Water Resour Manage (2012) 26:377–390
DOI 10.1007/s11269-011-9922-y

Effect of Well Radius on Drawdown Solutions Obtained with Laplace Transform and Green's Function

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Received: 30 July 2010 / Accepted: 20 September 2011 /
Published online: 7 October 2011

Abstract We have developed a drawdown solution for a partially penetrating well under constant flux pumping in a confined aquifer with finite thickness. The predictions of our solution diverge from the predictions of Hantush's solution (1961), particularly for problems with low ratios of well screen length to aquifer thickness. Furthermore, the predicted drawdown from Hantush's solution (1961) differs from that of Yang et al.'s solution Water Resour Res 42:W0552, (2006) only near the well and at small time values as indicated in Yang et al. Water Resour Res 42:W0552, (2006). Our solution is based on Green's function with a columnar source (sink) that represents pumping from a finite-radius well. Hantush's solution (1961) and Yang et al.'s solution Water Resour Res 42:W0552, (2006), however, were derived from Laplace transform techniques for pumping in a well with an infinitesimal and a finite radius, respectively.

Keywords Analytical solutions · Confined aquifer · Groundwater hydraulics · Partially penetrating well · Laplace transform · Green's function

Notations

B	Thickness of the aquifer [L]
D	$=K_z/S_s$, hydraulic diffusivity [L^2/T]
J_0, J_1	Bessel functions of order 0 and 1
K_r, K_z	Hydraulic conductivities in the radial and vertical directions [L/T]
Q	Pumping rate [L^3/T]
S_s	Specific storage [1/L]
$V'(z_D,\tau,\alpha)$	$= DW(z,t,\lambda)/\widehat{r}_w^2$
$\overline{V}'(\tau,\alpha)$	A function given by Eqs. B2 to B8

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$W(z,t,\lambda)$	A function given by Eqs. A7 to A13 [T]
b_1, b_2	Upper and lower vertical coordinates of the pumping well screen [L]
b'_1, b'_2	Upper and lower vertical coordinates of the observation well screen [L]
b_{1D}, b_{2D}	$= b_1/B, = b_2/B$, dimensionless variable
b'_{1D},b'_{2D}	$= b'_1/B, = b'_2/B$, dimensionless variable
r	Radial distance from the centerline of the pumping well [L]
r_D	$= \widehat{r}/\widehat{r}_w$, dimensionless radial distance from the centerline of the pumping well
r_w	Well radius [L]
$\widehat{r}$	$= \sqrt{\gamma}r$[L]
$\widehat{r}_w$	$= \sqrt{\gamma}r_w$[L]
$\widehat{r}_{wD}$	$= \widehat{r}_w/B$, dimensionless well radius
$\widehat{r}'$	Location for an instantaneous point source of strength unity generated [L]
s	Drawdown [L]
t	Time from the start of testing [T]
t'	Time for the instantaneous point source generated [T]
z, z'	Vertical distance positive downward from the upper impermeable layer [L]
z_D	$=z/B$, dimensionless vertical distance
α	$= \widehat{r}_w\lambda$, dimensionless variable
δ	$= \overline{\delta}_D/\gamma$, dimensionless average drawdown
δ_D	$= sDS_s(b_2 - b_1)/Q$, dimensionless drawdown
$\overline{\delta}_D$	Dimensionless average drawdown
φ	$= (b_2 - b_1)/B$, penetration ratio
γ	$=K_z/K_r$
λ	A dummy variable [1/L] given by performing the integration with respect to $\widehat{r}'$
θ, θ'	Angle of a point from the radial coordinate
τ	$= Dt/\widehat{r}_w^2$, dimensionless time

1 Introduction

Laplace transform techniques are commonly used to solve various groundwater flow problems which may involve well pumping or surface recharge (see, *e.g.*, Hantush 1964; Park and Zhan 2002; Wang and Yeh 2008; Moutsopoulos and Tsihrintzis 2009; Bansal and Das 2011). For pumping problems involving confined aquifers of horizontally infinite extent, there are two categories of drawdown solutions.

The first category includes solutions based on a two-dimensional (2-D) radial flow equation for an aquifer with a fully penetrating well and constant pumping rate. Such solutions include the Theis equation for an infinitesimal well radius and the solution presented in Hantush (1964, p. 318) for a well with a finite well radius. Wang and Yeh (2008) examined this type of drawdown solution for constant-head and constant-flux tests conducted in finite and infinite confined aquifers; they examined cases with and without well radius effects.

The second category includes solutions derived from a three-dimensional (3-D) radial flow equation for aquifers with a partially penetrating well. This category includes the solution presented in Hantush (1961) for an aquifer with a well of partial penetration and infinitesimal radius, the solutions presented in Cassiani and Kabala (1998) for an aquifer of semi-infinite thickness with a finite-radius well, and the solutions of Yang et al. (2006) for an aquifer of finite thickness with a finite-radius well. Unlike the solutions based on a 2-D radial flow equation, these 3-D drawdown solutions can approach steady state due to the semi-infinite thickness in the case of Cassiani and Kabala (1998) or small penetration ratio in the case of Yang et al. (2006).

Another solution technique commonly used to solve the diffusion equation for the heat flow and groundwater transport problems is Green's function (see, *e.g.*, Carslaw and Jaeger 1959; Beck et al. 1992; Shan 2006; Yeh and Yeh 2007). Carslaw and Jaeger (1959) presented Green's function solutions for heat released from a point source and a continuously spherical surface source in an infinite solid. Shan (2006) obtained a transient solution developed based on Green's function for a finite line source (sink) representing a well screen with a finite length of extraction (injection) in a vadose zone. Yeh and Yeh (2007) presented a solution derived from Green's function for the advective-dispersive equation with a constant point source. In a similar manner, the Theis equation can also be obtained from a heat problem in an infinite three-dimensional (3-D) space with a continuous line source using Green's function (Carslaw and Jaeger 1959, p.261; Loaiciga 2010).

Obviously, the Theis equation can be developed either from Green's function if the pumping is represented by a line source term or from a Laplace transform technique in which the pumping is treated as a flux boundary condition and the well radius approaches zero. Accordingly, one might expect that, if the well-radius effect were neglected, the solution presented in Hantush (1961) for pumping in an aquifer with a partial penetration well would be equal to Green's function solution for the aquifer with a finite line source. It may be of interest to delineate the differences of solutions based on different techniques if the effects of well radius and partial penetration were both considered in the pumping problems. This motivates us to develop a drawdown solution using Green's function for a partially penetrating well with a finite radius.

The objectives of this note are first to develop a drawdown solution that uses Green's function with a columnar sink representing a constant flux pumping from a finite-radius well under partial penetration condition in a finite thickness confined aquifer. This note compares the developed solution with Hantush's solution (Hantush 1961) and discusses the effects of well radius and partial penetration.

2 2 Mathematical Formulation of the Source Solution

Figure 1 depicts well and aquifer configurations for pumping at a partially penetrating well in a finite-thickness confined aquifer. The assumptions for the pumping problem are as follows: (1) the aquifer is homogeneous, anisotropic, infinite in extent, and of constant thickness (B); (2) the well is partially penetrated with a finite radius (r_w); (3) the pumping rate (Q) is maintained at a constant value throughout the whole test period; (4) the drawdown is initially assumed to be zero; (5) the drawdown is zero at an infinite distance from the well; (6) there is no vertical flow across the upper and lower impermeable boundaries; (7) the wellbore storage is negligibly small. Based on these assumptions, the governing equation of drawdown, $s(r,z,t)$, can then be written as

$$K_r \frac{\partial^2 s}{\partial r^2} + \frac{K_r}{r}\frac{\partial s}{\partial r} + K_z \frac{\partial^2 s}{\partial z^2} = S_s \frac{\partial s}{\partial t} \tag{1}$$

where K_r and K_z are the hydraulic conductivities in the radial and vertical directions [L/T], respectively; S_s is the specific storage [1/L]; r is the radial distance from the centerline of well [L]; z is the vertical distance (positive downward) from the upper impermeable layer

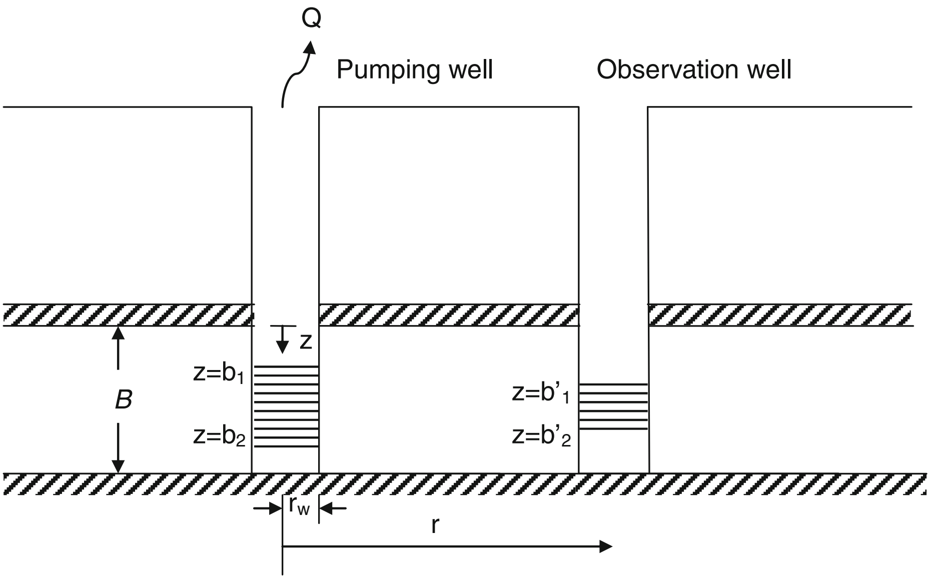


Fig. 1 Schematic representation of well and aquifer

[L] ; and t is the time from the start of testing [T]. Note that Darcian flow is also presumed in Eq. 1 and the problems of non-Darcian flow can be found in, *e.g.*, Wen et al. (2008). Equation 1 can be transformed into the diffusion equation by introducing the transformation $\widehat{r} = \sqrt{\gamma} r$ with $\gamma = K_z/K_r$ (Hantush 1964; Shan 2006),

$$\frac{\partial^2 s}{\partial \widehat{r}^2} + \frac{1}{\widehat{r}}\frac{\partial s}{\partial \widehat{r}} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{D}\frac{\partial s}{\partial t} \tag{2}$$

where D (=K_z/S_s) is the hydraulic diffusivity (L^2/T).

Assumptions (4)–(6) give the initial and boundary conditions for the problem, expressed as:

$$s(r, z, 0) = 0 \tag{3a}$$

$$s(\infty, z, t) = 0 \tag{3b}$$

$$\frac{\partial s(r, 0, t)}{\partial z} = 0 \tag{3c}$$

$$\frac{\partial s(r, B, t)}{\partial z} = 0 \tag{3d}$$

Appendix A describes the use of Green's function and the method of images for a finite aquifer thickness in the z-axis, to solve Eq. 2 subject to conditions (3a) to (3d). The result is

$$s(\widehat{r}, z, t) = \frac{Q}{2\pi \widehat{r}_w S_s (b_2 - b_1)} \int_0^\infty J_0(\lambda \widehat{r}) J_1(\lambda \widehat{r}_w) W(z, t, \lambda) d\lambda \tag{4}$$

where b_1 and b_2 are the upper and lower vertical coordinates of the well screen, respectively; λ is a dummy variable [1/L] given by integration with respect to $\hat{r}'$; J_0 and J_1 are respectively the Bessel functions of the first kind of order 0 and 1. The function $W(z,t,\lambda)$ is in terms of exponential, error and complementary error functions; it is given in Eqs. A8 to A13 of Appendix A, in which $W3$ to $W6$ are infinite series.

The drawdown solution can be expressed in dimensionless form using the dimensionless variables defined as

$$\delta_D = \frac{sDS_s(b_2 - b_1)}{Q}, \alpha = \hat{r}_w\lambda, \tau = \frac{Dt}{\hat{r}_w^2}, r_D = \frac{\hat{r}}{\hat{r}_w} \tag{5a}$$

$$z_D = \frac{z}{B}, b_{2D} = \frac{b_2}{B}, b_{1D} = \frac{b_1}{B}, b'_{2D} = \frac{b'_2}{B}, b'_{1D} = \frac{b'_1}{B}, \hat{r}_{wD} = \frac{\hat{r}_w}{B} \tag{5b}$$

Equation 4 becomes

$$\delta_D(r_D, z_D, \tau) = \frac{1}{2\pi}\int_0^\infty J_0(\alpha r_D)J_1(\alpha)V'(z_D, \tau, \alpha)d\alpha \tag{6}$$

where $V'(z_D, \tau, \alpha) = DW(z, t, \lambda)/\hat{r}_w^2$.

The average drawdown in an observation well screened from b'_1 to b'_2 can be obtained by integrating Eq. 6 with respect to z_D between the limits of b'_{1D} and b'_{2D} and dividing the result by (b'_{2D} - b'_{1D}). The dimensionless average drawdown is expressed as

$$\overline{\delta}_D(r_D, \tau) = \frac{1}{2\pi(b'_{2D} - b'_{1D})}\int_0^\infty J_0(\alpha r_D)J_1(\alpha)\overline{V}'(\tau, \alpha)d\alpha \tag{7}$$

where the function $\overline{V}'(\tau, \alpha)$ is defined in Appendix B. Another dimensionless drawdown, δ, is defined as $\delta = \overline{\delta}_D/\gamma$. The calculation of Eq. 7 requires integration of a Bessel function product from zero to infinity. Because of the nature of the Bessel functions, the integrand in Eq. 7 oscillates along the horizontal axis and approaches zero when α is close to infinity. Several numerical approaches, including a root search scheme, a numerical integration method and the Shank method, proposed by Peng et al. (2002), were used to evaluate the integral. Detailed calculation procedures can also be found in Yeh et al. (2003) and Yeh and Yang (2006). The function $\overline{V}'(\tau, \alpha)$ in the integrand of Eq. 7 contains four infinite series, *i.e.*, $V3$ in Eq. B4 to $V6$ in Eq. B7; these arise from infinite image wells due to the upper and lower impermeable boundaries. It was found that the difference of the estimated dimensionless drawdowns based on the first 50 terms and 70 terms of those infinite series is less than 10^{-5}. This result indicates that the use of the first 50 terms for those series yields enough accuracy for the problem. Note that for a fixed α value the function $\overline{V}'(\tau, \alpha)$ approaches a finite value as dimensionless time (τ) approaches infinity. The integral in Eq. 7 similarly approaches a finite value as dimensionless time approaches infinity.

3 3. Results and Discussion

Note that Hantush's solution (Hantush 1961) was developed by a Laplace transform technique for pumping in a partially penetrating well with infinitesimal radius in an aquifer of finite thickness. One might expect that Hantush's solution (Hantush 1961) would be equivalent to a Green's function solution for pumping in a finite-thickness aquifer with a finite line source. The present source solution has also been developed for the problem of pumping in a finite-thickness aquifer with a partially penetrating well. In contrast to Hantush's solution (Hantush 1961), our solution is derived from Green's function and treats the pumping well as a columnar source. The differences between our solution and Hantush's solution (Hantush 1961) are discussed below.

Figure 2 shows the curves of dimensionless drawdown versus dimensionless time predicted by the two solutions for γ=0.1, b_{1D}=b'_{1D}=0.2, b_{2D}=b'_{2D}=0.8 and r_D=1 and 5. This figure exhibits that the two solutions are in good agreement except τ>10^9 for which Hantush's solution (Hantush 1961) keeps increasing and our solution approaches steady state. The deviation at very large times reflect the fact that Hantush's solution (Hantush 1961) does not have steady-state result due to the problem of infinite boundary as discussed in Wang and Yeh (2008). On the other hand, the present solution developed based on Green's function along with the method of images gives steady-state results when the time goes very large as demonstrated in Shan (2006). Yang et al. (2006) mentioned that at small time values, for r_D=1, a discrepancy is observed between Yang et al.'s solution (2006) and Hantush's solution (Hantush 1961) because of Hantush's neglect of well-radius effects. However, the present solution is close to Hantush's solution (1961) at small time values; this shows that the influence of a columnar source on drawdown near the well is equal to that of a finite line source for a solution based on Green's function.

Because the real time for τ=10^{10} is about 273 years in the studied cases which seems useless for the real-world problems. Therefore, comparison of these two solutions is only up to 10^9 in the time frame for the following two figures. Figure 3 shows several curves of dimensionless drawdown distributions versus dimensionless distances at several dimensionless times, as predicted by our solution and Hantush's solution (Hantush 1961). This figure indicates that our solution agrees well with Hantush's solution (Hantush 1961). It further verifies the accuracy of our solution.

Yang et al. (2006) discussed the effects of various penetration ratios on drawdown. The present solution also displays a partial penetration trend similar to the one presented in Yang et al. (2006); this trend relates to the influences of the formation loss associated with horizontal flow and the loss associated with the converging vertical flows near the well screen. This study further investigates the difference between the present solution and Hantush's solution (Hantush 1961) under various penetration ratios, defined as

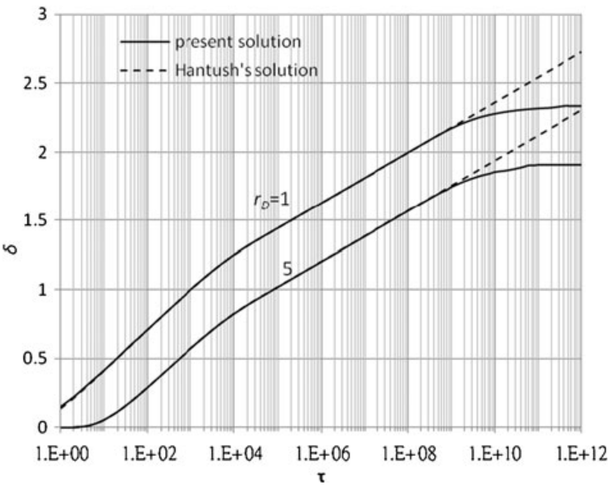
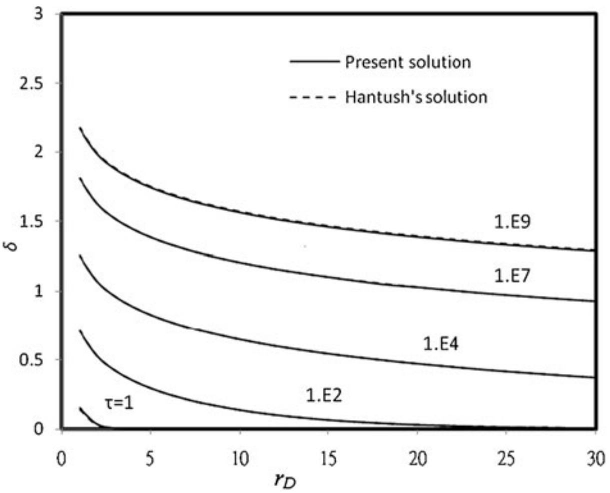


Fig. 2 Dimensionless drawdown (δ) versus dimensionless time (τ) for γ=0.1, b_{1D}=b'_{1D}=0.2 and b_{2D}=b'_{2D}=0.8 when r_D=1 or 5

Fig. 3 Dimensionless drawdown (δ) versus dimensionless distance (r_D) for γ=0.1, b_{1D}=b'_{1D}=0.2 and b_{2D}=b'_{2D}=0.8 when τ=1, 1.e2, 1.e4, 1.e7 or 1.e9

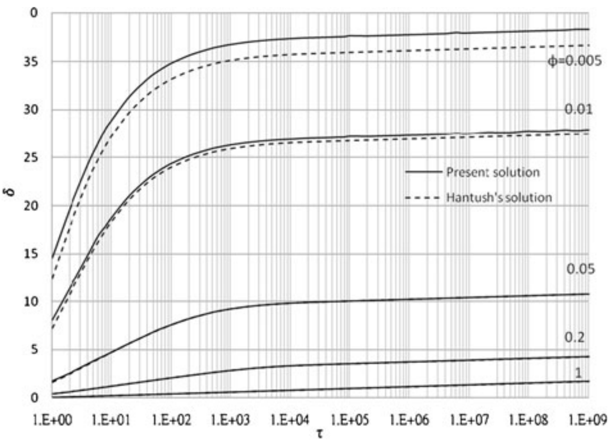


φ$(= (b_2 - b_1)/B)$, for the case in which the well screen is situated at the middle part of the aquifer. For well pumping or recharge problems (see, *e.g.*, Mishra and Majumdar 2010) φ= 1 represents the case of a fully penetrating well. Figure 4 presents the curves of dimensionless drawdown versus dimensionless time for γ=0.1 and r_D=1 as φ varies from 1 to 0.005. It shows that Hantush's solution (Hantush 1961) starts to deviate from our solution at early time values, *i.e.*, the well-radius effect becomes significant when φ ≤ 0.01, and the deviation persists over the whole time frame.

4 Concluding Remarks

This note discussed discrepancies in well drawdown solutions obtained from Laplace transform and Green's function techniques with and without considerations of well radius. We developed a new drawdown solution for a partially penetrating well under constant flux pumping in a finite-thickness confined aquifer based on Green's function with a cylindrical column source to represent finite well radius. Our solution was compared with Hantush's

Fig. 4 Dimensionless drawdown (δ) versus dimensionless time (τ) for γ=0.1 when the penetrating ratio (φ) ranges from 0.005 to 1



solution (Hantush 1961), which was developed for the case in which well radius is neglected. Results of the comparison show that the difference of these two solutions increases over the whole time frame for φ less than 0.01.

Acknowledgments This study was partly supported by the Taiwan National Science Council under the contract numbers NSC99-2221-E-035-030- and NSC99-2221-E-009-062-MY3.

Appendix A

The problem consisting of Eq. 2 and the conditions of (3a) and (3b) can be satisfied by (Carslaw and Jaeger 1959)

$$s(\widehat{r},\theta,z,t)=\frac{1}{8[\pi D(t-t')]^{3/2}}\exp\left[-\frac{\widehat{r}^2+\widehat{r}'^2+z^2-2\widehat{r}\,\widehat{r}'\cos(\theta-\theta')}{4D(t-t')}\right] \tag{A1}$$

which is a form of Green's function. The drawdown at ($\widehat{r}$,θ, z, t) is due to an instantaneous point source of strength unity that appears at point ($\widehat{r}'$,θ', z') at time t'. The aquifer drawdown is produced by continuously pumping at a constant rate from a finite-radius well whose well screen has a finite length; thus, a continuous, cylindrical columnar sink is presumed to be located at the centerline of the aquifer system.

Following Carslaw and Jaeger (1959, p.259 and p.260), integration of Eq. A1 with respect to θ', $\widehat{r}'$, z' and t', incorporating a constant sink strength, yields

$$s(\widehat{r},z,t)=\frac{Q}{2\pi^{3/2}D^{1/2}\widehat{r}_wS_s(b_2-b_1)}\int_0^\infty J_0(\lambda\widehat{r})J_1(\lambda\widehat{r}_w)\int_0^t(t-t')^{-1/2}\exp\left[-D\lambda^2(t-t')\right]\int_{b_1}^{b_2}\exp\left[-\frac{(z-z')^2}{4D(t-t')}\right]dz'\,dt'\,d\lambda \tag{A2}$$

which is obtained by assuming that the sink is located in an infinite 3-D space. However, the aquifer system for the problem is finite in its vertical direction, *i.e.*, it is bounded by upper and lower impermeable layers as represented by boundary conditions (3c) and (3d). To satisfy these boundary conditions, Eq. A2 is further modified by applying the method of images (Bear 1979; Beck et al. 1992):

$$s(\widehat{r},z,t)=\frac{Q}{2\pi^{3/2}D^{1/2}\widehat{r}_wS_s(b_2-b_1)}\int_0^\infty J_0(\lambda\widehat{r})J_1(\lambda\widehat{r}_w)\int_0^t(t-t')^{-1/2}\exp\left[-D\lambda^2(t-t')\right]\sum_{n=0}^{\infty}\int_{b_1}^{b_2}\left\{\exp\left[-\frac{(z-z_{1n})^2}{4D(t-t')}\right]+\exp\left[-\frac{(z-z_{2n})^2}{4D(t-t')}\right]\right\}dz'\,dt'\,d\lambda \tag{A3}$$

where the images are located at

$$z_{1n}=(-1)^n z'\pm 2nL\qquad(n=0,1,2,3,\ldots) \tag{A4a}$$

$$z_{2n}=(-1)^{n+1} z'\pm 2nL\qquad(n=0,1,2,3,\ldots) \tag{A4b}$$

The integration of Eq. A3 with respect to *z'* can then be obtained as

$$s(\widehat{r},z,t)=\frac{Q}{2\pi\widehat{r}_wS_s(b_2-b_1)}\int_0^\infty J_0(\lambda\widehat{r})J_a(\lambda\widehat{r}_w)\int_0^t Z(z,t,t')\exp\left[-D\lambda^2(t-t')\right]dt'd\lambda \tag{A5}$$

where

$$\begin{aligned}Z(z,t,t')=&-erf\left(\frac{z-b_2}{2\sqrt{D(t-t')}}\right)+erf\left(\frac{z-b_1}{2\sqrt{D(t-t')}}\right)\\&+erf\left(\frac{z+b_2}{2\sqrt{D(t-t')}}\right)-erf\left(\frac{z+b_1}{2\sqrt{D(t-t')}}\right)\\&+\sum_{n=1}^{\infty}\left[erf\left(\frac{z-b_1-2nL}{2\sqrt{D(t-t')}}\right)-erf\left(\frac{z-b_2-2nL}{2\sqrt{D(t-t')}}\right)+erf\left(\frac{z-b_1+2nL}{2\sqrt{D(t-t')}}\right)\right.\\&-erf\left(\frac{z-b_2+2nL}{2\sqrt{D(t-t')}}\right)+erf\left(\frac{z+b_2-2nL}{2\sqrt{D(t-t')}}\right)-erf\left(\frac{z+b_1-2nL}{2\sqrt{D(t-t')}}\right)\\&\left.+erf\left(\frac{z+b_2+2nL}{2\sqrt{D(t-t')}}\right)-erf\left(\frac{z+b_1+2nL}{2\sqrt{D(t-t')}}\right)\right]\end{aligned} \tag{A6}$$

The integration with respect to *t'* in Eq. A5 can also be obtained and the final result is presented in Eq. 4, in which

$$W(z,t,\lambda)=W1+W2+W3+W4+W5+W6 \tag{A7}$$

and

$$\begin{aligned}W1=&\int_0^t\left[erf\left(\frac{z-b_1}{2\sqrt{D(t-t')}}\right)-erf\left(\frac{z-b_2}{2\sqrt{D(t-t')}}\right)\right]\exp\left[-D\lambda^2(t-t')\right]dt'\\=&\frac{1}{D\lambda^2}\left[2-\left[erf\left(\frac{b_2-z}{2D^{1/2}t^{1/2}}\right)+erf\left(\frac{z-b_1}{2D^{1/2}t^{1/2}}\right)\right]\exp\left(-D\lambda^2t\right)\right]-\frac{1}{2D\lambda^2}\\&\left\{\exp[(b_2-z)\lambda]erfc\left(\frac{b_2-z}{2D^{1/2}t^{1/2}}+\lambda D^{1/2}t^{1/2}\right)\right.\\&+\exp[-(b_2-z)\lambda]erfc\left(\frac{b_2-z}{2D^{1/2}t^{1/2}}-\lambda D^{1/2}t^{1/2}\right)\\&+\exp[(z-b_1)\lambda]erfc\left(\frac{z-b_1}{2D^{1/2}t^{1/2}}+\lambda D^{1/2}t^{1/2}\right)\\&\left.+\exp[-(z-b_1)\lambda]erfc\left(\frac{z-b_1}{2D^{1/2}t^{1/2}}-\lambda D^{1/2}t^{1/2}\right)\right\}\end{aligned} \tag{A8}$$

$$
\begin{aligned}
W2 &= \int_0^t \left[erf\left(\frac{z+b_2}{2\sqrt{D(t-t')}} \right) - erf\left(\frac{z+b_1}{2\sqrt{D(t-t')}} \right) \right] \exp\left[-D\lambda^2(t-t')\right] dt' \\
&= \frac{1}{D\lambda^2} \left[erf\left(\frac{z+b_1}{2D^{1/2}t^{1/2}} \right) - erf\left(\frac{z+b_2}{2D^{1/2}t^{1/2}} \right) \right] \exp\left(-D\lambda^2 t\right) + \frac{1}{2D\lambda^2} \\
&\left\{ \exp[(z+b_1)\lambda] erfc\left(\frac{z+b_1}{2D^{1/2}t^{1/2}} + \lambda D^{1/2}t^{1/2} \right) \right. \\
&+ \exp[-(z+b_1)\lambda] erfc\left(\frac{z+b_1}{2D^{1/2}t^{1/2}} - \lambda D^{1/2}t^{1/2} \right) \\
&- \exp[(z+b_2)\lambda] erfc\left(\frac{z+b_2}{2D^{1/2}t^{1/2}} + \lambda D^{1/2}t^{1/2} \right) \\
&\left. - \exp[-(z+b_2)\lambda] erfc\left(\frac{z+b_2}{2D^{1/2}t^{1/2}} - \lambda D^{1/2}t^{1/2} \right) \right\}
\end{aligned} \tag{A9}
$$

$$
\begin{aligned}
W3 &= \sum_{n=1}^{\infty} \int_0^t \left[erf\left(\frac{z-b_1+2nL}{2\sqrt{D(t-t')}} \right) - erf\left(\frac{z-b_2+2nL}{2\sqrt{D(t-t')}} \right) \right] \exp\left[-D\lambda^2(t-t')\right] dt' \\
&= \sum_{n=1}^{\infty} \left\{ \frac{1}{D\lambda^2} \left[erf\left(\frac{z-b_2+2nL}{2D^{1/2}t^{1/2}} \right) - erf\left(\frac{z-b_1+2nL}{2D^{1/2}t^{1/2}} \right) \right] \exp\left(-D\lambda^2 t\right) + \frac{1}{2D\lambda^2} \right. \\
&\left[\exp[(z-b_2+2nL)\lambda] erfc\left(\frac{z-b_2+2nL}{2D^{1/2}t^{1/2}} + \lambda D^{1/2}t^{1/2} \right) \right. \\
&+ \exp[-(z-b_2+2nL)\lambda] erfc\left(\frac{z-b_2+2nL}{2D^{1/2}t^{1/2}} - \lambda D^{1/2}t^{1/2} \right) \\
&- \exp[(z-b_1+2nL)\lambda] erfc\left(\frac{z-b_1+2nL}{2D^{1/2}t^{1/2}} + \lambda D^{1/2}t^{1/2} \right) \\
&\left.\left. - \exp[-(z-b_1+2nL)\lambda] erfc\left(\frac{z-b_1+2nL}{2D^{1/2}t^{1/2}} - \lambda D^{1/2}t^{1/2} \right) \right] \right\}
\end{aligned} \tag{A10}
$$

$$
\begin{aligned}
W4 &= \sum_{n=1}^{\infty} \int_0^t \left[erf\left(\frac{z-b_1-2nL}{2\sqrt{D(t-t')}} \right) - erf\left(\frac{z-b_2-2nL}{2\sqrt{D(t-t')}} \right) \right] \exp\left[-D\lambda^2(t-t')\right] dt' \\
&= \sum_{n=1}^{\infty} \left\{ \frac{1}{D\lambda^2} \left[erf\left(\frac{b_1+2nL-z}{2D^{1/2}t^{1/2}} \right) - erf\left(\frac{b_2+2nL-z}{2D^{1/2}t^{1/2}} \right) \right] \exp\left(-D\lambda^2 t\right) + \frac{1}{2D\lambda^2} \right. \\
&\left[\exp[(b_1+2nL-z)\lambda] erfc\left(\frac{b_1+2nL-z}{2D^{1/2}t^{1/2}} + \lambda D^{1/2}t^{1/2} \right) \right. \\
&+ \exp[-(b_1+2nL-z)\lambda] erfc\left(\frac{b_1+2nL-z}{2D^{1/2}t^{1/2}} - \lambda D^{1/2}t^{1/2} \right) \\
&- \exp[(b_2+2nL-z)\lambda] erfc\left(\frac{b_2+2nL-z}{2D^{1/2}t^{1/2}} + \lambda D^{1/2}t^{1/2} \right) \\
&\left.\left. - \exp[-(b_2+2nL-z)\lambda] erfc\left(\frac{b_2+2nL-z}{2D^{1/2}t^{1/2}} - \lambda D^{1/2}t^{1/2} \right) \right] \right\}
\end{aligned} \tag{A11}
$$

$$W5=\sum_{n=1}^{\infty}\int_0^t\left[erf\left(\frac{z+b_2+2nL}{2\sqrt{D(t-t')}}\right)-erf\left(\frac{z+b_1+2nL}{2\sqrt{D(t-t')}}\right)\right]\exp\left[-D\lambda^2(t-t')\right]dt'$$
$$=\sum_{n=1}^{\infty}\left\{\frac{1}{D\lambda^2}\left[erf\left(\frac{z+b_1+2nL}{2D^{1/2}t^{1/2}}\right)-erf\left(\frac{z+b_2+2nL}{2D^{1/2}t^{1/2}}\right)\right]\exp\left(-D\lambda^2t\right)+\frac{1}{2D\lambda^2}\right.$$
$$\left[\exp[(z+b_1+2nL)\lambda]erfc\left(\frac{z+b_1+2nL}{2D^{1/2}t^{1/2}}+\lambda D^{1/2}t^{1/2}\right)\right.$$
$$+\exp[-(z+b_1+2nL)\lambda]erfc\left(\frac{z+b_1+2nL}{2D^{1/2}t^{1/2}}-\lambda D^{1/2}t^{1/2}\right)$$
$$-\exp[(z+b_2+2nL)\lambda]erfc\left(\frac{z+b_2+2nL}{2D^{1/2}t^{1/2}}+\lambda D^{1/2}t^{1/2}\right)$$
$$\left.\left.-\exp[-(z+b_2+2nL)\lambda]erfc\left(\frac{z+b_2+2nL}{2D^{1/2}t^{1/2}}-\lambda D^{1/2}t^{1/2}\right)\right]\right\} \tag{A12}$$

$$W6=\sum_{n=1}^{\infty}\int_0^t\left[erf\left(\frac{z+b_2-2nL}{2\sqrt{D(t-t')}}\right)-erf\left(\frac{z+b_1-2nL}{2\sqrt{D(t-t')}}\right)\right]\exp\left[-D\lambda^2(t-t')\right]dt'$$
$$=\sum_{n=1}^{\infty}\left\{\frac{1}{D\lambda^2}\left[erf\left(\frac{2nL-b_2-z}{2D^{1/2}t^{1/2}}\right)-erf\left(\frac{2nL-b_1-z}{2D^{1/2}t^{1/2}}\right)\right]\exp\left(-D\lambda^2t\right)+\frac{1}{2D\lambda^2}\right.$$
$$\left[\exp[(2nL-b_2-z)\lambda]erfc\left(\frac{2nL-b_2-z}{2D^{1/2}t^{1/2}}+\lambda D^{1/2}t^{1/2}\right)\right.$$
$$+\exp[-(2nL-b_2-z)\lambda]erfc\left(\frac{2nL-b_2-z}{2D^{1/2}t^{1/2}}-\lambda D^{1/2}t^{1/2}\right)$$
$$-\exp[(2nL-b_1-z)\lambda]erfc\left(\frac{2nL-b_1-z}{2D^{1/2}t^{1/2}}+\lambda D^{1/2}t^{1/2}\right)$$
$$\left.\left.-\exp[-(2nL-b_1-z)\lambda]erfc\left(\frac{2nL-b_1-z}{2D^{1/2}t^{1/2}}-\lambda D^{1/2}t^{1/2}\right)\right]\right\} \tag{A13}$$

The result of integration shown in Eq. A8 is based on the assumption that the screened portion of an observation well is within the upper and lower limits of the screened depths of the pumping well, *i.e.*, $b_1 \le z \le b_2$. This result should be modified if the observed depth (z) is not located between b_1 and b_2.

Appendix B

The function $\overline{V}'(\tau,\alpha)$ in Eq. 7 is written as

$$\overline{V}'(\tau,\alpha)=(V1+V2+V3+V4+V5+V6)\big|_{z_D=b'_{1D}}^{z_D=b'_{2D}} \tag{B1}$$

in which

$$V1 = \frac{1}{\alpha^2}\left\{2\left(b'_{2D} - b'_{1D}\right) - \exp\left(-\tau\alpha^2\right)\left[I1\left(\frac{b_{2D} - z_D}{2r_{wD}\tau^{1/2}}\right) - I1\left(\frac{z_D - b_{1D}}{2r_{wD}\tau^{1/2}}\right)\right]\right.$$
$$- \frac{1}{2}\left[I2\left(\frac{b_{2D} - z_D}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right) - I2\left(\frac{z_D - b_{1D}}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right)\right.$$
$$\left.\left. + I3\left(\frac{b_{2D} - z_D}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right) - I3\left(\frac{z_D - b_{1D}}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right)\right]\right\} \tag{B2}$$

$$V2 = \frac{1}{\alpha^2}\left\{\exp\left(-\tau\alpha^2\right)\left[-I1\left(\frac{z_D + b_{1D}}{2r_{wD}\tau^{1/2}}\right) + I1\left(\frac{z_D + b_{2D}}{2r_{wD}\tau^{1/2}}\right)\right]\right.$$
$$+ \frac{1}{2}\left[-I2\left(\frac{z_D + b_{1D}}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right) + I2\left(\frac{z_D + b_{2D}}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right)\right.$$
$$\left.\left. - I3\left(\frac{z_D + b_{1D}}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right) + I3\left(\frac{z_D + b_{2D}}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right)\right]\right\} \tag{B3}$$

$$V3 = \sum_{n=1}^{\infty}\left\langle\frac{1}{\alpha^2}\left\{\exp\left(-\tau\alpha^2\right)\left[I1\left(\frac{b_{1D} + 2n - z_D}{2r_{wD}\tau^{1/2}}\right) - I1\left(\frac{b_{2D} + 2n - z_D}{2r_{wD}\tau^{1/2}}\right)\right]\right.\right.$$
$$+ \frac{1}{2}\left[I2\left(\frac{b_{1D} + 2n - z_D}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right) - I2\left(\frac{b_{2D} + 2n - z_D}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right)\right.$$
$$\left.\left.\left. + I3\left(\frac{b_{1D} + 2n - z_D}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right) - I3\left(\frac{b_{2D} + 2n - z_D}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right)\right]\right\}\right\rangle \tag{B4}$$

$$V4 = \sum_{n=1}^{\infty}\left\langle\frac{1}{\alpha^2}\left\{\exp\left(-\tau\alpha^2\right)\left[-I1\left(\frac{z_D - b_{2D} + 2n}{2r_{wD}\tau^{1/2}}\right) + I1\left(\frac{z_D - b_{1D} + 2n}{2r_{wD}\tau^{1/2}}\right)\right]\right.\right.$$
$$+ \frac{1}{2}\left[-I2\left(\frac{z_D - b_{2D} + 2n}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right) + I2\left(\frac{z_D - b_{1D} + 2n}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right)\right.$$
$$\left.\left.\left. - I3\left(\frac{z_D - b_{2D} + 2n}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right) + I3\left(\frac{z_D - b_{1D} + 2n}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right)\right]\right\}\right\rangle \tag{B5}$$

$$V5 = \sum_{n=1}^{\infty}\left\langle\frac{1}{\alpha^2}\left\{\exp\left(-\tau\alpha^2\right)\left[I1\left(\frac{2n - b_{2D} - z_D}{2r_{wD}\tau^{1/2}}\right) - I1\left(\frac{2n - b_{2D} - z_D}{2r_{wD}\tau^{1/2}}\right)\right]\right.\right.$$
$$+ \frac{1}{2}\left[I2\left(\frac{2n - b_{2D} - z_D}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right) - I2\left(\frac{2n - b_{2D} - z_D}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right)\right.$$
$$\left.\left.\left. + I3\left(\frac{2n - b_{2D} - z_D}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right) - I3\left(\frac{2n - b_{2D} - z_D}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right)\right]\right\}\right\rangle \tag{B6}$$

$$V6 = \sum_{n=1}^{\infty} \left\langle \frac{1}{\alpha^2} \left\{ \exp\left(-\tau\alpha^2\right) \left[-I1\left(\frac{z_D + b_{1D} + 2n}{2r_{wD}\tau^{1/2}}\right) + I1\left(\frac{z_D + b_{2D} + 2n}{2r_{wD}\tau^{1/2}}\right) \right] \right. \right.$$
$$+ \frac{1}{2}\left[-I2\left(\frac{z_D + b_{1D} + 2n}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right) + I2\left(\frac{z_D + b_{2D} + 2n}{2r_{wD}\tau^{1/2}} + \alpha\tau^{1/2}\right) \right. \tag{B7}$$
$$\left. \left. \left. - I3\left(\frac{z_D + b_{1D} + 2n}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right) + I3\left(\frac{z_D + b_{2D} + 2n}{2r_{wD}\tau^{1/2}} - \alpha\tau^{1/2}\right) \right] \right\} \right\rangle$$

and

$$I1(\xi) = -2r_{wD}\tau^{1/2}\left[\xi erf\xi + \frac{1}{\sqrt{\pi}}\exp\left(-\xi^2\right)\right] \tag{B8}$$

$$I2(\xi) = -\frac{r_{wD}}{\alpha}\exp\left(-2\alpha^2\tau\right)\left[\exp\left(2\alpha\tau^{1/2}\xi\right) erfc\xi + \exp\left(\alpha^2\tau\right) erfc\left(\xi - \alpha\tau^{1/2}\right)\right] \tag{B9}$$

$$I3(\xi) = \frac{r_{wD}}{\alpha}\exp\left(-2\alpha^2\tau\right)\left[\exp\left(-2\alpha\tau^{1/2}\xi\right) erfc\xi + \exp\left(\alpha^2\tau\right) erfc\left(\xi + \alpha\tau^{1/2}\right)\right] \tag{B10}$$

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