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Minimization of Transient Groundwater Pumping Cost - Analytical and Practical Solutions

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Abstract In this paper, we study pumping cost minimization for any number and layout of wells under transient groundwater flow conditions in infinite confined aquifers and semi-infinite ones, to which the method of images applies. Moreover, we take into account additional steady-state flow, which is independent of the well system and results in non-horizontal initial hydraulic head level distribution. We prove analytically that, at any time, the instant pumping cost is minimum, when the following condition holds: the observed at that instant differences between hydraulic head values at the locations of the wells are equal to the half of the initial ones, which are due to the additional steady-state flow. Based on this proof, an analytical calculation procedure of the time-dependent optimal distribution of the required total flow rate to the individual wells is also presented. Moreover, as well flow rates usually remain constant over the pumping period, an approximate calculation of the optimal constant well flow rate distribution is outlined, based again on an analytical procedure.

Keywords Transient groundwater flow · Pumping cost · System of wells · Optimization · Analytical solution · Method of images

1 Introduction

Minimization of energy consumption, usually translated as pumping cost, is one of the most common problems in groundwater resources management. Simulation of groundwater flow is usually part of the optimization procedure and may determine the difficulty of the respective optimization problem (e.g. Ketabchi and Ataie-Ashtiani 2015; Moutsopoulos et al. 2017; Singh and Panda 2013; Sreekanth and Datta 2011). Moreover, there are often additional constraints to the optimization process, such as flow rate limits, due to pump capacities, or limits to hydraulic

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head drawdown in parts of the aquifer (e.g. Bayer et al. 2009). In other cases, pumping cost is examined together with other cost items, such as well or pipe network construction cost. Water quality considerations may also enter the optimization process (e.g. Mayer et al. 2002). In many cases, pumping cost is the main item in aquifer restoration problems (e.g. Kontos 2013; Matott et al. 2006). Moreover, optimal conjunctive management of surface and groundwater resources is often sought (e.g. Heydari et al. 2016; Mani et al. 2016). Parameter uncertainty may also increase the complexity of optimization problems (Sreekanth and Datta 2014).

Due to the importance of proper development of groundwater resources, many optimization methods (and combinations of them) have been used to tackle the respective problems (e.g. Bostan et al. 2016; Fowler et al. 2008; Khadem and Afshar 2015; Nicklow et al. 2010; Sidiropoulos and Tolikas 2008). On the other hand, some analytical solutions have led to more general results, which can serve as guidelines. Such results are outlined in the following paragraphs.

Katsifarakis (2008) studied steady flow in confined infinite aquifers, as well as in semi-infinite ones to which the method of images applies. He proved that the cost to pump a given total flow rate Q_T from any number and layout of wells is minimized, when hydraulic head levels at all wells are equal to each other, as long as flow is due to that system of wells only. Moreover, he presented an analytical calculation procedure of the optimal distribution of Q_T to the individual wells.

Katsifarakis and Tselepidou (2009) extended the aforementioned work to steady flows in aquifers with two zones of different transmissivities, to which the method of images applies. They proved that pumping cost is minimized, when hydraulic head levels at all wells are equal to each other, as long as flow is due to that system of wells only. Moreover, they outlined the analytical calculation of the optimal distribution of Q_T to the individual wells. Then they took into account regional flow, independent of the operation of the wells. They proved that in this general case, pumping cost is minimized, when final differences between hydraulic head values at the locations of the wells, resulting from superposition of the regional flow and the operation of the well system, are equal to the half of those, which are due to the regional flow only. They also presented an analytical calculation procedure of the optimal distribution of Q_T to the individual wells.

Ahlfeld and Laverty (2011), using a matrix formulation, have come up with similar results, even for non-infinite flow fields, if the groundwater flow equation is linear and boundary conditions are not head dependent. Their proof is based on the assumption that the response matrix of drawdown to pumping is symmetric. Its coefficients have to be calculated numerically, and symmetry of the numerical model coefficient matrix is a prerequisite. Moreover, their proof holds for transient flows with constant well flow rates, under the same assumptions. In a more recent paper, Ahlfeld and Laverty (2015) have tested their formulation on a hypothetical problem with complex hydrogeology and large drawdowns, which was developed from a field-scale problem in California.

Recently, there is a renewed interest in analytical solutions for groundwater flows. Saeedpanah and Golmohamadi Azar (2017), for instance, derived a new analytical expression for predicting the groundwater level and flow rate in a confined aquifer between two streams of varying water level boundaries and two constant head boundaries, while Bansal et al. (2016) investigated the influence of a thin clogging layer in aquifer-stream interaction.

In many cases, it is plausible to consider steady groundwater flow during parts of the examined period (Papadopoulou et al. 2007). Nevertheless, taking into account transient groundwater flow conditions is imperative in other applications of practical interest, e.g. when real-time management is sought (e.g. Bauser et al. 2012) or when variable power price has to be taken into account (Bauer-Gottwein et al. 2016). Our work is relevant to such cases. We study pumping cost minimization for any number and layout of wells under transient groundwater flow conditions in infinite confined aquifers and semi-infinite ones, to which the method of images applies. Moreover, we take into account additional steady state flow, which is independent of the well system and results in non-horizontal initial hydraulic head level distribution. We prove analytically that, at any time, the instant pumping cost is minimum, when the following condition holds: the observed at that instant differences between hydraulic head values at the locations of the wells are equal to the half of the initial ones, which are due to the additional steady-state flow. Based on this proof, an analytical calculation procedure of the time dependent optimal distribution of the required total flow rate Q_T to the individual wells is presented. Moreover, as well flow rates usually remain constant over the pumping period, an approximate calculation of the optimal constant flow rate distribution is outlined. To our knowledge, no such optimal analytical solutions exist for transient groundwater flows.

2 Formulation of the Optimization Problem

At any time, and for any aquifer type, pumping cost for a system of N wells can be defined as:

$$K_1 = A \cdot \sum_{J=1}^{N} Q_J \cdot h_J \tag{2.1}$$

where Q_J is the flow rate of well J, h_J is the distance between water level at well J and a predefined level (e.g. highest ground elevation) and A is a constant, depending on energy cost. Treating A as constant implies that: a) pump efficiencies are considered as constants and equal to each other and b) Energy price does not change with time.

In the simplest case, the initial undisturbed hydraulic head level is horizontal, namely flow will be due to the studied system of wells only. Here, we consider a more general case, involving an additional regional flow, which is independent of the well system, remains constant and results in non-horizontal initial hydraulic head level. Then, Eq. (2.1) can be written as:

$$K_1 = A \cdot \sum_{J=1}^{N} Q_J (s_J(t) + \delta_J) \tag{2.2}$$

where $s_J(t)$ is the time-dependent hydraulic head level drawdown (from then on simply drawdown) at well J at time t and δ_J is the distance between the initial hydraulic head level at well J and the reference level, the choice of which should guarantee that no δ_J value is negative. Constant A does not affect the optimization process, hence the objective function that should be minimized is:

$$K = \sum_{J=1}^{N} Q_J (s_J(t) + \delta_J) \tag{2.3}$$

The flow rates Q_J are the decision variables, while the terms $(s_J(t) + \delta_J)$ serve as coefficients. While δ_J values can be considered as known constants, $s_J(t)$ have to be calculated using the flow simulation model.

The decision variables, namely the well flow rates Q_J, should fulfill the basic constraint of the problem, namely:

$$\sum_{J=1}^{N} Q_J = Q_T \tag{2.4}$$

Moreover, in praxis, they should not obtain negative values, since such values correspond to recharge wells.

The problem will be solved first for infinite aquifers and then for semi-infinite ones, to which the method of images applies. Use of analytical solutions for the flow simulation model, allows analytical, namely more accurate solution of the optimization problem, at low overall computational load. Moreover, the possibility of superposition with additional steady-state flow, allows application to more complex cases. Finally, it should be kept in mind, that sophisticated models produce better results, only if they are supported by accurate and adequate field data, which are not always available.

3 Infinite Aquifers

3.1 The Combined Simulation-Optimization Model

Suppose that N wells start pumping at time $t_0 = 0$ with constant flow rates from a confined infinite aquifer. For any time $t_k > 0$, transient drawdown $s_J(t)$ at a point J of the aquifer with respect to the initial horizontal hydraulic head level is given as (Theis 1935):

$$s_J(t_k) = \frac{1}{4\pi T} \sum_{I=1}^{N} Q_I W(u_{IJ}) \tag{3.1}$$

where

$$W(u_{IJ}) = \int_{u_{ij}}^{\infty} \frac{e^{-y}}{y} dy = -\gamma - \ln u_{IJ} - \sum_{n=1}^{\infty} \frac{(-1)^n \cdot u_{IJ}^n}{n \cdot n!} \tag{3.2}$$

and

$$u_{IJ} = \frac{S r_{IJ}^2}{4 T t_k} \tag{3.3}$$

In Eqs. (3.1) to (3.3), T is the aquifer's transmissivity, Q_I the flow rate of well I, γ the Euler's constant, S the aquifer's storativity and r_{IJ} the distance between point J and well I. It is worth mentioning that $W(u_{IJ})$ decreases with increasing u_{IJ} (namely with increasing r_{IJ}, for any given t_k). The physical meaning is that the influence of pumping at well I on location J, decreases with the distance between I and J.

The superposition principle (e.g. Bear 1979) allows adding the result of pumping to that of the steady-state flow. Hence, for any t_k, and as long as the flow remains confined at any point of the flow field, the respective objective function K_k of the cost minimization problem can be written as:

$$K_k = \sum_{J=1}^{N} Q_J \sum_{I=1}^{N} \frac{Q_I}{4\pi T} W\left(\frac{S r_{IJ}^2}{4 T t_k}\right) + \sum_{J=1}^{N} Q_J \delta_J \tag{3.4}$$

where r_{IJ} is the distance between wells I and J (therefore $r_{IJ} = r_{JI}$). For I = J in particular, r_{IJ} is equal to the radius of the well r_0. The first and second terms of the right-hand side of (3.4) represent cost due to transient pumping and to additional steady-state flow, respectively.

To simplify notation, we use W_{IJk} instead of $W\left(S r_{IJ}^2/(4 T t_k)\right)$ in the rest of the paper (and W_{0k} for I = J). Since $r_{IJ} = r_{JI}$, $W_{IJk} = W_{JIk}$, too.

3.2 Analytical Solution of the Optimization Problem

Following the approach developed by Katsifarakis (2008) and Katsifarakis and Tselepidou (2009) for steady-state flow problems, we shall calculate the first derivatives of K_k with respect to the decision variables, namely the flow rates Q_I. First, we note that they are not independent of each other, since they are subject to the constraint (2.4). We can assume, without loss of generality, that the first N-1 of them are independent, while Q_N depends upon the rest, namely

$$Q_N = Q_T - \sum_{I=1}^{N-1} Q_I \tag{3.5}$$

It follows that, for any M ∈ [1, N-1]

$$\frac{\partial Q_N}{\partial Q_M} = -1 \tag{3.6}$$

Then, for any M ∈[1, N-1]

$$\frac{\partial}{\partial Q_M}\left(\sum_{J=1}^{N} Q_J C_J\right) = C_M - C_N \tag{3.7}$$

where C_J is a coefficient, e.g. W_{IJk} or δ_J. Applying this result to the objective function K_k, we get, for any M∈[1, N-1]:

$$\begin{aligned} \frac{\partial K_k}{\partial Q_M} &= \sum_{I=1}^{N} \frac{Q_I}{4\pi T} W_{IMk} - \sum_{I=1}^{N} \frac{Q_I}{4\pi T} W_{INk} + \sum_{J=1}^{N} Q_J \frac{W_{MJk}}{4\pi T} - \sum_{J=1}^{N} Q_J \frac{W_{NJk}}{4\pi T} + \delta_M - \delta_N \Rightarrow \\ \Rightarrow \frac{\partial K_k}{\partial Q_M} &= 2\sum_{I=1}^{N} \frac{Q_I}{4\pi T} W_{IMk} - 2\sum_{I=1}^{N} \frac{Q_I}{4\pi T} W_{INk} + \delta_M - \delta_N = 2(s_M - s_N) + \delta_M - \delta_N \end{aligned} \tag{3.8}$$

To derive (3.8), Eq. (3.1) and the equality $W_{IJk} = W_{JIk}$ have been used. Setting the derivative of K_k equal to zero, we get:

$$\frac{\partial K_k}{\partial Q_M} = 0 \Leftrightarrow s_M - s_N = \frac{\delta_N - \delta_M}{2} \tag{3.9}$$

Equation (3.9) holds for every M ∈ [1, N-1]. It follows, then, that (for any given t_k) a critical point of the objective function K_k occurs, when the following condition holds: the differences between hydraulic head values at the locations of the wells, which result from the superposition of transient operation of the wells and the steady-state flow, are equal to the half of the initial ones, which are due to the steady-state flow only.

The coordinates of the critical point, namely the corresponding set of Q_M values, can be found by solving a linear system of N equations and N unknowns. The first N-1 equations have the following form:

$$\sum_{I=1}^{N} \frac{Q_I}{4\pi T}(W_{MIk} - W_{INk}) = \frac{\delta_N - \delta_M}{2} \tag{3.10}$$

Equation (3.10) is another form of (3.9), namely it results from (3.8) setting the derivative of K_k equal to zero, and it can be written for every M ∈ [1, N-1]. The N-th equation, which completes the system, is the constraint (2.4), namely:

$$\sum_{J=1}^{N} Q_J = Q_T$$

The aforementioned linear system has one solution only, namely only one critical point P exists. To verify that P corresponds to the minimum of K_k, the respective second derivatives will be used. Starting from Eq. (3.8) and using Eq. (3.7) one gets, for any M ∈[1, N-1]:

$$\begin{aligned} \frac{\partial^2 K_k}{\partial Q_M^2} &= \frac{1}{2\pi T}\sum_{I=1}^{N}\frac{\partial Q_I}{\partial Q_M}W_{IMk} - \frac{1}{2\pi T}\sum_{I=1}^{N}\frac{\partial Q_I}{\partial Q_M}W_{INk} \Rightarrow \\ \frac{\partial^2 K_k}{\partial Q_M^2} &= \frac{1}{\pi T}\left(W_{0k} - W_{MNk}\right) \end{aligned} \tag{3.11}$$

The parenthesis of the right-hand side of Eq. (3.11) is positive, for the following reason: As mentioned in section 3, W_{IJ} decreases with increasing distance between wells. But W_{0k} corresponds to r_0, namely to the radius of each well, which is smaller than any distance r_{MN} between wells. Then, the value of the second derivative of K_k with respect to Q_M is positive, for every M ∈ [1, N-1]. This means that the critical point P corresponds to a minimum or to a saddle point.

Moreover, it is easily proved that all second derivatives of K_k with respect to well flow rates are constant. Then, according to the reasoning developed by Katsifarakis (2008) for steady-state flows, P is a minimum of K_k; and since it is the only critical point, P is the absolute minimum.

Finally, it should be mentioned that, as discussed in Katsifarakis and Tselepidou (2009) for steady-state flows and in Ahlfeld and Laverty (2015), solution of the aforementioned linear system may result in negative (namely recharge) flow rates for some of the wells, which have the largest δ_I values, at least for certain t_k values. In praxis, wells should not be used as long as the respective Q_I values are negative, and Q_T should be redistributed to the rest of the wells. This point is further discussed in the second example of section 6.

4 Semi-Infinite Aquifers

In the following paragraphs, we study the pumping cost minimization problem, described by Eqs. (2.1) to (2.3), in semi-infinite aquifers, to which the method of images applies. We take into account flow fields with a rectilinear impermeable boundary or with a rectilinear constant head boundary. The optimization procedure remains the same, but the flow simulation model is different for each case.

The method of images (e.g. Bear 1979), provides analytical solutions for fields with one (or more, under certain conditions) straight-line boundaries. Its basic concept is that a boundary can be "removed" by adding a number of fictitious (or image) wells, symmetrical of the real ones with respect to it. The relationship between the flow rate of each real well and that of its image depends on the boundary condition along the "removed" boundary and guarantees its observance. Application of the method of images to a flow field with one rectilinear boundary is shown in Fig. 1, where real and image wells are denoted with capital and lower case letters, respectively.

4.1 Flow Fields with a Straight-Line Impermeable Boundary

First, we study a system of N wells, which start pumping at time $t_0 = 0$ with constant flow rates from a confined semi-infinite aquifer, bounded by a rectilinear impermeable boundary. At any

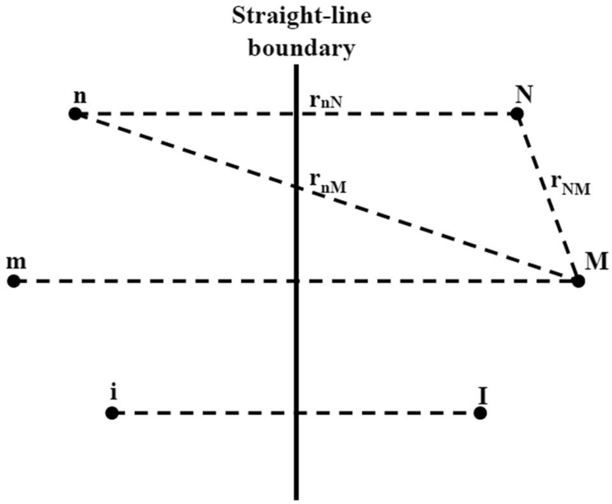

Fig. 1 Real and fictitious wells in a semi-infinite aquifer with one straight-line boundary

time $t_k > 0$, transient drawdown $s_J(t)$ at a point J of the aquifer with respect to the initial horizontal hydraulic head level is given as:

$$s_J(t_k) = \frac{1}{4\pi T}\sum_{I=1}^{N} Q_I\left(W\left(\frac{Sr_{IJ}}{4Tt_k}\right) + W\left(\frac{Sr_{iJ}}{4Tt_k}\right)\right) \tag{4.1}$$

Equation (4.1) should be introduced to Eq. (2.3) to derive the objective function. Invoking the superposition principle and the notation used in section 3, and assuming that the flow remains confined at any point of the flow field, the objective function K_k obtains the following form:

$$K_k = \sum_{J=1}^{N} Q_J \sum_{I=1}^{N} \frac{Q_I}{4\pi T}(W_{IJk} + W_{iJk}) + \sum_{J=1}^{N} Q_J\delta_J \tag{4.2}$$

In Eq. (4.2), we have used W_{iJk} instead of $W\left(Sr_{iJ}^2/(4Tt_k)\right)$, in addition to the notation, which has been already introduced in section 3.

Equations (3.5) to (3.7) hold in this case too, since they do not depend on the flow simulation model. Then, for any M∈[1, N-1], the derivative of the objective function K_k reads:

$$\begin{aligned}\frac{\partial K_k}{\partial Q_M} &= \sum_{I=1}^{N} \frac{Q_I}{4\pi T}(W_{IMk} + W_{iMk}) - \sum_{I=1}^{N} \frac{Q_I}{4\pi T}(W_{INk} + W_{iNk}) + \\ &+ \sum_{J=1}^{N} \frac{Q_J}{4\pi T}(W_{JMk} + W_{jMk}) - \sum_{J=1}^{N} \frac{Q_J}{4\pi T}(W_{JNk} + W_{jNk}) + \delta_M - \delta_N = 2(s_M - s_N) + \delta_M - \delta_N\end{aligned} \tag{4.3}$$

Setting the derivative of K_k equal to zero, we get, as in the infinite aquifer case:

$$\frac{\partial K_k}{\partial Q_M} = 0 \Leftrightarrow s_M - s_N = \frac{\delta_N - \delta_M}{2} \tag{4.4}$$

Since Eq. (4.4) holds for every M ∈ [1, N-1], a critical point of the objective function K_k occurs, when the following condition holds: the differences between hydraulic head values at the locations of the wells, which result from the superposition of transient operation of the wells and the steady-state flow, are equal to the half of the initial ones, which are due to the steady-state flow only.

The coordinates of the critical point, namely the corresponding set of Q_M values, can be found by solving a linear system of N equations and N unknowns. The first N-1 equations have the following form:

$$\sum_{I=1}^{N}\frac{Q_I}{4\pi T}(W_{IMk}+W_{iMk}-W_{INk}-W_{iNk})=\frac{\delta_N-\delta_M}{2} \tag{4.5}$$

Equation (4.5) is a more explicit form of Eq. (4.4). The N-th equation, which completes the system, is the constraint (2.4). The linear system has one solution only, namely only one critical point P exists. To verify that P corresponds to the minimum of K_k, the respective second derivatives will be used. Starting from Eq. (4.3) one gets, for any M ∈[1, N-1]:

$$\frac{\partial^2 K_k}{\partial Q_M^2}=\frac{1}{2\pi T}[2W_{0k}-2W_{MNk}-2W_{mNk}+W_{mMk}+W_{nNk}] \tag{4.6}$$

To prove that the term inside the brackets is positive, we rewrite it in the following way:

$$\begin{aligned}&[2W_{0k}-2W_{MNk}-2W_{mNk}+W_{mMk}+W_{nNk}]=\\&=(W_{0k}+W_{mMk})-(W_{MNk}+W_{mNk})+(W_{0k}+W_{nNk})-(W_{NMk}-W_{nMk})\end{aligned} \tag{4.7}$$

In the right-hand side of Eq. (4.7), the first parenthesis has larger value than the second, for the following reason: If they are multiplied by $Q_M/(4\pi T)$, the first parenthesis will be equal to the drawdown at the well M, while the second at the point N, when only Q_M is pumped. Similarly, if the values of the third and fourth parentheses are multiplied by $Q_N/(4\pi T)$, they will be equal to the drawdown at the well N and at the point M, respectively, when only Q_N is pumped. Therefore, the value of the third parenthesis is larger than that of the fourth. It follows, then, that the right-hand side of Eq. (4.7) and the second derivative of K_k are positive. Another proof, based on the definition of W(u), is given in appendix A.

The rest of the proof, regarding the nature of critical point P, is exactly the same as the one outlined for the infinite aquifer case.

4.2 Flow Fields with a Rectilinear Constant Head Boundary

In this section, we study a similar system of N wells, which start pumping at time $t_0 = 0$ with constant flow rates from a confined semi-infinite aquifer, bounded by a rectilinear constant head boundary. At any time $t_k > 0$, $s_J(t)$ at a point J of the aquifer with respect to the initial horizontal hydraulic head level is given as:

$$s_J(t_k)=\frac{1}{4\pi T}\sum_{I=1}^{N}Q_I\left(W\left(\frac{Sr_{IJ}}{4Tt_k}\right)-W\left(\frac{Sr_{iJ}}{4Tt_k}\right)\right) \tag{4.8}$$

Then, the objective function K_k of the pumping cost minimization problem, which results from introducing Eqs. (4.8) to (2.3), reads:

$$K_k=\sum_{J=1}^{N}Q_J\sum_{I=1}^{N}\frac{Q_I}{4\pi T}(W_{IJk}-W_{iJk})+\sum_{J=1}^{N}Q_J\delta_J \tag{4.9}$$

Invoking again Eqs. (3.5) to (3.7), we get, for any M∈[1, N-1]:

$$\begin{aligned}\frac{\partial K_k}{\partial Q_M}&=\sum_{I=1}^{N}\frac{Q_I}{4\pi T}(W_{IMk}-W_{iMk})-\sum_{I=1}^{N}\frac{Q_I}{4\pi T}(W_{INk}-W_{iNk})+\\&+\sum_{J=1}^{N}\frac{Q_J}{4\pi T}(W_{JMk}-W_{jMk})-\sum_{J=1}^{N}\frac{Q_J}{4\pi T}(W_{JNk}-W_{jNk})+\delta_M-\delta_N=2(s_M-s_N)+\delta_M-\delta_N\end{aligned} \tag{4.10}$$

Setting the derivative of K_k equal to zero, we get, as in section 4.1:

$$\frac{\partial K_k}{\partial Q_M} = 0 \Leftrightarrow s_M - s_N = \frac{\delta_N - \delta_M}{2} \tag{4.11}$$

Equation (4.11) is exactly the same with Eq. (4.4). Therefore, it leads to the same conclusions, regarding the existence and the properties of the critical point P of K_k. The coordinates of P, namely the corresponding set of Q_M values, can be found by solving a linear system of N equations and N unknowns. The first N-1 equations have the following form:

$$\sum_{I=1}^{N} \frac{Q_I}{4\pi T}\left(W_{IMk} - W_{iMk} - W_{INk} + W_{iNk}\right) = \frac{\delta_N - \delta_M}{2} \tag{4.12}$$

The N-th equation, which completes the system, is the constraint (2.4). The linear system has one solution only, namely only one critical point P exists. To verify that P corresponds to the minimum of K_k, we shall check the respective second derivatives. Starting from Eq. (4.10), and after proper term rearrangement, we get, for any M ∈[1, N-1]:

$$\frac{\partial^2 K_k}{\partial Q_M^2} = \frac{1}{2\pi T}\left[\left(W_{0k} - W_{mMk}\right) - \left(W_{MNk} - W_{mNk}\right) + \left(W_{0k} - W_{nNk}\right) - \left(W_{NMk} - W_{nMk}\right)\right] \tag{4.13}$$

As explained for the respective terms of Eq. (4.7), the first term of the right hand side of Eq. (4.13) is larger than the second, and the third larger than the fourth. Therefore, the second derivative of K_k is positive. The rest of the proof, regarding the nature of critical point P, is exactly the same as the one outlined for the infinite aquifer case.

5 Optimal Constant well Flow Rates Over the Pumping Period

Changing well flow rates continuously, or even every few minutes, is practically infeasible. It is quite useful then to come up with an approximate calculation of the optimal well flow rate distribution, given that it will not change during the pumping period T_P. To achieve this, T_P is divided in T1 equal intervals and the objective function K_C is written as:

$$K_C = \sum_{k=1}^{T1} K_k = \sum_{k=1}^{T1}\left(\sum_{J=1}^{N} Q_J \sum_{I=1}^{N} \frac{Q_I}{4\pi T} W_{IJk} + \sum_{J=1}^{N} Q_J \delta_J\right) \tag{5.1}$$

Rearranging terms in Eq. 5.1, we get:

$$K_C = \sum_{J=1}^{N} Q_J \sum_{I=1}^{N} \frac{Q_I}{4\pi T} \sum_{k=1}^{T1} W_{IJk} + T1 \sum_{J=1}^{N} Q_J \delta_J \tag{5.2}$$

The average value W_{IJav} of W_{IJk} over the pumping period can be approximated as:

$$W_{IJav} = \frac{1}{T1} \sum_{k=1}^{T1} W_{IJk} \tag{5.3}$$

Taking into account Eq. (5.3), Eq. (5.2) can be written as follows:

$$K_C = T1\left(\sum_{J=1}^{N} Q_J \sum_{I=1}^{N} \frac{Q_I}{4\pi T} W_{IJav} + \sum_{J=1}^{N} Q_J \delta_J\right) \tag{5.4}$$

It follows from Eq. (5.4) that K_C has the same decision variables and the same form as any K_k. Moreover, W_{IJav} have the same properties as W_{IJk}, namely they decrease with r_{IJ} and

$W_{IJav} = W_{JIav}$. Then, K_C behaves as any K_k, namely its minimum occurs when, for every J ∈ [1, N-1], the following equation holds:

$$s_{Jav} - s_{Nav} = \frac{\delta_N - \delta_J}{2} \tag{5.5}$$

where

$$s_{Jav} = \frac{1}{4\pi T} \sum_{I=1}^{N} Q_J W_{IJav} \tag{5.6}$$

As mentioned above, the calculation of the optimum is approximate. Its accuracy improves with that of W_{IJav}, depending on the value of T1, as in the case of calculating integrals using the Simpson rule. The optimal well flow rate distribution is calculated by solving the respective linear system of N equations and N unknowns, as in section 3.

6 Illustrative Examples

The aforementioned analytical procedure allows calculation of the optimal distribution of Q_T to N wells for any $t_k > 0$. Change of this distribution with time is discussed in the first of the following examples. Moreover, the influence of different initial hydraulic head levels δ_I is investigated in the second example.

Example 1 A total flow rate $Q_T = 200$ lit/s will be pumped by 8 wells, from an infinite confined aquifer, with transmissivity and storativity equal to $T = 0.002$ m^2/s and S = 0.001, respectively. Pumping will start at $t = 0$ and it will last for 18 h. The initial hydraulic head level is horizontal, namely there is no additional steady-state flow. The layout of the wells is shown in Fig. 2, while their coordinates x_I, y_I appear in the first line of Table 1. The radius of each well is $r_0 = 0.2$ m.

We are going to calculate the optimal well flow rate distribution for $t_k = 1, 2, 4, 6, 8, 10, 12, 14, 16$ and 18 h after the beginning of pumping. For each t_k, a system of 8 equations with 8 unknowns is solved. In all cases the constant terms of the Eqs. 1 to 7 are equal to zero, since the initial hydraulic head level is horizontal, namely all δ_I are equal to each other. For the 8th equation, the constant term is equal to Q_T, namely to 200, while the coefficients of the unknowns are equal to 1. All other coefficients depend on t_k.

Fig. 2 Well layout of the illustrative examples

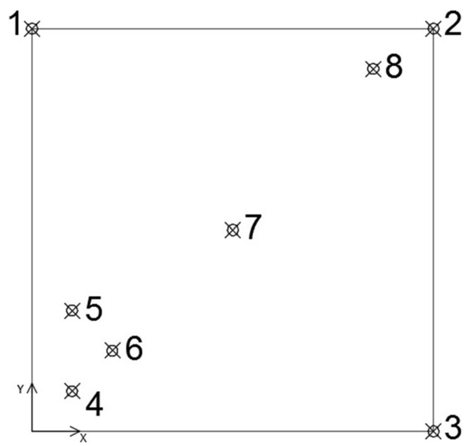

Table 1 Optimal flow rate distribution for different times (Q_T = 200 l/s)

well	1 (0, 1000)	2 (1000, 1000)	3 (1000, 0)	4 (100, 100)	5 (100, 300)	6 (200, 200)	7 (500, 500)	8 (850, 900)
t_k = 3600 s							K_k = 2620.8	
Q_I	25.498	25.152	25.498	24.570	24.570	24.061	25.498	25.152
s_I	13.104	13.104	13.104	13.104	13.104	13.104	13.104	13.104
t_k = 7200 s							K_k = 2826.4	
Q_I	26.098	25.193	26.098	24.062	24.049	23.248	26.063	25.190
s_I	14.132	14.132	14.132	14.132	14.132	14.132	14.132	14.132
t_k = 14,400 s							K_k = 3063.4	
Q_I	26.905	25.233	26.909	23.479	23.369	22.352	26.567	25.185
s_I	15.317	15.317	15.317	15.317	15.317	15.317	15.317	15.317
t_k = 21,600 s							K_k = 3221.6	
Q_I	27.458	25.314	27.483	23.173	22.932	21.830	26.638	25.172
s_I	16.108	16.108	16.108	16.108	16.108	16.108	16.108	16.108
t_k = 28,800 s							K_k = 3347.2	
Q_I	27.869	25.421	27.928	22.994	22.621	21.480	26.526	25.162
s_I	16.736	16.736	16.736	16.736	16.736	16.736	16.736	16.736
t_k = 36,000 s							K_k = 3455.4	
Q_I	28.185	25.538	28.287	22.883	22.388	21.225	26.339	25.153
s_I	17.277	17.277	17.277	17.277	17.277	17.277	17.277	17.277
t_k = 43,200 s							K_k = 3552.4	
Q_I	28.436	25.658	28.583	22.815	22.208	21.030	26.126	25.145
s_I	17.762	17.762	17.762	17.762	17.762	17.762	17.762	17.762
t_k = 50,400 s							K_k = 3641.6	
Q_I	28.640	25.776	28.831	22.773	22.062	20.874	25.909	25.134
s_I	18.208	18.208	18.208	18.208	18.208	18.208	18.208	18.208
t_k = 57,600 s							K_k = 3724.8	
Q_I	28.811	25.887	29.042	22.748	21.943	20.746	25.700	25.123
s_I	18.624	18.624	18.624	18.624	18.624	18.624	18.624	18.624
t_k = 64,800 s							K_k = 3803.2	
Q_I	28.956	25.992	29.225	22.734	21.843	20.638	25.503	25.110
s_I	19.016	19.016	19.016	19.016	19.016	19.016	19.016	19.016
Steady-state well flow rate distribution								
Q_I	31.009	28.031	31.869	23.158	20.610	19.205	21.638	24.480

Results are summarized in Table 1. For each t_k, optimal well flow rates are presented, together with the respective s_I values, which are equal to each other, as expected.

It can be seen that for t_k = 3600 s the well flow rates are almost equal to each other, since the interaction between wells is minimal. As time goes by, the transient optimal well flow rate distribution generally evolves towards that of the steady-state flow, shown also in Table 1 and calculated according to Katsifarakis (2008). For t = 64,800 s, the ratio of the largest to the smallest well flow rate Q_3/Q_6 exceeds 1.43 already, while for the steady-state flow it reaches 1.66.

It is interesting to study the evolution of drawdown at the wells, if the distributions of flow rates, given in Table 1 as optimal for particular t_k values, remain constant during the pumping period. This evolution is shown in Fig. 3.

Moreover, it is interesting to compare the s_I and K_k values, shown in Table 1, which correspond to instant optimal flow rate distributions, with those of the optimal constant flow rate distribution. The latter are shown in Table 2. The evolution of drawdowns at the wells is shown in Fig. 3d. Comparison reveals that constant flow rate distribution is very similar to the transient one for t_k = 28,800 s, namely for t close to, but smaller than $T_p/2$. This is due to the form of all $W_{IJ}(t)$, which increase with t at a diminishing rate.

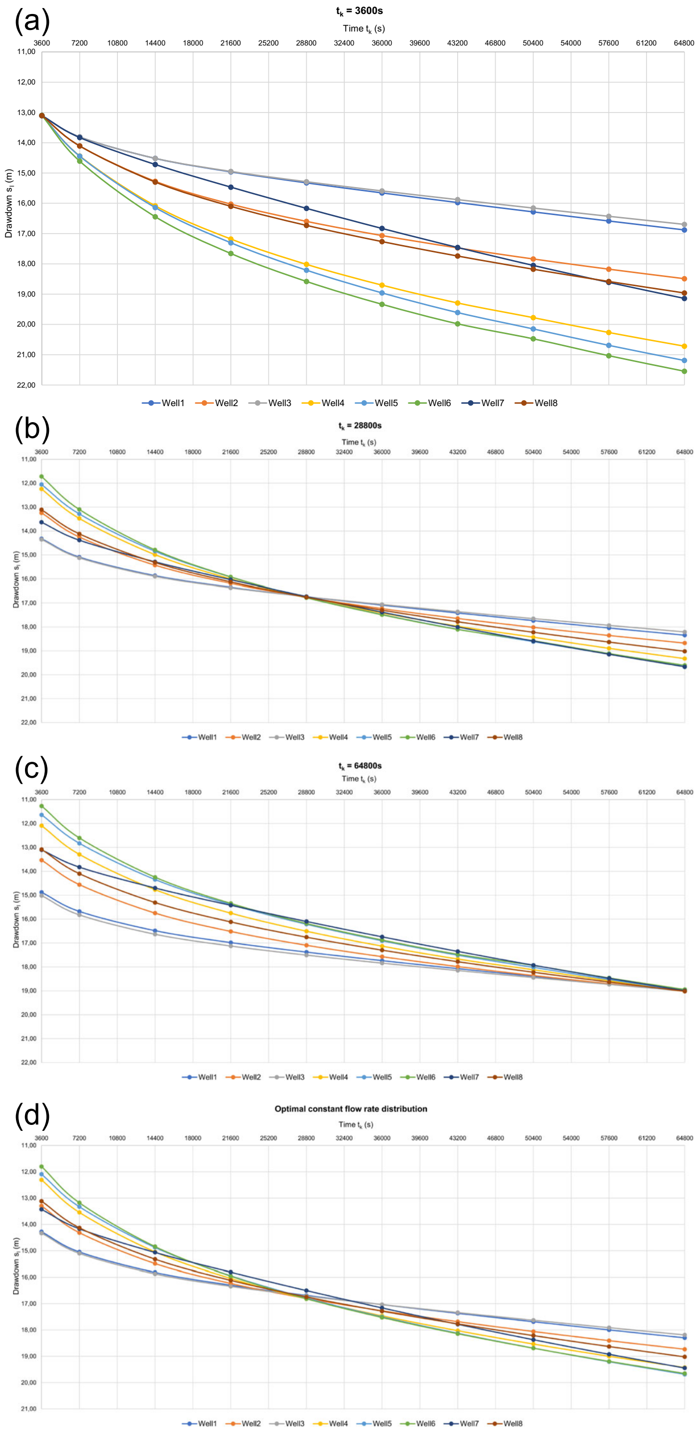

Fig. 3 (**a**-**d**) Evolution of the drawdown at the wells, for optimal Q_I distributions (Tables 1 and 2). (**a**) t_k = 3600 s (**b**) t_k = 28,800 s (**c**) t_k = 64,800 s (**d**) constant Q_I distribution

Table 2 Drawdown for practically optimal constant well flow rate distribution

well	1	2	3	4	5	6	7	8
Q_I	27.794	25.541	27.899	23.119	22.695	21.630	26.141	25.183
t_k = 3600 s							K_k = 2633.1	
s_I	14.284	13.304	14.338	12.313	12.098	11.805	13.435	13.123
t_k = 7200 s							K_k = 2833.1	
s_i	15.051	14.321	15.108	13.550	13.334	13.183	14.174	14.135
t_k = 14,400 s							K_k = 3065.4	
s_i	15.823	15.492	15.881	15.052	14.883	14.850	15.070	15.327
t_k = 21,600 s							K_k = 3222.1	
s_i	16.304	16.241	16.351	16.051	15.950	15.967	15.814	16.125
t_k = 28,800 s							K_k = 3347.4	
s_i	16.691	16.809	16.718	16.825	16.794	16.836	16.510	16.754
t_k = 36,000 s							K_k = 3455.9	
s_i	17.041	17.280	17.042	17.471	17.507	17.564	17.169	17.293
t_k = 43,200 s							K_k = 3553.6	
s_i	17.372	17.693	17.345	18.036	18.134	18.201	17.791	17.776
t_k = 50,400 s							K_k = 3643.6	
s_i	17.692	18.067	17.637	18.543	18.698	18.773	18.378	18.221
t_k = 57,600 s							K_k = 3727.8	
s_i	18.003	18.415	17.920	19.007	19.215	19.296	18.932	18.637
t_k = 64,800 s							K_k = 3807.1	
s_i	18.305	18.742	18.196	19.438	19.694	19.781	19.455	19.030

K_k values for the constant distribution are larger than those of the respective instant optimal distributions, as expected. The differences are smaller than 0.5%, though, allowing us to recommend use of the optimal constant well flow rate distribution for practical purposes.

Example 2 A total flow rate Q_T = 200 lit/s will be pumped for 18 h by 8 wells (with r_0 = 0.2 m), from an infinite confined aquifer, with transmissivity and storativity equal to $T = 0.0004$ m^2/s and S = 0.001, respectively. Pumping will start at $t = 0$ and it will last for 18 h. The layout of the wells is shown in Fig. 2, namely their coordinates are the same as the ones of example 1. One additional well, W_s, located at (60, 60), pumps continuously 30.0 l/s. This steady-state pumping results in different δ_I values at the locations of the 8 wells, which are given in the first line of Table 3.

We are going to calculate the optimal well flow rate distribution for t_k = 0.5, 1, 2, 4, 6, 8, 10, 12 and 18 h after the beginning of pumping. It turns out that, for t_k = 0.5 h, Q_4 is negative, while it is slightly larger than zero, for t_k = 1 h. The conclusion is that the well 4 should start pumping one hour later than the other 7 wells. Results are summarized in Table 3. For t_k = 0.5 and 1 h, both the initial and the corrected flow rate distribution appear. For t_k > 1 h, delayed start of pumping at well 4 is taken into account.

For t_k = 1800 s, K_k is smaller for the initial distribution than for the corrected one. This result is due to the calculation procedure, where negative flow rates result in "cost" reduction. In this case, the solution of the mathematical problem does not correspond to the solution of the physical one. If only we set the respective cost coefficient to 0, the term:

$$Q_4 \ (\delta_4 + s_4) = 1.029(47.42 - 0.421) = 48.36$$

will be added to the K_k value of the initial distribution, rendering it definitely larger.

Table 3 Optimal flow rate distribution for different times (Q_T = 200 l/s)

well (δ_I)	1 (13.84)	2 (9.72)	3 (13.84)	4 (47.42)	5 (30.0)	6 (32.42)	7 (18.79)	8 (11.42)
t_k = 1800 s - initial distribution							K_k = 6244.23	
Q_I	33.659	37.793	33.659	−1.029	16.906	14.403	28.569	36.040
s_I	16.369	18.429	16.369	−0.421	8.289	7.079	13.894	17.579
t_k = 1800 s - corrected distribution							K_k = 6244.81	
Q_I	33.513	37.647	33.513	–	16.761	14.248	28.424	35.895
s_I	16.298	18.358	16.298	–	8.218	7.008	13.823	17.508
t_k = 3600 s - initial distribution							K_k = 6436.62	
Q_I	33.432	36.955	33.432	0.169	17.274	14.845	28.616	35.278
s_I	17.181	19.241	17.181	0.391	9.101	7.891	14.706	18.391
t_k = 3600 s - corrected distribution							K_k = 6436.63	
Q_I	33.455	36.978	33.455	–	17.298	14.873	28.639	35.301
s_I	17.194	19.254	17.194	–	9.114	7.904	14.719	18.404
t_k = 7200 s							K_k = 6647.0	
Q_I	33.438	36.007	33.438	1.106	17.599	15.194	28.843	34.376
s_I	18.107	20.167	18.107	1.317	10.027	8.817	15.632	19.317
t_k = 14,400 s							K_k = 6880.15	
Q_I	33.695	35.093	33.698	1.762	17.839	15.384	29.088	33.440
s_I	19.181	21.241	19.181	2.391	11.101	9.891	16.706	20.391
t_k = 21,600 s							K_k = 7033.28	
Q_I	33.952	34.650	33.973	2.101	17.919	15.432	29.068	32.904
s_I	19.906	21.966	19.906	3.116	11.826	10.616	17.431	21.116
t_k = 28,800 s							K_k = 7154.84	
Q_I	34.171	34.406	34.222	2.342	17.952	15.450	28.917	32.540
s_I	20.488	22.548	20.488	3.698	12.408	11.198	18.013	21.698
t_k = 36,000 s							K_k = 7259.83	
Q_I	34.353	34.263	34.442	2.536	17.969	15.457	28.711	32.268
s_I	20.995	23.055	20.995	4.205	12.915	11.705	18.520	22.205
t_k = 43,200 s							K_k = 7354.52	
Q_I	34.506	34.179	34.635	2.701	17.979	15.460	28.487	32.054
s_I	21.455	23.515	21.455	4.665	13.375	12.165	18.980	22.665
t_k = 64,800 s							K_k = 7600.90	
Q_I	34.849	34.087	35.090	3.086	17.991	15.450	27.849	31.599
s_I	22.659	24.719	22.659	5.869	14.579	13.369	20.184	23.869
Steady-state well flow rate distribution								
Q_I	36.630	34.485	37.435	4.981	17.834	15.054	23.946	29.635

Based on this application example, the following procedure is recommended, when differences between δ_I values are substantial:

a) Check the optimal flow rate distribution for the end of the pumping period (in our example for t_k = 64,800 s). If there are flow rates with negative signs, the respective wells should not be used at all.
b) Check the optimal flow rate distribution to the system of the remaining wells for different t_k, starting with short values (in our example we have started with t_k = 1800 s). Exclude the wells with negative flow rates, for the period that remain negative. In our example, we have excluded well 4 for the first hour, as mentioned in the previous paragraphs.

7 Conclusions

In this paper, we have studied pumping cost minimization for any number and layout of wells under transient groundwater flow conditions in infinite confined aquifers and semi-infinite ones, to which the method of images applies. Moreover, we have taken into account additional steady-state flow, which is independent of the well system and results in non-horizontal initial hydraulic head level distribution. We have proved analytically that at any time, the instant pumping cost K_k is minimum, when the following condition holds: the observed at that instant differences between hydraulic head values at the locations of the wells are equal to the half of the initial ones, which are due to the additional steady-state flow. Moreover, we have presented the methodology of finding the time dependent optimal well flow rate distribution, by solving a linear system of N equations and N unknowns, N being the number of the pumping wells. In addition, we have discussed handling of negative flow rates that might appear when additional steady-state flow might overlap with the transient operation of the system of wells.

The aforementioned theoretical results have restricted practical importance, since it is infeasible to change well flow rates continuously, or even every few minutes. For this reason, we have presented an approximate calculation of the optimal well flow rate distribution, given that it will remain constant over the pumping period T_P. This distribution can be also calculated by solving a linear system of N equations and N unknowns, and can be used in many cases of practical interest.

Finally, we mention that analytical calculation of the optimal solutions, which in principle holds for any number of wells, is not computationally intensive. It is restricted only by the maximum number (N) of linear equations, or the matrix dimensions (N x N) that can be handled by the available computer.

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Compliance with Ethical Standards

Conflict of Interest The authors declare that they have no conflict of interest.

Appendix

Sign of the 2nd derivative of K_k for semi-infinite fields with one impermeable boundary.

The second derivative of K_k with respect to Q_M is given as:

$$\frac{\partial^2 K_k}{\partial Q_M^2} = \frac{1}{2\pi T}\left[2W_{0k} - 2W_{MNk} - 2W_{mNk} + W_{mMk} + W_{nNk}\right] \tag{4.6}$$

We know that W(u) is decreasing with u at a diminishing rate. Therefore, W_{0k} is larger than any W_{MNk}, since r_0 is smaller than any r_{MN}. Then, in order to prove that the second derivative of K_k with respect to Q_M is positive, it is enough to show that:

$$W_{mMk} + W_{nNk} - 2W_{mNk} > 0 \tag{A.1}$$

In the rest of the proof we drop index k, to simplify the notation.

If r_{mM} and r_{nM} are given, $r_{Mn} = r_{mN}$ obtains its smallest value when M and N are on the same perpendicular to the boundary, namely when the isosceles trapezoid MNnm (see Fig. 1) is reduced to a line segment. In this case, the following relationships hold:

$$r_{Mn} = r_{MN} + r_{Nn} \tag{A.2a}$$

$$\mathrm{r_{Mm}} = \mathrm{r_{Nn}} + 2r_{\mathrm{MN}} \tag{A.2b}$$

Invoking Eq. (3.2), namely the definition of W(u), we have:

$$W_{mM} = \int_{u_{mM}}^{\infty} \frac{e^{-y}}{y} dy = \int_{u_{nM}}^{\infty} \frac{e^{-y}}{y} dy - \int_{u_{nM}}^{u_{mM}} \frac{e^{-y}}{y} dy = W_{mN} - \int_{u_{nM}}^{u_{mM}} \frac{e^{-y}}{y} dy \tag{A.3}$$

$$W_{nN} = \int_{u_{mN}}^{\infty} \frac{e^{-y}}{y} dy + \int_{u_{nN}}^{u_{mN}} \frac{e^{-y}}{y} dy = W_{mN} + \int_{u_{nN}}^{u_{mN}} \frac{e^{-y}}{y} dy \tag{A.4}$$

Introducing Eqs. (A3) and (A4) to (A1) we get:

$$W_{mM} + W_{nN} - 2W_{mN} = \int_{u_{nN}}^{u_{mN}} \frac{e^{-y}}{y} dy - \int_{u_{nM}}^{u_{mM}} \frac{e^{-y}}{y} dy \tag{A.5}$$

Suppose that r_{nN}, and therefore u_{nN}, is given, and consider the function $F(r_{MN})$, defined as:

$$F(r_{MN}) = \int_{u_{nN}}^{u_{mN}} \frac{e^{-y}}{y} dy - \int_{u_{nM}}^{u_{mM}} \frac{e^{-y}}{y} dy \tag{A.6}$$

While u_{nN} is given, the values u_{mN} and u_{mM} depend on r_{MN}. Moreover,

$$\partial u / \partial r = Sr/(2Tt) \tag{A.7}$$

Therefore, according to the Leibniz rule and some trivial calculations:

$$\frac{dF}{dr_{MN}} = \frac{4}{e^{u_{mN}}(r_{nN} + r_{MN})} - \frac{4}{e^{u_{mM}}(r_{nN} + 2r_{MN})} > 0 \tag{A.8}$$

Therefore, $F(r_{MN})$ increases with r_{MN}. For $r_{MN} = 0$, it follows from Eq. (A6) that $F(r_{MN}) = 0$. Hence $F(r_{MN}) > 0$, for any positive r_{MN} value. This proves that the second derivative of K_k with respect to Q_M is positive.

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