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Groundwater Pumping Cost Minimization – an Analytical Approach

K. L. Katsifarakis

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Abstract In this paper, a groundwater resources management problem has been studied, namely pumping cost minimization for any number and layout of wells. Steady state flow in infinite and semi-infinite confined aquifers, to which the method of images applies, has been considered. It has been proved analytically that when pumping cost is minimized, hydraulic head is the same at all wells. Based on this proof, an analytical calculation procedure of the respective optimal distribution of the required total flow rate to the individual wells is also presented.

Keywords Pumping cost · System of wells · Optimization process · Analytical approach · Groundwater flow

Abbreviations

- h_J distance between water level et well J and the predefined reference level (m)
- N number of wells
- Q_J flow rate of well J (l/s) or (m^3/s)
- Q_T total flow rate pumped from the system of N wells (l/s)
- R radius of influence of the system of wells (m)
- r_0 well radius (m)
- r_{IJ} distance between wells I and J (m)
- s_J drawdown of hydraulic head at well J, due to the operation of the system of the wells (m)
- T aquifer transmissivity (m^2/s)
- δ distance between the initial horizontal level of the hydraulic head and the predefined reference level (m)

K. L. Katsifarakis (✉)
Department of Civil Engineering, Division of Hydraulics and Environmental Engineering,
A.U.Th, GR-54124 Thessaloniki, Macedonia, Greece
e-mail: klkats@civil.auth.gr

1 Introduction

One of the most common problems in groundwater resources management is minimization of the cost due to pumping (e.g. Sidiropoulos and Tolikas 2004). Quite often there are additional constraints to the optimization process, such as flow rate limits, due to pump capacities, or limits to hydraulic head drawdown in parts of the aquifer. In other cases pumping cost is examined together with other cost items, such as well or pipe network construction cost. Water quality considerations, like seawater intrusion, may also enter the optimization process (e.g. Katsifarakis et al. 1999, Petala 2004). In many cases, pumping cost is the main item in aquifer restoration problems (e.g. Matott et al. 2006, Papadopoulou et al. 2007).

Due to the importance of proper development of groundwater resources, many optimization methods have been used to tackle the respective problems. They range from linear and non-linear programming (e.g. Bear 1979; Rastogi 1989; Mylopoulos et al. 1999) to genetic algorithms and other evolutionary techniques (e.g. Ouazar and Cheng 2000, Mantoglou et al. 2004).

In this paper the following proposition is proved (for steady flow conditions): When the cost to pump a given total flow rate Q_T from any number and layout of wells in a confined aquifer is minimized, hydraulic head levels at all wells are equal to each other, as long as flow is due to that system of wells only. This proposition holds for infinite aquifers, as well for semi-infinite ones, to which the method of images applies.

Based on the aforementioned proof, an analytical calculation procedure of the respective optimal distribution of Q_T to the individual wells is also presented.

2 Mathematical Formulation of the Problem

Pumping cost, namely the objective function of the minimization problem, is defined as:

$$K_1 = A \cdot \sum_{J=1}^{N} \mathrm{Q}_J \cdot h_J \tag{2.1}$$

where N is the number of wells, Q_J is the flow rate of well J, h_J the distance between water level at well J and a predefined level (e.g. highest ground elevation) and A is a constant, depending on energy cost. Treating A as constant implies that pump efficiencies are considered as constants, too, and equal to each other.

Well flow rates Q_J should not obtain negative values, since such values correspond to recharge wells. Moreover, they should fulfill the basic constraint of the problem, namely:

$$\sum_{J=1}^{N} \mathrm{Q}_J = \mathrm{Q}_T \tag{2.2}$$

3 Infinite Aquifers

Since flow is due to the system of wells only, the initial hydraulic head level is horizontal. Then, Eq. 2.1 can be written as:

$$K_1 = A \cdot \sum_{J=1}^{N} \mathrm{Q}_J \cdot (s_J + \delta) \tag{3.1}$$

where s_J is the drawdown of the hydraulic head at well J, which is due to the operation of the system of the wells, and δ is the distance between the initial horizontal level of the hydraulic head and the predefined reference level. Since δ is the same everywhere, the function K that should actually be minimized is:

$$K = \sum_{J=1}^{N} \mathrm{Q}_J \cdot s_J \tag{3.2}$$

Since steady flow is considered, s_J for a system of N wells is given as (e.g. Bear, 1979):

$$s_J = -\frac{1}{2\pi T} \cdot \sum_{I=1}^{N} \mathrm{Q}_I \cdot 1n\frac{r_{IJ}}{R} \tag{3.3}$$

where T is aquifer's transmissivity, R the radius of influence of the system of wells and r_{IJ} the distance between wells I and J (therefore r_{IJ}=r_{JI}). The value of r_{JJ} in particular, is equal to the radius of well J, denoted by r_0 for all J. Then K, referred as "variable pumping cost" in the following paragraphs, can be written as:

$$K = -\frac{1}{2\pi T} \cdot \sum_{I=1}^{N} \mathrm{Q}_I \cdot \sum_{J=1}^{N} \mathrm{Q}_J \cdot 1n\frac{r_{IJ}}{R} \tag{3.4}$$

Well flow rates entering (3.4) are not independent of each other, since they are subject to the constraint (2.2). We can assume, though, without loss of generality, that the first $N-1$ of them are independent, while Q_N depends upon the rest, namely

$$\mathrm{Q}_N = \mathrm{Q}_T - \sum_{I=1}^{N-1} \mathrm{Q}_I \tag{3.5}$$

It follows that, for any $M \in [1, N-1]$

$$\frac{\partial \mathrm{Q}_N}{\partial \mathrm{Q}_M} = -1 \tag{3.6}$$

To investigate the conditions that minimize K, we calculate its first derivative with respect to Q_M. First we write K as:

$$\begin{aligned}
K &= -\frac{1}{2\pi T} \cdot \left(\sum_{I=1}^{N-1} \mathrm{Q}_I + \mathrm{Q}_N\right)\left(\sum_{J=1}^{N-1} \mathrm{Q}_J \cdot 1n\frac{r_{IJ}}{R} + \mathrm{Q}_N \cdot 1n\frac{r_{IN}}{R}\right) \\
&\Rightarrow K = -\frac{1}{2\pi T}\left[\sum_{I=1}^{N-1} \mathrm{Q}_I \cdot \sum_{J=1}^{N-1} \mathrm{Q}_J \cdot 1n\frac{r_{IJ}}{R} + \mathrm{Q}_N \cdot \sum_{i=1}^{N-1} \mathrm{Q}_I \cdot 1n\frac{r_{IN}}{R} + \mathrm{Q}_N \cdot \sum_{J=1}^{N-1} \mathrm{Q}_J \cdot 1n\frac{r_{NJ}}{R} + \mathrm{Q}_n^2 \cdot 1n\frac{r_0}{R}\right] \\
&\Rightarrow K = -\frac{1}{2\pi T}\left[\sum_{I=1}^{N-1} \mathrm{Q}_I \cdot \sum_{J=1}^{N-1} \mathrm{Q}_J \cdot 1n\frac{r_{IJ}}{R} + 2 \cdot \mathrm{Q}_N \sum_{J=1}^{N-1} \mathrm{Q}_J \cdot 1n\frac{r_{NJ}}{R} + \mathrm{Q}_N^2 \cdot 1n\frac{r_0}{R}\right]
\end{aligned} \tag{3.7}$$

To derive Eq. 3.7, the equality $r_{IJ}=r_{JI}$ has been used. Then, the derivative of K with respect to Q_M is:

$$\begin{aligned}\frac{\partial K}{\partial Q_M} &= -\frac{1}{2\pi T}\left(2\sum_{J=1}^{N-1} Q_J\cdot\ln\frac{r_{MJ}}{R}+2Q_N\cdot\ln\frac{r_{MN}}{R}+2\frac{\partial Q_N}{\partial Q_M}\cdot\sum_{J=1}^{N-1} Q_j\cdot\ln\frac{r_{NJ}}{R}+2Q_N\frac{\partial Q_N}{\partial Q_M}\cdot\ln\frac{r_0}{R}\right)\\ &= -\frac{2}{2\pi T}\left(\sum_{J=1}^{N-1} Q_J\cdot\ln\frac{r_{MJ}}{R}+Q_N\cdot\ln\frac{r_{MN}}{R}-\sum_{J=1}^{N-1} Q_J\cdot\ln\frac{r_{NJ}}{R}-Q_N\cdot\ln\frac{r_0}{R}\right)\\ &= -\frac{2}{2\pi T}\left(\sum_{J=1}^{N} Q_J\cdot\ln\frac{r_{MJ}}{R}-\sum_{J=1}^{N} Q_J\cdot\ln\frac{r_{NJ}}{R}\right)\Rightarrow\frac{\partial K}{\partial Q_M}=2\cdot(s_M-s_N)\end{aligned} \tag{3.8}$$

To derive Eq. 3.8, Eqs. 3.6 and 3.3 have been used. Setting the derivative of K equal to zero, one gets:

$$\frac{\partial K}{\partial \mathrm{Q}_M}=0\Leftrightarrow s_M=s_N \tag{3.9}$$

Moreover,

$$\frac{\partial K}{\partial \mathrm{Q}_M}=0\Rightarrow\sum_{J=1}^{N}\mathrm{Q}_J\cdot\left(1n\frac{r_{MJ}}{R}-\ln\frac{r_{JN}}{R}\right)=0 \tag{3.10}$$

Since Eq. 3.9 holds for every $M\in[1,N-1]$, a critical point of variable pumping cost K corresponds to equal hydraulic head drawdowns at all wells. Its coordinates, namely the corresponding set of Q_M values, can be found by solving a linear system of N equations and N unknowns. The first $N-1$ equations are obtained by writing Eq. 3.10, for every $M\in[1, N-1]$. The Nth equation, which completes the system, is Eq. 2.2, namely:

$$\sum_{J=1}^{N}\mathrm{Q}_J=\mathrm{Q}_T$$

The aforementioned linear system has one solution only, namely only one critical point P exists. To verify that P corresponds to the minimum of K, the respective second derivative conditions should be checked. Starting from Eq. 3.8 one gets:

$$\begin{aligned}\frac{\partial^2 K}{\partial \mathrm{Q}_M^2} &= -\frac{1}{\pi T}\left(\sum_{J=1}^{N}\frac{\partial \mathrm{Q}_J}{\partial \mathrm{Q}_M}\cdot\ln\frac{r_{MJ}}{R}-\sum_{J=1}^{N}\frac{\partial \mathrm{Q}_J}{\partial \mathrm{Q}_M}\cdot\ln\frac{r_{NJ}}{R}\right)\\ &= -\frac{1}{\pi T}\left(\ln\frac{r_{MM}}{R}-\ln\frac{r_{MN}}{R}-\ln\frac{r_{NM}}{R}+\ln\frac{r_{NN}}{R}\right)\Rightarrow\frac{\partial^2 K}{\partial \mathrm{Q}_M^2}\\ &= -\frac{2}{\pi T}\left(\ln\frac{r_0}{R}-\ln\frac{r_{MN}}{R}\right)\end{aligned} \tag{3.11}$$

The parenthesis of the right hand side of Eq. 3.11 is negative, since well radius r_0 is smaller than any distance between wells. Thus, the value of the second derivative of K with respect to Q_M is positive, for every $M\in[1, N-1]$. This means that the critical point P

corresponds to a minimum or to a saddle point. Suppose it corresponds to a saddle point, namely K exhibits a maximum at P along a certain "direction" Q_{EM} with $E\in[1, N-1]$. Since there is no other critical point, K should decrease continuously along Q_{EM}, as we move away from P. But K is positive, so its value should tend asymptotically to a minimum at infinity, implying that K is convex there, along Q_{EM}. Yet K is concave at P along Q_{EM}. Transition from concave to convex would require a zero of the second mixed derivative of K with respect to Q_M and Q_E, at a point between P and infinity. It results from Eq. 3.8 that this derivative has the following form:

$$\begin{aligned}\frac{\partial^2 K}{\partial \mathrm{Q}_E \partial \mathrm{Q}_M} &= -\frac{1}{\pi T}\left(\ln\frac{r_0}{R} + \ln\frac{r_{ME}}{R} - \ln\frac{r_{MN}}{R} - \ln\frac{r_{NE}}{R}\right) \\ &= -\frac{1}{\pi T}\left(\ln r_0 - \ln\frac{r_{MN}\cdot r_{NE}}{r_{ME}}\right)\end{aligned} \tag{3.12}$$

Namely its value (and the value of any other second derivative of K) is constant. Consequently, the assumption that P is a saddle point does not hold. Hence it is a local minimum. But, since P is the only critical point of K, it is the global minimum. Finally, it can be proved that the Q_J values, which correspond to the minimum variable pumping cost and result from the solution of the aforementioned linear system of N equations and N unknowns, are all positive. The proof is based on a physical argument: The lowest hydraulic head level (or the maximum respective drawdown, $s_{\max}$), always coincides with a pumped well, since water flows towards it and, if not pumped, it would accumulate. Since the solution of the system leads to equal s_J for all wells, all of them are equal to $s_{\max}$. So they correspond to pumped wells, namely to positive Q_J values.

3.1 Illustrative Example

The analytical procedure of calculating the optimal distribution of Q_T to N wells and the respective variable pumping cost is illustrated through the following example: Find the optimal distribution of a total required flow rate Q_T=500 l/s to 6 wells, pumping from an infinite confined aquifer, under steady state flow conditions. The radius of each well is r_0= 0.2 m, while the radius of influence of the system of wells is R=3,000 m. The coordinates of the wells are given in Table 1, while aquifer transmissivity T=0.0025 m^2/s.

As shown in Table 1, wells 1, 2 and 3 are close to each other, wells 4 and 5 form another cluster and well 6 is isolated. The distances between the wells are summarized in Table 2.

To find the optimal distribution of Q_T, a linear system of six equations and six unknowns (namely Q_I, $I\in$ [1, 6]), should be solved. The coefficients $A(I,J)$ of the five first equations result from Eq. 3.10. For example, the coefficients of the first equation (M=1) are summarized in Table 3. It can be seen that the diagonal coefficient, which includes r_0, is always different than zero.

Table 1 Well coordinates

Well	1	2	3	4	5	6
x_I	100	180	100	700	800	900
y_I	0	0	80	0	0	900

Table 2 Distances between wells

Well	1	2	3	4	5	6
1	0.2	80	80	600	700	1,204.16
2	80	0.2	113.14	520	620	1,152.56
3	80	113.14	0.2	605.31	704.56	1,145.60
4	600	520	605.31	0.2	100	921.96
5	700	620	704.56	100	0.2	905.54
6	1,204.16	1,152.56	1,145.60	921.96	905.54	0.2

The coefficients $A(6{,}J)$ are all equal to 1, while the constant terms of the equations are: $C(1) = C(2) = C(3) = C(4) = C(5) = 0$, according to Eq. 3.10, while $C(6) = Q_T = 500l/s$, according to Eq. 2.2. Solution of the equation system results in the following well flow rates (in l/s):

$$\mathrm{Q}_1 = 70.21 \quad \mathrm{Q}_2 = 69.92 \quad \mathrm{Q}_3 = 73.46 \quad \mathrm{Q}_4 = 80.32 \quad \mathrm{Q}_5 = 85.49 \quad \mathrm{Q}_2 = 120.60$$

It can be seen that the flow rate of the isolated well 6 is almost twice as large as the flow rate of well 2. Moreover, after transforming l/s to m^3/s, one gets from Eq. 3.3 that the drawdown of hydraulic head at any well J is s_j=99.27 m. Finally, it results from Eq. 3.2, that the variable pumping cost is K=0.5·99.27=49.635 m^4/s.

4 Semi-Infinite Aquifers

The aforementioned procedure can be used to study pumping cost minimization to semi-infinite aquifers, to which the method of images applies. Two cases are presented in the following paragraphs: (a) Flow fields with a rectilinear impermeable boundary and (b) Flow fields with a rectilinear constant head boundary.

4.1 Flow Fields with a Rectilinear Impermeable Boundary

According to the method of images, a straight-line impermeable boundary has the same effect and therefore it can be replaced by a set of N imaginary wells, symmetric of the real ones with respect to that boundary and of the same sign. Then s_J at any real well J is given as:

$$s_J = -\frac{1}{2\pi T} \cdot \sum_{I=1}^{N} \mathrm{Q}_I \cdot 1n \frac{r_{IJ} \cdot r_{iJ}}{R^2} \tag{4.1}$$

Table 3 Coefficients A(1,J)

Summary of coefficients
$A(1,1) = \ln\frac{r_0}{R} - \ln\frac{r_{16}}{R} = -8.703$
$A(1,2) = \ln\frac{r_{12}}{R} - \ln\frac{r_{26}}{R} = -2.677$
$A(1,3) = \ln\frac{r_{13}}{R} - \ln\frac{r_{36}}{R} = -2.662$
$A(1,4) = \ln\frac{r_{14}}{R} - \ln\frac{r_{46}}{R} = -0.4295$
$A(1,5) = \ln\frac{r_{15}}{R} - \ln\frac{r_{56}}{R} = -0.2575$
$A(1,6) = \ln\frac{r_{16}}{R} - \ln\frac{r_0}{R} = 8.703$

where small case indices refer to image wells, namely r_{iJ} is the distance between well J and the image of well I. Due to symmetry, $r_{iJ}=r_{Ji}$. Then, the variable pumping cost has the following form:

$$K = -\frac{1}{2\pi T}\cdot\sum_{I=1}^{N}\mathrm{Q}_I\cdot\sum_{J=1}^{N}\mathrm{Q}_J\cdot 1n\frac{r_{IJ}\cdot r_{Ij}}{R^2} \tag{4.2}$$

Assuming, as in the infinite aquifer case, that the first $N-1$ well flow rates Q_J are independent of each other, while Q_N depends upon the rest, the derivative of K with respect to any Q_M, M∈[1, $N-1$] can be finally written as:

$$\frac{\partial K}{\partial Q_M} = -\frac{2}{2\pi T}\left(\sum_{J=1}^{N}Q_J\cdot\ln\frac{r_{MJ}\cdot r_{Mj}}{R^2} - \sum_{J=1}^{N}Q_J\cdot\ln\frac{r_{NJ}\cdot r_{Nj}}{R^2}\right)$$

$$\Rightarrow \frac{\partial K}{\partial Q_M} = 2\cdot(s_M - s_N) \tag{4.3}$$

Setting the derivative of K equal to zero, one gets:

$$\frac{\partial K}{\partial Q_M} = 0 \Leftrightarrow s_M = s_N \tag{4.4}$$

Moreover,

$$\frac{\partial K}{\partial Q_M} = 0 \Rightarrow \sum_{J=1}^{N}Q_J\cdot\left(1n\frac{r_{MJ}\cdot r_{Mj}}{R^2} - 1n\frac{r_{JN}\cdot r_{jN}}{R^2}\right) = 0 \tag{4.5}$$

Since (4.4) holds for every M∈[1,$N-1$], a critical point of variable pumping cost K corresponds to equal hydraulic head drawdowns at all wells. Its coordinates, namely the corresponding set of Q_M values, can be found by solving a linear system of N equations and N unknowns. The first $N-1$ equations are obtained by writing Eq. 4.5, for every M∈ [1, $N-1$]. The Nth equation, which completes the system, is again Eq. 2.2. The aforementioned linear system has one solution only, namely only one critical point exists. To verify that this point corresponds to the minimum of K, the respective second derivative conditions should be checked. Starting from Eq. 4.3 one gets:

$$\begin{aligned}
\frac{\partial^2 K}{\partial \mathrm{Q}_M^2} &= -\frac{1}{\pi T}\left(\sum_{J=1}^{N}\frac{\partial \mathrm{Q}_J}{\partial \mathrm{Q}_M}\cdot\ln\frac{r_{MJ}\cdot r_{Mj}}{R^2} - \sum_{J=1}^{N}\frac{\partial \mathrm{Q}_J}{\partial \mathrm{Q}_M}\cdot\ln\frac{r_{NJ}\cdot r_{Nj}}{R^2}\right)\\
&= -\frac{1}{\pi T}\left(\ln\frac{r_{MM}\cdot r_{Mm}}{R^2} - \ln\frac{r_{MN}\cdot r_{Mn}}{R^2} - \ln\frac{r_{NM}\cdot r_{Nm}}{R^2} + \ln\frac{r_{NN}\cdot r_{Nn}}{R^2}\right)\\
&= -\frac{1}{\pi T}\left(2\cdot\ln\frac{r_0}{R} - 2\cdot\ln\frac{r_{MN}}{R} + \ln\frac{r_{Mm}\cdot r_{Nn}}{R^2} - \ln\frac{r_{Nm}^2}{R^2}\right) \Rightarrow \frac{\partial^2 K}{\partial \mathrm{Q}_M^2}\\
&= -\frac{2}{\pi T}\left(\ln\frac{r_0}{R} - \ln\frac{r_{MN}}{R}\right) - \frac{1}{\pi T}\left(\ln\frac{r_{Mm}\cdot r_{Nn}}{R^2} - \ln\frac{r_{Nm}^2}{R^2}\right)
\end{aligned} \tag{4.6}$$

The first parenthesis of the right hand side of Eq. 4.6 is negative, since well radius r_0 is smaller than any distance between wells. The second parenthesis is also negative, for the following reason: As shown in Fig. 1, wells M and N and their images m and N form an

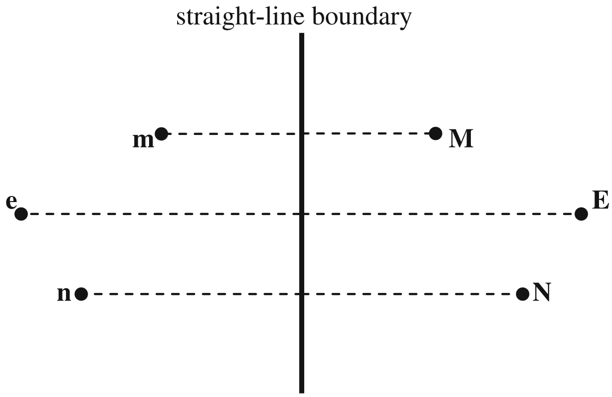

Fig. 1 Wells M, E and N and their images m, e and n

isosceles trapezoid, with bases r_{Mm} and r_{Nn}, while r_{Nm} is its diagonal. It follows from the Pythagorean theorem that

$$r_{Nm}^2 \geq \left(\frac{r_{Mm} + r_{Mn}}{2}\right)^2 \geq \frac{2 \cdot r_{Mm} \cdot r_{Nn} + 2 \cdot r_{Mm} \cdot r_{Nn}}{4} = r_{Mm} \cdot r_{Nn} \tag{4.7}$$

Since both parentheses are negative, the value of the second derivative of K with respect to Q_M is positive, for every $M \in [1, N-1]$. Thus, the value of the second derivative of K with respect to Q_M is positive, for every $M \in [1, N-1]$. This means that the critical point P corresponds to a minimum or to a saddle point. Suppose it corresponds to a saddle point, namely K exhibits a maximum at P along a certain "direction" Q_{EM} with $E \in [1, N-1]$. Since there is no other critical point, K should decrease continuously along Q_{EM}, as we move away from P. But K is positive, so its value should tend asymptotically to a minimum at infinity, implying that K is convex there, along Q_{EM}. Yet K is concave at P along Q_{EM}. Transition from concave to convex would require a zero of the second mixed derivative of K with respect to Q_M and Q_E, at a point between P and infinity. It results from (4.3) that this derivative has the following form:

$$\begin{aligned} \frac{\partial^2 K}{\partial \mathrm{Q}_E \partial \mathrm{Q}_M} &= -\frac{1}{\pi T}\left(\ln\frac{r_0 \cdot r_{Nn}}{R} + \ln\frac{r_{ME} \cdot r_{Me}}{R} - \ln\frac{r_{MN} \cdot r_{Mn}}{R} - \ln\frac{r_{NE} \cdot r_{Ne}}{R}\right) \\ &= -\frac{1}{\pi T}\ln\frac{r_0 \cdot r_{Nn} \cdot r_{ME} \cdot r_{Me}}{r_{MN} \cdot r_{Mn} \cdot r_{NE} \cdot r_{Ne}} \end{aligned} \tag{4.8}$$

Namely its value is constant and, consequently, the assumption that P is a saddle point does not hold. Hence it is a local minimum. But, since P is the only critical point of K, it is the global minimum.

4.2 Flow Fields with a Rectilinear Constant Head Boundary

According to the method of images, a straight line constant head boundary has the same effect and therefore it can be replaced by a set of N imaginary wells, symmetric of the real ones with respect to that boundary and with opposite sign (namely recharge wells are the images of pumped wells). Then s_J at any real well J is given as:

$$s_J = -\frac{1}{2\pi T} \cdot \sum_{I=1}^{N} Q_I \cdot 1n \frac{r_{IJ}}{r_{iJ}} \tag{4.10}$$

where small case indices refer again to image wells, and, due to symmetry, $r_{iJ}=r_{Ji}$. Then, the variable pumping cost has the following form:

$$K = -\frac{1}{2\pi T} \cdot \sum_{I=1}^{N} Q_I \cdot \sum_{J=1}^{N} Q_J \cdot 1n \frac{r_{IJ}}{r_{Ij}} \tag{4.11}$$

Assuming, as in the previous cases, that the first $N-1$ well flow rates Q_J are independent of each other, while Q_N depends upon the rest, the derivative of K with respect to Q_M can be finally written as:

$$\frac{\partial K}{\partial Q_M} = -\frac{2}{2\pi T}\left(\sum_{J=1}^{N} Q_J \cdot \ln\frac{r_{MJ}}{r_{Mj}} - \sum_{J=1}^{N} Q_J \cdot \ln\frac{r_{NJ}}{r_{Nj}}\right) \Rightarrow \frac{\partial K}{\partial Q_M} = 2 \cdot (s_M - s_N) \tag{4.12}$$

Setting the derivative of K equal to zero, one gets:

$$\frac{\partial K}{\partial Q_M} = 0 \Leftrightarrow s_M = s_N \tag{4.13}$$

Moreover,

$$\frac{\partial K}{\partial Q_M} = 0 \Rightarrow \sum_{J=1}^{N} Q_J \cdot \left(1n\frac{r_{MJ}}{r_{Mj}} - 1n\frac{r_{JN}}{r_{jN}}\right) = 0 \tag{4.14}$$

Since (4.13) holds for every $M\in[1, N-1]$, a critical point of variable pumping cost K corresponds to equal hydraulic head drawdowns at all wells. Its coordinates, namely the corresponding set of Q_M values, can be found by solving a linear system of N equations and N unknowns. The first $N-1$ equations are obtained by writing Eq. 4.14, for every $M\in[1, N-1]$. The Nth equation, which completes the system, is again Eq. 2.2. The aforementioned linear system has one solution only, namely only one critical point exists. To verify that this point corresponds to the minimum of K, the respective second derivative conditions should be checked. Starting from (4.12) one gets:

$$\begin{aligned}
\frac{\partial^2 K}{\partial Q_M^2} &= -\frac{1}{\pi T}\left(\sum_{J=1}^{N} \frac{\partial Q_J}{\partial Q_M} \cdot \ln\frac{r_{MJ}}{r_{Mj}} - \sum_{J=1}^{N} \frac{\partial Q_J}{\partial Q_M} \cdot \ln\frac{r_{NJ}}{r_{Nj}}\right) \\
&= -\frac{1}{\pi T}\left(\ln\frac{r_{MM}}{r_{Mm}} - \ln\frac{r_{MN}}{r_{Mn}} - \ln\frac{r_{NM}\cdot}{r_{Nm}} + \ln\frac{r_{NN}}{r_{Nn}}\right) \Rightarrow \frac{\partial^2 K}{\partial Q_M^2} \\
&= -\frac{1}{\pi T}\left(\ln r_0^2 - \ln\frac{r_{MN}^2 \cdot r_{Mm} \cdot r_{Nn}}{r_{Mn}^2}\right)
\end{aligned} \tag{4.15}$$

The value of the second derivative is positive, if the value of the parenthesis is negative, which is equivalent to

$$r_0^2 \leq \frac{r_{MN}^2 \cdot r_{Mm} \cdot r_{Nn}}{r_{Mn}^2} \tag{4.16}$$

But Eq. 4.16 holds, as long as $2{\cdot}r_0$ is smaller than any well distance r_w. The proof is based on the triangle inequality. Applying it to triangle *Mmn* of Fig. 1, one gets that at least one of r_{mn} (which is equal to r_{MN}) and r_{Mm} is larger than $r_{Mn}/2$. Similarly, from triangle *MnN* one gets that at least one of r_{nN} and r_{MN} is larger than $r_{Mn}/2$. Therefore:

$$\frac{r_{MN}^2 \cdot r_{Mm} \cdot r_{Nn}}{r_{Mn}^2} > \frac{r_{Mn}^2 \cdot r_{w1} \cdot r_{w2}}{4 \cdot r_{Mn}^2} = \frac{r_{w1} \cdot r_{w2}}{4} \tag{4.17}$$

where r_{w1}=min $[r_{mn}, r_{Mm}]$ and r_{w2}=min$[r_{MN}, r_{Nn}]$. Therefore, if $2{\cdot}r_0$ is smaller than any r_W, Eq. 4.16 holds a fortiori, and the value of the second derivative of K is positive, for any $M \in [1, N-1]$. This means that the critical point P corresponds to a minimum or to a saddle point. The rest of the proof is similar to that of the previous case, taking into account that any second order mixed derivative of K (e.g. with respect to Q_M and Q_E), has the following form:

$$\frac{\partial^2 K}{\partial \mathrm{Q}_E \partial \mathrm{Q}_M} = -\frac{1}{\pi T}\left(\ln\frac{r_0}{r_{Nn}} + \ln\frac{r_{ME}}{r_{Me}} - \ln\frac{r_{MN}}{r_{Mn}} - \ln\frac{r_{NE}}{r_{Ne}}\right) \tag{4.18}$$

5 Summary and Discussion

In this paper a common water resources management problem has been studied, namely minimization of pumping cost in confined aquifers under steady state flow conditions. It has been proved that when the cost to pump a given total flow rate Q_T from any number and layout of wells is minimized, hydraulic head levels at all wells are equal to each other, as long as flow is due to that system of wells only. This proposition has been proved first for infinite aquifers and then for semi-infinite ones, to which the method of images applies.

The aforementioned result leads to an analytical calculation procedure of the respective optimal distribution of Q_T, when one considers pumping cost only. But it can also be used as a quality criterion for solutions of more complex problems, in which the optimization process could be trapped to suboptimal solutions. If, for instance, optimization of both the layout of the wells and the distribution of total flow rate to them is sought, differences between s_i values indicate that the optimum is not reached. Local fine-tuning of the solution could probably lead to better results in this case, without guarantying that the global optimum is reached. In other cases, discrepancies from the s_i equality might be justified by reductions in other cost items and should be evaluated accordingly.

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