Kassam (1979), evapotranspiration deficits more than 50% are invalid in any growth stage. The following constraint is used to constrain evapotranspiration deficits to be less than 50% in the GSF:

$$\sum_{t \in g} ETa_t \leq \sum_{t \in g} 0.5etm_t \quad (16)$$

The same restriction applies to the improved formulation, but the constraint is valid for each soil-moisture balance included in the optimization model as follows:

$$\sum_{t \in g} ETa_{tw} \leq \sum_{t \in g} 0.5etm_t \quad (17)$$

3 Solution Procedure

The two model formulations of the intra-seasonal water allocation were developed using the General Algebraic Modelling System (GAMS) (GAMS Development Corporation 2017) and solved with MINOS (Murtagh et al. 1998). The GSF of the complete water allocation model includes the objective function (1), Constraints 2–6 and Constraints 14–16. The objective function and crop yield calculation are the only non-linear equations in the model. The improved formulation consists of the objective function (1), Constraint 7, the square root representation of Constraints 8 and 9, Constraints 14–15 and Constraint 17. The constraint set in the improved model is highly non-linear due to the square root approximation of the MIN functions used in Constraints 8 and 9, which makes it more difficult to solve.

4 Data and Analysis

The data used to compare the two models originate from the SAPWAT model (Van Heerden and Walker 2016), which is the South African equivalent of CROPWAT (Smith 1992). Data were extracted for a short growing variety of maize that is planted in the semiarid region of Douglas where full irrigation is practised. Centre pivot irrigation with an application rate of 12 mm/day is dominant in the area according to a local irrigation scheduling consultant. The

soils are characterized as clay-loam soils with a soil-moisture holding capacity of 130 mm/m. The crop data for the optimization models are summarized in Table 1. All the economic parameters were obtained from the local cooperative in the region.

The water production function for the improved model formulation was derived by fixing the irrigated area to the size of the pivot and varying the water allocation. A CU of 85% was assumed to model the impact of non-uniform water applications. Percolation of water below the root zone is an important output of the model as it is an indication of the efficiency with which irrigation water applications are managed. The improved model specification does not include a variable to quantify DP_{tw} . The amount of water that percolates below the root zone is calculated with Eq. 18 using the optimized results:

$$DP_{tw} = \text{MAX}(SM_{t-1w} + rain_{t-1} + sf_w IR_{t-1} - ET a_{t-1w} + \Delta sm_{t-1w} - (fc - pwp), 0) \quad (18)$$

The drainage amount calculated for no water stress conditions with the improved formulation was subsequently used to calibrate the model for the GSF to yield the same drainage level when expressed as a proportion of irrigation. Accordingly, e_a was calculated as 80.5%. The calibrated model was used to derive the water production function for the GSF by varying the water allocation without allowing for changes in the irrigated area. Two points on each derived production function were selected to evaluate the intraseasonal distribution of water within the season. The selected points correspond to the full water allocation under conditions of no water stress and a 20% reduction in the water supply. The preceding analyses ignore the opportunity costs of water, as the irrigated area was assumed to be fixed. The opportunity cost of water was included by solving the two models while allowing the model to determine the area irrigated in conjunction with the irrigation schedule for a specified water supply.

5 Results and Discussion

5.1 Water Production Functions

Water production functions, which define the relationship between applied water and crop yield, form the basis of water allocation models, the results of which are shown in Fig. 1.

The results show large differences between the two model specifications. The production function for the GSF is comprised of two more or less linear segments between 267 and 340 and 340 to 646 mm where it reaches the maximum maize yield of 17 tons/ha. The improved

Table 1 Summarized crop data for the optimization models

Crop developmental stages				Water stress sensitivity		
Stage	Duration (days)	Kc-values ¹ (fractions)	etm (mm)	Stage	Duration (days)	Ky-values ² (fractions)
Initial	21	0.3	185	Vegetative	50	0.3
Development	26	0.3–1.15	102	Flowering	15	1.05
Mid-season	63	1.15	258	Yield formation	45	0.5
Late season	10	1.15–0.1	23	Ripening	10	0.2

¹ Crop coefficient, which relates reference evapotranspiration to the potential evapotranspiration of maize

² Crop water stress sensitivity factor for maize

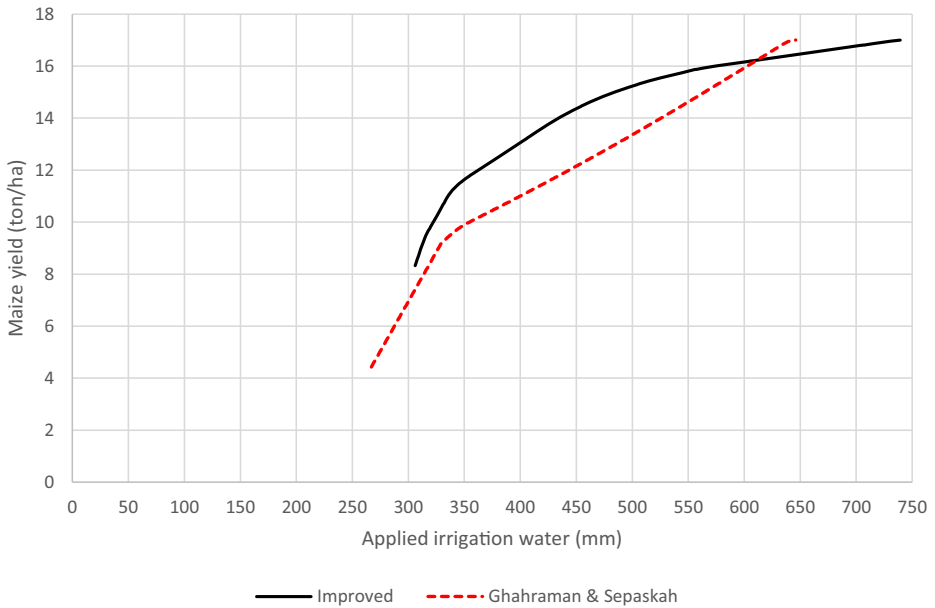


Fig. 1 Water production functions generated with the Ghahraman and Sepaskhah and improved formulations

formulation exhibits much more curvature, and for water applications below 609 mm, the maize yields for the improved model specification are higher than for the other model formulation. The curvature is the direct result of incorporating nonuniform water applications into the analysis. The linear segments of the GSF more closely follow the optimal crop water production function, which defines the relationship between consumptive water use and crop yield, because the inefficiencies are assumed to be constant. The form of the water production function has some serious implications for water allocation. In the next section, within season water distribution is evaluated to determine the ability of the two model specifications to model the soil-moisture balance components correctly.

5.1.1 Soil-Moisture Balance Evaluation: Full Water Allocation

The optimized soil-moisture balance components for the GSF and improved formulation for a full water allocation under no crop water stress are shown in Fig. 2 and Fig. 3, respectively.

Figure 2 shows that no ET deficits occurred for 646 mm of irrigation because the SM_t exceeded the critical soil-moisture content $((1-p)(fc-pwp))$ for the entire season. The results clearly highlight the inability of the GSF to correctly model SM_t . Deep percolation below the root zone can only occur if the inflow into the soil is more than the soil-moisture holding capacity in the root zone ($fc-pwp$). However, Constraint 6 forces DP_t to occur with every irrigation event. As a result, Fig. 2 shows DP_t to occur even though the SM_t is not close to the $fc-pwp$ moisture content. In total, 126 mm of the applied irrigation water percolated below the root zone, resulting in an irrigation efficiency of 80.5%.

In contrast to the GSF, the average DP_{tw} of the improved model formulation was in complete agreement with the water status of the soil-moisture store. The results of all three soil-moisture balances that were used to model nonuniform water applications with the improved model formulation are included in Fig. 3. To avoid crop water stress, enough water

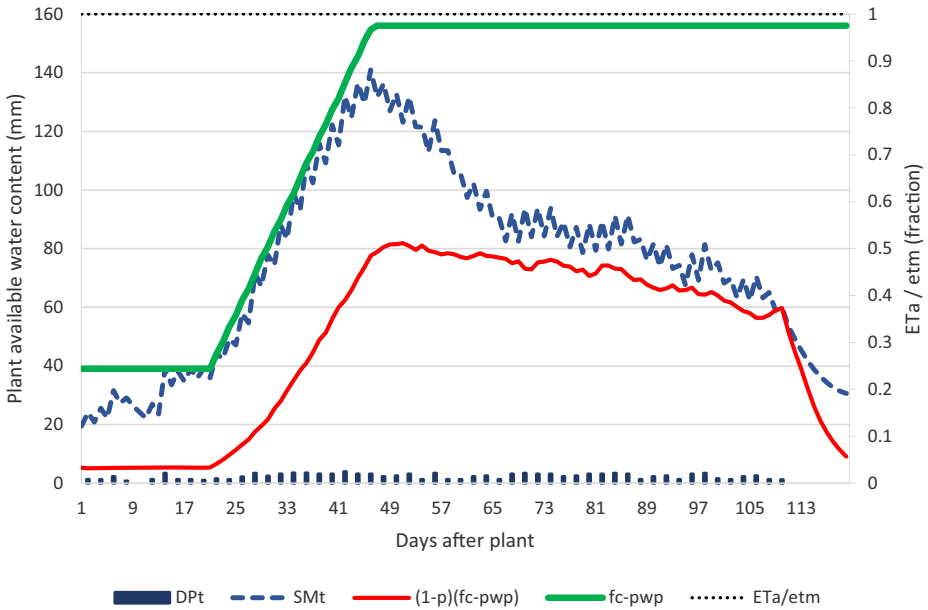


Fig. 2 Optimized soil-moisture budget components for the Ghahraman and Sepaskhah formulation for a water allocation of 646 mm/ha

should be applied to avoid water stress in the third of the field that received the least water ($SM_{t\ lower}$). Consequently, other parts of the field will be over irrigated, which may result in DP_{tw} . The part of the field that received the actual amount of irrigation ($SM_{t\ actual}$) and the part

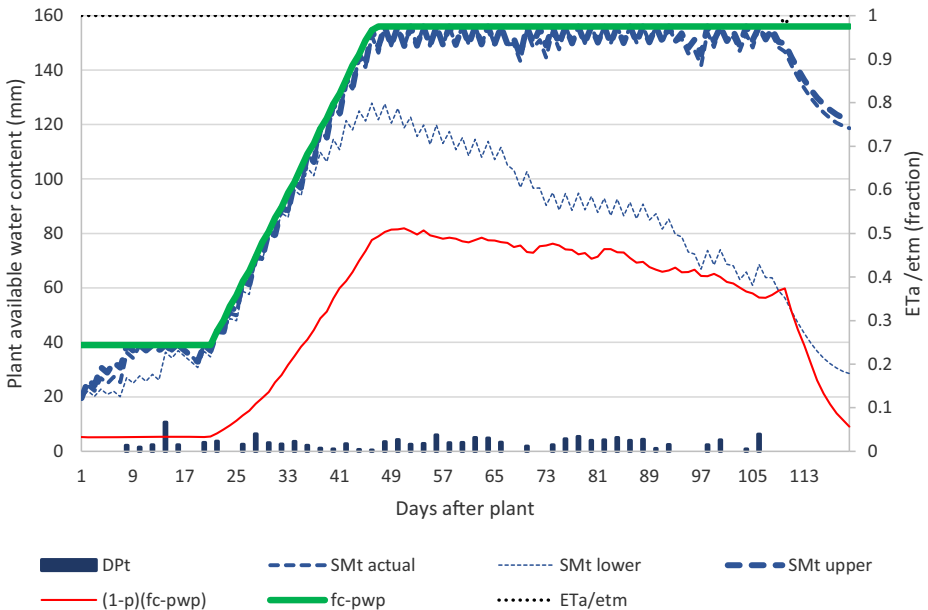


Fig. 3 Optimized soil-moisture balance components for the improved formulation with a water allocation of 739 mm/ha

that received more than the actual amount of irrigation ($SM_{t\ upper}$) are indeed close to the $fc\text{-pwp}$ moisture content when DP_{tw} occurs. Furthermore, the amount of DP_{tw} varies depending on the irrigation amounts in conjunction with the status of the soil-moisture store. Interestingly, 93 mm more irrigation water is needed to achieve nearly nonstressed production levels when compared with the GSF. The increase in irrigation water required to sustain nearly non-stressed crop yields is the direct result of including non-uniform water applications in the model. Decreasing the CU causes larger deviations in the water applications of the two thirds of the field that do not receive the average water application. Larger deviations require larger irrigation applications on average to ensure no crop water stress in the portion of the field that receives the least water. Higher irrigation applications increase the probability of percolation. In total, 144 mm of the 739 mm applied irrigation water percolated below the root zone during the season, resulting in an irrigation application efficiency of 80.5%.

5.1.2 Soil-Moisture Balance Evaluation: Limited Water Allocation

Next, the soil-moisture balance components were evaluated at a point on each production function characterized by a 20% reduction in water supply. The results are shown in Fig. 4 and Fig. 5 for the GSF and improved formulations, respectively.

The results show that inappropriate levels of DP_t , which are not in harmony with the status of the soil-moisture balance, are not alleviated under limited water supply conditions. The same proportion of applied irrigation water drains below the root zone, even though deficit irrigation reduces the chance of deep percolation.

Conditions of limited water supply highlight another problem with the GSF in that zero ET_t values are possible even though SM_t exceeds the critical soil-moisture content ($(1-p)(fc\text{-pwp})$) during which crop water stress commences. The zero ET_t levels during the

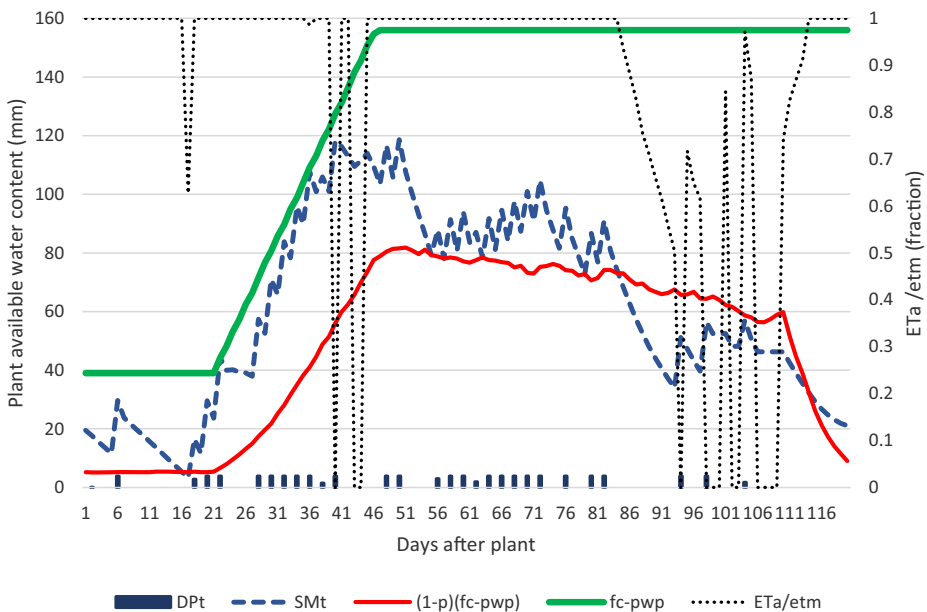


Fig. 4 Optimized soil-moisture balance components for the Ghahraman and Sepaskhah formulation with a water allocation of 517 mm/ha

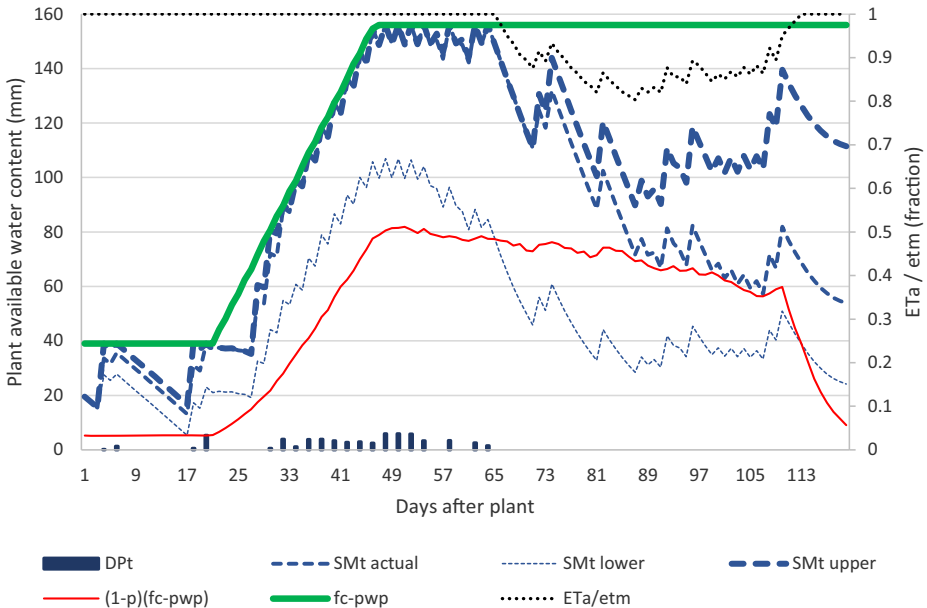


Fig. 5 Optimized soil-moisture balance components for the improved formulation with a water allocation of 591 mm/ha

vegetative stage when the crop is not that sensitive to water stress implies that the SM_t will be higher. Consequently, more water is available for growth in subsequent crop growth stages where the crop is more sensitive to water stress compared to the vegetative stage. Thus, zero ET_t is optimal but illogical in terms of the SM_t in relation to the critical soil-moisture content. The wrong calculation of ET_t is the direct result of the free formulation used in Constraints 3 and 4 to model ET_t , as a zero ET_t satisfies both constraints. The fact that time periods are dynamically linked causes the optimizer to acknowledge that it is more valuable to have water available during later stages where the impact of crop water stress is more severe. The necessary assumption that ET_t will be driven upwards is thus not met, which results in the malfunctioning of Constraints 3 and 4. The improved formulation overcomes the problem by forcing the model to assign appropriate values to ET_t based on the status of the soil-moisture content using the square root approximation of the MIN function.

Figure 5 shows that under limited water, it is optimal to practice deficit irrigation during yield formation and ripening of the crop. Water applications are undertaken such that only a third of the field will suffer water stress. Furthermore, the results show a considerable reduction in DP_{tw} when deficit irrigation is practised. Beyond day 65, no DP_{tw} occurs because deficit irrigation causes the water content of all three soil-moisture balances to be reduced, thereby reducing the chance of DP_{tw} during irrigation.

5.2 Optimal Water Allocation

In the current context, the water allocation problem under limited water supply is characterized by a decision regarding the area to irrigate, which will determine the water application per unit area. The answer to the problem is dependent on whether water or land is the most limiting

Table 2 Optimized applied irrigation water, percolation, maize yield and increase in irrigated hectares for two model formulations in the water limiting phase of production

	Model formulation	
	Ghahraman and Sepaskhah	Improved
Applied irrigation water (mm/ha)	639.42	457.23
Percolation (mm)	124.85	21.65
Maize yield (ton/ha)	16.95	14.51
Decrease in hectares (ha/30.1 mm)	0.047	0.066

factor of production. In the land limiting phase, the most profitable decision is to irrigate all the land. As long as land is the most limiting factor, reducing water supply will only reduce water applications per hectare with reduced crop yields per hectare. Consequently, production levels will follow the production functions derived in Fig. 1 without changing the area irrigated. Irrigated area will be reduced once water becomes so limited that production per hectare is not profitable, and the only way to make production profitable is to reduce the irrigated area to increase production levels per hectare. During the water limiting phase, irrigated area will be reduced at a constant rate to maintain profitable production levels when total water supply is reduced at a constant rate. Table 2 shows the onset of the water limiting phase and the profitable production levels for the two model formulations.

Results for the GSF shows that the onset of the water limiting phase occurs when water availability is such that water applications of 639 mm/ha are achieved, which is very close to full production. The irrigated area needs to be reduced by 0.047 ha in the water limiting phase if the water allocation is reduced by 30.1 mm to maintain water application rates of 639 mm/ha to achieve a maize yield of 16.95 tons/ha. In contrast, the level of deficit irrigation that is profitable during the water limiting phase with the improved model formulation is much higher. With the improved formulation, it is optimal to apply only 457 mm/ha, which signifies a substantial level of deficit irrigation as indicated by optimal crop yields of 14.51 tons/ha. The results also show that deficit irrigation will reduce percolation to 21.65, thereby increasing the efficiency of water use.

6 Conclusions

The closed form square root approximation of the MIN function used to model soil-moisture content in the root zone and evapotranspiration proved to be superior to the free form formulation of GSF. This is especially true for conditions of limited water supply. The highly non-linear model was solved with ease using MINOS (Murtagh et al. 1998), while the solution time was not much longer than the GSF with its linear constraint set. Furthermore, the approximation error around the truncation point was smaller than 0.01 mm, which is deemed negligible.

The improved formulation has the benefit that equalities are used, which forces the answer to take on a value based on the level of the arguments of the function. Consequently, deep percolation losses could be modelled explicitly as a function of applied irrigation water in relation to the soil-moisture content in the root zone. Correct modelling of deep percolation allows for the incorporation of non-uniform water applications as a source of inefficiency through the inclusion of multiple soil-moisture balances. This resulted in inefficiencies being

modelled as a function of irrigation scheduling decisions. Increasing efficiencies under deficit irrigation are therefore modelled endogenously.

Results from the analysis clearly demonstrated the importance of modelling increasing efficiency under deficit irrigation more appropriately, as the conclusions drawn from the two models differ substantially. Notwithstanding the fact that the GSF is not correct under limited water supply conditions, the model indicated that deficit irrigation is not profitable and that it is better to reduce irrigated area to attain near maximum crop yields. On the other hand, the improved model specification with a more explicit method of modelling inefficiencies indicated that deficit irrigation is profitable.

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