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ON THE NONSTANDARD MAXIMUM PRINCIPLE AND ITS APPLICATION FOR CONSTRUCTION OF MONOTONE FINITE-DIFFERENCE SCHEMES FOR MULTIDIMENSIONAL QUASILINEAR PARABOLIC EQUATIONS

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We consider the difference maximum principle with input data of variable sign and its application to the investigation of the monotonicity and convergence of finite-difference schemes (FDSs). Namely, we consider the Dirichlet initial-boundary-value problem for multidimensional quasilinear parabolic equations with unbounded nonlinearity. Unconditionally monotone linearized finite-difference schemes of the second-order of accuracy are constructed on uniform grids. A two-sided estimate for the grid solution, which is completely consistent with similar estimates for the exact solution, is obtained. These estimates are used to prove the convergence of FDSs in the grid L_2-norm. We also present a study aimed at constructing second-order monotone difference schemes for the parabolic convection-diffusion equation with boundary conditions of the third kind and unlimited nonlinearity without using the initial differential equation on the domain boundaries. The goal is a combination of the assumption of existence and uniqueness of a smooth solution and the regularization principle. In this case, the boundary conditions are directly approximated on a two-point stencil of the second order.

1. Introduction

In some cases, the Maximum Principle permits one not only to determine the uniqueness of a solution, which continuously depends on the input data for elliptic and parabolic equations but also to get estimates for the uniform norm of the solution with *a priori* upper bounds for the analyzed problems of arbitrary dimension with nonself-adjoint elliptic operator, which is the opposite of the energy inequality method [35, p. 500]. Moreover, the Maximum Principle helps us to establish the consistency of the finite-difference solution with the input data and its convergence in studying the uniform norm in the theory of finite-difference schemes. Monotonicity is widely recognized as a finite-difference method satisfying the grid maximum principle [32, p. 228; 33, p. 296]. In order to solve problems related to multidimensional linear convection diffusion equations (see, e.g., [34, p. 35]), various classes of monotone finite-difference schemes have been improved. The well-conditioning of the systems of algebraic equations based on the monotone difference schemes made these schemes extremely important for computational practice [9]. In addition, the monotone difference schemes enable us to get numerical approximations without oscillations, even in the case of nonsmooth (including discontinuous) solutions [8, 31]. Moreover, preliminary estimates of the error in the uniform norm can be obtained. Numerous specialized works available from the literature were focused on the construction and analysis of monotone FDS for linear partial differential equations of mathematical physics with different boundary conditions; see, e.g., the monographs [32, 34] and

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articles [17, 20, 30]. As for the investigation of monotonous FDS in the nonlinear case, we can especially mention the works [7, 8, 10, 19, 21, 22, 27–29].

In the investigations of the properties of numerical methods used for the solution of problems with unbounded nonlinearity, it is necessary to show that the position of the solution grid must be in the vicinity of values of the exact solution [16, 18]. Thus, the importance of lower (or, in general, two-sided) estimates of the solutions of differential-difference problems is equally widely accepted. In the linear case, by using these estimates, it is possible to find the range of values of the desired solution to the input data problem (the coefficients and right-hand side of the equation, as well as the initial and boundary conditions). Thus, these estimates allow us to find the range of values of the required solution to the input data problem in the linear case (the coefficients and right-hand side of the equation, as well as the initial and boundary conditions). This enables one both to demonstrate the nonnegative solution, which is significant for the physical cases, and to identify conditions imposed on the input data for elliptic or parabolic equations. A good example is that the gamma equation [10–12] was obtained by transforming the nonlinear Black–Scholes equation for the option price into the quasilinear parabolic equation for the second derivative of the option price in financial mathematics and can be considered by one. In this case, it is acceptable that the estimates should be as sharp as possible. Accordingly, a noteworthy technique associated with changes in variables and minimization (or maximization) of some functions depending on a parameter was mentioned in the classical monograph [14, p. 22].

The theory of finite-difference schemes [32, p. 229] is based on the method, advanced for linear problems, of the grid maximum principle, which provides two-sided estimates for approximate solutions. The last estimates for the solutions of finite-difference problems are less precise [3] than the corresponding estimates for the solutions of differential problems [14, p. 22]. Farago, et al. identified similar estimates for the finite-element technique for linear problems and problems with bounded nonlinearity (see, e.g., [2, 4]). Note that, to state the grid maximum principle, it is normally required that the input data of the problem must have a constant sign. However, the advanced two-sided estimation of the grid solution for the input data of the problem in the absence of assumption concerning its constant sign was applied in [23, 24] to establish a generalized finite-difference scheme in the canonical form. Note that the method recommended by O. A. Ladyzhenskaya in her pioneering study [13] was enhanced and applied in [26] to get two-sided estimates for the solution of the grid schemes, as being entirely consistent with the differential problem.

The main features of convection-diffusion problems, which are fundamental in continuum mechanics, are associated with the fact that their operators may have an indeterminate sign. The finite-difference schemes formed from the simplest approximations of the first derivative by using directional differences on homogenous grids for the one-dimensional convection-diffusion equation are unconditionally monotonic but, as a rule, only for the first order of approximations. In addition, they are constructed to approximate the convective term by using central difference relations. Despite the quadratic approximation, the monotonicity property of these schemes is in perfect agreement with the restrictions on grid steps in the space. With the help of the regularization principle [32, 34] on uniform grids, unconditionally monotone finite-difference schemes of quadratic approximation were constructed for convection-diffusion problems. The principle of regularization on uniform grids also plays a significant role in constructing unconditionally monotone finite-difference schemes of quadratic approximation for convection-diffusion problems.

It is essential to preserve quadratic accuracy when constructing monotonic difference schemes approximating the parabolic convection-diffusion equation with boundary conditions of the third type. The application of the initial differential equation on the boundary of the region (e.g., in the case of a p-dimensional parallelepiped; see [1, 6]) often causes an increase in the order of approximation of the boundary conditions. On the other hand, it is not easy to demonstrate the convergence in the uniform norm to quadratic with the help of this classical approach. Therefore, an approach proposed in [15] was used to form monotone finite-difference schemes for linear differential problems with boundary conditions of the second and third types without applying the main differential equation on the boundary of the region, which preserves both the quadratic approximation and accuracy.

The boundary conditions were approximated to quadratic on a two-point stencil according to the main idea based on the hypothesis of existence and uniqueness of a smooth solution in some sufficiently minor neighborhood of the domain of definition of the problem. If the equation is assumed to be significant at the boundary nodes, then the fourth-order difference schemes can be also established on homogenous grids [15]. Furthermore, the problems of existence and uniqueness of continuous solutions of the problem in some sufficiently small neighborhoods of the domain of definition are not discussed here because they deserve to be considered separately, e.g., on the basis of the famous Cauchy–Picard theorems [5].

In the present study, we construct monotone difference schemes of quadratic accuracy on uniform grids for multidimensional quasilinear parabolic equations with unbounded nonlinearity [25]. It is crucial to demonstrate the relationships between the exact solutions and neighboring approximate solutions in the theoretical study of the properties of difference solutions in the unbounded nonlinear case. By using these connections, in the present work, we establish two-sided estimates of the numerical solution with regard for the input data of the problem. The evidence of monotonicity of the difference solution and the *a priori* estimates obtained in the maximum norm are based on the subsequent development of the maximum-principle technology for any nonsign-constant input data of the problem. Based on the further expansion of the maximum-principle approach for all input data of the problem that are not sign-constant [18, 23, 24, 31], the evidence of monotonicity of the difference solution and the *a priori* estimates was obtained in the maximum norm. A new second-order monotone finite-difference scheme has been also proposed by applying the regularization principle. This scheme approximates the initial-boundary value problem (IBVP) for multidimensional parabolic convection-diffusion equation with boundary condition of the third type and unbounded nonlinearity on the basis of the assumption of existence and uniqueness of a smooth solution in some sufficiently small neighborhood of the domain of definition of the problem. A significant drawback of this approach is that it cannot be applied in the case of nonsmooth input data. Moreover, this method fails to get *a priori* information about the approximate solution at fictitious grid nodes lying outside the domain of the problem. As a result, the monotonicity of the scheme, as well as the two-sided and *a priori* estimates of the approximate solution depending only on the initial and boundary conditions and on the right-hand side are demonstrated.

The paper is organized as follows. In Section 2, we formulate the maximum principle for difference schemes with variable-sign input data and its application to a weakly coupled system of two linear parabolic equations. In Section 3, we establish two-side estimates for the exact solution to a multidimensional quasilinear parabolic equation. On uniform grids, we construct monotone linearized difference schemes of the second-order accuracy and establish two-sided estimates of the difference solution entirely consistent with the estimation of solutions of the corresponding differential problem in Section 4. The proof of quadratic convergence of the finite-difference solution in the grid L_2-norm is given in Section 5 and based on the energy inequality method. Finally, in Section 6, we formulate the IBVP for a multidimensional quasilinear parabolic equation of convection-diffusion type with boundary condition of the third type and indicate the intrinsic properties of the problem with unlimited nonlinearity.

2. Maximum Principle for Difference Schemes with Variable-Sign Input Data

Assume that finitely many points of the grid Ω_h are given in the n-dimensional Euclidean space. Each point $x \in \Omega_h$ is associated with one and only one stencil $\mathcal{M}(x)$, which is a subset of Ω_h, containing this point. The set $\mathcal{M}'(x) = \mathcal{M}(x) \setminus x$ is called a *neighborhood* of the point x. Let the functions $A(x)$, $B(x,\xi)$, and $F(x)$ be given for $x \in \Omega_h$ and $\xi \in \Omega_h$ and take real values. Further, each point $x \in \Omega_h$ corresponds to one and only one equation of the form [32]

$$A(x)y(x) = \sum_{\xi \in \mathcal{M}'(x)} B\left(x,\xi\right) y\left(\xi\right) + F(x), \quad x \in \Omega_h, \tag{1}$$

which is called the *canonical form* of the finite-difference scheme [32, p. 226]. We assume that the coefficients of this equation satisfy the following ordinary positivity conditions:

$$A(x) > 0, \qquad B\,(x,\xi) > 0 \quad \text{for all} \quad \xi \in \mathcal{M}'(x), \tag{2}$$

$$D(x) = A(x) - \sum_{\xi \in \mathcal{M}'(x)} B(x,\xi) > 0. \tag{3}$$

Lemma 1 [18, 23, 24, 31]**.** *Assume that conditions (2) and (3), guaranteeing that the coefficients are positive, are satisfied. Then the maximum and minimum values of the solution of the finite-difference scheme (1) belong to the following range of input data:*

$$\min_{x\in\Omega_h} \frac{F(x)}{D(x)} \le y(x) \le \max_{x\in\Omega_h} \frac{F(x)}{D(x)}, \quad x \in \Omega_h. \tag{4}$$

Corollary 1 [32, p. 231]**.** *Assume that the conditions of Lemma 1 are satisfied. Then, in the grid analog of the C-norm, the solution of finite-difference problem (1) satisfies the estimate:*

$$\|y\|_C = \max_{x\in\Omega_h} \left|y(x)\right| \le \left\|\frac{F}{D}\right\|_C.$$

3. Statement of the Problem and Two-Sided Estimate of the Exact Solution

Consider the following problem for a quasilinear parabolic equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(u)\frac{\partial u}{\partial x_\alpha}\right) + f(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{5}$$

in the parallelepiped $\bar{Q}_T = \bar{\Omega} \times [0,T]$, where

$$\Omega = \left\{x = (x_1, x_2, \ldots, x_p) \colon 0 < x_\alpha < l_\alpha,\ \alpha = 1,2,\ldots,p\right\},$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{6}$$

and the Dirichlet boundary conditions

$$u(x,t) = \mu(x,t), \quad (x,t) \in \partial\Omega \times [0,T]. \tag{7}$$

Assume that the functions $k_\alpha = k_\alpha(u),\ \alpha = 1,2,\ldots,p$, are sufficiently smooth, the functions $f,\ u_0$, and μ are continuous, and the corresponding matching conditions are satisfied. Let $u(x,t)$ be a solution of problem (5)–(7) and let $D_u = [m_1, m_2]$ be a closed interval containing the range of the solution, i.e., $m_1 \le u(x,t) \le m_2$. Since the functions $k_\alpha = k_\alpha(u),\ \alpha = 1,2,\ldots,p$, are smooth, we conclude that there exist constants $k_{\alpha,1},\ k_{\alpha,2}$, and L_α such that

$$\left|k'_\alpha(u)\right| \le L_\alpha, \quad 0 < k_{\alpha,1} \le k_\alpha(u) \le k_{\alpha,2}, \qquad u \in D_u, \quad (x,t) \in \bar{Q}_T, \quad \alpha = 1,2,\ldots,p. \tag{8}$$

In what follows, we assume that there exists a unique solution of problem (5)–(7) and that the desired function has continuous bounded derivatives of the order required to proceed with our presentation. We set

$$\bar{Q}_{t_1} = \big\{(x,t) \in \bar{Q}_T : \, t \le t_1\big\}.$$

Then the following assertion is true:

Theorem 1. *The classical solution* $u(x,t)$ *of problem (5)–(7) satisfies the following two-sided estimate for every* $t_1 \in [0,T]$ *:*

$$u(x,t_1) \ge m_1 = \sup_{\lambda>0} \min\left\{0,\ \min_{\bar{Q}_{t_1}} \{\mu(x,t), u_0(x)\} e^{\lambda(t_1-t)},\ \frac{1}{\lambda} \min_{\bar{Q}_{t_1}} \left(f(x,t) e^{\lambda(t_1-t)}\right)\right\}, \tag{9}$$

$$u(x,t_1) \le m_2 = \inf_{\lambda>0} \max\left\{0,\ \max_{\bar{Q}_{t_1}} \{\mu(x,t), u_0(x)\} e^{\lambda(t_1-t)},\ \frac{1}{\lambda} \max_{\bar{Q}_{t_1}} \left(f(x,t) e^{\lambda(t_1-t)}\right)\right\}. \tag{10}$$

We outline the proof of the upper bound (10) in a form convenient for our subsequent presentation because similar arguments will be used in what follows in the finite-difference case. To prove (10), we transform the function $u(x,t)$ into a new function $v(x,t)$ given by the formula

$$u(x,t) = v(x,t) e^{\lambda t},$$

where λ is an arbitrary number. The function $v(x,t)$ satisfies the equation

$$\frac{\partial v}{\partial t} + \lambda v - \sum_{\alpha=1}^{p} k_\alpha \left(v e^{\lambda t}\right) \frac{\partial^2 v}{\partial x_\alpha^2} - \sum_{\alpha=1}^{p} \frac{\partial k_\alpha \left(v e^{\lambda t}\right)}{\partial x_\alpha} \frac{\partial v}{\partial x_\alpha} = f e^{-\lambda t} \tag{11}$$

with the following initial and boundary conditions:

$$v(x,0) = u_0(x), \quad x \in \Omega,$$

$$v(x,t) = \mu(x,t) e^{-\lambda t}, \quad (x,t) \in \partial\Omega \times [0,T].$$

We take an arbitrary t_1 from $(0,T)$. The following three cases are possible for the function $v(x,t)$:

(i) $\max_{\bar{Q}_{t_1}} v(x,t)$ is nonpositive (i.e., $v(x,t) \le 0,\ (x,t) \in \bar{Q}_{t_1}$);

(ii) $\max_{\bar{Q}_{t_1}} v(x,t)$ is located either on the base $t=0$ or on the boundary (i.e., the following inequality holds:

$$v(x,t) \le \max_{\bar{Q}_{t_1}} e^{-\lambda t} \big\{\mu_1(t), \mu_2(t), u_0(x)\big\}, \qquad (x,t) \in \bar{Q}_{t_1}\Big);$$

(iii) a positive maximum is attained at an interior point $\left(x^0, t^0\right) \in \Omega \times (0, t_1]$:

$$\max_{\bar{Q}_{t_1}} v(x,t) = v\big(x^0, t^0\big).$$

In the last case, by using the relations at the maximum point $\left(x^0, t^0\right)$, i.e.,

$$\frac{\partial v\left(x^0, t^0\right)}{\partial t} \geq 0, \qquad \frac{\partial v\left(x^0, t^0\right)}{\partial x_\alpha} = 0, \qquad \frac{\partial^2 v\left(x^0, t^0\right)}{\partial x_\alpha^2} \leq 0, \quad \alpha = 1, 2, \ldots, p,$$

from Eq. (11), we obtain

$$v(x,t) \leq v\left(x_0, t_0\right) \leq \frac{f\left(x_0, t_0\right) e^{-\lambda t^0}}{\lambda} \leq \max_{\bar{Q}_{t_1}} \frac{f(x,t) e^{-\lambda t}}{\lambda}, \quad \lambda > 0.$$

Combining cases (i)–(iii) and returning to the original function u, we get the upper bound (10). Similar arguments used for the minimum point give us the lower estimate (9).

4. Two-Sided Estimates for the Solutions of Finite-Difference Schemes

In the parallelepiped $\bar{Q}_T$, we introduce the uniform grid $\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau$,

$$\bar{\omega}_\tau = \left\{t_n = n\tau,\ n = 0, 1, \ldots, N_0,\ \tau N_0 = T\right\}, \quad \bar{\omega}_h = \omega_h \cup \gamma_h,$$

where the set of inner nodes is defined by

$$\omega_h = \left\{x = \left(x_1^{(i_1)}, x_2^{(i_2)}, \ldots, x_p^{(i_p)}\right) : x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha,\ h_\alpha N_\alpha = l_\alpha,\ i_\alpha = \overline{1, N_\alpha - 1},\ \alpha = \overline{1, p}\right\}$$

and γ_h is the set of boundary nodes.

Further, we use the following notation from the theory of difference schemes [32]:

$$v^{(\pm 1_\alpha)} = v\left(x_1^{(i_1)}, \ldots, x_{\alpha-1}^{(i_\alpha - 1)}, x_\alpha^{(i_\alpha)} \pm h_\alpha, x_{\alpha+1}^{(i_\alpha + 1)}, \ldots, x_p^{(i_p)}, t_n\right),$$

$$y = y(x, t_n), \qquad y_t = \frac{\hat{y} - y}{\tau}, \quad \hat{y} = y\left(x, t_{n+1}\right),$$

$$y_{\bar{x}_\alpha} = \frac{y - y^{(-1_\alpha)}}{h_\alpha}, \qquad y_{x_\alpha} = \frac{y^{(+1_\alpha)} - y}{h_\alpha}.$$

On the uniform grid $\bar{\omega}$ considered in the domain $\bar{Q}_T$, we approximate the differential problem (5)–(7) by a difference scheme

$$\begin{gathered} y_t = \sum_{\alpha=1}^{p} \left(a_\alpha(y) \hat{y}_{\bar{x}_\alpha}\right)_{x_\alpha} + \hat{f}, \\ y(x,0) = u_0(x), \quad x \in \bar{\omega}_h, \qquad \hat{y}|_{\gamma_h} = \mu(x,t), \qquad x \in \gamma_h, \quad t \in \omega_\tau. \end{gathered} \tag{12}$$

As usual, the stencil functionals

$$a_\alpha(y) = 0.5\left(k_\alpha\left(y^{(-1_\alpha)}\right) + k_\alpha(y)\right), \quad \alpha = 1, 2, \ldots, p, \tag{13}$$

are chosen from the second-order consistency condition [32, p. 140]

$$\big(a_\alpha(u)\hat{u}_{\bar{x}_\alpha}\big)_{x_\alpha} - \frac{\partial}{\partial x_\alpha}\left(k_\alpha(u)\frac{\partial u}{\partial x_\alpha}\right) = O\big(h_\alpha^2 + \tau\big)$$

for the elliptic operator with respect to the space variables.

Theorem 2. *The solution $y(x,t)$ of problem (12) satisfies the following two-sided estimate at any point $(x,t_n)\in\omega$:*

$$y(x,t_n) \ge m_1^n = \sup_{\lambda>0}\min\left\{0,\ \min_{\omega_{t_n}} e^{\lambda(t_n-t)}\big\{\mu(x,t),u_0(x)\big\},\ \frac{\tau}{e^{\lambda\tau}-1}\min_{\omega_{t_n}} f(x,t)e^{\lambda(t_n-t)}\right\}, \tag{14}$$

$$y(x,t_n) \le m_2^n = \inf_{\lambda>0}\max\left\{0,\ \max_{\omega_{t_n}} e^{\lambda(t_n-t)}\big\{\mu(x,t),u_0(x)\big\},\ \frac{\tau}{e^{\lambda\tau}-1}\max_{\omega_{t_n}} f(x,t)e^{\lambda(t_n-t)}\right\}. \tag{15}$$

Proof. We prove the upper bound (15). To do this, we rewrite the finite-difference scheme (12) in the canonical form (1)

$$A\hat{y} = \sum_{\alpha=1}^{p}\left(B_\alpha\hat{y}^{(+1_\alpha)} + C_\alpha\hat{y}^{(-1_\alpha)}\right) + F, \quad (x,t)\in\omega.$$

The coefficients of the canonical form are as follows:

$$B_\alpha = \frac{\tau}{h_\alpha^2}a_\alpha^{(+1_\alpha)}(y), \qquad a_\alpha^{(+1_\alpha)}(y) = 0.5\left(k_\alpha(y) + k_\alpha\left(y^{(+1_\alpha)}\right)\right),$$

$$C_\alpha = \frac{\tau}{h_\alpha^2}a_\alpha(y), \qquad A = 1 + \sum_{\alpha=1}^{p}(B_\alpha + C_\alpha), \qquad F = y + \tau\hat{f}.$$

The difference schemes satisfying the maximum principle are called monotone schemes. We now show that the developed scheme is monotone. To this end, it is necessary to prove that $y\in D_u$ with the help of an auxiliary grid function

$$z(x,t_n) = ye^{-\lambda t_n}, \quad \lambda\neq 0.$$

The function $z(x,t_n)$ satisfies the following finite-difference equation:

$$\bar{A}\hat{z} = \sum_{\alpha=1}^{p}\left(\bar{B}_\alpha\hat{z}^{(+1_\alpha)} + \bar{C}_\alpha\hat{z}^{(-1_\alpha)}\right) + Kz + \bar{F}, \quad (x,t)\in\omega,$$

where

$$\bar{B}_\alpha = e^{\lambda\tau}B_\alpha, \qquad \bar{C}_\alpha = e^{\lambda\tau}\bar{C}_\alpha, \quad K = 1,$$

$$\bar{A} = e^{\lambda\tau} + \sum_{\alpha=1}^{p}\big(\bar{B}_\alpha + \bar{C}_\alpha\big), \qquad \bar{F} = \tau\hat{f}e^{-\lambda t_n}.$$

We introduce the coefficients $\bar{D}$ as follows:

$$\bar{D} = \bar{A} - K - \sum_{\alpha=1}^{p} \left(\bar{B}_\alpha + \bar{C}_\alpha\right) = e^{\lambda\tau} - 1 > 0$$

for all $\lambda\tau > 0$. We take an arbitrary $t_n \in \omega_\tau$. The following three cases are possible for the function $z(x,t)$:

(i) $\max_{\omega_{t_n}} z(x,t)$ is nonpositive (i.e., $z(x,t) \le 0,\ (x,t) \in \omega_{t_n}$);

(ii) $\max_{\omega_{t_n}} z(x,t)$ is located either on the base $t = 0$ or on the boundary (i.e., the inequality $z(x,t) \le \max_{\omega_{t_n}} e^{-\lambda t}\{\mu(x,t), u_0(x)\},\ (x,t) \in \omega_{t_n}$, holds);

(iii) a positive maximum is attained at an interior point $\left(x^0, t^0\right)$: $z(x,t) \le z\left(x^0, t^0\right) = \max_{\omega_{t_n}} z(x,t)$.

Obviously, for $n = 0$, we have $y^0 = u_0 \in D_u$. Assume that, for any n, the inclusion $y = y^n \in D_u$ is also true. We need to prove that $\hat{y} = y^{n+1} \in D_u$ is true. From this assumption, we obtain $\bar{A} > 0,\ \bar{B} > 0$, and $\bar{C} > 0$. According to Lemma 1, in view of estimate (4) in case (iii), we get

$$z(x,t_n) \le z(x^0,t^0) \le \frac{\tau}{e^{\lambda\tau} - 1} f(x^0,t^0) e^{-\lambda t^0} \le \max_{\omega_{t_n}} \frac{\tau f(x,t) e^{-\lambda t}}{e^{\lambda\tau} - 1}, \quad \lambda > 0.$$

Thus, in all cases (i)–(iii), the function $z(x,t)$ satisfies the estimate

$$z(x,t_n) \le \max\left\{0,\ \max_{\omega_{t_n}} e^{-\lambda t}\{\mu(x,t), u_0(x)\},\ \max_{\omega_{t_n}} \frac{\tau f(x,t) e^{-\lambda t}}{e^{\lambda\tau} - 1}\right\},$$

and, hence, this implies that the upper bound (15) is true. In a similar way, we obtain the lower bound (14). Since

$$\frac{\tau}{e^{\lambda\tau} - 1} \le \frac{1}{\lambda} \quad \text{for all} \quad \lambda, \tau > 0,$$

we conclude that estimates (9), (10) and (14), (15) imply the inequalities $m_1 \le m_1^n,\ m_2^n \le m_2$. In this sense, the finite-difference estimates inherit the properties of the differential problem. Thus, if $\max_{\omega_{t_{n+1}}} z(x,t) = z\left(x^0, t^0\right)$, then $\hat{y} \in D_u$. Otherwise, if

$$\max_{\omega_{t_{n+1}}} z(x,t) > z\left(x^0, t^0\right),$$

i.e., a positive maximum is attained at an interior point (x, t_{n+1}), then, once again, with the help of Lemma 1, we obtain

$$z(x,t_{n+1}) \le \max\left\{0,\ \max_{\omega_{t_{n+1}}} e^{-\lambda t}\left\{\mu_1(t), \mu_2(t), u_0(x)\right\},\ \max_{\omega_{t_{n+1}}} \frac{\tau f(x,t) e^{-\lambda t}}{e^{\lambda\tau} - 1}\right\},$$

which implies that

$$y(x,t_{n+1}) \le m_2^{n+1} = \inf_{\lambda>0} \max\left\{0,\ \max_{\omega_{t_{n+1}}} e^{\lambda(t_{n+1}-t)}\{\mu_1(t), \mu_2(t), u_0(x)\},\ \max_{\omega_{t_{n+1}}} \frac{\tau f(x,t) e^{\lambda(t_{n+1}-t)}}{e^{\lambda\tau} - 1}\right\}.$$

In a similar way, we obtain the lower bound

$$y(x,t_{n+1}) \geq m_1^{n+1} = \sup_{\lambda>0} \min \left\{ 0, \min_{\omega_{t_{n+1}}} e^{\lambda(t_{n+1}-t)} \{\mu_1(t), \mu_2(t), u_0(x)\}, \min_{\omega_{t_{n+1}}} \frac{\tau f(x,t) e^{\lambda(t_{n+1}-t)}}{e^{\lambda\tau}-1} \right\}.$$

Thus, in this case, $\hat{y} \in D_u$. Moreover, since all positivity conditions for the coefficients (2), (3) are satisfied, the difference scheme (12) is monotone for all h and τ (i.e., unconditionally monotone).

The theorem is proved.

5. Convergence in the Grid L_2-Norm

If we manage to get two-sided estimates for the solutions of finite-difference schemes, then the convergence analysis of linearized numerical algorithms lead to a linear problem for the error $z = y - u$ of the method. In this section, we additionally assume that the exact solution of problem (5)–(7) is sufficiently smooth; namely, $u(x,t) \in C^{4,2}(Q_T)$. We use the energy-inequality method to obtain estimates of the error and convergence results in the discrete L_2-norm. We define the approximation error $\hat{\psi}$ at the interior nodes as follows:

$$\hat{\psi} = -u_t + \sum_{\alpha=1}^{p} \left(a_\alpha(u)\hat{u}_{\bar{x}_\alpha}\right)_{x_\alpha} + \hat{f}.$$

Thus, the grid-function error z is the solution of the following discrete problem:

$$z_t = \sum_{\alpha=1}^{p} \left(a_\alpha(y)\hat{y}_{\bar{x}_\alpha} - a_\alpha(u)\hat{u}_{\bar{x}_\alpha}\right)_{x_\alpha} + \hat{\psi} \tag{16}$$

with initial and boundary conditions

$$z(x,0) = 0, \quad x \in \omega_h, \tag{17}$$

$$z(x,t) = 0, \quad (x,t) \in \gamma_h \times \bar{\omega}_\tau. \tag{18}$$

It is easy to see that the approximation errors have the following order: $O\left(h^2+\tau\right)$, $h^2 = h_1^2 + \ldots + h_p^2$, at all nodes. We now define the following inner products and the corresponding norms:

$$(u,v) = \sum_{x\in\omega_h} h_1 \ldots h_p u(x) v(x), \quad \|u\| = \sqrt{(u,u)},$$

$$(u,v]_\alpha = \sum_{x\in\omega_{h,\alpha}^+} h_1 \ldots h_p u(x) v(x), \quad \|u]|_\alpha = \sqrt{(u,u]_\alpha},$$

where

$$\omega_{h,\alpha}^+ = \omega_h \cup \{x_\alpha = l_\alpha\}, \quad \alpha = 1, \ldots, p.$$

The following results are used in proving the convergence of the scheme. Actually, for this purpose, we apply the formula of summation by parts [32], and Gronwall's inequality [33].

Lemma 2. *(Summation by Parts). For any grid functions u and v defined in ω_h and vanishing at the boundary points $x \in \gamma_h$, the following identity holds:*

$$(u_{x_\alpha}, v) = -(u, v_{\bar{x}_\alpha}]_\alpha. \tag{19}$$

Lemma 3. *(Gronwall's Inequality). Let ε_n and f_n be nonnegative discrete functions defined on the grid $\omega_t = \{t_n = n\tau,\ n = 0, 1, \ldots\}$ and let $\rho > 0$ be a constant such that the following inequalities are satisfied:*

$$\varepsilon_{n+1} \le \rho\varepsilon_n + f_n, \quad n = 0, 1, \ldots.$$

Then the following estimate holds:

$$\varepsilon_{n+1} \le \rho^{n+1}\varepsilon_0 + \sum_{k=0}^{n} \rho^{n-k} f_k.$$

Theorem 3. *The solution of scheme (16)–(18) satisfies the estimate*

$$\|\hat{z}\| \le C(h^2 + \tau), \quad h^2 = h_1^2 + \ldots + h_p^2,$$

where C is a positive constant independent of the discretization parameters.

Proof. Multiplying (16) scalarly by $2\tau\hat{z}$, we obtain

$$2\tau\,(z_t, \hat{z}) = 2\tau \sum_{\alpha=1}^{p} \big(\hat{z}, (a_\alpha(y)\hat{y}_{\bar{x}_\alpha} - a_\alpha(u)\hat{u}_{\bar{x}_\alpha})_{x_\alpha}\big) + 2\tau(\hat{z}, \hat{\psi}). \tag{20}$$

We apply the identity $\hat{z} = 0.5(\hat{z} + z) + 0.5\tau z_t$ to represent the left-hand side of Eq. (20) in the form

$$2\tau\,(z_t, \hat{z}) = \|\hat{z}\|^2 - \|z\|^2 + \tau^2\|z_t\|^2.$$

We apply the summation-by-parts formula (19) to the first term on the right-hand side of Eq. (20) and obtain

$$2\tau \sum_{\alpha=1}^{p} \Big(\hat{z}, \big(a_\alpha(y)\hat{y}_{\bar{x}_\alpha} - a_\alpha(u)\hat{u}_{\bar{x}_\alpha}\big)_{x_\alpha}\Big) = -2\tau \sum_{\alpha=1}^{p} \Big(\hat{z}_{\bar{x}_\alpha}, a_\alpha(y)\hat{y}_{\bar{x}_\alpha} - a_\alpha(u)\hat{u}_{\bar{x}_\alpha}\Big]_\alpha.$$

Substituting these relations in Eq. (20), we get

$$\|\hat{z}\|^2 - \|z\|^2 + \tau^2\|z_t\|^2 = -2\tau \sum_{\alpha=1}^{p} \Big(\hat{z}_{\bar{x}_\alpha}, a_\alpha(y)\hat{y}_{\bar{x}_\alpha} - a_\alpha(u)\hat{u}_{\bar{x}_\alpha}\Big]_\alpha + 2\tau(\hat{z}, \hat{\psi}). \tag{21}$$

Since

$$a_\alpha(y)\hat{y}_{\bar{x}_\alpha} - a_\alpha(u)\hat{u}_{\bar{x}_\alpha} = a_\alpha(y)\hat{z}_{\bar{x}_\alpha} + \big(a_\alpha(y) - a_\alpha(u)\big)\hat{u}_{\bar{x}_\alpha}, \quad \alpha = 1, 2, \ldots, p,$$

we arrive at the representation

$$\big(\hat{z}_{\bar{x}_\alpha}, a_\alpha(y)\hat{y}_{\bar{x}_\alpha} - a_\alpha(u)\hat{u}_{\bar{x}_\alpha}\big]_\alpha = \big(\hat{z}_{\bar{x}_\alpha}, a_\alpha(y)\hat{z}_{\bar{x}_\alpha}\big]_\alpha + \big(\hat{z}_{\bar{x}_\alpha}, \big(a_\alpha(y) - a_\alpha(u)\big)\hat{u}_{\bar{x}_\alpha}\big]_\alpha.$$

Since $a_\alpha(y) \geq k_{\alpha,1} > 0$ for any $y \in D_u$, by virtue of (13), we conclude that, in view of conditions (8),

$$\big(\hat{z}_{\bar{x}_\alpha}, a_\alpha(y)\hat{z}_{\bar{x}_\alpha}\big]_\alpha \geq k_{\alpha,1}\|\hat{z}_{\bar{x}_\alpha}]|^2_\alpha.$$

For the functions $k_\alpha,\ \alpha = 1, 2, \ldots, p,$ there exist constants $L_\alpha,\ \alpha = 1, 2, \ldots, p,$ such that

$$\big|a_\alpha(y) - a_\alpha(u)\big| \leq L_\alpha |z|_{\alpha,(0.5)},$$

where

$$|z|_{\alpha,(0.5)} = \frac{|z| + \big|z^{(-1_\alpha)}\big|}{2}.$$

Hence, we obtain the inequality

$$\Big(|\hat{z}_{\bar{x}_\alpha}|, \big|a_\alpha(y) - a_\alpha(u)\big||\hat{u}_{\bar{x}_\alpha}|\Big]_\alpha \leq L_\alpha\big(|\hat{z}_{\bar{x}_\alpha}|, |z|_{\alpha,(0.5)}|\hat{u}_{\bar{x}_\alpha}|\big]_\alpha.$$

The solution $u(x,t)$ of problem (5)–(7) is sufficiently smooth and, hence, we get the estimate

$$|\hat{u}_{\bar{x}_\alpha}| \leq \frac{1}{h_\alpha} \int\limits_{x_\alpha^{(i_\alpha - 1)}}^{x_\alpha^{(i_\alpha)}} \left|\frac{\partial \hat{u}}{\partial x_\alpha}\right| dx_\alpha \leq c, \quad \alpha = 1, 2, \ldots, p.$$

We now apply the generalized Cauchy–Schwarz inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \quad a, b \in R,$$

for $a = |\hat{z}_{\bar{x}_\alpha}|$ and $b = |z|_{\alpha,(0.5)}$ and obtain

$$L_\alpha\Big(|\hat{z}_{\bar{x}_\alpha}|, |z|_{\alpha,(0.5)}\, |\hat{u}_{\bar{x}_\alpha}|\Big]_\alpha \leq cL_\alpha\varepsilon_\alpha\|\hat{z}_{\bar{x}_\alpha}]|^2_\alpha + \frac{cL_\alpha}{4\varepsilon_\alpha}\|z\|^2.$$

Here and in what follows, $\varepsilon_i > 0,\ i = 1, 2, \ldots.$ Thus, for the first term on the right-hand side of (21) we get the following estimate:

$$- 2\tau \sum_{\alpha=1}^{p} \Big(\hat{z}_{\bar{x}_\alpha}, a_\alpha(y)\hat{y}_{\bar{x}_\alpha} - a_\alpha(u)\hat{u}_{\bar{x}_\alpha}\Big]_\alpha$$

$$\leq -2\tau \sum_{\alpha=1}^{p} (k_{\alpha,1} - cL_\alpha\varepsilon_\alpha)\,\|\hat{z}_{\bar{x}_\alpha}]|^2_\alpha + \tau \sum_{\alpha=1}^{p} \frac{cL_\alpha}{2\varepsilon_\alpha}\|z\|^2.$$

Applying again the generalized Cauchy–Schwarz inequality, the last term on the right-hand side in (21) satisfies the estimate

$$2\tau(\hat{z}, \hat{\psi}) = 2\tau\big(\tau z_t + z, \hat{\psi}\big) \leq 2\tau^2\varepsilon_{p+1}\|z_t\|^2 + \frac{\tau^2}{2\varepsilon_{p+1}}\|\hat{\psi}\|^2 + 2\tau\varepsilon_{p+2}\|z\|^2 + \frac{\tau}{2\varepsilon_{p+2}}\|\hat{\psi}\|^2.$$

We now take into account all these inequalities and, in view of the right-hand side of (21), arrive at the inequality

$$\|\hat{z}\|^2 + \tau^2 \left(1 - 2\varepsilon_{p+1}\right) \|z_t\|^2 + 2\tau \sum_{\alpha=1}^{p} \left(k_{\alpha,1} - cL_\alpha \varepsilon_\alpha\right) ||\hat{z}_{\bar{x}_\alpha}]|_\alpha^2$$

$$\leq \left(1 + \tau \left(2\varepsilon_{p+2} + \sum_{\alpha=1}^{p} \frac{cL_\alpha}{2\varepsilon_\alpha}\right)\right) \|z\|^2 + \tau \left(\frac{1}{2\varepsilon_{p+1}} + \frac{1}{2\varepsilon_{p+2}}\right) \|\hat{\psi}\|^2.$$

Consequently,

$$\|\hat{z}\|^2 + \tau^2(1 - 2\varepsilon_{p+1})\|z_t\|^2 + 2\tau \sum_{\alpha=1}^{p} \left(k_{\alpha,1} - cL_\alpha \varepsilon_\alpha\right) \|\hat{z}_{\bar{x}_\alpha}]|_\alpha^2 \leq (1 + \tau\bar{c}) \|z\|^2 + \tau\bar{c}(h^2 + \tau)^2.$$

We take sufficiently small values $\varepsilon_i,\ i = 1, 2, \ldots, p+1$, i.e., such that the inequalities $1 - 2\varepsilon_{p+1} > 0$ and $k_{\alpha,1} - cL_\alpha\varepsilon_\alpha > 0,\ \alpha = 1, 2, \ldots, p,$ are true. As a result, we arrive at the final estimate

$$\|\hat{z}\|^2 \leq e^{\tau\bar{c}}\|z\|^2 + \tau\bar{c}(h^2 + \tau)^2.$$

Applying the finite-difference analog of the Gronwall lemma to the last inequality, we obtain the desired estimate.

The theorem is proved.

6. Third Boundary-Value Problem for a Quasilinear Parabolic Equation of the Convection-Diffusion Type

In this section, for the multidimensional quasilinear convection-diffusion equation with boundary conditions of the third kind, we construct a monotone finite-difference scheme and establish two-sided estimates for both exact and difference solutions.

In the cylindrical domain $\bar{Q}_T$, we consider the third boundary-value problem for the p-dimensional convection-diffusion equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(u)\frac{\partial u}{\partial x_\alpha}\right) + \sum_{\alpha=1}^{p} v_\alpha(u)\frac{\partial u}{\partial x_\alpha} - q(x)u + f(x,t), \quad x \in \Omega, \quad t \in (0, T], \tag{22}$$

with boundary conditions

$$k_\alpha \frac{\partial u}{\partial x_\alpha} = \sigma_{-\alpha}u - \mu_{-\alpha}, \quad x_\alpha = 0, \quad -k_\alpha \frac{\partial u}{\partial x_\alpha} = \sigma_{+\alpha}u - \mu_{+\alpha}, \quad x_\alpha = l_\alpha, \quad t \in (0, T], \quad \alpha = \overline{1, p}, \tag{23}$$

and initial condition

$$u(x, 0) = u_0(x), \quad x \in \bar{\Omega}, \tag{24}$$

where $\sigma_{\pm\alpha} = \sigma_{\pm\alpha}(x,t) > 0$ and $\mu_{\pm\alpha} = \mu_{\pm\alpha}(x,t)$ are given functions, $q(x) \geq c_0 > 0,\ x \in \Omega,\ \left|v_\alpha(u)\right| \leq c_1,$ $0 < k_{\alpha,1} \leq k_\alpha(u) \leq k_{\alpha,2},\ u \in D_u = [m_3, m_4]$, and $\alpha = \overline{1,p}$. Further, we assume that a unique solution of problem (22)–(24) exists and can be continuously extended into the h-neighborhood of the domain of definition of the problem

$$\bar{\Omega}_h = \{x = (x_1, \dots, x_p)\colon -h \le x_\alpha \le l_\alpha + h,\ \alpha = \overline{1,p}\}.$$

By using the transformation of the function $u(x,t)$ into a new function $v(x,t)$ (see the proof of Theorem 1), we get the following assertion:

Theorem 4. *The solution $u(x,t)$ of problem (22)–(24) at any point $(x,t_1) \in \bar{Q}_T$ satisfies the following two-sided estimate:*

$$u(x,t_1) \ge m_3 = \sup_{\lambda > -c_0} \min\left\{0,\ e^{\lambda t_1} u_0,\ \min_{\bar{Q}_{t_1}} \left\{\frac{\mu_{\pm\alpha}}{\sigma_{\pm\alpha}} e^{\lambda(t_1 - t)}\right\},\ \frac{1}{\lambda + c_0} \min_{\bar{Q}_{t_1}} \left(f e^{\lambda(t_1 - t)}\right)\right\}, \tag{25}$$

$$u(x,t_1) \le m_4 = \inf_{\lambda > -c_0} \max\left\{0,\ e^{\lambda t_1} u_0,\ \max_{\bar{Q}_{t_1}} \left\{\frac{\mu_{\pm\alpha}}{\sigma_{\pm\alpha}} e^{\lambda(t_1 - t)}\right\},\ \frac{1}{\lambda + c_0} \max_{\bar{Q}_{t_1}} \left(f e^{\lambda(t_1 - t)}\right)\right\}. \tag{26}$$

Parallel with the previously introduced uniform grid in the time variable $\bar{\omega}_\tau$, in the domain $\bar{\Omega}_h$, we introduce a uniform discrete space grid

$$\bar{\omega}_h^* = \Big\{x_{i_1 \dots i_p} = (x_1^{i_1}, \dots, x_p^{i_p}),\ x_\alpha^{i_\alpha} = (i_\alpha - 1/2) h_\alpha,$$
$$i_\alpha = \overline{0, N_\alpha + 1},\ h_\alpha N_\alpha = l_\alpha,\ h_\alpha/2 \le h,\ \alpha = \overline{1,p}\Big\}.$$

By using the regularization principle [32], we approximate the initial problem on the grid $\bar{\omega}^* = \bar{\omega}_\tau \times \bar{\omega}_h^*$ by the following difference scheme of the second-order approximation:

$$y_t = \sum_{\alpha=1}^p \varkappa_\alpha (a_\alpha \hat{y}_{\bar{x}_\alpha})_{x_\alpha} + \sum_{\alpha=1}^p b_\alpha^+ a_\alpha^{(+1_\alpha)} \hat{y}_{x_\alpha} + \sum_{\alpha=1}^p b_\alpha^- a_\alpha \hat{y}_{\bar{x}_\alpha} - q\hat{y} + \hat{f}, \quad (x,t_n) \in \omega^*, \tag{27}$$

$$a_\alpha \hat{y}_{\bar{x}_{\alpha,1}} - \hat{\sigma}_{-\alpha}(x) \frac{\hat{y}_{0_\alpha} + \hat{y}_{1_\alpha}}{2} = -\hat{\mu}_{-\alpha}(x), \qquad x_\alpha = 0, \quad t \in (0,T], \quad \alpha = \overline{1,p},$$

$$-a_\alpha \hat{y}_{\bar{x}_{\alpha,N_\alpha+1}} - \hat{\sigma}_{+\alpha}(x) \frac{\hat{y}_{N_\alpha} + \hat{y}_{N_\alpha+1}}{2} = -\hat{\mu}_{+\alpha}(x), \qquad x_\alpha = l_\alpha, \quad t \in (0,T], \quad \alpha = \overline{1,p}, \tag{28}$$

$$y(x,0) = u_0(x), \quad x \in \bar{\omega}_h^*, \tag{29}$$

where

$$\varkappa_\alpha = \frac{1}{1 + R_\alpha}, \qquad R_\alpha = 0.5 h_\alpha \frac{|v_\alpha(y)|}{k_\alpha(y)}, \qquad b_\alpha^\pm = \frac{v_\alpha^\pm(y)}{k_\alpha(y)}, \qquad v_\alpha^\pm(y) = 0.5\big(v_\alpha(y) \pm |v_\alpha(y)|\big).$$

Representing the difference scheme (27)–(29) in the canonical form (1), we prove that the positivity conditions hold for coefficients (2), (3) at all internal points $x \in \omega^*$ for any grid steps. In order that these conditions be satisfied for the boundary nodes, it is necessary to require that the space steps h_α, $\alpha = \overline{1,p}$, must satisfy inequality (30). Then the following theorem can be proved in a similar way:

Theorem 5. *Let the condition*

$$\max_{\alpha}\{h_\alpha\} \leq \frac{2\min\limits_{\alpha}\{k_{\alpha,1}\}}{\max\limits_{\alpha}\{\sigma_{\pm\alpha}\}} \tag{30}$$

be satisfied. Then the finite-difference scheme (27)–(29) is unconditionally monotone (without restrictions on the steps τ and $h_\alpha,\ \alpha=\overline{1,p}$) and its solution satisfies the following two-sided estimate at any point $(x,t_n)\in\bar{\omega}^*$:

$$y(x,t_n) \geq m_3^n = \sup_{\lambda>0}\min\left\{0,\ e^{\lambda t_n}u_0,\ \min_{\omega_{t_n}}\left\{\frac{\mu_{\pm\alpha}}{\sigma_{\pm\alpha}}e^{\lambda(t_n-t)}\right\},\ \frac{\tau\min\limits_{\omega_{t_n}}\left(fe^{\lambda(t_n-t)}\right)}{(1+\tau c_0)\,e^{\lambda\tau}-1}\right\}, \tag{31}$$

$$y(x,t_n) \leq m_4^n = \inf_{\lambda>0}\max\left\{0,\ e^{\lambda t_n}u_0,\ \max_{\omega_{t_n}}\left\{\frac{\mu_{\pm\alpha}}{\sigma_{\pm\alpha}}e^{\lambda(t_n-t)}\right\},\ \frac{\tau\max\limits_{\omega_{t_n}}\left(fe^{\lambda(t_n-t)}\right)}{(1+\tau c_0)\,e^{\lambda\tau}-1}\right\}. \tag{32}$$

Based on the maximum principle, in a standard way, we obtain the following important *a priori* estimate in the strong C-norm:

Theorem 6. *Assume that conditions (30) are satisfied. Then the solution of the finite-difference scheme (27)–(29) satisfies the following a priori estimate for any $t_n\in\omega_\tau$:*

$$\left\|y(t_{n+1})\right\|_{\bar{C}} \leq \max\left\{\|u_0\|_{\bar{C}},\ \max_{1\leq k\leq n+1}\left\|\frac{\mu_{\pm\alpha}(t_k)}{\sigma_{\pm\alpha}(t_k)}\right\|_{C_\gamma}\right\} + t_{n+1}\max_{1\leq k\leq n+1}\left\|f(t_k)\right\|_{C}.$$

Remark 1. Since

$$\frac{\tau}{(1+\tau c_0)e^{\lambda\tau}-1} \leq \frac{1}{\lambda+c_0} \quad \text{for all} \quad \tau,\lambda>0,$$

we see that estimates (25), (26) and (31), (32) imply that $m_3\leq m_3^n$ and $m_4^n\leq m_4$ and, in this sense, it is possible to say that the finite-difference estimates inherit the properties of the differential problem.

7. Conclusions

A new unconditionally monotone quadratic (in space) linearized finite-difference scheme on a uniform grid that approximates the IBVP for multidimensional quasilinear equation with unbounded nonlinearity is proposed according to the results of the present study. The insignificant bilateral point estimates are obtained for the solution of the scheme. They are in perfect agreement with the corresponding estimates for the differential problem. The dependence of these estimates is confirmed only on the initial and boundary conditions and on the right-hand side. A lower bound and an upper bound are established for the solution of the appropriate differential problem in Ladyzhenskaya's book [14]. Similar results by using similar methods for the linearized finite-difference scheme of the problem on a uniform grid are obtained in the present study. Bilateral estimates for the problems with unbounded nonlinearity are not obtained by using the standard technique of grid maximum principle [32] in the nonlinear case because the discreteness of the solution in the vicinity of the exact solution was not proved. These estimates simultaneously allow verifying the nonnegativity of the exact solutions, which is essential in the problems of physics, and determination of acceptable conditions for the input data in the nonlinear problem, which is parabolic. The convergence of finite-difference schemes in the grid L_2-norm is proved by using the difference maximum principle with input data of variable sign.

We formulate the IBVP for the multidimensional quasilinear parabolic equation of convection-diffusion type with boundary condition of the third type and indicate the intrinsic properties of the problem with unlimited nonlinearity. We construct second-order monotone difference schemes for the parabolic convection-diffusion equation with boundary condition of the third kind and unlimited nonlinearity without using the initial differential equation on the domain boundaries. The boundary conditions in this case are directly approximated on a two-point stencil of the second order. All theoretical results are obtained under the assumption that some conditions imposed only on the input data of the differential problem are satisfied.

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