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ROTATIONAL INTERVAL EXCHANGE TRANSFORMATIONS

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We prove the equivalence of two possible definitions of rotational interval exchange transformations: by the first definition, this is the first return map for the rotation of a circle onto a union of finitely many circle arcs, whereas by the second definition, this is an interval exchange with a scheme (in a sense of interval rearrangement ensemble) whose dual is also an interval exchange scheme.

1. Introduction

In [1], we have proposed a new concept of interval rearrangement ensembles (IREs) that generalizes the construction of interval exchange transformations classical for the theory of dynamical systems [2–7]. The cornerstone of our concept is the duality involution in the space of schemes (i.e., discrete components) of IREs that produces a dual IRE scheme for every given IRE scheme, and this duality reverses time for the Rauzy–Veech-type induction. Interval exchange schemes form a partial case of IRE schemes and, as it was shown, their dual schemes may be or may be not interval exchange schemes themselves. In a certain sense, the space of all IRE schemes is an extension of the space of all (multisegment) interval exchange schemes with respect to the duality operation. On the other hand, in the space of all interval exchange schemes, there is a subspace formed by the schemes whose duals are also interval exchange schemes. Interval exchange transformations with schemes from this subspace are called rotational because these exchanges are related to circle rotations, namely, they are the first return maps for circle rotations onto a union of finitely many its arcs. In fact, we speak about two approaches to the definition of the same object (in a certain sense, they are equivalent): the first approach is based on the duality of IRE schemes, while the second approach is based on the use of first return maps on the circle. The aim of the present work is to give the exact description of the relationship between these approaches. Our results are formulated in the form of three statements in Theorem 1 and proved in the corresponding sections of our paper.

We believe that the investigation of rotational interval exchanges within the framework of the IRE concept opens a way for getting new results in the solution of still open problems of the rigidity theory for circle diffeomorphisms with multiple breaks similar to the results obtained earlier for circle diffeomorphisms with single break (see [8, 9]). This is explained by the fact that the most promising tool in the investigation of circle rotations with special points is the renormalization group approach, which replaces the initial map by a sequence of first return maps onto the unions of small neighborhoods of special points renormalized from exponentially small to macroscopic lengths. In a sequence of the first return maps, the next map is obtained from the previous one by applying the Rauzy–Veech-type induction and, therefore, the duality allowing to reverse time in this process serves as an important tool in our subsequent studies.

If special points of an irrational circle diffeomorphism are nondegenerate (i.e., the left derivative is not equal to the right derivative but both are positive), then the renormalized first return maps approach certain finite-dimensional spaces as the lengths of the analyzed neighborhoods decrease. As was first shown in [10], these

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spaces consist of linear-fractional maps. In the case where the product of all break sizes is equal to 1, the limit spaces consist of affine maps as demonstrated, in particular, in [11, 12].

The next logical step is to extend the duality from the space of IRE schemes to the corresponding spaces of linear-fractional maps. We currently work on this problem and plan to publish the corresponding results elsewhere.

The structure of this paper is as follows: In Section 2, we recall basic notions of our theory of interval rearrangement ensembles. In Section 3, we formulate our main result in the form of a theorem with three statements, and then prove Statements 1–3 of the indicated theorem in Sections 4–6, respectively.

2. Basic Notions for IREs and Interval Exchanges

In the present section, we recall basic notions in theory of interval rearrangement ensembles presented in [1].

Let $\mathcal{A}$ be an alphabet with $d \geq 1$ symbols; these are labels for intervals in our rearrangement ensemble. Consider a *doubled alphabet* $\bar{\mathcal{A}} = \mathcal{A} \times \{\mathrm{b}, \mathrm{e}\}$ (here, the letters "b" and "e" come from the words "beginning" and "ending," respectively) and any permutation σ of this doubled alphabet, i.e., a bijective map from $\bar{\mathcal{A}}$ onto $\bar{\mathcal{A}}$. We call this permutation a *scheme* of an interval rearrangement ensemble (i.e., an "IRE scheme"), while an *interval rearrangement ensemble* (i.e., an "IRE") itself is a pair $(\sigma, \mathbf{x})$ in which the scheme σ is equipped with a *vector of endpoints* $\mathbf{x} \in \mathbb{R}^{\bar{\mathcal{A}}}$ whose coordinates satisfy the equalities

$$x_{\alpha\mathrm{b}} + x_{\alpha\mathrm{e}} - x_{\sigma(\alpha\mathrm{b})} + x_{\sigma(\alpha\mathrm{e})} = 0 \quad \text{for all} \quad \alpha \in \mathcal{A}. \tag{1}$$

A vector $\mathbf{x}$ satisfying (1) is called *allowed* by the scheme σ. For a given IRE $(\sigma, \mathbf{x})$, the *vector of lengths* $\mathbf{v} \in \mathbb{R}^{\mathcal{A}}$ is defined coordinate-wise as follows:

$$v_\alpha = x_{\sigma(\alpha\mathrm{b})} - x_{\alpha\mathrm{b}} = x_{\alpha\mathrm{e}} - x_{\sigma(\alpha\mathrm{e})}, \quad \alpha \in \mathcal{A}. \tag{2}$$

Two IREs are called *shift equivalent* if both their schemes and vectors of lengths are identical. A vector $\mathbf{v} \in \mathbb{R}^{\mathcal{A}}$ is called a vector of lengths *allowed* by the scheme σ if there exists a vector of endpoints allowed by this scheme and satisfying (2). A pair $(\sigma, \mathbf{v})$ in which the vector of lengths $\mathbf{v}$ is allowed by the scheme σ is called a *floating IRE*, unlike the "fixed" IRE $(\sigma, \mathbf{x})$. A floating IRE can be regarded as an equivalence class of shift-equivalent fixed IREs.

In the present paper, we mainly work with floating IREs and regularly apply the following simple criterion of admissibility of a vector of lengths for the analyzed scheme. The key fact in this case is that a scheme σ, as an arbitrary permutation, can be decomposed into $N \geq 1$ disjoint cycles of the form

$$c = c(\bar{\xi}) = (\bar{\xi}, \sigma(\bar{\xi}), \ldots, \sigma^k(\bar{\xi})), \qquad \bar{\xi} \in \bar{\mathcal{A}},$$

where $k \geq 0,\ \sigma^{k+1}(\bar{\xi}) = \bar{\xi},\ \sigma^i(\bar{\xi}) \neq \bar{\xi}$ for $0 < i \leq k$, and the cycles $c(\bar{\xi})$ and $c(\sigma^i(\bar{\xi}))$ are regarded as identical.

Proposition 1. *A vector of lengths $\mathbf{v}$ is allowed by a scheme σ if and only if, for any its cycle $c = c(\bar{\xi})$, $\bar{\xi} \in \bar{\mathcal{A}}$, the following equality is true :*

$$\sum_{\alpha:\ \alpha\mathrm{b} \in c} v_\alpha = \sum_{\alpha:\ \alpha\mathrm{e} \in c} v_\alpha. \tag{3}$$

Proof. Assume that the vector of lengths $\mathbf{v}$ is allowed by the scheme σ. Then there exists a vector of endpoints $\mathbf{x}$ such that equalities (2) hold. According to these equalities, $x_{\sigma(\bar{\eta})} - x_{\bar{\eta}} = v_\alpha$ for $\bar{\eta} = \alpha\mathrm{b}$, and $x_{\sigma(\bar\eta)} - x_{\bar\eta} = -v_\alpha$ for $\bar\eta = \alpha\mathrm{e}$. Equalities (3) follow from the equivalences

$$\sum_{\bar\eta \in c(\bar\xi)} (x_{\sigma(\bar\eta)} - x_{\bar\eta}) = (x_{\sigma(\bar\xi)} - x_{\bar\xi}) + (x_{\sigma^2(\bar\xi)} - x_{\sigma(\bar\xi)}) + \ldots + (x_{\bar\xi} - x_{\sigma^k(\bar\xi)}) = 0$$

along each cycle $c(\bar\xi)$.

We now assume that the vector $\mathbf{v}$ satisfies all equalities (3). For each cycle $c(\bar\xi)$ in the scheme σ, we take an arbitrary number as a coordinate of an endpoint $x_{\bar\xi}$ and determine the remaining endpoints by using the following algorithm: If the value of $x_{\bar\eta}$ is already determined but the value of $x_{\sigma(\bar\eta)}$ is not yet known, then we set

$$x_{\sigma(\bar\eta)} = x_{\bar\eta} + v_\alpha \quad \text{for} \quad \bar\eta = \alpha\mathrm{b} \qquad \text{or} \qquad x_{\sigma(\bar\eta)} = x_{\bar\eta} - v_\alpha \quad \text{for} \quad \bar\eta = \alpha\mathrm{e}.$$

In view of equalities (3), the same relations hold for $\bar\eta = \sigma^k(\bar\xi)$. Therefore, the vector of endpoints $\mathbf{x}$ is allowed and $\mathbf{v}$ satisfies (2).

Proposition is proved.

The cornerstone of our theory of IREs is the notion of duality, which reverses time in the application of the Rauzy–Veech-type induction to the IRE schemes (in Sec. 5.1 in what follows, we recall the definitions of elementary steps of this induction). Two IRE schemes σ and σ^* are called *dual* to each other if

$$\sigma^*(\alpha\mathrm{b}) = \sigma(\alpha\mathrm{e}), \qquad \sigma^*(\alpha\mathrm{e}) = \sigma(\alpha\mathrm{b}) \quad \text{for all} \quad \alpha \in \mathcal{A}. \tag{4}$$

By using this duality, we define rotational interval exchange schemes in the next section.

An IRE scheme is called *irreducible*, if the equality $\sigma(\bar{\mathcal{A}}_0) = \bar{\mathcal{A}}_0$, where $\bar{\mathcal{A}}_0 = \mathcal{A}_0 \times \{\mathrm{b}, \mathrm{e}\}$ and $\mathcal{A}_0 \subset \mathcal{A}$, implies that $\mathcal{A}_0 \in \{\varnothing, \mathcal{A}\}$. It is easy to see from the definition of duality (4) that $\sigma(\bar{\mathcal{A}}_0) = \sigma^*(\bar{\mathcal{A}}_0)$ for any subset $\mathcal{A}_0 \subset \mathcal{A}$ and, hence, the irreducibility of σ implies the irreducibility of σ^*, and vice versa. If a scheme is not irreducible, then an IRE with this scheme is efficiently decomposed into two or more totally independent IREs and, for this reason, in analyzing their dynamics, it is reasonable to consider only IREs with irreducible schemes.

It follows from Proposition 2 in [1] that, in the case of an irreducible scheme σ, exactly $N-1$ out of N equalities (3) are linearly independent (the sum of all equalities (3) is a trivial equivalence; hence, any $N-1$ of these inequalities are linearly independent).

An IRE is called *positive* if its vector of lengths is positive. An IRE scheme is called positive, if it allows a positive vector of lengths.

A positive IRE should be regarded as an ensemble of $2d$ intervals coupled into d pairs with labels $\alpha \in \mathcal{A}$. In each of these pairs, the *beginning interval* $I_{\alpha\mathrm{b}} = [x_{\alpha\mathrm{b}}, x_{\sigma(\alpha\mathrm{b})})$ and the *ending interval* $I_{\alpha\mathrm{e}} = [x_{\sigma(\alpha\mathrm{e})}, x_{\alpha\mathrm{e}})$ with the same label α have the same length v_α. According to cycles in the scheme, all beginning and ending intervals with the corresponding subscripts are connected at their endpoints into N closed chains, i.e., one-dimensional polygonal lines. It is possible to consider a closed polygonal chain of this kind as a path going from $x_{\bar\xi}$ to $x_{\sigma(\bar\xi)}$, then from $x_{\sigma(\bar\xi)}$ to $x_{\sigma^2(\bar\xi)}$, and so on, until returning to $x_{\bar\xi}$. On this path, every beginning interval is passed from left to right and every ending interval is passed from right to left. A parallel translation of any of these N closed one-dimensional polygonal lines as a whole, obviously, does not affect equalities (2) and does not change the lengths of the intervals; this is the reason why we say that a pair $(\sigma, \mathbf{v})$ is a "floating IRE," while the corresponding fixed IRE are called "shift equivalent".

We now consider a special case where every cycle in the scheme σ of a positive IRE can be split into two arcs one of which consists solely of the beginning intervals, and the other is formed only by the ending intervals (i.e., every cycle has the form $c = (\alpha_1\mathrm{b}, \ldots, \alpha_m\mathrm{b}, \beta_n\mathrm{e}, \ldots, \beta_1\mathrm{e})$ for some labels $\alpha_1, \ldots, \alpha_m,\ \beta_1, \ldots, \beta_n \in \mathcal{A}$, $n, m \geq 1$). It is natural to call this IRE *interval exchange* and associate it with a discrete dynamical system (mapping) on a disjoint union of segments $J_1, \dots, J_N$ corresponding to the cycles $c_1, \dots, c_N$ (namely, a segment

$$J = [x_{\alpha_1 \mathrm{b}}, x_{\beta_n \mathrm{e}}) = \bigcup_{i=1}^{m} I_{\alpha_i \mathrm{b}} = \bigcup_{i=1}^{n} I_{\beta_i \mathrm{e}}$$

corresponds to the cycle c given above); this mapping shifts every beginning interval onto the ending interval with the same label from the alphabet $\mathcal{A}$. In this case, Proposition 1 says that a vector of lengths is allowed for a scheme if and only if, for any segment J, the sum of lengths of all beginning segments contained in it is equal to the sum of lengths of all ending segments contained in it. We use the same name "interval exchange" for a pair $(\sigma, \mathbf{x})$, as well as for an associated mapping and an induced one-dimensional dynamical system. In this case, σ is called an *interval exchange scheme*. By analogy with the general IREs, we can consider *floating interval exchanges* $(\sigma, \mathbf{v})$ by focusing our attention only on the lengths of intervals and allowing the segments $J_1, \dots, J_N$ to shift freely along the axis, i.e., factorizing the space of all interval exchanges w.r.t. the shift equivalence.

For cycles of the indicated form $(\alpha_1\mathrm{b}, \dots, \alpha_m\mathrm{b}, \beta_n\mathrm{e}, \dots, \beta_1\mathrm{e})$, it is convenient to use a clearer "two-row notation," namely,

$$c = \begin{bmatrix} \alpha_1 & \dots & \alpha_m \\ \beta_1 & \dots & \beta_n \end{bmatrix}$$

(the entire interval exchange scheme can be also represented in the two-row form as the set of all its cycles $\sigma = \{c_1, \dots, c_N\}$, $N \geq 1$). In the terminology of [1], this "two-row" cycle is called a cycle with zero twist number. To be precise, the *twist number* of a cycle in an IRE scheme σ is the number of positions in this cycle, where a beginning element is followed by an ending element, i.e., $\sigma(\alpha\mathrm{b}) = \beta\mathrm{e}$ for some $\alpha, \beta \in \mathcal{A}$, minus one. If a cycle is formed either solely by the beginning elements or solely by the ending elements, then its twist number is -1. However the condition of positivity makes this case impossible. The twist number $T(\sigma)$ of a scheme is the sum of the twist numbers of all its cycles. Therefore, in this terminology, an *interval exchange scheme* is a positive IRE scheme with twist number zero, and an *interval exchange* itself is a positive IRE with scheme of this kind.

The definition presented above determines a classical interval exchange transformation if its scheme consists of a single cycle and the left endpoint of the corresponding segment lies at the origin. In our opinion, these restrictions do not look natural and, hence, we use the term "interval exchange" only for a more general structure on multiple segments. In Sec. 3 of [1], prior to the definition of IRE, the author formulated the definition of a multi-segment interval exchange transformation in the classical form. Thus, it is easy to see that the definition of interval exchange given within the framework of the IRE concept is much simpler than the classical-like definition.

3. Rotational Schemes and the Main Result

For an interval-exchange scheme σ, its dual σ^* is not necessarily an interval exchange scheme (but can be a scheme of this kind) and the schemes possessing this property form an important special class, which is studied in the present paper. Thus, we say that an interval exchange scheme is *rotational* if its dual is also an interval exchange scheme. An interval exchange with rotational scheme is called a *rotational interval exchange*.

The *twist total* of an IRE scheme σ (see [1]) is the number $T(\sigma) + T(\sigma^*)$, i.e., the sum of the twist numbers of the schemes σ and σ^*. In these terms, a *rotational* scheme is an IRE scheme, which is positive together with its dual and has zero twist total. At the same time, a *rotational interval exchange* is a positive IRE with a scheme of this kind. According to Sec. 10 in [1], these are exactly the schemes whose positive natural extensions generate translation surfaces of genus $g = 1$, i.e., 2D tori without singular points.

It is clear that the class of all rotational schemes is closed w.r.t. this duality operation. Note that we call these interval exchanges and their schemes "rotational" because they are directly related to circle rotations. The exact formulation of this relationship is the main result of the present paper given in what follows.

A rotation of a circle of length $L > 0$ by a distance M is described by a map of the form

$$R_{L,M}\colon\ a \mapsto a + M, \quad a \in \mathbb{R}/L\mathbb{Z}, \tag{5}$$

where the factor space $\mathbb{R}/L\mathbb{Z}$ is actually a circle of length L. A circle rotation is called *irrational* if its *rotation number* $\rho = \{M/L\} \notin \mathbb{Q}$. Here, $\{\,\cdot\,\}$ denotes the fractional part of a real number.

Alternatively, we can interpret a circle rotation as its projection onto an arbitrarily chosen half open segment $[x_0, x_0 + L),\ x_0 \in \mathbb{R}$, i.e., as a piecewise linear map

$$R_{L,M}\colon\ x \mapsto x + M - \left[\frac{x + M - x_0}{L}\right] L, \quad x \in [x_0, x_0 + L). \tag{6}$$

It is easy to see that this map is itself a rotational exchange of (two) intervals. Here, $[\,\cdot\,]$ denotes the integer part of a real number.

Any half-open segment on a circle is called an *arc*. In what follows, we consider the unions of finitely many arcs and, according to the alternative definition (6), interpret them as the union of finitely many half-open segments of the real line. If this *union of arcs* (we omit the words "finitely many" because we consider only finite unions) does not cover the entire circle, then it is natural to choose x_0 as the projection onto $\mathbb{R}$ of any circle point that does not belong to the interior of any arc. It is also natural to consider, as separate segments of this union, the maximal segments (i.e., if two arcs overlap or touch at the endpoints, then their projections onto $\mathbb{R}$ should be combined into a single segment). In the case where the union of arcs is the entire circle, the endpoint x_0 can be chosen arbitrarily and the analyzed union of arcs can be regarded as the entire real segment $[x_0, x_0 + L)$.

Finally, we can formulate our main result.

Note that the third statement of this theorem is its main part, while the first two are, in fact, additional because the qualitative (discrete) data prove to be more important than the quantitative (real) data because the space of allowed lengths is determined by its IRE scheme, but not vice versa. However, the chosen order of three statements of the theorem is explained by the logic of their proving: the proof of the third statement is based on the proofs of the first two statements and, moreover, a part the third statement is a direct consequence of the first two results (see Sec. 6, for more details). The proof of this theorem is constructive in a sense that the existence of all mentioned objects is established by their algorithmic construction. In particular, it is necessary to mention the canonical form of rotational interval exchange introduced in Sec. 5.7, connected with the so-called dynamical partitions of a circle (see [9]), and playing the role of the most convenient "intermediary" between the general rotational interval exchanges and circle rotations.

Theorem 1.

1. *For any irrational circle rotation, the first return map onto any subset, which is a union of arcs, is an irreducible rotational interval exchange.*

2. *For any irreducible rotational interval exchange, there exists a first return map for a circle rotation onto the union of arcs, which is shift equivalent to the indicated interval exchange.*

3. *An irreducible interval exchange scheme is rotational if and only if there exists a first return map for an irrational circle rotation onto the union of arcs, which is an interval exchange with this scheme.*

Remark 1. The theorem is formulated for irreducible schemes and interval exchanges, and a (single) circle rotation corresponds to each of these cases. If an interval exchange scheme consists of several irreducible components, then the dynamical system splits into the same number of dynamically independent components, and the first return map should be considered for the union of the same number of circle rotations. Hence, Statements 2 and 3 of the theorem can be reformulated by omitting the requirement of irreducibility and replacing a circle rotation by the union of circle rotations (irrational circle rotations in Statement 3) whose number is equal to the number of irreducible components of the scheme.

Remark 2. It is necessary to explain why the fact of irrationality of a circle rotation is mentioned in Statements 1 and 3 but not mentioned in Statement 2. To do this, we indicate that if an interval exchange is the first return map onto the union of arcs for an irrational circle rotation, then small perturbations of the parameters of this system produce an interval exchange with the same scheme, which is the first return map onto a union of arcs for a rational circle rotation (close to the original irrational rotation). Due to this fact, every scheme of this kind (and, due to the third statement of the theorem, this is true for all rotational schemes) admits both irrational and rational (in the above-mentioned sense) interval exchanges. However, on the other hand, there exist interval exchange schemes allowing only rational interval exchanges, namely, the schemes containing a chain of cycles of the form

$$\left\{\begin{bmatrix}\alpha_1\\ \\ \alpha_2\end{bmatrix};\ \begin{bmatrix}\alpha_2\\ \\ \alpha_3\end{bmatrix};\ \ldots;\ \begin{bmatrix}\alpha_m\\ \\ \alpha_1\end{bmatrix}\right\}$$

or a single cycle of the form $\begin{bmatrix}\alpha\\ \alpha\end{bmatrix}$. Actually, these periodic chains of cycles in the scheme are characteristic of rational interval exchanges: an arc moves along a certain path over the circle and finally returns onto itself. The scheme dual to this IRE scheme contains a couple of cycles

$$\left\{\begin{bmatrix}\alpha_m & \ldots & \alpha_1\\ & & \\ & \varnothing & \end{bmatrix};\ \begin{bmatrix} & \varnothing & \\ & & \\ \alpha_m & \ldots & \alpha_1\end{bmatrix}\right\}$$

(here, $\varnothing$ denotes the absence of elements in a row) and, clearly, is not positive. The interval exchange schemes of this kind have negative twist total and their natural extensions do not form any translation surfaces. Therefore, we do not call these schemes (and interval exchanges with these schemes) rotational, despite the fact that these interval exchanges can be the first return maps for the circle rotations (only rational) onto certain unions of arcs.

4. Proof of the First Part of the Theorem

4.1. First Return Maps Are Finite. Recall the definition of a first return map for a dynamical system with discrete time onto a subset of its phase space. Assume that the analyzed dynamical system is given by a map $f\colon X \to X$ and a nonempty subset $\Gamma \subset X$ has the following property: for any point $x \in \Gamma$, there exists a positive integer n such that $f^n(x) \in \Gamma$, i.e., the trajectory of the point x returns to the set Γ after a certain time n. Denote by $n(x, f, \Gamma)$ the *first return time* of x under the action of f to Γ, i.e., the smallest of the following parameters: $f^i(x) \notin \Gamma$ for all $1 \le i < n(x, f, \Gamma)$, and $f^{n(x,f,\Gamma)}(x) \in \Gamma$. The map

$$f_\Gamma\colon x \mapsto f^{n(x,f,\Gamma)}(x)$$

is called the *first return map* for f into Γ. Clearly, this map defines on the indicated subset a new (induced) dynamical system $f_\Gamma\colon \Gamma \to \Gamma$.

According to this definition, for a given number $n \ge 1$, all points $x \in \Gamma$ such that $n(x, f, \Gamma) = n$ form a set

$$\Gamma_n = \Gamma \cap f^{-n}(\Gamma) \backslash f^{-(n-1)}(\Gamma) \backslash \ldots \backslash f^{-1}(\Gamma).$$

Moreover, we have a disjoint splitting

$$\Gamma = \bigcup_{n=1}^{+\infty} \Gamma_n, \qquad \Gamma_n \cap \Gamma_m = \varnothing$$

for $n \neq m$ and, moreover, $f_\Gamma(x) = f^n(x)$ for all $x \in \Gamma_n,\ n \geq 1$.

We now show that, in the case of a circle rotation $R = R_{M,L}$, the first return map to any union of finitely many arcs Γ is always *finite*, i.e., the set of first return times for all points of this union $\{n(x, f, \Gamma) | x \in \Gamma\}$ is bounded.

If the rotation number is rational, i.e., $\rho = M/L = p/q$, where p and q are mutually prime positive integers, then the equivalence $R^q(x) \equiv x$ holds. Therefore, $n(x, R, \Gamma) \leq q$ for all x on the circle and, hence, the first return map for a rational circle rotation to any subset (not only to a union of arcs) is always defined and finite.

Further, assume that the rotation number ρ is irrational. In this case, the trajectory of any point x is everywhere dense on the circle and, therefore, for any $\delta > 0$, there exists a positive integer n_0 such that the circle does not contain any arc of length δ free from the points of a finite segment of this trajectory $(R^i(x))_{i=1}^{n_0}$. Moreover, since all trajectories of the circle rotation have identical spacing, this n_0 does not depend on x. Taking the length of the shortest arc from the union Γ as δ, we obtain the following bound: $n(x, R, \Gamma) \leq n_0$ for all x on the circle.

Thus, we have shown that the first return map for R to Γ is indeed finite:

$$\Gamma = \bigcup_{n=1}^{n_0} \Gamma_n$$

for finite n_0. Further, since both the intersection and union of any two finite unions of arcs are also finite unions of arcs, we conclude that each set Γ_n is a finite union of arcs, as well. On each of these arcs, the first return map R_Γ is a shift (by a distance nM). Interpreting circle arcs as intervals of the real line inside a certain segment $[x_0, x_0 + L)$, we obtain the following statement:

Proposition 2. *For any circle rotation, the first return map onto any finite union of arcs is an interval exchange transformation.*

If a circle rotation is irrational, then the obtained interval exchange is irreducible due to minimality of this rotation as a dynamical system.

4.2. First Return Maps Are Rotational. Consider an irrational circle rotation (5), (6) and a finite union of arcs Γ interpreted as a set of pairwise disconnected half open segments of the real line contained in $[x_0, x_0 + L)$. By Proposition 2, the first return map $R_\Gamma : \Gamma \to \Gamma$ is an interval exchange. We now show that the scheme of this interval exchange, regarded as an IRE, is rotational by definition, i.e., that its dual IRE scheme is also an interval exchange scheme.

In order to determine the IRE scheme for the interval exchange R_Γ, we denote every interval in this exchange by its own symbol, thus forming an alphabet $\mathcal{A}$. The set Γ is, on the one hand, the union of all pairwise disjoint intervals $I_{\alpha\mathrm{b}},\ \alpha \in \mathcal{A}$ and, on the other hand, the union of their (also pairwise disjoint) images $I_{\alpha\mathrm{e}} = \Gamma(I_{\alpha\mathrm{b}})$, $\alpha \in \mathcal{A}$. In its turn, each disconnected segment in Γ is also split, on the one hand, into several beginning intervals $I_{\alpha_1\mathrm{b}}, \dots, I_{\alpha_m\mathrm{b}}$ (count from the left to the right) and, on the other hand, into several ending intervals $I_{\beta_1\mathrm{e}}, \dots, I_{\beta_n\mathrm{e}}$ (also count in the left–right direction); in the scheme σ, this segment is encoded by a cycle

$$(\alpha_1\mathrm{b}, \dots, \alpha_m\mathrm{b}, \beta_n\mathrm{e}, \dots, \beta_1\mathrm{e}), \quad \{\alpha_i\}_{i=1}^m, \{\beta_j\}_{j=1}^n \subset \mathcal{A}, \quad n, m \geq 1.$$

The total array of endpoints $\mathbf{x} \in \mathbb{R}^{\bar{\mathcal{A}}}$, where $x_{\alpha\mathrm{b}}$ is the left endpoint of a beginning interval $I_{\alpha\mathrm{b}}$ and $x_{\beta\mathrm{e}}$ is the right endpoint of an ending interval $I_{\beta\mathrm{e}},\ \alpha, \beta \in \mathcal{A}$, forms a real component of the IRE $(\sigma, \mathbf{x})$.

All endpoints of a given interval exchange are naturally split into four types: type L (left) containing the left endpoints of disconnected segments in the composition of Γ (in the example presented above, this is the point $x_{\alpha_1\mathrm{b}}$); type R (right) containing the right endpoints of disconnected segments in Γ (in the example presented above, this is $x_{\beta_1\mathrm{e}}$); type MB (middle beginning) containing the points corresponding to the connection of two neighboring beginning intervals (in the example presented above, these are $x_{\alpha_i\mathrm{b}},\ i \neq 1$), and type ME (middle ending) containing the points corresponding to the connection of two neighboring ending intervals (in the example presented above, these are $x_{\beta_j\mathrm{e}},\ j \neq 1$). From the IRE point of view, the indicated types are actually specified not for the real numbers $x_{\bar{\xi}}$ themselves but for the symbols attached, namely, $\bar{\xi} \in \bar{\mathcal{A}}$ corresponds to type L if $\bar{\xi} = \xi\mathrm{b}$ and $\sigma^{-1}(\bar{\xi}) = \eta\mathrm{e}$; to type R if $\bar{\xi} = \xi\mathrm{e}$ and $\sigma^{-1}(\bar{\xi}) = \eta\mathrm{b}$; to type MB if $\bar{\xi} = \xi\mathrm{b}$ and $\sigma^{-1}(\bar{\xi}) = \eta\mathrm{b}$; and to type ME if $\bar{\xi} = \xi\mathrm{e}$ and $\sigma^{-1}(\bar{\xi}) = \eta\mathrm{e}$ for some $\xi, \eta \in \mathcal{A}$.

It is worth noting, that every disconnected segment in the composition of Γ (corresponding to a disconnected cycle in the permutation σ) possesses exactly one endpoint of type L and exactly one endpoint of type R (these are actually the left and right endpoints of Γ, respectively). At the same time, the number of endpoints of types MB and ME lying on this set can be arbitrary (including zero).

We consider an arbitrary type-MB endpoint $x_{\xi_1\mathrm{b}}$ (hence, $\sigma^{-1}(\xi_1\mathrm{b}) = \eta_1\mathrm{b}$ and $\xi_1, \eta_1 \in \mathcal{A}$) and trace the dynamic trajectories of its left $x^-_{\xi_1\mathrm{b}} = (x_{\xi_1\mathrm{b}} - \varepsilon, x_{\xi_1\mathrm{b}}),\ \varepsilon \to 0$, and right $x^+_{\xi_1\mathrm{b}} = (x_{\xi_1\mathrm{b}}, x_{\xi_1\mathrm{b}} + \varepsilon),\ \varepsilon \to 0$, infinitesimally small half neighborhoods (it is obvious that, for sufficiently small $\varepsilon > 0$, the consecutive images of these half neighborhoods under the action of R_Γ do not cover any endpoint for sufficiently long time; in particular, they do not split into smaller intervals and, therefore, the indicated infinitesimal consideration is well defined).

The interval $I_{\xi_1\mathrm{b}}$ with the left endpoint $x_{\xi_1\mathrm{b}}$ is mapped by R_Γ onto the interval $I_{\xi_1\mathrm{e}}$. Hence, a right half neighborhood of $x^+_{\xi_1\mathrm{b}}$ is mapped onto a right half neighborhood of the left endpoint of $I_{\xi_1\mathrm{e}}$, i.e., onto $x^+_{\sigma(\xi_1\mathrm{e})}$. Thus, there are two possible cases: this endpoint is either of type ME (and, hence, $\sigma(\xi_1\mathrm{e}) = \eta_*\mathrm{e},\ \eta_* \in \mathcal{A}$) or of type L (and, hence, $\sigma(\xi_1\mathrm{e}) = \xi_2\mathrm{b},\ \xi_2 \in \mathcal{A}$). In the first case, we stop. However, in the second case, we continue to trace the trajectory of the chosen half neighborhood. By analogy with the initial step, $x^+_{\sigma(\xi_1\mathrm{e})} = x^+_{\xi_2\mathrm{e}}$ is mapped onto $x^+_{\sigma(\xi_2\mathrm{e})}$, and we again get the following two cases: the endpoint $x_{\sigma(\xi_2\mathrm{e})}$ is either of type ME (and, hence, $\sigma(\xi_2\mathrm{e}) = \eta_*\mathrm{e},\ \eta_* \in \mathcal{A}$) or of type L (and, hence, $\sigma(\xi_2\mathrm{e}) = \xi_3\mathrm{b},\ \xi_3 \in \mathcal{A}$). In the first case, we stop, whereas in the second case, we continue to trace the trajectory. At a certain step of the algorithm, the process terminates because the number of type-L endpoints is finite and the map R_Γ does not have periodic trajectories due to the irrationality of the original circle rotation. Therefore, after termination, we get a sequence of symbols

$$\xi_1\mathrm{b},\ \xi_2\mathrm{b} = \sigma(\xi_1\mathrm{b}),\ \ldots,\ \xi_m\mathrm{b} = \sigma(\xi_{m-1}\mathrm{b}),\ \eta_*\mathrm{e} = \sigma(\xi_m\mathrm{b})$$

and the corresponding sequence of half neighborhoods

$$x^+_{\xi_1\mathrm{b}},\ x^+_{\xi_2\mathrm{b}} = R_\Gamma(x^+_{\xi_1\mathrm{b}}),\ \ldots,\ x^+_{\xi_m\mathrm{b}} = R_\Gamma(x^+_{\xi_{m-1}\mathrm{b}}),\ x^+_{\eta_*\mathrm{e}} = R_\Gamma(x^+_{\xi_m\mathrm{b}}),$$

where the endpoints $x_{\xi_1\mathrm{b}},\ x_{\xi_2\mathrm{b}}, \ldots, x_{\xi_m\mathrm{b}},\ x_{\eta_*\mathrm{e}}$ are of types MB, L, …, L, ME respectively; $\xi_1, \ldots, \xi_m$, and $\eta_* \in \mathcal{A},\ m \geq 1$.

We now consider the trajectory of the left half neighborhood of the same starting endpoint of type MB . The interval $I_{\eta_1\mathrm{b}}$ with right endpoint $x_{\xi_1\mathrm{b}}$ is mapped by R_Γ onto the interval $I_{\eta_1\mathrm{e}}$. Therefore, the left half neighborhood of $x^-_{\xi_1\mathrm{b}}$ is mapped onto the left half neighborhood of the right endpoint of $I_{\eta_1\mathrm{e}}$, i.e., in fact, on $x^-_{\eta_1\mathrm{e}}$. There are two possible cases: The analyzed endpoint is either of type ME (and, hence, $\eta_1\mathrm{e} = \sigma(\xi_*\mathrm{e}),\ \xi_* \in \mathcal{A}$) or of type R (and, hence, $\eta_1\mathrm{e} = \sigma(\eta_2\mathrm{b}),\ \eta_2 \in \mathcal{A}$). In the first case, we stop. In the second case, we continue to trace the trajectory of the chosen half neighborhood. By analogy with the initial step, the interval $I_{\eta_2\mathrm{b}}$ with the right endpoint $x_{\eta_1\mathrm{e}}$ is mapped onto the interval $I_{\eta_2\mathrm{e}}$ and, hence, $x^-_{\eta_1\mathrm{e}}$ is mapped onto $x^-_{\eta_2\mathrm{e}}$. We again have the following two cases: $x_{\eta_2\mathrm{e}}$ is either of type ME (and, hence, $\eta_2\mathrm{e} = \sigma(\xi_*\mathrm{e}),\ \xi_* \in \mathcal{A}$) or of type R (and, hence, $\eta_2\mathrm{e} = \sigma(\eta_3\mathrm{b}),\ \eta_3 \in \mathcal{A}$).

In the first case, we stop. In the second case, we continue the procedure of tracing. After a certain number of steps, the process stops because R_Γ does not have periodic trajectories. Finally, we get a sequence of symbols

$$\xi_1\mathrm{b},\ \eta_1\mathrm{b} = \sigma^{-1}(\xi_1\mathrm{b}),\ \eta_2\mathrm{b} = \sigma^{-1}(\eta_1\mathrm{e}),\ \ldots,\ \eta_n\mathrm{b} = \sigma^{-1}(\eta_{n-1}\mathrm{b})$$

and a sequence of half neighborhoods

$$x^-_{\xi_1\mathrm{b}},\ x^-_{\eta_1\mathrm{e}} = R_\Gamma(x^-_{\xi_1\mathrm{b}}),\ x^-_{\eta_2\mathrm{e}} = R_\Gamma(x^-_{\eta_1\mathrm{e}}),\ \ldots,\ x^-_{\eta_n\mathrm{e}} = R_\Gamma(x^-_{\eta_{n-1}\mathrm{e}}),$$

where the endpoints $x_{\xi_1\mathrm{b}},\ x_{\eta_1\mathrm{e}}, \ldots, x_{\eta_n\mathrm{e}}$ are of types MB, R, ..., R, ME, respectively, and $\eta_1, \ldots, \eta_n \in \mathcal{A}$, $n \geq 1$.

It remains to show that (necessarily) $x_{\eta_n\mathrm{e}} = x_{\eta_*\mathrm{e}}$ and, therefore, $\eta_* = \eta_n$. This is true because R_Γ is the first return map for a circle rotation to the union of arcs Γ. A circle rotation R is a continuous map and, hence, the left and right half neighborhoods of any point never separate under its action. Therefore, at the time when an inner point (such as a type-MB endpoint) of the set Γ returns to the inner point (endpoint of type ME) of the set Γ for the first time, its left and right half neighborhoods meet each other again, although prior to this they could, at certain times, appear on the boundary (endpoints of types L and R) of the set Γ. Moreover, the total return times (in terms of the number of iterations of R) of two half neighborhoods measured from their splitting at the point $x_{\xi_1\mathrm{b}}$ till their reunion at the point $x_{\eta_n\mathrm{e}}$ are identical for both these neighborhoods. This means the following: We recall that any beginning interval $I_{\alpha\mathrm{b}}$ of the exchange R_Γ is included in a certain set Γ_k of points returning to Γ exactly after k iterations of a circle rotation R, $1 \leq k \leq k_0$, and denote this time by k_α, $\alpha \in \mathcal{A}$. Hence, for the investigated trajectory of the right half neighborhood $x^+_{\xi_1\mathrm{b}}$, the total time of its coming to the point $x_{\eta_n\mathrm{e}}$ measured in terms of the number of iterations of R is equal to $k_{\xi_1} + \ldots + k_{\xi_m}$, while, for the left half neighborhood $x^-_{\xi_1\mathrm{b}}$ this time is equal to $k_{\eta_1} + \ldots + k_{\eta_n}$. As shown above, these times are necessarily equal and, therefore, we get

$$\sum_{i=1}^{m} k_{\xi_i} = \sum_{j=1}^{n} k_{\eta_j}$$

for any endpoint of type MB of the interval exchange R_Γ.

We can now explicitly present an interval exchange with IRE scheme dual to σ. This dual scheme σ^* consists of all cycles $(\xi_1\mathrm{b}, \ldots, \xi_m\mathrm{b}, \eta_n\mathrm{e}, \ldots, \eta_1\mathrm{e})$ constructed for all endpoints of type MB in the interval exchange R_Γ. All these cycles are disjoint because the analyzed interval exchange is bijective. On the other hand, any element of $\bar{A}$ belongs to one of these cycles because we take into account all endpoints of type MB; the number of endpoints of type ME is the same as the number of type-MB endpoints, and a half neighborhood of any type-L or type-R endpoint necessarily hits a type-ME endpoint after a certain number of iterations; therefore, we have also taken into account all other endpoints. The duality of σ^* to σ can be easily checked by using definition (4) in view of the relations obtained in the investigation of the trajectories of half neighborhoods of the endpoint $x_{\xi_1\mathrm{b}}$. In order to show that the scheme σ^* is positive, we set

$$y_{\xi_i\mathrm{b}} = \sum_{s=1}^{i-1} k_{\xi_s}, \quad 1 \leq i \leq m, \qquad \text{and} \qquad y_{\eta_j\mathrm{e}} = \sum_{t=1}^{j} k_{\eta_t}, \quad 1 \leq j \leq n,$$

for any cycle presented above. This enables us to determine the vector of endpoints $\mathbf{y}$ allowed by the scheme σ^* due to the equalities

$$\sum_{i=1}^{m} k_{\xi_i} = \sum_{j=1}^{n} k_{\eta_j}.$$

The corresponding vector of lengths consists of positive (and even integer-valued) components k_ξ, $\xi \in \mathcal{A}$. Hence, the IRE $(\sigma^*, \mathbf{y})$ is indeed an interval exchange.

Statement 1 of Theorem 1 is proved.

Remark 3. To prove this statement, for an interval exchange induced on a subset of a circle by its rotation, we constructed a dual dynamical system of interval exchange with the use of time intervals exchanged as in the case of space intervals.

5. Proving the Second Part of the Theorem

In this section, we prove Statement 2 of Theorem 1 by using an algorithm for the construction of an irrational circle rotation and the union of arcs such that the corresponding first return map is shift-equivalent to a given rotational interval exchange.

Thus, we assume that a rotational interval exchange $(\sigma, \mathbf{x})$ is given. Since our aim is to analyze shift-equivalent interval exchanges, it suffices to consider, from the very beginning, a floating IRE $(\sigma, \mathbf{v})$, i.e., restrict ourselves to the analysis of the lengths of intervals without taking into account the coordinates of endpoints.

For this IRE, we successively apply two types of operations: the first type is the reverse step of induction (which produces a new dynamical system for which the initial system is the first return map). The second type of operations is merging of two neighboring intervals into a single interval (this does not change the dynamical system at all but the number of exchanging intervals decreases).

5.1. Elementary Steps of Induction. In this section, we recall four elementary steps of induction $\Pi^{\mathrm{rb}}_{\alpha\beta}$, $\Pi^{\mathrm{re}}_{\alpha\beta}$, $\Pi^{\mathrm{lb}}_{\alpha\beta}$, and $\Pi^{\mathrm{le}}_{\alpha\beta}$, which were defined in [1]. They are the operations of transformation, which can be applied either to the IREs (and, in particular, to interval exchanges) or separately to their schemes generalizing the classical steps of the Rauzy–Veech induction. According to their action, these four steps are called "cropping a beginning interval on the right," "cropping an ending interval on the right," "cropping a beginning interval on the left," and "cropping an ending interval on the left," respectively. The general formulas for the four elementary steps of induction were presented in Sec. 7 of the work [1], where it was also explained that the steps $\Pi^{\mathrm{rb}}_{\alpha\beta}$ and $\Pi^{\mathrm{re}}_{\alpha\beta}$ can be applied to an IRE scheme σ under the condition $\sigma(\alpha\mathrm{b}) = \beta\mathrm{e}$, whereas the steps $\Pi^{\mathrm{lb}}_{\alpha\beta}$ and $\Pi^{\mathrm{le}}_{\alpha\beta}$ are applicable under the condition $\sigma(\beta\mathrm{e}) = \alpha\mathrm{b}$. In terms of cycles in the permutation σ, these operations act as follows: the step $\Pi^{\mathrm{rb}}_{\alpha\beta}$ (the steps $\Pi^{\mathrm{re}}_{\alpha\beta}$, $\Pi^{\mathrm{lb}}_{\alpha\beta}$, and $\Pi^{\mathrm{le}}_{\alpha\beta}$) moves an element $\beta\mathrm{e}$ (elements $\alpha\mathrm{b}$, $\beta\mathrm{e}$, and $\alpha\mathrm{b}$) from its current position into the position right in front of $\alpha\mathrm{e}$ (right after $\beta\mathrm{b}$, right after $\alpha\mathrm{e}$, and right in front of $\beta\mathrm{b}$). In the vector of lengths $\mathbf{v}$, one component changes: the steps $\Pi^{\mathrm{rb}}_{\alpha\beta}$ and $\Pi^{\mathrm{lb}}_{\alpha\beta}$ subtract the quantity v_β from the component v_α, while the steps $\Pi^{\mathrm{re}}_{\alpha\beta}$ and $\Pi^{\mathrm{le}}_{\alpha\beta}$ subtract the quantity v_α from the component v_β. The reverse steps $(\Pi^{\mathrm{rb}}_{\alpha\beta})^{-1}$, $(\Pi^{\mathrm{re}}_{\alpha\beta})^{-1}$, $(\Pi^{\mathrm{lb}}_{\alpha\beta})^{-1}$, or $(\Pi^{\mathrm{le}}_{\alpha\beta})^{-1}$ are applicable to the IRE scheme σ under the conditions $\sigma(\beta\mathrm{e}) = \alpha\mathrm{e}$, $\sigma(\beta\mathrm{b}) = \alpha\mathrm{b}$, $\sigma(\alpha\mathrm{e}) = \beta\mathrm{e}$, or $\sigma(\alpha\mathrm{b}) = \beta\mathrm{b}$, respectively. For the lengths: the reverse steps $(\Pi^{\mathrm{rb}}_{\alpha\beta})^{-1}$ and $(\Pi^{\mathrm{lb}}_{\alpha\beta})^{-1}$ add the quantity v_β to the component v_α, while the steps $(\Pi^{\mathrm{re}}_{\alpha\beta})^{-1}$ and $(\Pi^{\mathrm{le}}_{\alpha\beta})^{-1}$ add the quantity v_α to the component v_β.

Further, we now demonstrate how these steps work for (floating) interval exchanges with the help of two-row notation for the cycles in their schemes. In fact, the classical Rauzy–Veech induction works in exactly the same way. However, it acts on a single segment and only at its right end but we consider interval exchanges on multiple segments and apply induction steps at both ends of the segments.

In order to guarantee that an interval exchange remains an interval exchange after applying an induction step or a reverse induction step, it is necessary to guarantee that (in addition to the conditions listed above) all components of the vector of lengths remain positive and that the twist total of the scheme is equal to zero (in other words, the twist number of every cycle must be equal to zero and, hence, the cycle must remain "two-row;" see the definition in Sec. 2).

Consider the step $\Pi^{\mathrm{rb}}_{\alpha\beta}$ in more detail. The condition of its applicability $\sigma(\alpha\mathrm{b}) = \beta\mathrm{e}$ means that the beginning interval $I_{\alpha\mathrm{b}}$ and the ending interval $I_{\beta\mathrm{e}}$ lie at the right edge of the same segment J, which means that, in the two-row notation, one cycle of the scheme has the form $\begin{bmatrix} \dots & \alpha \\ \dots & \beta \end{bmatrix}$. Since $\Pi^{\mathrm{rb}}_{\alpha\beta}$ subtracts the length v_β from the length v_α (by cutting the ending interval $I_{\beta\mathrm{e}}$ from the segment and, hence, by "cropping the beginning interval $I_{\alpha\mathrm{b}}$ on the right" by v_β), the resulting vector of lengths remains positive if and only if $v_\alpha > v_\beta$, i.e., $I_{\alpha\mathrm{b}}$ must be longer than $I_{\beta\mathrm{e}}$. If this condition is satisfied, then the interval $I_{\beta\mathrm{e}}$ cannot be the sole ending interval on the segment J because the sum of lengths of the ending intervals on a segment is always equal to the sum of lengths of the beginning intervals and, hence, some other ending interval $I_{\gamma\mathrm{e}}$ lies on J straight to the left of $I_{\beta\mathrm{e}}$, i.e., $\sigma(\beta\mathrm{e}) = \gamma\mathrm{e}$ for some $\gamma \in \mathcal{A}$. In the two-row notation, the scheme is transformed as follows (we show only the cycles engaged in the transformation; all other cycles do not change):

$$\Pi^{\mathrm{rb}}_{\alpha\beta}: \begin{bmatrix} \dots & \alpha \\ \dots & \gamma\,\beta \end{bmatrix}, \begin{bmatrix} & \dots & \\ \dots & \alpha & \dots \end{bmatrix} \mapsto \begin{bmatrix} \dots & \alpha \\ \dots & \gamma \end{bmatrix}, \begin{bmatrix} & \dots & \\ \dots & \alpha\,\beta & \dots \end{bmatrix}.$$

Here, the first cycle corresponds to the segment J, while the second cycle corresponds to a segment containing the ending interval $I_{\alpha\mathrm{e}}$; the role of the latter can be played by the same segment J. In this case, we have

$$\Pi^{\mathrm{rb}}_{\alpha\beta}: \begin{bmatrix} & \dots & & \alpha \\ \dots & \alpha & \dots & \gamma\,\beta \end{bmatrix} \mapsto \begin{bmatrix} & \dots & & \alpha \\ \dots & \alpha\,\beta & \dots & \gamma \end{bmatrix}.$$

If $\gamma = \alpha$, then the scheme σ does not change under the action of $\Pi^{\mathrm{rb}}_{\alpha\beta}$, and only its length v_α changes.

In all cases, under the action of $\Pi^{\mathrm{rb}}_{\alpha\beta}$, the segment J is cropped on the right by v_β, which is the length of the interval $I_{\beta\mathrm{e}}$. The intervals $I_{\alpha\mathrm{b}}$ and $I_{\alpha\mathrm{e}}$ are also cropped on the right by v_β and the interval $I_{\beta\mathrm{e}}$ moves to a new position straight to the right of the interval $I_{\alpha\mathrm{e}}$ (cropped by the length of $I_{\beta\mathrm{e}}$). We also see that, in all cases, all cycles remain untwisted (the twist number is equal to zero), and this is true for all (straight!) elementary steps of induction.

It is easy to see that the dynamical system of interval exchange obtained under the described action of $\Pi^{\mathrm{rb}}_{\alpha\beta}$ upon $(\sigma, \mathbf{v})$ is nothing else but the first return map for the original dynamical system on the union of segments, where the segment J is cropped on the right by v_β. Indeed, all points in this union return to it for time 1, except the points of the interval $I_{\beta\mathrm{b}}$, which return for time 2: First they hit the cut-out interval $I_{\beta\mathrm{e}}$ of the original dynamical system, and then the map brings them to the interval $I_{\beta\mathrm{e}}$ of the new dynamical system (in the original system, this interval was the right part of the interval $I_{\alpha\mathrm{e}}$).

The remaining three steps of induction act similarly, and every time the result of their action is the first return map for the original dynamical system onto a union of segments one of which is cropped either from the right or from the left. For the sake of clarity, we describe their action on the schemes just as this has been done above for the step $\Pi^{\mathrm{rb}}_{\alpha\beta}$ and note that the step $\Pi^{\mathrm{lb}}_{\alpha\beta}$ transforms an interval exchange into an interval exchange if and only if $v_\alpha > v_\beta$, whereas the steps $\Pi^{\mathrm{re}}_{\alpha\beta}$ and $\Pi^{\mathrm{le}}_{\alpha\beta}$ do this if and only if $v_\alpha < v_\beta$:

$$\Pi^{\mathrm{re}}_{\alpha\beta}: \begin{bmatrix} \dots & \gamma\,\alpha \\ \dots & \beta \end{bmatrix}, \begin{bmatrix} \dots & \beta & \dots \\ & \dots & \end{bmatrix} \mapsto \begin{bmatrix} \dots & \gamma \\ \dots & \beta \end{bmatrix}, \begin{bmatrix} \dots & \beta\,\alpha & \dots \\ & \dots & \end{bmatrix},$$

$$\Pi^{\mathrm{lb}}_{\alpha\beta}: \begin{bmatrix} \alpha & \dots \\ \beta\,\gamma & \dots \end{bmatrix}, \begin{bmatrix} & \dots & \\ \dots & \alpha & \dots \end{bmatrix} \mapsto \begin{bmatrix} \alpha & \dots \\ \gamma & \dots \end{bmatrix}, \begin{bmatrix} & \dots & \\ \dots & \beta\,\alpha & \dots \end{bmatrix},$$

$$\Pi^{\mathrm{le}}_{\alpha\beta}: \begin{bmatrix} \alpha\,\gamma & \dots \\ \beta & \dots \end{bmatrix}, \begin{bmatrix} \dots & \beta & \dots \\ & \dots & \end{bmatrix} \mapsto \begin{bmatrix} \gamma & \dots \\ \beta & \dots \end{bmatrix}, \begin{bmatrix} \dots & \alpha\,\beta & \dots \\ & \dots & \end{bmatrix}.$$

Each of these steps decreases the larger of the two lengths v_α and v_β by the smaller of these lengths and does not affect all other lengths.

The elementary reverse steps of induction $(\Pi^{\rm rb}_{\alpha\beta})^{-1}$, $(\Pi^{\rm re}_{\alpha\beta})^{-1}$, $(\Pi^{\rm lb}_{\alpha\beta})^{-1}$, or $(\Pi^{\rm le}_{\alpha\beta})^{-1}$ are applicable to an IRE scheme σ under the conditions $\sigma(\beta{\rm e}) = \alpha{\rm e}$, $\sigma(\beta{\rm b}) = \alpha{\rm b}$, $\sigma(\alpha{\rm e}) = \beta{\rm e}$, or $\sigma(\alpha{\rm b}) = \beta{\rm b}$, respectively (see Proposition 3 in [1]). The scheme obtained from an interval exchange scheme σ by applying the reverse steps of induction $(\Pi^{\rm rb}_{\alpha\beta})^{-1}$, $(\Pi^{\rm re}_{\alpha\beta})^{-1}$, $(\Pi^{\rm lb}_{\alpha\beta})^{-1}$, or $(\Pi^{\rm le}_{\alpha\beta})^{-1}$ has zero twist total if and only if there exists an element $\gamma \in \mathcal{A}$ such that $\sigma(\alpha{\rm b}) = \gamma{\rm e}$, $\sigma^{-1}(\beta{\rm e}) = \gamma{\rm b}$, $\sigma^{-1}(\alpha{\rm b}) = \gamma{\rm e}$, or $\sigma(\beta{\rm e}) = \gamma{\rm b}$, respectively. Since the action of the steps $(\Pi^{\rm rb}_{\alpha\beta})^{-1}$ and $(\Pi^{\rm lb}_{\alpha\beta})^{-1}$ upon the real component of the IRE $\mathbf{v}$ adds the length v_β to v_α and the action of the steps $(\Pi^{\rm re}_{\alpha\beta})^{-1}$ and $(\Pi^{\rm le}_{\alpha\beta})^{-1}$ adds the length v_α to v_β, the positivity of the original IRE yields the positivity of the resulting IRE, and one length only increases.

Proposition 3. *An interval exchange obtained from a rotational interval exchange by applying an elementary step of induction, is also rotational.*

Proof. According to Proposition 6 in [1], the induction steps do not change the twist total of a scheme. Therefore, in this case it remains equal to zero. According to Theorem 1 in [1], in the case where an IRE scheme is transformed under the action of an elementary step of induction, its dual scheme transforms under the action of a certain reverse step. Further, since the action of a reverse step of induction on the positive IRE only increases one length by the value of another length, the dual scheme remains positive. Proposition is proved.

A similar statement for reverse steps of induction is, general speaking, false, as shown by the following **counterexample:**

Consider a scheme

$$\sigma = \left\{ \begin{bmatrix} \gamma & \alpha & \delta \\ \delta & \beta & \end{bmatrix}, \begin{bmatrix} \beta & \\ \alpha & \gamma \end{bmatrix} \right\}.$$

It has zero twist number and is positive: Thus, the vector of lengths $\mathbf{v} = (v_\alpha, v_\beta, v_\gamma, v_\delta) = (1, 2, 1, 1)$ is allowed. The dual scheme

$$\sigma^* = \left\{ \begin{bmatrix} \alpha & \beta & \\ \gamma & \beta & \delta \end{bmatrix}, \begin{bmatrix} \delta & \gamma \\ \alpha & \end{bmatrix} \right\}$$

is also untwisted and positive. For example, the vector of lengths $\mathbf{w} = (w_\alpha, w_\beta, w_\gamma, w_\delta) = (2, 1, 1, 1)$ is allowed. Therefore, the IRE $(\sigma, \mathbf{v})$ is a rotational interval exchange. The reverse step of induction $(\Pi^{\rm le}_{\gamma\alpha})^{-1}$ is applicable, and the obtained scheme

$$\sigma' = \left\{ \begin{bmatrix} \alpha & \delta \\ \delta & \beta \end{bmatrix}, \begin{bmatrix} \gamma & \beta \\ \alpha & \gamma \end{bmatrix} \right\}$$

is also untwisted and positive, as the vector of lengths $\mathbf{v} = (1, 2, 1, 1)$ is transformed into $\mathbf{v}' = (2, 2, 1, 1)$. Therefore, the obtained IRE $(\sigma', \mathbf{v}') = (\Pi^{\rm le}_{\gamma\alpha})^{-1}(\sigma, \mathbf{v})$ is an interval exchange. However, this interval exchange is not rotational. Indeed, the scheme

$$(\sigma')^* = \Pi^{\rm le}_{\alpha\gamma}\sigma^* = \left\{ \begin{bmatrix} \beta & & \\ \gamma & \beta & \delta \end{bmatrix}, \begin{bmatrix} \delta & \alpha & \gamma \\ \alpha & & \end{bmatrix} \right\}$$

dual to σ' also has zero twist number but, clearly, is not positive because the vectors of lengths $\mathbf{u} = (u_\alpha, u_\beta, u_\gamma, u_\delta)$ allowed by this scheme are determined by the condition $u_\gamma + u_\delta = 0$.

5.2. Operation of Merging Intervals. If an interval exchange has two neighboring intervals shifted by the same distance, then it is natural to merge them into a single interval; as a result, the dynamical system does not change. This situation takes place if $\sigma(\alpha\mathrm{b}) = \beta\mathrm{b}$ and $\sigma(\beta\mathrm{e}) = \alpha\mathrm{e}$ for some $\alpha \neq \beta$. We now define the operation of *merging intervals* $\Sigma_{\alpha\beta}$ applicable to an IRE $(\sigma, \mathbf{v})$ under the above-mentioned condition imposed on the scheme σ and acting in the following way: the symbol β is removed from the alphabet $\mathcal{A}$, the elements $\beta\mathrm{b}$ and $\beta\mathrm{e}$ are removed from the corresponding cycles, and the length v_α increases by v_β.

Formally, $\Sigma_{\alpha\beta}(\sigma, \mathbf{v}) = (\sigma', \mathbf{v}')$, where σ' is a permutation of the reduced double alphabet $\bar{\mathcal{A}}' = \mathcal{A}' \times \{\mathrm{b}, \mathrm{e}\}$, $\mathcal{A}' = \mathcal{A}\backslash\{\beta\}$, given by the following equalities: In the case $\sigma(\beta\mathrm{b}) \neq \beta\mathrm{e}$, these are $\sigma'(\alpha\mathrm{b}) = \sigma(\beta\mathrm{b})$, $\sigma'(\sigma^{-1}(\beta\mathrm{e})) = \alpha\mathrm{e}$, and $\sigma'(\bar{\xi}) = \sigma(\bar{\xi})$ for $\bar{\xi} \in \bar{\mathcal{A}}'\backslash\{\alpha\mathrm{b}, \sigma^{-1}(\beta\mathrm{e})\}$. At the same time, in the case $\sigma(\beta\mathrm{b}) = \beta\mathrm{e}$, these are $\sigma(\alpha\mathrm{b}) = \alpha\mathrm{e}$ and $\sigma'(\bar{\xi}) = \sigma(\bar{\xi})$ for $\bar{\xi} \in \bar{\mathcal{A}}'\backslash\{\alpha\mathrm{b}\}$; the new vector of lengths $\mathbf{v}' \in \mathbb{R}^{\mathcal{A}'}$ is given by $v'_\alpha = v_\alpha + v_\beta$, and $v'_\xi = v_\xi$ for $\xi \in \mathcal{A}'\backslash\{\alpha\}$.

Note that, for a rotational scheme σ, the case $\sigma(\beta\mathrm{b}) = \beta\mathrm{e}$ is impossible because this equality implies the relation $\sigma^*(\beta\mathrm{e}) = \beta\mathrm{e}$ for the dual scheme σ^*, which makes the latter definitely nonpositive because every vector of lengths $\mathbf{w}$ allowed by the scheme σ^* contains a component $w_\beta = 0$.

The condition $\sigma(\alpha\mathrm{b}) = \beta\mathrm{b}$, $\sigma(\beta\mathrm{e}) = \alpha\mathrm{e}$, for the applicability of $\Sigma_{\alpha\beta}$ to σ is equivalent to the condition $\sigma^*(\alpha\mathrm{e}) = \beta\mathrm{b}$, $\sigma^*(\beta\mathrm{b}) = \alpha\mathrm{e}$, for the dual scheme σ^*. This means that σ^* contains a two-element cycle $\begin{bmatrix}\beta\\ \alpha\end{bmatrix}$. Applying the operation of merging intervals $\Sigma_{\alpha\beta}$ to σ, we remove this two-element cycle from σ^* and replace $\beta\mathrm{e}$ by $\alpha\mathrm{e}$ in the cycle of σ^* containing the element $\beta\mathrm{e}$.

Proposition 4. *The operation of merging intervals applied to a rotational interval exchange leaves it rotational.*

Proof. It is clear from the definition of this operation that the twist numbers of cycles do not change and only one cycle (untwisted, as all these cycles) in the dual scheme completely disappears. Therefore, the twist total remains equal to zero. One of the lengths increases by the value of another length. Hence, the vector of lengths remains positive. Prior to applying the operation, the dual scheme contained an allowed positive vector of lengths $\mathbf{w}$ such that $w_\alpha = w_\beta$. Therefore, the vector $\mathbf{w}'$ with the same components (only the component w_β is removed) is obviously positive and allowed by the scheme dual to the scheme obtained by applying the operation $\Sigma_{\alpha\beta}$.

Proposition is proved.

5.3. Idea of the Algorithm. We now describe an algorithm of consecutive transformations of an arbitrary irreducible rotational interval exchange, which eventually leads to the construction of the first return map on a circle sought in Statement 2 of Theorem 1. It proves to be convenient to operate with dual schemes. The idea is as follows: We consecutively apply elementary induction steps to one cycle in the dual scheme. This cycle is eventually reduced to a two-element cycle and removed from consideration as a result of merging of the corresponding intervals. Performing this procedure, in turn, for each cycle of the dual scheme, as long as the number of cycles is greater than one, we eventually get a dual scheme that consists of a single cycle. At this point, the algorithm stops, we analyze the resulting (very special) interval exchange and show how to construct a circle rotation and a union of arcs for which this interval exchange is the first return map.

It is important to keep the dual scheme positive after each new transformation (not to get the effect described in the counterexample presented at the end of Sec. 5.1). In this case, by Proposition 3, the scheme always remains rotational and, therefore, the transformed interval exchange also remains rotational.

The following two lemmas are necessary for what follows.

5.4. Lemma on Unsplittability. We call an interval exchange scheme σ *splittable* (for a cycle c_0) if its alphabet $\mathcal{A}$ can be split into two nonempty subalphabets $\mathcal{A}_1$ and $\mathcal{A}_2$ ($\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}$, $\mathcal{A}_1 \neq \varnothing$, and $\mathcal{A}_2 \neq \varnothing$)

such that the following property holds: one cycle

$$c_0 = (\bar{\xi}, \sigma(\bar{\xi})), \ldots, \sigma^k(\bar{\xi})), \quad \bar{\xi} \in \bar{\mathcal{A}}, \quad k \geq 1, \quad \sigma^{k+1}(\bar{\xi}) = \bar{\xi},$$

in the scheme $\sigma = \{c_0, c_1, \ldots, c_n\}$, $n \geq 0$, can be split into two nonempty arcs $a_1 = (\bar{\xi}, \sigma(\bar{\xi}), \ldots, \sigma^i(\bar{\xi}))$ and $a_2 = (\sigma^{i+1}(\bar{\xi}), \ldots, \sigma^k(\bar{\xi}))$, $0 \leq i < k$, and the remaining cycles can be split into two sets $\tau_1 = \{c_1, \ldots, c_j\}$ and $\tau_2 = \{c_{j+1}, \ldots, c_n\}$, $0 \leq j \leq n$, such that all elements of a_1 and cycles from τ_1 belong to $\bar{\mathcal{A}}_1 = \mathcal{A}_1 \times \{\mathrm{b}, \mathrm{e}\}$, while all elements of a_2 and cycles from τ_2 belong to $\bar{\mathcal{A}}_2 = \mathcal{A}_2 \times \{\mathrm{b}, \mathrm{e}\}$. Otherwise, the scheme is called *unsplittable*.

It is easy to see that, for an interval exchange with splittable scheme, the sum of lengths of all beginning intervals in the arc a_1 is equal to the sum of lengths of all ending intervals in the arc a_1, and the same is true for the intervals in the arc a_2. This follows from the fact that, by Proposition 1, the indicated property (the sum of lengths of all beginning intervals from a certain collection is equal to the sum of lengths of all ending intervals from the same collection) holds, on one hand, for each particular cycle and, therefore, for the set of all intervals indexed by the elements of cycles from τ_1; on the other hand, this property also holds for the collection of all intervals indexed by the elements of $\bar{\mathcal{A}}_1$. Moreover, these two collections of intervals differ exactly by the set of all intervals from the arc a_1.

In particular, the property established above implies that any arc, a_1 or a_2, in a splittable interval exchange scheme σ cannot consist solely of the beginning elements or solely of the ending elements because, in this case, the sum of the corresponding lengths allowed by the scheme would be equal to zero and, hence, the scheme would not be positive. Since the cycle c_0 has zero twist, one of these arcs necessarily starts from an ending element and ends at a beginning element, whereas the other arc starts from a beginning element and ends at an ending element. For definiteness, we assume that the first arc is a_1 and the second arc is a_2. Thus, we have $\bar{\xi} = \alpha\mathrm{e}$, $\sigma^i(\bar{\xi})) = \beta\mathrm{b}$, $\sigma^{i+1}(\bar{\xi}) = \gamma\mathrm{b}$, and $\sigma^k(\bar{\xi}) = \delta\mathrm{e}$ for some $\alpha, \beta \in \mathcal{A}_1$ and $\delta, \gamma \in \mathcal{A}_2$.

The property of splittability of a scheme becomes quite clear if we consider the corresponding floating interval exchange $(\sigma, \mathbf{v})$. This simply means that the segments of this interval exchange can be fixed in positions such that every interval of the fixed interval exchange $(\sigma, \mathbf{x})$ thus obtained is entirely contained either in the half line $(-\infty, 0)$ or in the half line $[0, +\infty)$; the labels of all intervals from the first set belong to $\mathcal{A}_1$ and the labels of all intervals from the second set belong to $\mathcal{A}_2$. Moreover, precisely for one segment (corresponding to the cycle c_0) the origin is its inner point and $x_{\alpha\mathrm{e}} = x_{\gamma\mathrm{b}} = 0$ for α and γ determined in the previous paragraph. In fact, the entire set of intervals in $(\sigma, \mathbf{x})$ is separated by the point zero into two arrays: the intervals lying to the left of zero are indexed by the elements of $\bar{\mathcal{A}}_1$, while the intervals lying to the right of zero are indexed by the elements of $\bar{\mathcal{A}}_2$.

Lemma 1. *If an interval exchange scheme is splittable, then it cannot be rotational.*

Proof. According to the above-mentioned properties of a splittable scheme σ, the sets of elements $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$ are connected with this scheme at (exactly) two places, namely, there exist elements $\alpha, \beta \in \mathcal{A}_1$ and $\delta, \gamma \in \mathcal{A}_2$ such that $\sigma(\beta\mathrm{b}) = \gamma\mathrm{b}$ and $\sigma(\delta\mathrm{e}) = \alpha\mathrm{e}$, whereas for all $\bar{\xi} \in \bar{\mathcal{A}}_1 \backslash \{\beta\mathrm{b}\}$ and all $\bar{\eta} \in \bar{\mathcal{A}}_2 \backslash \{\delta\mathrm{e}\}$, we have $\sigma(\bar{\xi}) \in \bar{\mathcal{A}}_1$ and $\sigma(\bar{\eta}) \in \bar{\mathcal{A}}_2$. The corresponding property also holds for the dual scheme σ^*, namely, we have $\sigma^*(\beta\mathrm{e}) = \gamma\mathrm{b}$ and $\sigma^*(\delta\mathrm{b}) = \alpha\mathrm{e}$ but, for all $\bar{\xi} \in \bar{\mathcal{A}}_1 \backslash \{\beta\mathrm{e}\}$ and all $\bar{\eta} \in \bar{\mathcal{A}}_2 \backslash \{\delta\mathrm{b}\}$, we get $\sigma^*(\bar{\xi}) \in \bar{\mathcal{A}}_1$ and $\sigma^*(\bar{\eta}) \in \bar{\mathcal{A}}_2$. Therefore, one cycle in the scheme σ^* can be split into two nonempty arcs $a_1' = (\alpha\mathrm{e}, \ldots, \beta\mathrm{e})$ and $a_2' = (\gamma\mathrm{b}, \ldots, \delta\mathrm{b})$, while the remaining cycles can be split into two sets τ_1' and τ_2' such that all elements of a_1' and cycles from τ_1' belong to $\bar{\mathcal{A}}_1$ and all elements of a_2' and cycles from τ_2' belong to $\bar{\mathcal{A}}_2$. Let $\mathbf{w} = (w_\xi)_{\xi \in \mathcal{A}}$ be a vector of lengths allowed by the scheme σ^*. The sum of lengths of all beginning intervals in any cycle is equal to the sum of lengths of all ending intervals in this cycle, and the same is obviously true for a set of cycles. Hence, we get the equality

$$\sum_{\xi\,:\ \xi\mathrm{b} \in \tau_1'} w_\xi = \sum_{\xi\,:\ \xi\mathrm{e} \in \tau_1'} w_\xi$$

(here, a somewhat incorrect but demonstrative notation $\xi\mathrm{b} \in \tau_1'$ means that the element $\xi\mathrm{b}$ belongs to a cycle from the set τ_1' and the same is true for $\xi\mathrm{e}$). Since

$$\sum_{\xi:\ \xi\mathrm{b}\in\bar{\mathcal{A}}_1} w_\xi = \sum_{\xi\in\mathcal{A}_1} w_\xi = \sum\nolimits_{\xi:\ \xi\mathrm{e}\in\bar{\mathcal{A}}_1} w_\xi,$$

and the set of all elements of the cycles from τ_1' differs from $\bar{\mathcal{A}}_1$ exactly by the set of all elements of the arc a_1', subtracting the former equality from the latter, we get

$$\sum_{\xi:\ \xi\mathrm{b}\in a_1'} w_\xi = \sum_{\xi:\ \xi\mathrm{e}\in a_1'} w_\xi.$$

If we assume that the splittable interval exchange scheme σ is rotational, then the dual scheme σ^* must be positive and untwisted. However, if all cycles in σ^* have zero twist, then the nonempty arc a_1' determined above contains only ending elements. Therefore,

$$\sum_{\xi:\ \xi\mathrm{e}\in a_1'} w_\xi = 0$$

and the scheme σ^* is not positive.

Lemma is proved.

5.5. Lemma on Unequal Lengths. For a given IRE scheme σ, we say that the lengths v_α and v_β, $\alpha, \beta \in \mathcal{A}$, are *equal with necessity* if the equality $v_\alpha = v_\beta$ holds for any allowed vector of lengths $\mathbf{v}$. It is clear that all relations between the lengths are determined by equalities (3). However, if the scheme is sufficiently complicated, then the equality with necessity can be not evident for a certain pair of lengths.

Consider a situation in which, for some interval exchange, the beginning interval $I_{\alpha\mathrm{b}}$ and the ending interval $I_{\beta\mathrm{e}}$ are both adjacent either to the left end or to the right end of the same segment *J*. In the two-row notation, the scheme contains a cycle c_0, which has the following form: $\begin{bmatrix}\alpha & \dots\\ \beta & \dots\end{bmatrix}$ or $\begin{bmatrix}\dots & \alpha\\ \dots & \beta\end{bmatrix}$, i.e., $\sigma(\beta\mathrm{e}) = \alpha\mathrm{b}$ or $\sigma(\alpha\mathrm{b}) = \beta\mathrm{e}$, $\alpha, \beta \in \mathcal{A}$, respectively, and this cycle is not a two-element cycle. In the general case, the lengths v_α and v_β can be equal with necessity but not in the case of a rotational interval exchange, as shown in the next statement.

Lemma 2. *If, for an interval exchange scheme σ, we have $\sigma(\beta\mathrm{e}) = \alpha\mathrm{b}$ or $\sigma(\alpha\mathrm{b}) = \beta\mathrm{e}$ for some $\alpha, \beta \in \mathcal{A}$ and the corresponding cycle c_0 is not two-element but the lengths v_α and v_β are equal with necessity, then this scheme cannot be rotational.*

Proof. We restrict ourselves to the case where $\sigma(\beta\mathrm{e}) = \alpha\mathrm{b}$ (in the case where $\sigma(\alpha\mathrm{b}) = \beta\mathrm{e}$, the proof is similar). In fact, we prove that, under the conditions listed above, the scheme σ is splittable and, hence, according to Lemma 1, it cannot be rotational. First, we note that, for $\alpha = \beta$, the scheme σ is obviously splittable (at the cycle c_0 with subalphabets $\mathcal{A}_1 = \{\alpha\}$ and $\mathcal{A}_2 = \mathcal{A}\backslash\{\alpha\}$). Thus, in the remaining part of the proof we assume that $\alpha \neq \beta$.

Let $\mathbf{v}$ be a positive vector of lengths allowed by the scheme σ. If there exists a (looped) sequence of labels $\beta_1 = \beta, \beta_2, \dots, \beta_k, \beta_{k+1} = \beta$, $k \geq 1$, distinct from α and such that $\beta_{i+1}\mathrm{e} \in c(\beta_i\mathrm{b})$ for all $1 \leq i \leq k$, then a simultaneous increase in all lengths v_{β_i}, $1 \leq i \leq k$, by the same positive quantity does not affect the validity of all equalities (3) and, therefore, the lengths v_α and v_β are not equal with necessity. Since this contradicts the conditions of the lemma, the indicated looped sequence does not exist.

We now consider a set of cycles τ_1 gathered according to the following algorithm: First, we include the cycle $c_1 = c(\beta\mathrm{b})$. Then we add all cycles $c(\gamma\mathrm{b})$ for which $\gamma \neq \alpha$ and the element $\gamma\mathrm{e}$ belongs to one of the cycles that have been already added to τ_1. In fact, the set τ_1 is formed by cycles of the form $c(\beta_k\mathrm{b})$ for every existing finite sequence of labels $\beta_1 = \beta, \beta_2, \ldots, \beta_k,\ k \geq 1$, distinct from α and such that $\beta_{i+1}\mathrm{e} \in c(\beta_i\mathrm{b})$ for all $1 \leq i < k$. The cycle c_0 does not belong to the set τ_1 according to the result obtained in the previous paragraph.

Consider the set $\mathcal{A}_0 \subset \mathcal{A}$ of all labels $\gamma \neq \alpha$ such that $\gamma\mathrm{e} \in \tau_1$ (as earlier, this somewhat incorrect notation means that a given element belongs to a cycle from the set τ_1). According to our construction, we have $\gamma\mathrm{b} \in \tau_1$ for all $\gamma \in \mathcal{A}_0$. Since c_0 does not belong to the set τ_1, we have $\beta\mathrm{e} \notin \tau_1$ and $\alpha\mathrm{b} \notin \tau_1$. If $\alpha\mathrm{e} \notin \tau_1$, then the set of labels of all ending elements of cycles from the set τ_1 is $\mathcal{A}_0$, while the set of labels of all beginning elements of these cycles includes the set $\mathcal{A}_0$ and contains (at least) one more label $\beta \notin \mathcal{A}_0$, which contradicts Proposition 1 for a positive scheme. Hence, $\alpha\mathrm{e} \in \tau_1$ and the set of labels of all ending elements of cycles from the set τ_1 is $\mathcal{A}_0 \cup \{\alpha\}$, while the set of labels of all beginning elements of these cycles contains the set $\mathcal{A}_0 \cup \{\beta\}$ and, in fact, coincides with this set due to Proposition 1 and the equality of lengths v_α and v_β.

Hence, we can see that, in the case where $\alpha \neq \beta$, the interval exchange scheme σ is splittable (at the cycle c_0 with sub-alphabets $\mathcal{A}_1 = \mathcal{A}_0 \cup \{\alpha, \beta\}$ and $\mathcal{A}_2 = \mathcal{A} \backslash \mathcal{A}_1$) and, hence, it is not rotational.

Lemma is proved.

5.6. Realization of the Algorithm. Consider a given floating rotational interval exchange $(\sigma, \mathbf{v})$ and the rotational interval exchange scheme σ^* dual to σ. If this scheme contains two-element cycles, then the appropriate operations of merging intervals are applied to $(\sigma, \mathbf{v})$. As a result of these operations, the dual scheme σ^* does not contain two-element cycles any longer. If the scheme σ^* contains more than one cycle, then we choose one of these cycles arbitrarily and consecutively apply to this cycle elementary steps of induction of the form $\Pi^{\mathrm{lb}}_{\alpha\beta}$ or $\Pi^{\mathrm{le}}_{\alpha\beta}$, where α and β are the leftmost labels in the two-row notation of this cycle

$$c_0 = \begin{bmatrix} \alpha & \ldots \\ \beta & \ldots \end{bmatrix},$$

the top and bottom, respectively (i.e., $\sigma^*(\beta\mathrm{e}) = \alpha\mathrm{b}$), until this cycle would become two-element. In this process, the interval exchange $(\sigma, \mathbf{v})$ is transformed under the action of reverse induction steps, $(\Pi^{\mathrm{re}}_{\alpha\beta})^{-1}$ or $(\Pi^{\mathrm{le}}_{\beta\alpha})^{-1}$, according to Theorem 1 in [1]. It is also necessary to check that the scheme σ^* remains an interval exchange scheme (i.e., in the analyzed case, simply remains positive). Indeed, according to Proposition 3, it remains rotational in this case and, hence, the interval exchange $(\sigma, \mathbf{v})$ also remains rotational under the corresponding transformations. We now describe the choice, in each case, of one of two steps, $\Pi^{\mathrm{lb}}_{\alpha\beta}$ or $\Pi^{\mathrm{le}}_{\alpha\beta}$, applied to the dual scheme σ^* (note that the labels α and β vary according to the changes in the transformed cycle c_0; we do not want to make our notation too complicated and preserve the notation $(\sigma, \mathbf{v}),\ \sigma^*,\ c_0,\ \alpha$, and β for the objects that are actually variable in the course of operation of the algorithm). We immediately note that $\alpha \neq \beta$; otherwise, σ^* is not rotational.

The following four situations are now possible:

Situation 1: $\alpha\mathrm{e} \notin c_0$ and $\beta\mathrm{b} \notin c_0$. If this is a two-element cycle, then we merge the corresponding intervals in $(\sigma, \mathbf{v})$, remove this cycle, and pass to the next cycle. If this is not true, then, by Lemma 2, there exists a positive vector of lengths $\mathbf{w}$ allowed by the scheme σ^* and such that $w_\alpha \neq w_\beta$. If $w_\alpha > w_\beta$, then we apply the step $\Pi^{\mathrm{lb}}_{\alpha\beta}$ to σ^*. At the same time, if $w_\alpha < w_\beta$, then we apply the step $\Pi^{\mathrm{le}}_{\alpha\beta}$. This leaves the scheme σ^* positive and, therefore, rotational, thus decreasing the number of elements in the cycle c_0; in the first case, the element $\beta\mathrm{e}$ is moved to the cycle containing $\alpha\mathrm{e}$; in the second case, the element $\alpha\mathrm{b}$ is moved to the cycle containing $\beta\mathrm{b}$.

Situation 2: $\alpha\mathrm{e} \notin c_0$ and $\beta\mathrm{b} \in c_0$. We choose any positive vector of lengths $\mathbf{w}$ allowed by the scheme σ^*. Since the length w_β appears in exactly one equality (3) written for $(\sigma^*, \mathbf{w})$ both from the left and from the right, there are no restrictions imposed on this length and, therefore, it can be replaced by any positive number smaller than w_α (e.g., by setting $w_\beta = w_\alpha/2$). Thus, there exists a positive vector of lengths $\mathbf{w}$ allowed by the scheme σ^* in which $w_\alpha > w_\beta$. We now apply the step $\Pi^{\mathrm{lb}}_{\alpha\beta}$ to σ^*. This leaves the scheme σ^* positive and, hence, rotational, and move the element $\beta\mathrm{e}$ from the cycle c_0 to the cycle that contains $\alpha\mathrm{e}$.

Situation 3: $\alpha\mathrm{e} \in c_0$ and $\beta\mathrm{b} \notin c_0$. This situation is similar to Situation 2 but it is now allowed to arbitrarily change the length w_α in the positive vector of lengths $\mathbf{w}$ allowed by the scheme σ^*. In particular, there exists a vector satisfying the inequality $w_\alpha < w_\beta$. Therefore, the application of the step $\Pi^{\mathrm{le}}_{\alpha\beta}$ leaves the scheme σ^* positive and, therefore, rotational, and moves the element $\alpha\mathrm{b}$ from the cycle c_0 to the cycle containing $\beta\mathrm{b}$.

Situation 4: $\alpha\mathrm{e} \in c_0$ and $\beta\mathrm{b} \in c_0$. Consider a cycle c_0 in the two-row notation. Assume that there are $m \geq 1$ labels in the bottom row to the left of α and $n \geq 1$ labels in the top row to the left of β, i.e.,

$$c_0 = \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n & \beta & \dots \\ \beta_1 & \beta_2 & \dots & \beta_m & \alpha & \dots \end{bmatrix}, \qquad \alpha_1 = \alpha, \quad \beta_1 = \beta.$$

As in Situations 2 and 3, the lengths w_α and w_β in the positive vector of lengths $\mathbf{w}$ allowed by the scheme σ^* can be chosen arbitrarily and, hence, the application of any of the two steps $\Pi^{\mathrm{lb}}_{\alpha\beta}$ or $\Pi^{\mathrm{le}}_{\alpha\beta}$ leaves the scheme σ^* rotational. However, in this case, the number of elements in the cycle c_0 does not decrease. At the same time, if we apply $\Pi^{\mathrm{lb}}_{\alpha\beta}$ (or $\Pi^{\mathrm{le}}_{\alpha\beta}$), then the labels appearing in the bottom row to the left of $\alpha\mathrm{e}$ (in the top row to the left of $\beta\mathrm{b}$) are cyclically rearranged:

$$\Pi^{\mathrm{lb}}_{\alpha\beta_1}: \begin{bmatrix} \alpha & & \dots & & \dots \\ \beta_1 & \beta_2 & \dots & \beta_m & \alpha & \dots \end{bmatrix} \mapsto \begin{bmatrix} \alpha & & \dots & & \dots \\ \beta_2 & \dots & \beta_m & \beta_1 & \alpha & \dots \end{bmatrix},$$

$$\Pi^{\mathrm{le}}_{\alpha_1\beta}: \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n & \beta & \dots \\ \beta & & \dots & & \dots \end{bmatrix} \mapsto \begin{bmatrix} \alpha_2 & \dots & \alpha_n & \alpha_1 & \beta & \dots \\ \beta & & \dots & & \dots \end{bmatrix}.$$

If among the labels $\beta_i,\ 1 < i \leq m$, there is a label such that $\beta_i\mathrm{b} \notin c_0$, then we consecutively apply the induction steps $\Pi^{\mathrm{lb}}_{\alpha\beta_1}, \dots, \Pi^{\mathrm{lb}}_{\alpha\beta_{i-1}}$ and arrive at Situation 3. Similarly, if among the labels $\alpha_j,\ 1 < j \leq n$, there is a label such that $\alpha_j\mathrm{e} \notin c_0$, then we consecutively apply the induction steps $\Pi^{\mathrm{le}}_{\alpha_1\beta}, \dots, \Pi^{\mathrm{le}}_{\alpha_{j-1}\beta}$ and arrive at Situation 2. In both cases, the next step decreases the number of elements in the cycle c_0. We now assume that all elements $\alpha_1\mathrm{e}, \dots, \alpha_n\mathrm{e}$ and all elements $\beta_1\mathrm{b}, \dots, \beta_m\mathrm{b}$ belong to the cycle c_0. The sets $\{\alpha_2, \dots, \alpha_n\}$ and $\{\beta_2, \dots, \beta_m\}$ cannot coincide because, in this case, the scheme σ^* would be either splittable (by the cycle c_0) with subalphabets $\mathcal{A}_1 = \{\alpha_2, \dots, \alpha_n, \alpha, \beta\}$ and $\mathcal{A}_2 = \mathcal{A}\backslash\mathcal{A}_1$ and, hence, not rotational according to Lemma 1 or reducible (if c_0 does not contain elements not included in $\mathcal{A}_1 \times \{\mathrm{b}, \mathrm{e}\}$). Since the scheme σ^* is rotational and irreducible, at least one of the sets $\{\beta_2, \dots, \beta_m\}\backslash\{\alpha_2, \dots, \alpha_n\}$ and $\{\alpha_2, \dots, \alpha_n\}\backslash\{\beta_2, \dots, \beta_m\}$ is nonempty. If the first of these sets is nonempty, i.e., there exists $\beta_i \notin \{\alpha_2, \dots, \alpha_n\},\ 1 < i \leq m$, then we consecutively apply the steps $\Pi^{\mathrm{lb}}_{\alpha\beta_1}, \dots, \Pi^{\mathrm{lb}}_{\alpha\beta_{i-1}}$ and finally arrive at Situation 4 with elevated n (in this case, this is the number of labels in the top row to the left of β_i, which is larger than the former n because β_i is located to the right of β). If the set $\{\beta_2, \dots, \beta_m\}\backslash\{\alpha_2, \dots, \alpha_n\}$ is empty, then the set $\{\alpha_2, \dots, \alpha_n\}\backslash\{\beta_2, \dots, \beta_m\}$ is nonempty and we arrive, in a similar way at Situation 4, but this time with elevated m. Since n and m are bounded, this process cannot be continued infinitely and, therefore, we eventually necessarily arrive at Situations 2 or 3.

In all cases, summarizing the outlined algorithm of actions for a chosen cycle c_0 containing more than two elements, we reduce the number of its elements by one after finitely many induction steps. Hence, if we continue to apply the algorithm to the chosen cycle, then we eventually transform it into a two-element cycle. After this, we apply the operation of merging the corresponding intervals to the interval exchange $(\sigma, \mathbf{v})$ and get a new scheme σ^* with the number of cycles smaller than for the previous scheme by one.

If we continue to choose cycles in σ^* and apply the described algorithm to these cycles, then we successively get rid of them (one by one) and stop when only one cycle remains in the scheme σ^*. According to our construction, the resulting interval exchange (obtained as a result of consecutive application of the corresponding reverse steps of induction and operations of merging intervals to $(\sigma, \mathbf{v})$) is rotational and, as a dynamical system, the original interval exchange is the first return map to the corresponding segments for the resulting interval exchange.

5.7. Canonical Form of the Rotational Interval Exchange. Hence, we reach the situation in which the dual scheme σ^* consists of a single cycle. A specific property of interval exchange with a scheme of this kind is that among all its endpoints there is only one of type L and only one of type R (see their classification in Subsection 4.2), whereas all remaining endpoints are either of type MB or of type ME. Hence, for the interval exchange $(\sigma, \mathbf{v})$, the set of all its endpoints contains only one endpoint of type MB and only one endpoint of type ME, while all other endpoints are of type L or of type R. This means that we have only one site with $\sigma(\alpha\mathrm{b}) = \beta\mathrm{b}$ and only one site with $\sigma(\gamma\mathrm{e}) = \delta\mathrm{e},\ \alpha, \beta, \gamma, \delta \in \mathcal{A}$ (among these four labels, the only possible equalities are $\alpha = \gamma$ or $\beta = \delta$; any other equality is impossible because the schemes σ and σ^* are positive); at all other sites, an ending element is followed by a beginning element, and a beginning element is followed by an ending element.

There are two possible cases: If $\alpha\mathrm{b}$ and $\gamma\mathrm{e}$ belong to two different cycles in the permutation σ, then these cycles have the form $\begin{bmatrix} \alpha & \beta \\ \kappa & \end{bmatrix}$ and $\begin{bmatrix} \lambda & \\ \delta & \gamma \end{bmatrix}$, $\lambda, \kappa \in \mathcal{A}$ (the equality $\lambda = \kappa$ is possible), and all remaining cycles are two-element. In the opposite case, we have a cycle $\begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}$, and all other cycles are two-element.

In view of relations (3), we can easily see that the positivity of the schemes σ and σ^* takes place only if the set of all two-element cycles in σ (in the first case) can be split into the following three finite sequences:

$$\begin{bmatrix} \alpha_{i+1} \\ \alpha_i \end{bmatrix}, \quad 1 \le i < m, \qquad \text{where} \quad \alpha_1 = \alpha, \quad \alpha_m = \gamma, \quad m \ge 1,$$

$$\begin{bmatrix} \beta_{j+1} \\ \beta_j \end{bmatrix}, \quad 1 \le j < n, \qquad \text{where} \quad \beta_1 = \beta, \quad \beta_n = \delta, \quad n \ge 1,$$

$$\begin{bmatrix} \lambda_{k+1} \\ \lambda_k \end{bmatrix}, \quad 1 \le k < s, \qquad \text{where} \quad \lambda_1 = \lambda, \quad \lambda_s = \kappa, \quad s \ge 1$$

(any of these sequences can be empty). In the second case, we have only the first two sequences. A single cycle in the dual scheme σ^* can be written as

$$\begin{bmatrix} \beta_1 & \ldots & \beta_n & \lambda_1 & \ldots & \lambda_s & \alpha_1 & \ldots & \alpha_m \\ \alpha_1 & \ldots & \alpha_m & \lambda_1 & \ldots & \lambda_s & \beta_1 & \ldots & \beta_n \end{bmatrix}$$

in the first case or as

$$\begin{bmatrix} \beta_1 & \ldots & \beta_n & \alpha_1 & \ldots & \alpha_m \\ \alpha_1 & \ldots & \alpha_m & \beta_1 & \ldots & \beta_n \end{bmatrix}$$

in the second case.

Equalities (3) imply, in particular, that $v_\alpha = v_\gamma$ and $v_\beta = v_\delta$ in both cases and yield an additional relation $v_\lambda = v_\kappa = v_\alpha + v_\beta$ in the first case.

The first case $\Big($i.e., the case where a single cycle in σ^* has the form $\begin{bmatrix}\beta_1 & \dots & \beta_n & \lambda_1 & \dots & \lambda_s & \alpha_1 & \dots & \alpha_m\\ \alpha_1 & \dots & \alpha_m & \lambda_1 & \dots & \lambda_s & \beta_1 & \dots & \beta_n\end{bmatrix}$ with $s \geq 1\Big)$, is reduced to the second case by consecutive application of the induction steps $\Pi^{\text{le}}_{\beta_1\alpha_1}, \dots, \Pi^{\text{le}}_{\beta_n\alpha_1}$ to σ^*. As in Situation 4 from the previous section, the labels to the left of α_1 in the top row are cyclically rearranged (in this case, the scheme remains positive and, therefore, rotational, due to the absence of restrictions imposed on lengths) and, finally, form a sequence $\lambda_1, \dots, \lambda_s, \beta_1, \dots, \beta_n$, which coincides with the sequence in the top row to the right of α_m.

Hence, in all cases, we finally get at a situation in which the rotational scheme σ takes the *canonical form*

$$\sigma_{\text{can}} = \left\{ \begin{bmatrix}\alpha_1 & \beta_1\\ \\ \beta_n & \alpha_m\end{bmatrix};\ \begin{bmatrix}\alpha_{i+1}\\ \\ \alpha_i\end{bmatrix},\ 1 \leq i < m;\ \begin{bmatrix}\beta_{j+1}\\ \\ \beta_j\end{bmatrix},\ 1 \leq j < n \right\} \tag{7}$$

for a certain set of pairwise different labels $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$ and certain positive integers m and n. Taken together with an allowed positive vector of lengths $\mathbf{v} = \mathbf{v}_{\text{can}}$, the rotational scheme in the canonical form constitutes a rotational interval exchange $(\sigma_{\text{can}}, \mathbf{v}_{\text{can}})$ in the *canonical form.* The components of the vector of lengths satisfy the relations $v_{\alpha_1} = \dots = v_{\alpha_m}$ and $v_{\beta_1} = \dots = v_{\beta_m}$. We denote these two lengths simply by v_α and v_β, respectively (it is not impossible for them to be equal).

Thus, in the last two sections, we have, in fact, presented a constructive proof of the following proposition:

Proposition 5. *Any irreducible rotational interval exchange can be transformed into the canonical form by the consecutive application of finitely many reverse elementary steps of induction and operations of merging intervals. Moreover, at each step of this process, the transformed IRE remains an irreducible rotational interval exchange.*

Remark 4. As follows from the presented algorithm, in order to transform a rotational interval exchange into the canonical form it is, in fact, sufficient to restrict ourselves to the use only of reverse elementary steps of induction of two types, namely $(\Pi^{\text{le}}_{\alpha\beta})^{-1}$ and $(\Pi^{\text{re}}_{\alpha\beta})^{-1}$. Similarly, it would be sufficient to apply only the other two types of reverse elementary steps of induction, namely, $(\Pi^{\text{lb}}_{\alpha\beta})^{-1}$ and $(\Pi^{\text{rb}}_{\alpha\beta})^{-1}$. According to Theorem 1 in [1], in the application to the dual rotational scheme, the indicated two reverse steps correspond to the elementary induction steps $\Pi^{\text{re}}_{\alpha\beta}$ and $\Pi^{\text{rb}}_{\beta\alpha}$, respectively.

5.8. Construction on a Circle. For the rotational interval exchange $(\sigma_{\text{can}}, \mathbf{v}_{\text{can}})$ in the canonical form (7) obtained from the original rotational interval exchange $(\sigma, \mathbf{v})$, we take sufficiently large integers k_1 and k_2 and construct a circle rotation $R_{L,M}$ with $M = v_\beta + k_2 v_\alpha$ and $L = v_\alpha + k_1 M$. This circle rotation is considered in its projection onto the segment $[-v_\alpha, k_1 M)$ according to (6) with $x_0 = -v_\alpha$, i.e, as the map

$$R_{L,M} \colon x \mapsto \begin{cases} x + M, & x \in [-v_\alpha, (k_1 - 1)M), \\ \\ x + M - L, & x \in [(k_1 - 1)M, k_1 M). \end{cases}$$

We now mark on $[-v_\alpha, k_1 M)$ the points $a_i = R^i_{L,M}(0)$, $0 \leq i < q$, of a trajectory segment of length $q = 1 + k_1 + k_2 k_1$ starting from the point $a_0 = 0$ under the action of $R_{L,M}$. It is easy to see that the indicated $1 + k_1 + k_2 k_1$ points are ordered as follows (from left to right): $a_{k_1} = -v_\alpha$, $a_0 = 0$, then we place the array of k_2+1 points $a_{k_2k_1+1} = v_\beta,\ a_{(k_2-1)k_1+1} = v_\beta + v_\alpha, \ldots, a_{k_1+1} = v_\beta + (k_2-1)v_\alpha,\ a_1 = M$ and then, consecutively, k_1-1 more arrays of this kind shifted by $1 \le j < k_1$ rotations $R_{L,M}$, i.e., the arrays of k_2+1 points $a_{k_2k_1+1+j} = v_\beta + jM,\ a_{(k_2-1)k_1+1+j} = v_\beta + v_\alpha + jM, \ldots, a_{k_1+1+j} = v_\beta + (k_2-1)v_\alpha + jM,\ a_{1+j} = (1+j)M$, where the last point $a_{1+(k_1-1)} = k_1M$ in the last array (for $j = k_1-1$) is the right endpoint of the segment $[-v_\alpha, k_1M)$ whose projection coincides with its left endpoint $a_{k_1} = -v_\alpha$ already included in the list. We also note that $a_q = R^q_{L,M}(0) = v_\beta - v_\alpha$.

In view of this order and the equalities $a_{k_1} = -v_\alpha,\ a_0 = 0$, and $a_{k_2k_1+1} = v_\beta$, it is easy to see that the points $a_i,\ 0 \le i < q$, split the circle $[-v_\alpha, k_1M)$ into k_1 arcs of length v_β, namely, the arcs $R^i_{L,M}[0, v_\beta),\ 0 \le i < k_1$, and k_2k_1+1 arcs of length v_α, namely, the arcs $R^i_{L,M}[-v_\alpha, 0),\ 0 \le i < k_2k_1+1$. Any two of these $1+k_1+k_2k_1$ arcs do not overlap, and their union covers the entire circle. Moreover, the arcs

$$R^{k_1}_{L,M}[0, v_\beta) = [a_{k_1}, a_q) = [-v_\alpha, v_\beta - v_\alpha)$$

and

$$R^{k_2k_1+1}_{L,M}[-v_\alpha, 0) = [a_q, a_{k_2k_1+1}) = [v_\beta - v_\alpha, v_\beta)$$

also do not overlap and there union is the arc $[-v_\alpha, v_\beta)$, which is also the union of the arcs $[0, v_\beta)$ and $[-v_\alpha, 0)$.

Having in mind this construction realized on the circle, we select an arc $[-v_\alpha, v_\beta)$, any $m-1$ arcs among $R^i_{L,M}[-v_\alpha, 0),\ 0 \le i < k_2k_1+1$, and any $n-1$ arcs among $R^i_{L,M}[0, v_\beta),\ 0 \le i < k_1$, in such a way that there are no pairs of selected arcs touching by their endpoints (this is, clearly, possible if k_1 and k_2 are sufficiently large). According to our construction, the dynamical system determined by the first return map for the circle rotation $R_{L,M}$ to the chosen union of $n+m-1$ arcs is identical to the dynamical system of the rotational interval exchange in the canonical form $(\sigma_{\mathrm{can}}, \mathbf{v}_{\mathrm{can}})$. Since the indicated interval exchange in the canonical form is obtained by the consecutive application of reverse induction steps and operations of merging intervals to the original rotational interval exchange $(\sigma, \mathbf{v})$, the dynamical system of this interval exchange is, in turn, determined by the first return map to a certain finite union of segments in the phase space of the dynamical system $(\sigma_{\mathrm{can}}, \mathbf{v}_{\mathrm{can}})$. If we choose a union of arcs on the circle corresponding to the indicated union of segments, then we get a finite union of arcs such that the first return map to this union for the circle rotation $R_{L,M}$ is shift equivalent to the original irreducible rotational interval exchange $(\sigma, \mathbf{x})$.

Statement 2 of Theorem 1 is thus proved.

6. Proof of Statement 3 of Theorem 1

In the third part of Theorem, we formulate a criterion for an interval exchange scheme to be rotational in terms of the first return map to the union of arcs for an irrational circle rotation. In fact, this statement almost follows from the first two statements of the theorem, which have been already proved. Indeed, the first statement implies that if the indicated first return map exists, then the corresponding irreducible IRE scheme is rotational. The second statement implies that, for an irreducible rotational scheme there exists a first return map with this scheme. Actually, it remains to show that, for an irreducible rotational scheme, there exists the required first return map just for the irrational circle rotation. To do this, we return to the algorithm of transformation of a rotational interval exchange to the canonical form used in the previous section.

Thus, we assume that an irreducible rotational interval exchange scheme σ is given. To this scheme, we add an arbitrary allowed vector of lengths $\mathbf{v}$ and obtain a rotational interval exchange $(\sigma, \mathbf{v})$. It is transformed into the canonical form $(\sigma_{\mathrm{can}}, \mathbf{v}_{\mathrm{can}})$ according to Proposition 5. In Section 5.8, it is shown that the canonical interval exchange with lengths v_α and v_β is the first return map for the circle rotation $R_{L,M}$ with $M = v_\beta + k_2v_\alpha$ and $L = v_\alpha + k_1M$ for certain positive integers k_1 and k_2. The rotation number of this circle rotation $\rho = M/L = 1/(k_1 + 1/(k_2 + \rho_0))$, where $\rho_0 = v_\beta/v_\alpha$, is either rational, or irrational, depending on rationality or irrationality of the number ρ_0. Hence, if the lengths v_α and v_β are incommensurable, then the required irrational circle rotation is already constructed.

Assume that the lengths v_α and v_β are commensurable, i.e., that $\rho_0 = v_\beta/v_\alpha \in \mathbb{Q}$. In this case, we simply change these lengths by adding small perturbations guaranteeing that they are no longer commensurable, e.g., by replacing the lengths v_β with $v'_\beta = v_\beta + \varepsilon$, where $0 < \varepsilon \ll 1$, $\varepsilon/v_\beta \notin \mathbb{Q}$. Further, we realize the entire algorithm of transformation into the canonical form, but in the opposite direction. The interval exchange $(\sigma_{\mathrm{can}}, \mathbf{v}_{\mathrm{can}})$ was obtained from $(\sigma, \mathbf{v})$ by applying finitely many reverse steps of induction and operations of merging intervals. Thus, moving backward, we consecutively apply the corresponding direct induction steps and operations of splitting the intervals (into specified parts). Clearly, each of these operations is robust in a sense that, after its application, small perturbations of the real components (lengths) of an interval exchange remain small and, therefore, discrete components (schemes) remain unchanged. Hence, starting from the perturbed (as indicated above) canonical interval exchange $(\sigma_{\mathrm{can}}, \mathbf{v}'_{\mathrm{can}})$ and applying the algorithm of transformation in the opposite direction, we obtain a perturbed interval exchange $(\sigma, \mathbf{v}')$ with the same scheme σ specified at the beginning, and this perturbed interval exchange is actually the first return map for the irrational circle rotation with rotation number $\rho' = 1/(k_1 + 1/(k_2 + \rho'_0))$, where $\rho'_0 = (v_\beta + \varepsilon)/v_\alpha \notin \mathbb{Q}$.

Statement 3 of Theorem 1 is proved.

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REFERENCES

1. A. Teplinsky, "Interval rearrangement ensembles," *Ukr. Math. Zh.*, **75**, No. 2, 247–268 (2023); ***English translation:*** *Ukr. Math. J.*, **75**, No. 2, 282–304 (2023).
2. M. Keane, "Interval exchange transformation," *Math. Z.*, **141**, 25–31 (1975).
3. W. A. Veech, "Interval exchange transformations," *Anal. Math.*, **33**, 222–272 (1978).
4. G. Rauzy, "Échanges d'intervalles et transformations induites," *Acta Arith.*, **34**, 315–328 (1979).
5. M. S. Keane and G. Rauzy, "Stricte ergodicité des échanges d'intervalles," *Math. Z.*, **174**, 203–212 (1980).
6. H. Masur, "Interval exchange transformations and measured foliations," *Ann. Math. (2)*, **115**, 169–200 (1982).
7. W. A. Veech, "Gauss measures for transformations on the space of interval exchange maps," *Ann. Math. (2)*, **115**, 201–242 (1982).
8. A. Teplinsky, "Hyperbolic horseshoe for circle diffeomorphisms with break," *Nonlin. Oscillat.*, **11**, No. 1, 114–134 (2008).
9. K. Khanin and A. Teplinsky, "Renormalization horseshoe and rigidity for circle diffeomorphisms with breaks," *Comm. Math. Phys.*, **320**, 347–377 (2013).
10. K. M. Khanin and E. B. Vul, "Circle homeomorphisms with weak discontinuities," *Dynamical Systems and Statistical Mechanics*, Moscow, 57–98 (1991); ***English translation:*** *Adv. Soviet Math.*, **3**, American Mathematical Society, Providence, RI (1991).
11. K. Cunha and D. Smania, "Renormalization for piecewise smooth homeomorphisms on the circle," *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **30**, 441–462 (2013).
12. K. Cunha and D. Smania, "Rigidity for piecewise smooth homeomorphisms on the circle," *Adv. Math.*, **250**, 193–226 (2014).