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Image Segmentation Using Some Piecewise Constant Level Set Methods with MBO Type of Projection*

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Abstract. In this work, we are trying to propose fast algorithms for Mumford-Shah image segmentation using some recently proposed piecewise constant level set methods (PCLSM). Two variants of the PCLSM will be considered in this work. The first variant, which we call the binary level set method, needs a level set function which only takes values ± 1 to identify the regions. The second variant only needs to use one piecewise constant level set function to identify arbitrary number of regions. For the Mumford-Shah image segmentation model with these new level set methods, one needs to minimize some smooth energy functionals under some constrains. A penalty method will be used to deal with the constraint. AOS (additive operator splitting) and MOS (multiplicative operator splitting) schemes will be used to solve the Euler-Lagrange equations for the minimization problems. By doing this, we obtain some algorithms which are essentially applying the MBO scheme for our segmentation models. Advantages and disadvantages are discussed for the proposed schemes.

Keywords: level set method, image segmentation, total variation regularization, phase field model

1. Introduction

Recently, some piecewise constant level set methods (PCLSM) were proposed in Lie et al. (2005, 2004, 2003) for image segmentation and other interface problems. The binary level set method of Lie et al. (2004) is closely related to the phase field models (Du et al. 2004; Modica and Mortola 1977; Samson et al. 2000; Evans et al. 1992; Aubert and Kornprobst 2002; Rubinstein et al. 1989, 1993). For applications to image segmentation, it extends the ideas used in Gibou and Fedkiw (2002); Song and Chan (2002). The method proposed in Lie et al. (2003) seems to be a novel approach for tracing interfaces separating a domain into subdomains. It just needs to use one level set function to identify arbitrary number of regions. In Lie et al. (2004, 2003, 2005), the PCLSM was used for Mumford-Shah image segmentation. A smooth energy functional needs to be minimized under some constraint. The augmented

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Lagrangian method is used to deal with the constraint. For numerical implementations, Uzawa type of gradient iteration was used to find the minimizer of the cost functional. The iteration number is usually high. The purpose of this work is to develop some fast algorithms for these piecewise constant level set methods (PCLSM). We shall still concentrate on image segmentation in this work, but the method can be used for other problems that need to partition a domain into subdomains to minimize some energy functional.

We shall mainly try to use two technical devices to accelerate the convergence of the PCLSM. The first one is to use the MBO type of projection to deal with the constraint. The MBO scheme of Merriman et al. (1994) is used in a phase field model for motion by mean curvature.

Once the constraint has been handled by the MBO projection, we then propose AOS (additive operator splitting) schemes or multiplicative operator splitting (MOS) schemes to solve the Euler-Lagrange equations for the minimization of the energy functional. These splitting schemes could treat the time stepping in an implicit or semi-implicit manner and thus allow large time steps to be used. In addition, the computing cost is kept at a lower level due to the fact that tri-diagonal matrices can be solved exactly with low computing cost (Lu et al., 1991; Weickert et al., 1998).

We want to emphasize that we are trying to use MBO type of projection for two different piecewise constant level set methods. One of them, i.e. the binary level set method, is closely related to phase field models. The other just needs one level set function to identify arbitrary number of regions. It shall be mentioned that it is not new to use phase field for image segmentation problems, for example see Ambrosio and Tortorelli (1990); Aubert and Kornprobst (2002); Samson et al. (2000); March (1992). However, the approach of Lie et al. (2003) seems to offer a new technique to trace interfaces. In some recent works Esedoglu and Tsai (2004); Shen (2005), similar efforts have been devoted to use the phase field model or the binary level set idea to get efficient image segmentation algorithms. One purpose of this work is to show that we can use MBO type of projection for the binary level set method. Moreover, such a technique can also be extended to the single PCLSM of Lie et al. (2003). Numerical experiments show that we can indeed use one level set function to identify multiple regions and the efficiency is rather high compared with the traditional approaches of Chan and Vese (2001); Vese and Chan (2002); Lie et al. (2003, 2004); Weickert and Kühne (2003). In addition, our methods are truly variational. It avoids the re-initialization procedure and the connection with Heaviside type of non-differentiable functions. Moreover, the traditional level set method is trying to move a curve and this is not the case for our models. The model proposed here has advantages in treating some geometries, for example in situations where inside "holes" need to be identified. We also note that both the Chan-Vese model and our model can be extended to shape recognition using the framework of Cremers et al. (2004, 2003).

There also exists fast level set methods without solving PDEs. In a recent paper Shi and Karl (2005) present a method whose foundation is the direct use of an optimality condition for the final curve location based on the speed field.

This work is organized as follows: Some splitting algorithms are introduced in §2. The splitting algorithms are used later to solve the Euler-Lagrange equations for the minimization problem for the piecewise constant level set methods. In §3, the MBO scheme of, Merriman et al. (1994) is presented. The binary level set method of Lie et al. (2004) is outlined in §4. After introducing the MBO scheme and the binary level set method, we try to combine them in §5. Once we have applied the MBO projection for the binary level set method, it is easy to see that we can extend it to the single level set approach of Lie et al. (2003) using a slightly different projection, see §6. A number of algorithms are summarized here using different splitting techniques to improve the efficiency. Algorithm 1 is a recall of the MBO scheme. Algorithms 2 and 3 are an explanation of MBO as a splitting scheme for a phase field model. Algorithms 5 and 6 are then the applications of Algorithms 2 and 3 together with the dimensional splitting to the piecewise constant level set functions for multi-phase image segmentation. Intensive numerical experiments are presented in §7. Our numerical experiments indicate that Algorithm 6 is mostly recommended.

2. Sequential and Parallel Splitting Algorithms

For a given function space V and an operator (linear or nonlinear) defined in V, we often need to solve the following time dependent equation:

$$\frac{\partial \phi}{\partial t} + A(\phi) = f(t), \quad t \in [0, T], \quad \phi(0) = \hat{\phi} \in V. \tag{1}$$

In case that the operator A and the function f can be split in the following way:

$$A = A_1 + A_2 + \cdots A_m, \quad f = f_1 + f_2 + \cdots f_m, \quad (2)$$

then some splitting schemes can be used to approximate the solution of (1). Normally, the operators A_i are simpler and easier to solve. The first scheme is called the parallel splitting scheme or additive operator splitting (AOS) scheme. First we choose a time step τ and set $\phi^0 = \hat{\phi}$. At each time level $t_j = j\tau$, we compute $\phi^{j+\frac{i}{2m}}$ in parallel for $i = 1, 2, \cdots, m$ from:

$$\frac{\phi^{j+\frac{i}{2m}} - \phi^j}{m\tau} + A_i(\phi^{j+\frac{i}{2m}}) = f_i(t_j), \text{ and then set}$$
$$\phi^{j+1} = \frac{1}{m}\sum_{i=1}^{m} \phi^{j+\frac{i}{2m}}. \quad (3)$$

Note that all the subproblems for the operators A_i use the same initial value ϕ^j. This algorithm was first proposed in Lu et al. (1991, 1992). It was discovered independently later in Weickert et al. (1998) and used in a different context for image processing (Weickert and Kühne, 2003; Steidl et al., 2004; Barash et al., 2003; Barash, 2005). This scheme is locally second order of accuracy and globally first order of accuracy, i.e.

$$e^j = \phi^j - \phi(t_j) = O(\tau). \quad (4)$$

See Lu et al. (1992) for a proof of this error estimate. The advantage of the above scheme is that all the subproblems can be computed in parallel. Another advantage of the scheme is that it treats all the operators A_i in the same way. For image processing problems, the operators A_i are differential operators in the x_i directions. Thus this scheme will treat all the spatial variables in a symmetrical way and avoid the artifacts produced by treating the spatial variables in nonsymmetric ways.

The following sequential scheme, sometimes also called the multiplicative operator splitting (MOS) scheme can also be used to approximate the solution of (1):

$$\frac{\phi^{j+\frac{i}{m}} - \phi^{j+\frac{i-1}{m}}}{\tau} + A_i(\phi^{j+\frac{i}{m}}) = f_i(t_j),$$
$$i = 1, 2, \cdots, m. \quad (5)$$

The above scheme uses different initial values for the A_i operators and thus must be computed sequentially for $i = 1, 2, \cdots, m$. This scheme also has the first order convergence as stated in (4). Both schemes (3) and (5) are absolutely stable for some differential operators, see Marchuk (1990); Lions and Mercier (1979).

In case that the equation (1) has a steady state, then the steady state satisfies

$$A(\phi) = f. \quad (6)$$

Both schemes (3) and (5) can be used to compute the solution of (6). However, the parameter τ should not be regarded as a time step, but as a relaxation parameter. For the stability and convergence analysis of (3) and (5) for solving eq. (6), we refer to Lu et al. (1991, 1992).

3. The MBO Scheme

Merriman, Bence, and Osher introduced a very interesting scheme to approximate the motion of an interface by its mean curvature (Merriman et al., 1994). Suppose we wish to follow an interface moving with a normal velocity equal to its mean curvature. The MBO scheme for the case of two regions is given as an algorithm below:

Algorithm 1. (MBO scheme for two regions)
Choose initial value $\phi(0) = \pm 1$ and the time step τ. For $n = 0, 1, 2, \cdots$ and $t_n = n\tau$,

- Solve $\tilde{\phi}(t)$, $t \in [t_n, t_{n+1}]$ from

$$\tilde{\phi}_t = \Delta\tilde{\phi}, \quad \tilde{\phi}(t_n) = \phi(t_n) \text{ in } \Omega, \quad \frac{\partial\tilde{\phi}}{\partial n} = 0 \text{ on } \partial\Omega. \quad (7)$$

- Set

$$\phi(t_{n+1}) = \begin{cases} -1 & \text{if } \tilde{\phi}(t_{n+1}) < 0, \\ 1 & \text{if } \tilde{\phi}(t_{n+1}) \geq 0. \end{cases} \quad (8)$$

In the original paper (Merriman et al., 1994), the phase function ϕ is taking values 0 or 1. Here we use ± 1 to be consistent with our notation. To apply the above scheme for mean curvature motion of multiphase symmetric junctions, one just needs to use multiple phase functions ϕ_i, $i = 1, 2, \cdots r$ and project the largest value of ϕ_i to 1 and the others to -1 (See Ruuth (1998)).

The connection between the MBO scheme and the splitting algorithm is revealed in Esedoglu and Tsai (2004); Evans et al. (1992); Rubinstein et al. (1989, 1993); Glowinski et al. (2003) by interpreting it as a phase field method. Let u be the solution of

$$u_t = \epsilon \Delta u - \frac{1}{\epsilon} W'(u), \tag{9}$$

with $W(s) = (s^2 - 1)^2/2$. It is known that the rescaled solution $u(x, \frac{t}{\epsilon})$ is the solution of the mean curvature motion in the limit when $\epsilon \to 0^+$, c.f. Evans et al. (1992); Modica and Mortola (1977); Rubinstein et al. (1989, 1993).

If we use the splitting scheme (5) to solve (9), we would need to solve the following two equations on $[t_n, t_{n+1}]$:

$$a)\ \phi_t = \epsilon \Delta \phi, \qquad b)\ \phi_t = -\frac{1}{\epsilon} W'(\phi). \tag{10}$$

The rescaled solution $\phi(x, t_n/\epsilon)$ of (10.a) is exactly the solution of (7). When $\epsilon \to 0^+$, the rescaled solution $\phi(x, t_n/\epsilon)$ of (10.b) has three values, i.e. $1, 0, -1$. We drop the nonstable solution 0 and get (8). This splitting scheme bears some of the natures of the algorithm that have been analyzed in Lions and Mercier (1979).

4. The Binary Level Set Method

The binary level set method was originally introduced in Lie et al. (2004). To introduce the main idea, let us first assume that the interface is enclosing $\Omega_1 \subset \Omega \subset R^d$. For the standard level set methods, we need to use a distance function ϕ and the interior of Ω_1 is represented by points $\vec{x} : \phi(\vec{x}) > 0$, and the exterior of Ω_1 is represented by points $\vec{x} : \phi(\vec{x}) < 0$. For the binary level set method, we instead use a discontinuous level set function ϕ, with $\phi(\vec{x}) = 1$ if $\vec{x}$ is an interior point of Ω_1 and $\phi(\vec{x}) = -1$ if $\vec{x}$ is an exterior point of Ω_1, i.e.

$$\phi(\vec{x}) = \begin{cases} 1 & \text{if } \vec{x} \in \text{int}(\Omega_1), \\ -1 & \text{if } \vec{x} \in \text{ext}(\Omega_1). \end{cases} \tag{11}$$

Thus Γ is implicitly defined as the discontinuity of ϕ. This representation can be used for various applications where subdomains need to be identified. We shall use this idea for image segmentation. Let us assume that u_0 is an image consisting of two distinct regions Ω_1 and Ω_2, and that we want to construct a piecewise constant approximation u to u_0. Let $u(\vec{x}) = c_1$ in Ω_1, and $u(\vec{x}) = c_2$ in Ω_2. If $\phi(\vec{x}) = 1$ in Ω_1, and $\phi(\vec{x}) = -1$ in Ω_2, u can be written as the sum

$$u = \frac{c_1}{2}(\phi + 1) - \frac{c_2}{2}(\phi - 1). \tag{12}$$

The formula (12) can be generalized to represent functions with more than two constant values by using multiple functions $\{\phi_i\}$ following the essential ideas of the level set formulation used in Chan and Tai (2003); Vese and Chan (2002). A function having four constant values can be associated with two level set functions $\{\phi_i\}_{i=1}^2$ satisfying $\phi_i^2 = 1$. More precisely, a function given as

$$\begin{aligned} u = {} & \frac{c_1}{4}(\phi_1 + 1)(\phi_2 + 1) - \frac{c_2}{4}(\phi_1 + 1)(\phi_2 - 1) \\ & - \frac{c_3}{4}(\phi_1 - 1)(\phi_2 + 1) + \frac{c_4}{4}(\phi_1 - 1)(\phi_2 - 1), \end{aligned} \tag{13}$$

is a piecewise constant function of the form

$$u(\vec{x}) = \begin{cases} c_1, & \text{if } \phi_1(\vec{x}) = 1, \quad \phi_2(\vec{x}) = 1, \\ c_2, & \text{if } \phi_1(\vec{x}) = 1, \quad \phi_2(\vec{x}) = -1, \\ c_3, & \text{if } \phi_1(\vec{x}) = -1, \quad \phi_2(\vec{x}) = 1, \\ c_4, & \text{if } \phi_1(\vec{x}) = -1, \quad \phi_2(\vec{x}) = -1. \end{cases}$$

Introducing basis functions ψ_i as in the following

$$\begin{aligned} \psi_1 &= \frac{1}{4}(\phi_1 + 1)(\phi_2 + 1), \\ \psi_2 &= (-1)\frac{1}{4}(\phi_1 + 1)(\phi_2 - 1) \\ \psi_3 &= (-1)\frac{1}{4}(\phi_1 - 1)(\phi_2 + 1), \\ \psi_4 &= \frac{1}{4}(\phi_1 - 1)(\phi_2 - 1), \end{aligned} \tag{14}$$

we see that u can be written as

$$u = \sum_{i=1}^{4} c_i \psi_i. \tag{15}$$

For more general cases, we can use N level set functions to represent 2^N phases. To simplify the notation, we define the vectors $\vec{\phi} = \{\phi_1, \phi_2, \ldots, \phi_N\}$ and $\vec{c} = \{c_1, c_2, \ldots, c_{2^N}\}$. For $i = 1, 2, \ldots, 2^N$, let

$(b_1^{i-1}, b_2^{i-1}, \ldots, b_N^{i-1})$ be the binary representation of $i-1$, where $b_j^{i-1} = 0$ or 1. Furthermore, set

$$s(i) = \sum_{j=1}^{N} b_j^{i-1}, \tag{16}$$

and write ψ_i as the product

$$\psi_i = \frac{(-1)^{s(i)}}{2^N} \prod_{j=1}^{N} (\phi_j + 1 - 2b_j^{i-1}). \tag{17}$$

Then a function u having 2^N constant values can be written as the weighted sum

$$u = \sum_{i=1}^{2^N} c_i \psi_i. \tag{18}$$

If the level set functions ϕ_i satisfy $\phi_i^2 = 1$ and ψ_i are defined as in (15) or (17), then $\mathrm{supp}(\psi_i) = \Omega_i$, $\psi_i = 1$ in Ω_i, and $\mathrm{supp}(\psi_i) \cap \mathrm{supp}(\psi_j) = \emptyset$ when $j \neq i$. This ensures non-overlapping phases, and in addition $\bigcup_i \mathrm{supp}(\psi_i) = \Omega$, which prevents vacuums. It is clear that ψ_i is the characteristic function of the set Ω_i.

If the level set functions satisfy $\phi_i^2 = 1$, then we can use the basis functions ψ_i to calculate the length of the boundary of Ω_i and the area inside Ω_i, i.e.

$$|\partial\Omega_i| = \int_\Omega |\nabla\psi_i| dx, \quad \text{and} \quad |\Omega_i| = \int_\Omega \psi_i dx. \tag{19}$$

The first equality of (19) shows that the length of the boundary of Ω_i equals the total variation of ψ_i. See Ziemer (1989) for more explanations about the total variations of functions that might have discontinuities. In numerical computations, we use the approximation

$$\int_\Omega |\nabla\psi_i| dx \doteq \int_\Omega \sqrt{|\nabla\psi_i|^2 + \epsilon}\, dx, \tag{20}$$

for a small ϵ and the gradient $\nabla\psi$ is approximated by finite differences.

We have now introduced a way to represent a piecewise constant function u by using the binary level set functions. Based on this we propose to minimize the following Mumford-Shah functional to find a segmentation of a given image u_0 (Mumford and Shah, 1989):

$$F_{ms}(\vec{\phi}, \vec{c}) = \frac{1}{2}\int_\Omega |\, u - u_0 \,|^2 dx + \beta \sum_{i=1}^{2^N} \int_\Omega |\nabla\psi_i| dx. \tag{21}$$

In the above, β is a nonnegative parameter controlling the regularizing, u is a piecewise constant function depending on $\vec{\phi}$ and $\vec{c}$, as in (18). The first term of (21) is a least square functional, measuring how well the piecewise constant image u approximates u_0. The second term is a regularizer measuring the length of the edges in the image u_0. It is easy to see that

$$\begin{aligned} c_1(N) \sum_{i=1}^{N} \int_\Omega |\nabla\phi_i| dx &\leq \sum_{i=1}^{2^N} \int_\Omega |\nabla\psi_i| dx \\ &\leq c_2(N) \sum_{i=1}^{N} \int_\Omega |\nabla\phi_i| dx, \end{aligned} \tag{22}$$

where $c_1(N)$ and $c_2(N)$ only depend on N. Thus, we can replace the regularization term by an equivalent one and get the following simplified cost functional:

$$F(\vec{\phi}, \vec{c}) = \frac{1}{2}\int_\Omega |\, u - u_0 \,|^2 dx + \beta \sum_{i=1}^{N} \int_\Omega |\nabla\phi_i| dx. \tag{23}$$

Considering the constraints imposed on the level set functions, we find that the segmentation problem is the following constrained minimization problem

$$\boxed{\min_{\vec{\phi}, \vec{c}} F(\vec{\phi}, \vec{c}), \quad \text{subject to} \quad \phi_i^2 = 1, \ \forall i.} \tag{24}$$

Recall that $\vec{\phi}$ is a vector having N elements ϕ_i. For notational simplicity, we introduce a vector $\vec{K}(\vec{\phi})$ of the same dimension as $\vec{\phi}$ with $K_i(\vec{\phi}) = \phi_i^2 - 1$. It is easy to see that

$$\phi_i^2 = 1, \ \forall i \ \Leftrightarrow \ \vec{K}(\vec{\phi}) = \vec{0}. \tag{25}$$

In the next section, we try to use the MBO scheme to solve this minimization problem and point out a relationship between our scheme and the scheme of Esedoglu and Tsai (2004).

5. Combining the MBO Projection with the Binary Level Set Method

In order to make the relation clear, we shall consider the two dimensional two-phase model here, that is, we need to solve

$$\min_{\phi, \vec{c}} F(\phi, \vec{c}), \quad \text{subject to} \quad \phi^2 = 1 \tag{26}$$

The minimization functional in this case is:

$$F(\phi, \vec{c}) = \frac{1}{2}\int_\Omega |\, u(\phi, \vec{c}) - u_0 \,|^2 dx + \beta \int_\Omega |\nabla \phi| dx. \tag{27}$$

For the two-phase problem, we have

$$\psi_1 = \frac{1}{2}(1-\phi), \qquad \psi_2 = \frac{1}{2}(1+\phi). \tag{28}$$

In case that $\phi^2 = 1$, we can use relation (28) to show that the minimization functional of (26) is exactly

$$F_\mu(\phi, \vec{c}) = \frac{1}{2} \int\limits_{\Omega(\phi=1)} |c_1 - u_0\,|^2 dx + \frac{1}{2} \int\limits_{\Omega(\phi=-1)} |\, c_2 - u_0 \,|^2 dx + \beta \int\limits_\Omega |\nabla\phi| dx.$$

If we use a penalization term to tackle the constraint $\phi^2 = 1$, we need to choose a small μ and solve:

$$\min_{\phi, \vec{c}} F_\mu(\phi, \vec{c}), \tag{29}$$

where

$$F_\mu(\phi,\, \vec{c}) = \frac{1}{2}\int_\Omega |\, u(\phi,\, \vec{c}) - u_0 \,|^2 dx + \beta \int_\Omega |\nabla\phi| dx + \frac{1}{\mu}\int_\Omega W(\phi) dx, \tag{30}$$

with $u(\phi, \vec{c}) = c_1\psi_1(\phi) + c_2\psi_2(\phi)$ and $W(\phi) = (\phi^2 - 1)^2$. In order to find a minimizer for (29), we shall find $\vec{c}$ and ϕ that satisfy

$$a)\ \frac{\partial F_\mu}{\partial \vec{c}} = 0, \qquad b)\ \frac{\partial F_\mu}{\partial \phi} = 0. \tag{31}$$

As u is linear with respect to $\vec{c}$, we see that F_μ is quadratic with respect to $\vec{c}$. For a given ϕ^n, the minimizer of F_μ with respect to $\vec{c}$ satisfies

$$\sum_{j=1}^{2} \int_\Omega \psi_i(\phi^n)\psi_j(\phi^n) c_i^n = \int_\Omega u_0 \psi_i(\phi^n), \quad \forall i, i = 1, 2. \tag{32}$$

For a fixed $\vec{c}$, the steepest decent method in ϕ for the energy functional (30) gives the following equation for the level set function ϕ:

$$\phi_t = \beta\nabla\cdot\left(\frac{\nabla\phi}{|\nabla\phi|}\right) - (u(\phi,\ \vec{c}) - u_0)\frac{\partial u}{\partial\phi} - \frac{1}{\mu}W'(\phi), \tag{33}$$

with boundary condition

$$\frac{\nabla\phi}{|\nabla\phi|}\cdot\vec{n} = 0 \text{ on } \partial\Omega.$$

To use the MOS splitting scheme (5) to compute ϕ for a given $\vec{c}^n$, we choose a $\tau > 0$ and an initial value ϕ^0 and then solve

$$\frac{\phi^{n+1/2} - \phi^n}{\tau} = \frac{\partial F}{\partial \phi}(\phi^{n+1/2},\ \vec{c}^n). \tag{34}$$

Afterwards, we need to solve

$$\frac{\phi^{n+1} - \phi^{n+1/2}}{\tau} = -\frac{1}{\mu}W'(\phi^{n+1}). \tag{35}$$

For the parallel splitting scheme (3), we will need to solve

$$\frac{\phi^{n+1/4} - \phi^n}{2\tau} = -\frac{1}{\mu}W'(\phi^{n+1/4}). \tag{36}$$

and

$$\frac{\phi^{n+1/2} - \phi^n}{2\tau} = \frac{\partial F}{\partial \phi}(\phi^{n+1/2},\ \vec{c}^n). \tag{37}$$

In the end, we set

$$\phi^{n+1} = \frac{1}{2}(\phi^{n+1/4} + \phi^{n+1/2}).$$

For simplicity, we define the function for the MBO projection to be

$$\mathcal{P}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}. \tag{38}$$

If we replace the solving of (35) and (36) by the MBO projection, we get the following two algorithms.

Algorithm 2. (MOS MBO scheme) For $n = 0, 1, 2, \cdots$

- Solve $\phi^{n+1/2}$ from

$$\frac{\phi^{n+1/2} - \phi^n}{\tau} = \beta \nabla \cdot \left(\frac{\nabla \phi^{n+1/2}}{|\nabla \phi^{n+1/2}|} \right) - (u(\phi^{n+1/2}, \vec{c}^n) - u_0) \frac{\partial u}{\partial \phi}(\phi^{n+1/2}, \vec{c}^n). \tag{39}$$

- Set

$$\phi^{n+1} = \mathcal{P}(\phi^{n+1/2}). \tag{40}$$

- Compute $\vec{c}^n$ from (32).

Algorithm 3. (AOS MBO scheme) For $n = 0, 1, 2, \cdots$

- Solve $\phi^{n+1/2}$ from

$$\frac{\phi^{n+1/2} - \phi^n}{2\tau} = \beta \nabla \cdot \left(\frac{\nabla \phi^{n+1/2}}{|\nabla \phi^{n+1/2}|} \right) - (u(\phi^{n+1/2}, \vec{c}^n) - u_0) \frac{\partial u}{\partial \phi}(\phi^{n+1/2}, \vec{c}^n). \tag{41}$$

- Set

$$\phi^{n+1} = (\mathcal{P}(\phi^n) + \phi^{n+1/2})/2. \tag{42}$$

- Compute $\vec{c}^n$ from (32).

For the above two algorithms, we may not need to update the values for c^n at each iteration.

If we replace the total variation regularization term $\int_\Omega |\nabla \phi| dx$ by $\int_\Omega |\nabla \phi|^2 dx$ (which is not suggested though), we need to replace the curvature term $\nabla \cdot (\frac{\nabla \phi}{|\nabla \phi|})$ by $\Delta \phi$ (i.e the Laplacian of ϕ) in all the algorithms. If we do this for Algorithm 2, we will get essentially the same algorithms as Esedoglu and Tsai (2004). For clarity, we write this scheme in the following:

Algorithm 4. (The scheme of Esedoglu and Tsai (2004)) For $n = 0, 1, 2, \cdots$

- Solve $\phi^{n+1/2}$ from

$$\frac{\phi^{n+1/2} - \phi^n}{\tau} = \beta \Delta \phi^{n+1/2} - (u(\phi^{n+1/2}, \vec{c}^n) - u_0) \frac{\partial u}{\partial \phi}(\phi^{n+1/2}, \vec{c}^n). \tag{43}$$

- Set

$$\phi^{n+1} = \mathcal{P}(\phi^n). \tag{44}$$

When we need to identify more than two subdomains, we need to use multiple level set functions ϕ_i. The iterations for the multiple level set functions are essentially the same as for the one level set function case. The interplay between the different level set functions are through the values of $u(\vec{\phi})$ which depends on all the level set functions.

6. A Piecewise Constant Level Set Method

The binary level set method presented in §4 needs to use more than one level set function ϕ when more than two phases are needed in the segmentation. Here we shall introduce a method that just needs one level set function to represent multiphase segmentation. This idea was originally introduced in Lie et al. (2003). Assume that we need to find N regions $\{\Omega_i\}_{i=1}^N$ which form a partition of Ω. In order to find the regions, we want to find a piecewise constant function which takes values

$$\phi = i \text{ in } \Omega_i, \quad i = 1, 2, \ldots, N. \tag{45}$$

With this approach we just need one function to identify all the phases in Ω. The basis functions ψ_i associated with ϕ are defined in the following form:

$$\psi_i = \frac{1}{\alpha_i} \prod_{\substack{j=1 \\ j \neq i}}^{N} (\phi - j) \quad \text{and} \quad \alpha_i = \prod_{\substack{k=1 \\ k \neq i}}^{N} (i - k). \tag{46}$$

It is clear that the function u given by $u = \sum c_i \psi_i$ is a piecewise constant function and $u = c_i$ in Ω_i if ϕ is as in (45). The function u is a polynomial of order $N-1$ in ϕ. Each ψ_i is expressed as a product of linear factors of the form $(\phi - j)$, with the ith factor omitted.

Thereupon $\psi_i(\mathbf{x}) = 1$ for $\mathbf{x} \in \Omega_i$, and $\psi_i(\mathbf{x})$ equals zero elsewhere as long as (45) holds.

To ensure that different values of ϕ should correspond to different function values of $u(\phi, \vec{c})$ at convergence, we introduce

$$K(\phi) = (\phi - 1)(\phi - 2)\cdots(\phi - N) = \prod_{i=1}^{N}(\phi - i). \quad (47)$$

If a given function $\phi : \Omega \mapsto R$ satisfies

$$K(\phi) = 0, \quad (48)$$

there exists a unique $i \in \{1, 2, \ldots, N\}$ for every $x \in \Omega$ such that $\phi(x) = i$. Thus, each point $x \in \Omega$ can belong to one and only one phase if $K(\phi) = 0$. The constraint (48) is used to guarantee that there is no vacuum and overlap between the different phases. In Zhao et al. (1996) some other constraints for the classical level set methods were used to avoid vacuum and overlap.

In order to segment a given image, we shall solve the following constrained minimization problem:

$$\min_{K(\phi)=0} \frac{1}{2}\int_\Omega |\, u(\phi, \vec{c}) - u_0\,|^2 dx + \beta \int_\Omega |\nabla\phi| dx. \quad (49)$$

Above, $u(\phi, \vec{c}) = \sum c_i \psi_i$ and ψ_i are given as in (46). The minimization variables are ϕ and $\vec{c}$. In Lie et al. (2003), the length of the subdomain boundaries were used as the regularization term. Here we replace the regularization term by the total variation of ϕ which is equivalent to the regularization term up to a constant, c.f. (22).

For the new level set method, the function $W(\phi)$ is defined as $W(\phi) = |K(\phi)|^2$. If we use a penalization method to deal with the constraint $K(\phi) = 0$, then the penalization functional for this case will be:

$$F_\mu(\phi, \vec{c}) = \frac{1}{2}\int_\Omega |\, u(\phi, \vec{c}) - u_0\,|^2 dx + \beta \int_\Omega |\nabla\phi| dx + \frac{1}{\mu}\int_\Omega |K(\phi)|^2 dx. \quad (50)$$

If we split $A(\phi) = \frac{\partial F_\mu}{\partial\phi}$ into a sum of $B(\phi) = \frac{\partial F}{\partial\phi}$, $C(\phi) = \frac{1}{\mu}W'(\phi)$ and use the splitting schemes and MBO projections for such a splitting, we will get two algorithms for this piecewise constant level set method. We will omit the details of these algorithms as they look rather similar to Algorithms 2 and 3. The only difference is the MBO projection. For the level set method presented in this section, the MBO projection is given by:

$$\mathcal{P}(x) = \begin{cases} 1 & \text{if } x \le 1.5 \\ i & \text{if } x \in (i - 0.5, i + 0.5]. \\ N & \text{if } x > N - 0.5 \end{cases} \quad (51)$$

In order to further simplify the computations, we shall split B into a sum of

$$B_i(\phi) = D_i\left(\frac{D_i\phi}{|\nabla\phi|}\right) + \frac{1}{d}(u(\phi, \vec{c}) - u_0)\frac{\partial u}{\partial\phi}(\phi, \vec{c}),$$
$$i = 1, 2, \cdots d.$$

Above D_i denotes the partial derivative with respect to x_i and d is the dimension of Ω, i.e. $\Omega \subset R^d$. We see that

$$A = B_1 + B_2 + \cdots B_d + C.$$

If we use scheme (3) for such a splitting, we will get the following algorithm if the penalization is replaced by the MBO projection (51).

Algorithm 5. (Dimensional AOS MBO scheme) For $n = 0, 1, 2, \cdots$

- Solve $\phi^{n+i/2d}$ in parallel for $i = 1, 2, \cdots d$ from

$$\frac{\phi^{n+i/2d} - \phi^n}{d\tau} = \beta D_i \cdot \left(\frac{D_i\phi^{n+i/2d}}{|\nabla\phi^{n+i/2d}|}\right) - \frac{1}{d}(u(\phi^{n+i/2d}, \vec{c}^n) - u_0)\frac{\partial u}{\partial\phi}(\phi^{n+i/2d}, \vec{c}^n). \quad (52)$$

- Set

$$\phi^{n+1} = \frac{1}{d+1}\left(\mathcal{P}(\phi^n) + \sum_{i=1}^{d}\phi^{n+i/2d}\right). \quad (53)$$

- Compute $\vec{c}^n$ from (32).

However, if we solve the subproblems associated with the operators B_i by the parallel splitting scheme (3) and do the MBO projection in a sequential fashion, then the algorithm will look like:

Algorithm 6. (Dimensional MOS MBO scheme) All the other steps are the same as for Algorithm 5, only replace the MBO projection by:

- Set

$$\phi^{n+1} = \mathcal{P}\left(\frac{1}{d}\sum_{i=1}^{d}\phi^{n+i/2d}\right). \qquad (54)$$

Normally, Algorithm 6 is faster than Algorithm 5. Due to the fact that all the dimensional variables are treated in a symmetrical manner, it avoids the artifacts of the dimensional variables. Each subproblem is a one dimensional problem on the lines parallel to the axes and they can be solved efficiently using exact solver for tri-diagonal matrices (Lu et al., 1991, 1992; Weickert et al., 1998). We have tested all the proposed algorithms, and it was found that Algorithm 6 combined with the level set method of §6 is the favorable due to its efficiency and simplicity to implement.

7. Numerical experiments

It should be noted that the basis functions ψ_i and the MBO projection operator $\mathcal{P}$ are different for the binary level set method of section §4 and the piecewise constant level set method of section §6. Thus one should use the correct forms in the different algorithms.

For Algorithms 5 and 5, the subproblems (39) and (41) are nonlinear. We use the following Picard iteration to solve these nonlinear equations:

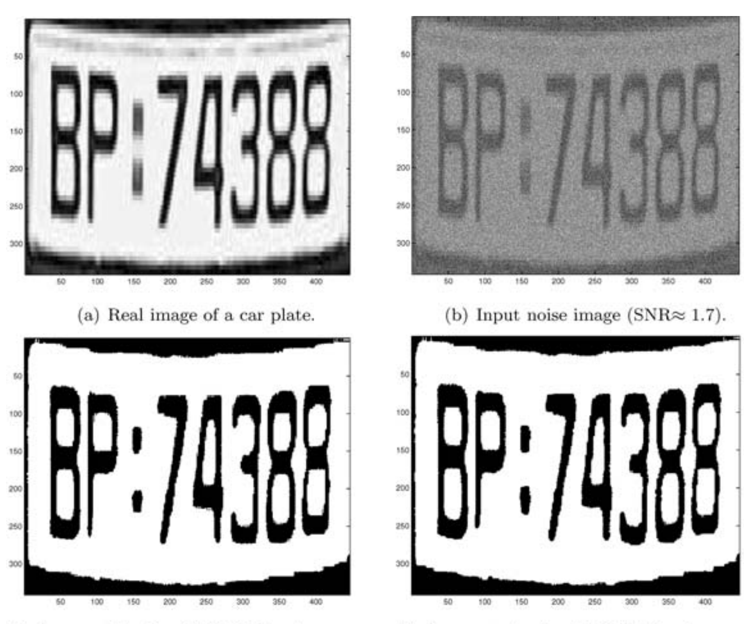
$$\begin{aligned}\frac{\phi^{new}-\phi^n}{m\tau} &= \beta\nabla\cdot\left(\frac{\nabla\phi^{new}}{|\nabla\phi^{old}|}\right)\\ &\quad -(u(\phi^{old},\vec{c}^n)-u_0)\frac{\partial u}{\partial\phi}(\phi^{old},\vec{c}^n). \qquad (55)\end{aligned}$$

Normally, we start with the initial value $\phi^{old}=\phi^n$. A CG method can be used to get a ϕ^{new} through the above linear equation. We use this ϕ^{new} as the initial values again and get another ϕ^{new} to be used as initial value. We do a fixed number of iterations for the Picard process. In all the experiments shown later, this iteration number is even set to be 1.

We use the same strategy to solve the nonlinear equation (52), i.e.

$$\begin{aligned}\frac{\phi_i^{new}-\phi^n}{m\tau} &= \beta D_i\cdot\left(\frac{D_i\phi_i^{new}}{|\nabla\phi_i^{old}|}\right)\\ &\quad -\frac{1}{d}(u(\phi_i^{old},\vec{c}^n)-u_0)\frac{\partial u}{\partial\phi}(\phi_i^{old},\vec{c}^n). \qquad (56)\end{aligned}$$

Similar to the solving of (55), we use ϕ^n as the initial value to get a ϕ_i^{new} through the above equation, and then use this ϕ_i^{new} again as the initial value to get



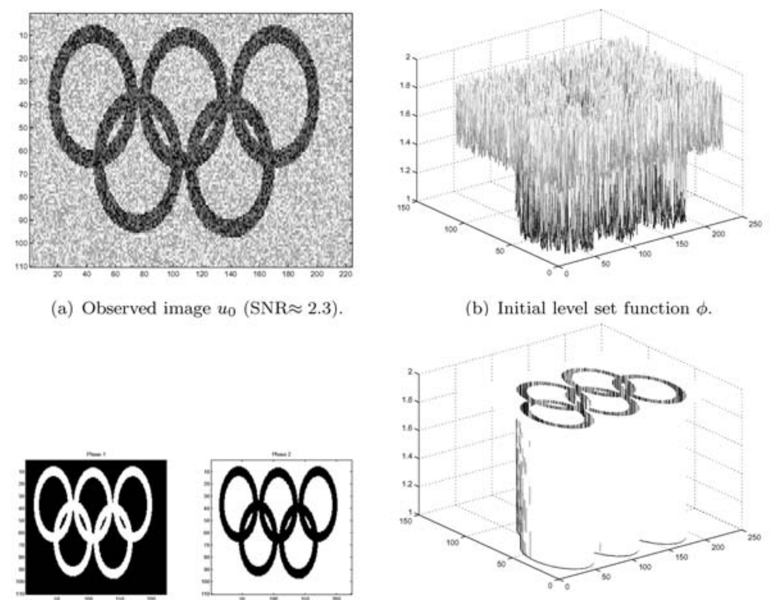
(a) Real image of a car plate.

(b) Input noise image (SNR≈ 1.7).

(c) Segmented using MOS MBO scheme. $\tau = 0.25$ and $\beta = 5$.

(d) Segmented using AOS MBO scheme. $\tau = 0.1$ and $\beta = 5$.

Figure 1. Character and number segmentation from a car plate.

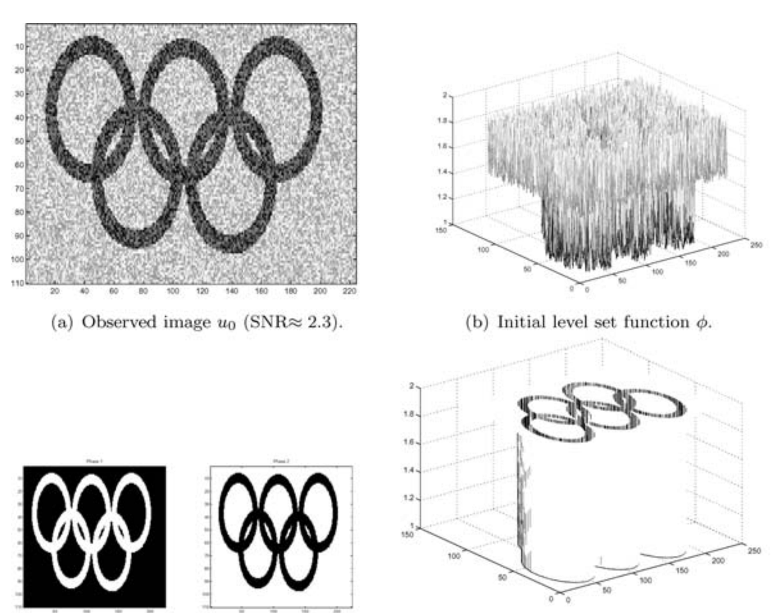


(a) Observed image u_0 (SNR≈ 2.3). (b) Initial level set function ϕ.

(c) Different phases using MOS MBO scheme. (d) At convergence ϕ approaches 2 constant values.

Figure 2. Segmentation of the Olympic rings using the MOS MBO scheme. $\tau = 0.5$, $\beta = 0.08$.

another ϕ_i^{new} to be used as the initial value again. As for (55), we just do one such iteration in all the experiments shown later. Note that the equation (56) reduces to some one dimensional equations in the x_i-direction and thus can be efficiently solved by direct solvers for tri-diagonal matrices (Lu et al., 1991, 1992; Weickert et al., 1998). Moreover, all these one dimensional problems can be solved in parallel.



(a) Observed image u_0 (SNR≈ 2.3). (b) Initial level set function ϕ.

(c) Different phases using MOS MBO scheme. (d) At convergence ϕ approaches 2 constant values.

Figure 3. Segmentation of the Olympic rings using the AOS MBO scheme. $\tau = 0.5$, $\beta = 0.08$.

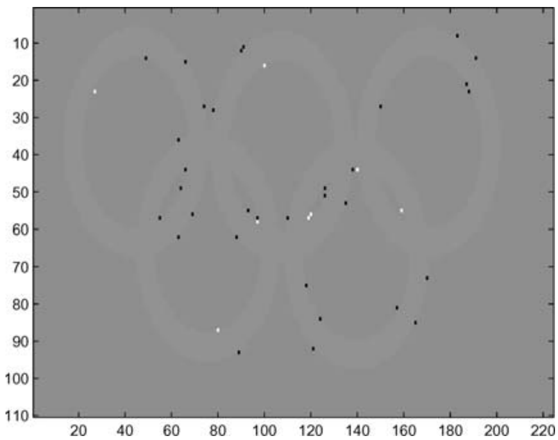


Figure 4. Difference image between the image segmented by the *MOS MBO* scheme and the image segmented by the *AOS MBO* scheme.

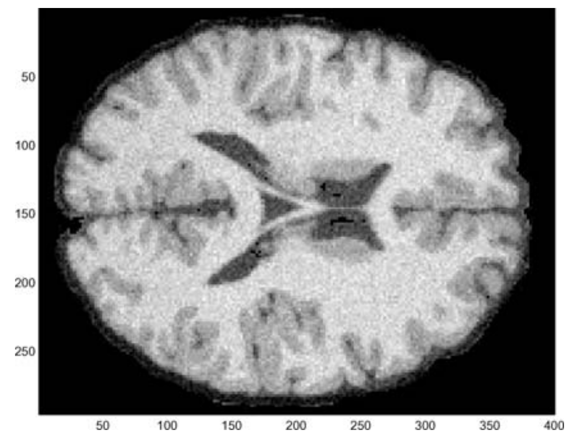
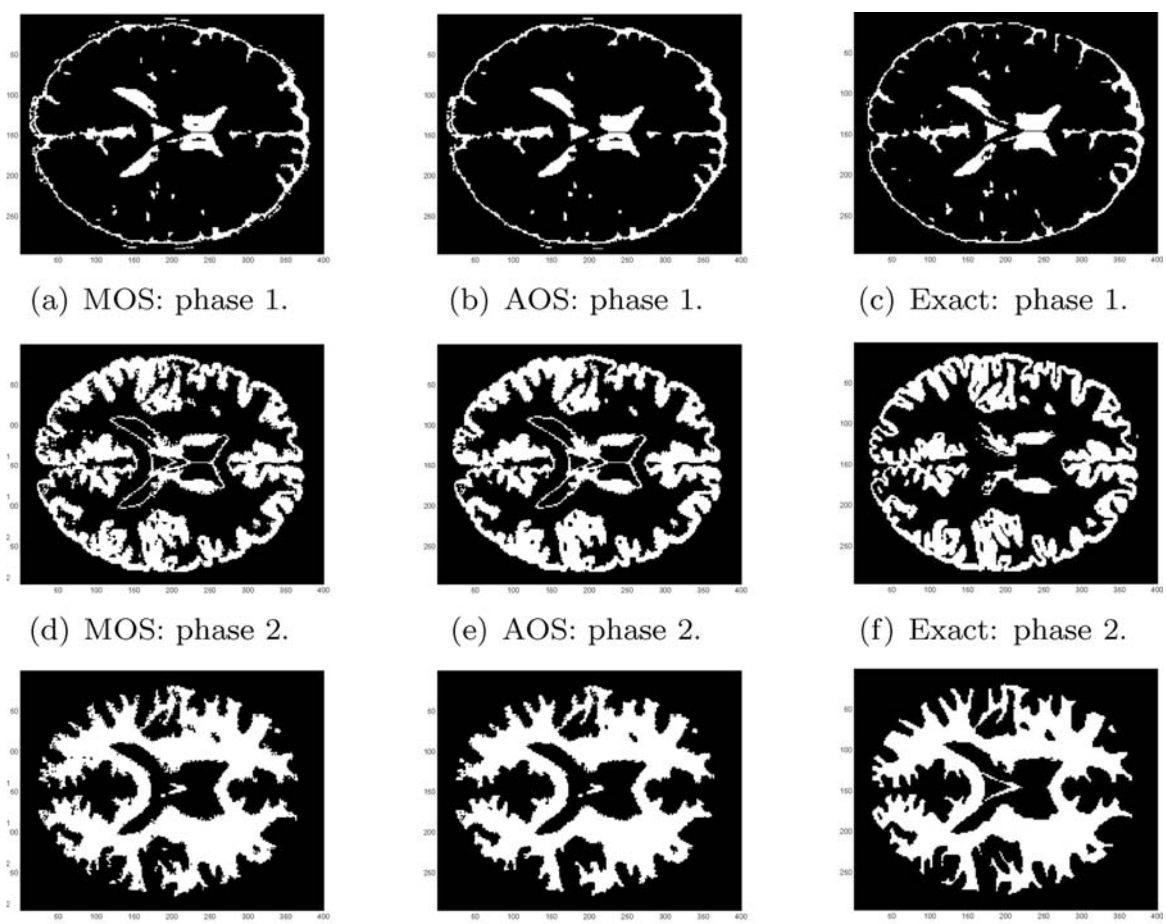


Figure 5. MRI image with a change in the intensity values going from left to right caused by the non-uniform RF-puls.

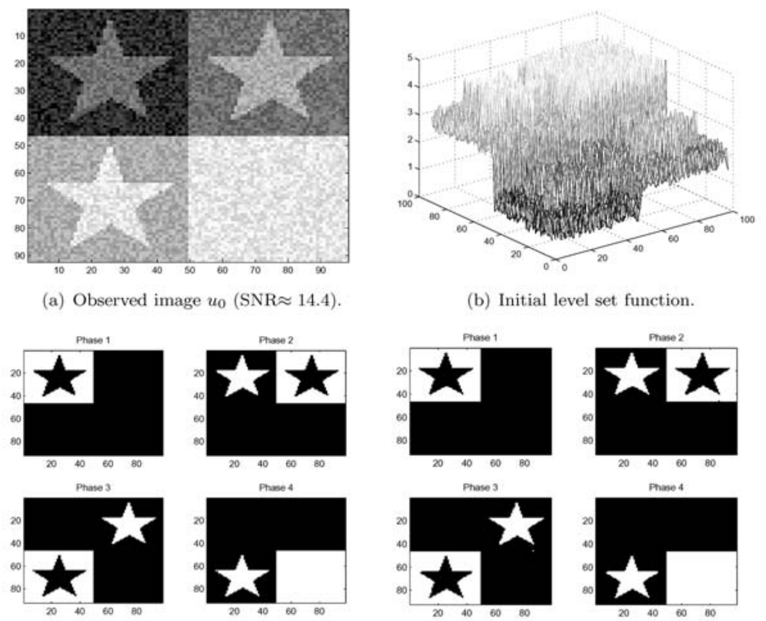


(a) MOS: phase 1. (b) AOS: phase 1. (c) Exact: phase 1.

(d) MOS: phase 2. (e) AOS: phase 2. (f) Exact: phase 2.

(g) MOS: phase 3. (h) AOS: phase 3. (i) Exact: phase 3.

Figure 6. Segmentation of MRI phantom using *MOS MBO* scheme and *AOS MBO* scheme. Last column shows exact segmentation.



(a) Observed image u_0 (SNR≈ 14.4). (b) Initial level set function.

(c) The four phases from *MOS MBO*. CPU time: 0.32 sec. (d) The four phases from *AOS MBO*. CPU time: 0.49 sec.

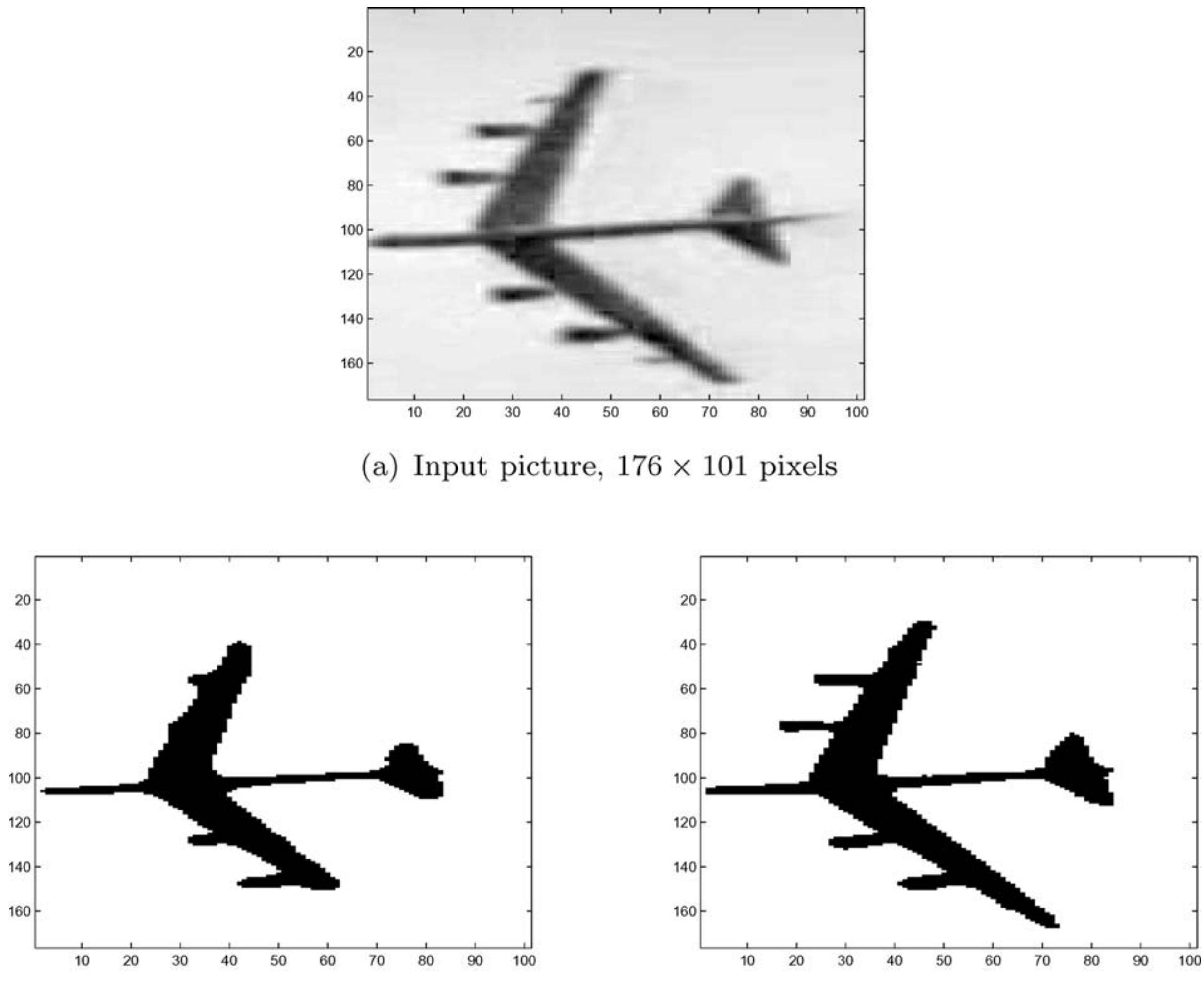
Figure 7. 4-phase segmentation using MOS and AOS MBO scheme.

We validate and compare the (dimensional) *MOS MBO* scheme, Algorithm 5, and the (dimensional) *AOS MBO* scheme, Algorithm 6. We consider only two-dimensional cases and restrict ourself to gray-scale images, but the schemes can handle any dimension and can be extended to vector-valued images as well. Synthesized images, natural images and an MR image are evaluated. The original image is known for some cases which we evaluate her. Thereupon it is trivial to find the perfect segmentation result. To complicate such a segmentation process we typically expose the original image with Gaussian distributed noise and use the polluted image as the observation data u_0.

All implementation are done in Matlab, and as the initial ϕ function we use the input image scaled between one and the number of phases. In all examples the iteration is terminated when the relative change in the level set function ϕ in L^2-norm is less then 10^{-8}. All tests are run on a 2.8GHz Pentium 4 processor.

In the first example we illustrate a 2-phase segmentation on a real car plate image. Locating and reading car plates is a well known problem, and there are a number of commercial software available. Below we demonstrate that the *MOS MBO* scheme and the *AOS MBO* scheme can be used for this segmentation. The real image is 341×446 pixels and is shown in Fig.1(a). We challenge these two segmentation techniques by adding Gaussian distributed noise to the real image and use the polluted image in Fig. 1(b) as the observation data. The difficult part is to find the optimal choice for τ and β, and we observe that we need different τ for the two methods. Both methods perform well, see Fig. 1(c) and Fig. 1(d). However, with this amount of noise we miss some details along the edges for the characters and numbers, even though we have large regularization parameters. The value we have used is $\beta = 5$ for both schemes. For the *MOS MBO* scheme the number of iterations is 14, and the CPU time is 54 seconds. For the *AOS MBO* scheme the number of iterations is 30, and the CPU time is 116 seconds. The *MOS MBO* is the faster one in all our results.

In the next example we illustrate a 2-phase segmentation on a noisy synthetic image containing 5 rings as shown in Fig. 2(a). The size of the image is 110×224 pixels. The image is segmented using both the *MOS MBO* scheme and the *AOS MBO* scheme. The results are shown in Figs. 2 and 3 respectively. In Fig. 2(b) we have shown the initial ϕ function, which is the observed image u_0 scaled between one and two, i.e. the number of phases. For the *MOS MBO* scheme the number of iterations is 2, and the CPU time is 1 second. For the *AOS MBO* scheme the number of iterations is 9,



(a) Input picture, 176×101 pixels

(b) Segmented image u, $\tau = 0.03$.

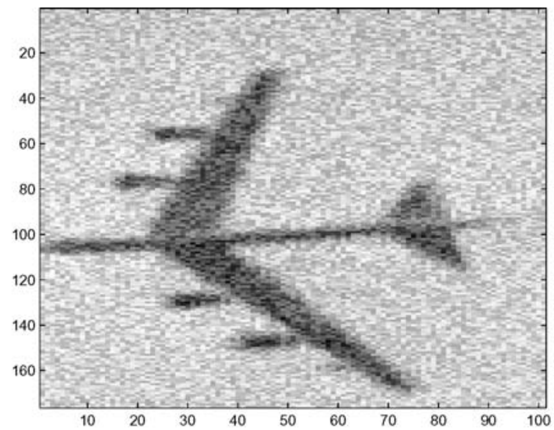
(c) Segmented image u, $\tau = 0.01$.

Figure 8. A small difference in the time step τ results in very different segmentations.

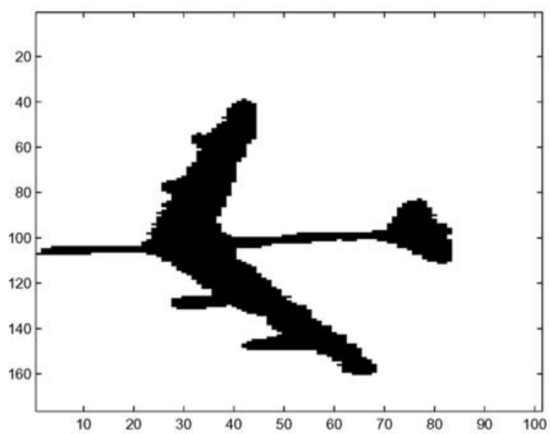
and the CPU time is 4.2 seconds. The input data u_0 is given in Fig. 2(a) and Fig. 3(a). In Fig. 2(d) and Fig. 3(d) the ϕ functions are depicted at convergence. ϕ approaches the predetermined constants $\phi = 1$ or 2. Each of these constants represents one unique phase as seen in Fig. 2(c) and Fig. 3(c).

The MOS MBO scheme has some artifacts caused by dimensional splitting. However, the artifacts is hardly noticeable by human eyes. Fig. 4 shows the difference between the image segmented by the *MOS MBO* scheme and the image segmented by the *AOS MBO* scheme.

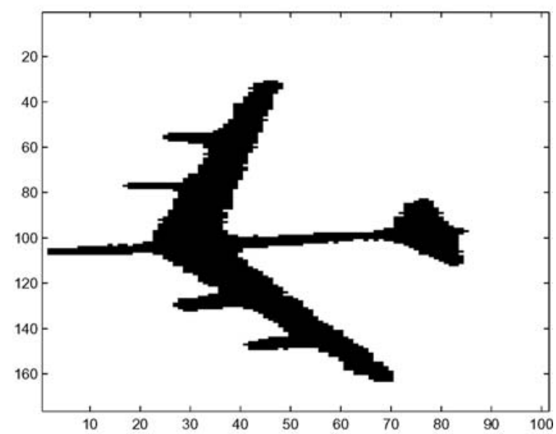
In our next example segmentation of an MR image is demonstrated. The image in Fig. 5 is available to the public at *http://www.bic.mni.mcgill.ca/brainweb/*. The size of the image is 296×400 pixels. These realistic MRI data are used by the neuroimaging community to evaluate the performance of various image analysis methods in a setting where the truth is known. For the image used in this test the noise level is 7% and the non-uniformity intensity level of the RF-puls is 20 %, for details concerning the noise level percent and the intensity level see *http://www.bic.mni.mcgill.ca/brainweb/*. Both the *MOS MBO* scheme and the *AOS MBO* scheme are used to segment the MRI phantom and the results are depicted in Fig. 6. There are three tissue classes that should be identified; phase 1: cerebrospinal fluid, phase 2: gray matter, phase 3: white matter. Because



(a) Input image (SNR$\approx$ 3.4).



(b) Segmented image u, $\tau = 0.03$.



(c) Segmented image u, $\tau = 0.01$.

Figure 9. A small difference in the time step τ results in very different segmentations.

of this, 4-phase segmentation was used, but we do not depict the background phase here. We have used $\beta = 0.52$, $\tau = 0.7$ in the *MOS MBO* scheme and $\beta = 0.52$, $\tau = 0.35$ in the *AOS MBO* scheme. For the *MOS MBO* scheme the number of iterations is 13, and the CPU time is 42 seconds. For the *AOS MBO* scheme the number of iterations is 23, and the CPU time is 80 seconds.

In Fig. 7 we show the results from a 4-phase segmentation of a star image, using the *MOS MBO* scheme and the *AOS MBO* scheme, respectively. The size of the image is 92×98 pixels and in both cases $\beta = 0.1$, and $\tau = 1$.

For the *MOS MBO* scheme the number of iterations is 2, and the CPU time is 0.32 seconds. For the *AOS MBO* scheme the number of iterations is 4, and the CPU time is 0.49 seconds. In this example, both the τ and β that give the best result were much easier to find than for the MR image.

As mentioned, the MBO scheme is very sensitive to the regularization parameter β and the time step τ. In some cases a large β is needed in order to keep the boundary of a phase smooth, and Fig. 8 shows the effect from a small change in τ. Here $\beta = 30$. The difference in τ in Fig. 8(b) and (c) is very small, yet it leads to quite different results. Because of the sensitivity to the time step illustrated here, finding the good parameters sometimes requires quite an effort. In these cases we reach convergence after 7 iterations taking 3.2 seconds resulting in Fig. 8(b) and 10 iterations taking 4.6 seconds, Fig. 8 (c). Fig. 9 illustrates the same, but here we have added noise to the input image. Convergence is reached after 9 iterations taking 4.1 seconds, Fig. 9(b), and 15 iterations taking 6.9 seconds, Fig. 9(c).

8. Conclusions and remarks

In this work, we propose to combine the MBO scheme of Merriman et al. (1994) with the piecewise constant level set methods of Lie et al. (2004, 2003). Numerical experiments show the success of these schemes. The scheme combining the binary level set method of Lie et al. (2004) with the MBO scheme of Merriman et al. (1994) is rather similar to the scheme proposed in Esedoglu and Tsai (2004), see §4. The schemes using the piecewise constant level set method of Lie et al. (2003) and the MBO scheme of Merriman et al. (1994) in a fashion with the AOS or MOS seem to be new compared with other proposed schemes. The numerical experiments show that these schemes are fast and give good results. Note that only one level set function is needed to segment any number of phases. The schemes are rather sensitive to the choice of the time step τ, some further researches need to be done in order to find a systematical strategy to choose the time step for different applications.

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