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FRACTAL EMBEDDED BOXES OF BIFURCATIONS

Christian Mira UDC 517.9

This descriptive text is essentially based on Sharkovsky's and Myrberg's publications on the ordering of periodic solutions *(cycles)* generated by a $\mathrm{Dim}\,1$ unimodal smooth map $f(x,\lambda)$. Taking $f(x,\lambda) = x^2 - \lambda$ as an example, it was shown in a paper published in 1975 that the bifurcations are organized in the form of a sequence of *well-defined fractal embedded "boxes"* (parameter λ intervals) each of which is associated with a basic cycle of period k and a symbol j permitting to distinguish cycles with the same period k. Without using the denominations *Intermittency* (1980) and *Attractors in Crisis* (1982), this new text shows that the notion of *fractal embedded "boxes"* describes the properties of each of these two situations as the *limit of a sequence of well-defined boxes* (k,j) as $k \to \infty$.

1. Introduction

In 1967 and some years after, I had a regular correspondence with A. N. Sharkovsky. In this framework, he sent me the text of his famous paper *"Coexistence of Cycles of a Continuous Mapping of the Line into Itself"* [37], *giving an ordering of these cycles based on their period* and other papers in Russian and Ukrainian. At that time, this outstanding text was not known in western countries, where the knowledge was limited to a particular case of the T. Y. Li and A. J. Yorke result *"Period Three Implies Chaos"* [38]. Then I met Sharkovsky at the *VIIth International Conference on Nonlinear Oscillation* (Berlin, Sept. 1975), where I submitted a text in French [18] dealing with the notion of *embedded boxes* generated by the bifurcations generated by a Dim1 quadratic map. This text cited three Sharkovsky's publications (*Ukr. Math. J.*, **16**, No. 1 (1964), **17**, No. 3 (1965), **18**, No. 2 (1966)) at the origin of the project [18]. Later, the periodic *European Conferences on Iteration Theory* (ECIT) gave rise to other meetings.

The Sharkovsky's cycles ordering is general in a sense that it concerns general forms of continuous Dim1 maps with any extrema but cannot discern between the cycles having the same period k. This situation induced me to think about a possible refining of the general ordering in the particular case where the map (T) is *smooth and unimodal* (i.e., defined by a function $f(x,\lambda)$ having only one extremum, e.g., $f(x,\lambda) = x^2 - \lambda$). Then it was easy to see that *the number N_k of cycles having the same period k and the bifurcations number $N_\lambda(k)$ generating these cycles increase quickly with k*. For example, $N_2 = 1,\ N_\lambda(2) = 1,\ N_3 = 2,\ N_\lambda(3) = 1,$ $N_4 = 2,\ N_\lambda(4) = 2,\ N_5 = 6,\ N_\lambda(5) = 3, \ldots,\ N_{20} = 52,377,\ N_\lambda(20) = 26,214, \ldots,\ N_{30} = 35,790,267,$ $N_\lambda(30) = 17,895,679, \ldots,\ N_{37} = 3,714,566,310$ (details for obtaining these results are given in [23, p. 93–97]). *These cycles with the same period k differ from each other in different permutations* (cyclic transfer) of their points by successive iterations of T. Hence, each period k-cycle can be identified by a symbolism $(k;j)$ with j being an index characterizing this *permutation. This symbolism imposes a convention*: $k \neq 2,\ 2^1 \neq 2,\ 2^2 \neq 4, \ldots;$ thus, a cycle $(2^1;j)$ exists, two cycles with different properties $(2^2;j)$ and $(4;j)$ exist, $\ldots$. For this purpose, Myrberg [34] defined the index j by using a *binary code* made up of a sequence [r] of $k-2$ signs $[+,-]$ that I called *"Myrberg's binary rotation sequence"* (for its properties, see [23, p. 103–183]). Ten years after Myrberg, independently, Metropolis, et al. [17] used an equivalent code made up of $k-1$ symbols $[R,L]$. Moreover,

Groupe d'Etude des Systèmes Non Linéaire et Applications, INSA Toulouse, France; e-mail: christian.mira@orange.fr.

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considering a basic cycle $(k; j)$ created from a *fold bifurcation* (i.e., the cycle *multiplier* — or *eigenvalue* — is $S = +1$), Myrberg called *"spectrum" the parameter interval of period doubling bifurcations* with its limit value $\lambda_{ks},\ k = 1, 3, 4, \ldots$ (the limit of the sequence by period doubling). This phenomenon of *period doubling* is now wrongly attributed to Feigenbaum [7], however, the author who, as compared with Myrberg, added a new result related to the existence of *two universal constants*.

The *box-within-a-box bifurcation structure* (in French, *"structure boîtes emboîtées," embedded boxes from* [9]) is based on the fundamental *Myrberg's ordering law* [34, p. 106–107] of [23]). This organization of *bifurcations* consists in an ordering of the Myrberg's *spectra (cascades of bifurcations by period doubling)*. In this respect, it is worth noting that the most famous contemporary authors have ignored Myrberg's fundamental contribution. As far as I know, Marek Kuczma was the first who mentioned Myrberg, more precisely, in the very large bibliography of his book *"Functional Equations in a Single Variable"* (Polish Sci. Publ., Warsaw (1968)). From 1974, opening the way to the Myrberg's German papers (1958 to 1968), Kuczma's references oriented researches leading to several publications [10] and following ones, which led to the concept of fractal *box-within-a-box organization* (in French *"structure boîtes emboîtées"* [10, 19, 20]). Later, quoting the French results [9], used the better term *"embedded boxes"* that I have adopted. Ignorance of the fundamental Myrberg's contribution became more evident from 1975, when the number of publications on $\mathrm{Dim}\,1$ real maps underwent an impressive increase. Since some 30 years, we can find rare quotations of [34], often of the second hand. Due to the first publications in French (around 1975), the notion of *box-within-a-box bifurcation structure* was rediscovered by speaking about *renormalization* with respect to a parameter interval and *compound windows* (cf. [16] for Hénon's map). About this point, Lorenz however quoted [2, 17–19] Mira's results about the *"foliated"* structure of the *parameter plane* made up of *embedded boxes* located on different sheets of this plane but with a possibility of communication via local plane structures called *cross-road area*, *spring-area*, *saddle area* (cf. [3, 4, 28], and the whole Chapter 6 of [23] devoted to this topic).

Section 2 is devoted to Myrberg's results on quadratic maps. It is followed by a description and study of bifurcations organized in the form of *embedded boxes* (Section 3). Sections 4 and 5 deal with the *intermittency* and *attractors in crisis* phenomena, seen as limits of sequences of infinitely many *boxes*.

2. Basic Myrberg's Results on Quadratic Maps

The first basic paper, which led to guess a *fractal bifurcations structure* generated by a quadratic map was [36] (at that time, the notions *fractal* and *chaos* did not exist). Indeed, considering $f(x, \lambda) = x^2 - \lambda,\ \lambda = 2$, Pulkin was the first author to show that the presence of *infinitely many unstable cycles with increasing period* $k \to \infty$ may generate *complex oscillating iterated sequences* (i.e., now *chaos*) giving rise to what he called *completely invariant sets*. These sets are related to the existence of *limit sets* of different *classes*. Hence, infinitely many limit points of the set of unstable period k cycles, with $k \to \infty$, lead to the notion of *class* 1 *limit set*. The limit sets of *class* 1 generate *limit sets of class* 2, and so on, until the *limit sets of class* ∞. Then Pulkin [36] and Myrberg [34] gave me the reference points, which (step-by-step) permitted to identify the *fractal organization of bifurcations* generated by smooth maps defined by a function with only one extremum. The simplest form of the smooth map T is a quadratic map (the map extremum is a minimum $x_e = 0$):

$$T\colon\ x' = f(x, \lambda) = x^2 - \lambda \quad \text{or, as a recurrrence,} \quad x_{n+1} = f(x_n, \lambda) = x_n^2 - \lambda, \quad n = 0, 1, 2, \ldots. \tag{1}$$

Here, x is a real variable and λ is a real parameter. The more general case of a smooth map $x' = f(x, \lambda)$ (with only one extremum and negative Schwarzian derivative) has *qualitatively* the same properties as the quadratic map. The map T is *noninvertible* [26]. It has two fixed points $q_i,\ i = 1, 2$, with a *multiplier* (eigenvalue) $S = 2x(q_i)$, where $x(q_1) = 1/2 + \sqrt{1/4 + \lambda}$ and $x(q_2) = 1/2 - \sqrt{1/4 + \lambda}$. The fixed point q_1 is always *unstable* $(S > 1)$, and q_2 is *stable* for $-1/4 < \lambda < 3/4$ $(-1 \leq S < 1)$. A period k cycle of T is a solution of $T^k x = x$ with $T^m x \neq x$ for $m < k$. It is *stable* when its *multiplier* (or eigenvalue) S is such that $|S| \leq 1$. For the initial condition $x_0 = x_e = 0$, n iterations of (1) lead to a polynomial $G_n(\lambda)$ with $G_1(\lambda) = -\lambda$, $G_2(\lambda) = \lambda^2 - \lambda$, $G_3(\lambda) = (\lambda^2 - \lambda)^2 - \lambda, \ldots$, and, hence, to the recursive process

$$G_{n+1}(\lambda) = G_n^2(\lambda) - \lambda \quad \text{from} \quad G_1(\lambda) = -\lambda. \tag{2}$$

Myrberg [32] showed that the cycles of period k with $S = 0$ (i.e., they are always stable) correspond to a set of parameter values λ that are *real zeros* of $G_k(\lambda)$ defined from the recursive process (2) by taking into account that the complete set of real zeros of $G_k(\lambda)$ corresponds not only to the λ-values of all period k cycles with $S = 0$ but also to period q cycles, where q is a divisor of k, including $\lambda = 0$, which is always a zero of $G_k(\lambda)$ (see Section 3.2 in [23]). Myrberg called these real zeros *primitive roots*. From $G_n(\lambda) = 0$, a formal representation of the solution is

$$\lambda = \sqrt{\lambda \pm \sqrt{\lambda \pm \ldots \pm \sqrt{\lambda}}}, \quad \text{and we write} \quad \lambda = \mathrm{r}(\lambda), \tag{3}$$

the first $\sqrt{}$ is numbered by 0, the second by $1, \ldots,$ and the last by $k-2$.

The first sign prior to the radical $\sqrt{}$ numbered by "1" in (3) is always $+$ and [r] completely characterizes the cyclic transfer of points of a given period k cycle by successive applications of T (Section 3.3.2 in [23]). Associated with (3), Myrberg [34] defined [r] (called *rotation sequence* in [11, 14, 15, 22, 23]) by a sequence of $k-2$ symbols $(+-)$, slope signs of $f(x, \lambda)$ at the points $q^{(2)}, q^{(3)}, \ldots, q^{(k-1)}$ with $q^{(i+1)} = Tq^{(i)}$, $i = 1, 2, \ldots, k-1$):

$$[\mathrm{r}] = m_1^+ n_1^- m_2^+ n_2^- \ldots m_s^+ n_s^-.$$

In [r], m_i^+, $i = 1, 2, \ldots, s$, is the number of consecutive signs $+$ and n_i^- is the number of consecutive signs $-$.

The radical equation (3) is associated with the central problem: *What are the sequences of signs* $(+,-)$ *corresponding to real "primitive" roots of* $G_k(\lambda) = 0$ (*the roots not corresponding to period* q cycles, where q is a divisor of k). The simplest example $m = 3$ leads to two forms of (3), namely, $\lambda = \sqrt{\lambda + \sqrt{\lambda}}$ and $\lambda = \sqrt{\lambda - \sqrt{\lambda}}$ but only $\lambda = \sqrt{\lambda + \sqrt{\lambda}}$ is the *real primitive solution* of $G_3(\lambda) = 0$ corresponding to $\lambda_{S=0} \simeq 1.7548776662$. For $m = 4$, relation (3) gives

$$\lambda = \sqrt{\lambda + \sqrt{\lambda + \sqrt{\lambda}}}, \quad \lambda = \sqrt{\lambda + \sqrt{\lambda - \sqrt{\lambda}}}, \quad \lambda = \sqrt{\lambda - \sqrt{\lambda + \sqrt{\lambda}}}, \quad \lambda = \sqrt{\lambda - \sqrt{\lambda - \sqrt{\lambda}}}.$$

Only the first two of these relations are real *"primitive"* roots of $G_4(\lambda) = 0$ with $\lambda_{S=0} \simeq 1.9407998065$ ($[\mathrm{r}] = ++$) and $\lambda_{S=0} \simeq 1.3107026413$ ($[\mathrm{r}] = +-$), this value being that the period 2^2 cycle resulting from the period doubling of the fixed point q_2.

A necessary and sufficient condition (cf. [34, p. 6]) for the existence of a real solution of (3) is as follows: All terms under the radical $\sqrt{}$ must be positive, which leads to the inequality

$$\lambda \geq \mathrm{r}_\nu = m_\nu^+ n_\nu^- \ldots m_s^+ n_s^-, \quad \nu = 2, 3, \ldots, s.$$

Then [34, p. 7] the radicals or the sequences of signs (3) are organized in the form of a *monotonically increasing sequence* whose *last term* contains only signs $+$, $[\mathrm{r}] = ++\ldots++$. Among all possible permutations of $k-2$

signs in (3), only some of them are real *"primitive"* roots of $G_k(\lambda) = 0$ (with extremum abscissa $x_e = 0$). Their permutations are associated with all possible period k cycles with *multiplier* (eigenvalue) $S = 0$.

For map (2), Myrberg stated the following properties [33, 34]:

(a) all the real zeros of $G_k(\lambda)$ are located in the interval $0 \leq \lambda < 2$;

(b) between two real zeros of $G_{k-2}(\lambda)$ and $G_{k-1}(\lambda)$, one can always find at least one zero of $G_k(\lambda)$ for $\lambda > 1$;

(c) between two real zeros λ_{k-2} of $G_{k-2}(\lambda)$ and λ_k of $G_k(\lambda)$, there exists at least one zero of $G_{k+1}(\lambda)$ for $\lambda > 5/4$;

(d) let m and p be two incommensurable integers and let $m > p$.

Between the real zeros λ_m of $G_m(\lambda)$ and λ_p of $G_p(\lambda)$, there exists at least one real zero of $G_l(\lambda)$, $l = m\mu + 1$. This leads to the *Myrberg's ordering law* of the cycles (k, j) [34] (ordering formulation can be found in Chapter 4 of the book [23, p. 106–109]).

Myrberg calls *"singular values of* λ", the λ-values, which are limit points of the real roots of $G_k(\lambda) = x_e$ as $k \to \infty$. Then [r] is made up of a set of periodically repeated signs: [r] $= (m_1^+ n_1^- m_2^+ n_2^- \ldots m_s^+ n_s^-)^\infty$. In Section 4.5 of the book [23], three types of *singular parameter values* are considered.

3. Fractal Embedded Boxes of Bifurcations (Unimodal Smooth Maps)

3.1. Some Basic Properties. Consider (1) as a smooth unimodal map, where T is a Dim1 *noninvertible map* and $x' = f(x, \lambda)$ (λ is a parameter). In this case, the λ-axis is made up of two open intervals: Z_2, each point of which has two distinct rank-one preimages (obtained from the inverse map T^{-1}) and Z_0 each point of which has no real preimage. This map is said to be of $(Z_0 - Z_2)$ type (for a generalization of this notion, see [24, p. 143–166]). The fractal *embedded boxes of bifurcations* was first identified in the case of unimodal maps with negative Schwarzian derivative [10, 19]. A more complete presentation was given in the books [11, 12, 23, 24]. A reduced version was given in Section 7.2 of Chapter 8 in [25].

As for the fractal bifurcation organization generated by smooth unimodal maps, it is easier to start directly from the above quadratic *Myrberg's map* $T\colon\ x' = x^2 - \lambda$. The inverse map T^{-1} is defined by $x = \pm\sqrt{x' + \lambda}$. Hence, the x-axis is made up of the intervals Z_2 ($x' > -\lambda$; a point has two real preimages), Z_0 ($x' < -\lambda$; a point has no preimages). The rank-one image $C = T(C_{-1})$ of the $f(x, \lambda)$ minimum C_{-1} ($x = 0$) is the *rank-one critical point* (in the Julia–Fatou sense), $x(C) = -\lambda$. This point has two merging rank-one preimages at $T^{-1}(C) = C_{-1}$ and C is the boundary separating Z_0 and Z_2. A rank-r *critical point* C_{r-1} is obtained after r iterations of C_{-1} (or, equivalently, $r-1$ iterations of C; the rank-one critical point $C_0 \equiv C$). The *set of increasing rank critical points* is denoted by E_c and its limit set is denoted by E'_c (derived set of E_c).

The map T is characterized by the following properties:

(a) Let $\lambda_{(1)_0} = -1/4$ $(q_1 \equiv q_2)$ and let $\lambda_1^* = 2$ $(q_1 \equiv C_1)$. The parameter interval

$$\Omega_1 = [\lambda_{(1)_0}, \lambda_1^*]$$

called *overall box* contains all the bifurcations values generated by the map (1). In the interval $-1/4 < \lambda < 2$, *the map possesses a unique attractor*, which is, in simple cases, either an asymptotically stable (or attracting) fixed point, or a stable period-k cycle, or a periodic chaotic attractor. The value $\lambda_{(1)_0} = -1/4$ corresponds to a *fold bifurcation* giving rise to two fixed points q_i, $i = 1, 2$, with a multiplier (or eigenvalue) $S = 2x(q_i)$; i.e., with q_2 being stable for $-1 \leq S < 1$, and q_1 being always unstable $(S > 1)$. If $\lambda < \lambda_{(1)_0} = -1/4$, then *no real* *fixed point or cycles exist*. The value $\lambda = \lambda_1^* = 2$ is a basic *nonclassical bifurcation* related to the merging of the unstable fixed point q_1 with the rank-two critical point $C_1 = T(C) = T^2(C_{-1})$. For this parameter value, we have

$$x(C_1) = x(q_1) = 2, \qquad x(C) = x(q_1^{-1}) = -2, \qquad \text{and} \qquad T^{-1}(q_1) = q_1 \cup q_1^{-1}.$$

An *absorbing segment* (d') is defined as a set bounded by two critical points such that the increasing rank p images T^p, $p = 1, 2, 3, \ldots$, of any point of its neighborhood $U(d')$, from *a finite number of iterations*, enter into (d') and cannot get away after entering.

For $0 < \lambda < \lambda_1^*$, the *invariant segment* $[q_1^{-1}, q_1]$ is the *closure of the basin of the absorbing segment* $\overline{CC_1}$ containing the unique attractor. For the parameter value $0 < \lambda < \lambda_1^* = 2$ the segment $\overline{CC_1}$ is *absorbing*. For $\lambda = \lambda_1^* = 2$, the segment $\overline{CC_1}$ merges with $[q_1^{-1}, q_1]$. Then it is *chaotic* (the situation of [36]); *all the possible period k cycles with their limit set as $k \to \infty$ have been created, they are unstable, and belong to* $\overline{CC_1}$, which is *invariant but not absorbing*. The unstable cycles and their limit sets as $k \to \infty$ form a set E, which is dense in the whole interval $[-2, 2]$ (as well as their preimages of any rank), that is the *derived set* (the set of limit points) $E' = [-2, 2]$ is perfect in the Julia sense [13] .

If $\lambda > 3/4$, then the fixed point q_2 *is always unstable with* $S(q_2) < -1$. As the values of λ increase and pass through $\lambda = \lambda_{b1} = 3/4$, we observe a *flip bifurcation*, i.e., the fixed point q_2 becomes unstable and gives rise to a stable period-2^1 cycle (q_{2^1}) with *multiplier* (i.e., eigenvalue) $S(q_{2^1}) = 4 - 4\lambda$, $\big(|S(q_{2^1})| \le 1$ if $3/4 \ge \lambda < 5/4\big)$. New increasing values of λ generate a sequence of *flip bifurcations* when $\lambda = \lambda_{bm}$, $m = 1, 2, 3, \ldots$, related to the birth of a stable period 2^m cycle (q_{2^m}), $m = 1, 2, \ldots$; the period 2^{m-1} cycle $(q_{2^{m-1}})$ becomes unstable for $\lambda > \lambda_{bm}$. This leads to the accumulation of bifurcation values $\lim_{m\to\infty} \lambda_{bm} = \lambda_{1s} \simeq 1.401155189$. For a particular bifurcation value $\lambda = \lambda_{1s}$ (the limit set of the *flip-bifurcations* sequence), the corresponding stable set is an *invariant set* called the *critical attractor* A_{cr} (see, among others, [9]). For $\lambda < \lambda_{1s}$ *the number of unstable cycles is finite*, each unstable cycle has a period 2^m created after passing through the value λ_{bm}. For $\lambda = \lambda_{1s} + \varepsilon$, $\varepsilon > 0$, $\varepsilon \to 0$, *there exist infinitely many unstable period-2^i cycles*, $i = 0, 1, 2, \ldots$ (cycles created via the above sequence of *flip bifurcations*). The parameter interval $\omega_1 \equiv [\lambda_{(1)_0}; \lambda_{1s}]$ was called *spectrum* by Myrberg (denomination adopted in this text). It corresponds to the sequence (cascade) of period doubling bifurcations from the fixed point q_2 $(i = 0)$, $x(q_2) = (1 - \sqrt{1 + 4\lambda})/2$, born from $\lambda \ge \lambda_{(1)_0} = -1/4$ with a *multiplier (eigenvalue)* $S < 1$, followed by period doubling bifurcations $(\lambda = \lambda_{2^i} = \lambda_{bi})$ of a period 2^{i-1}cycle into a period 2^i cycle with $S < 1$ $(i = 1$ corresponds to the fixed point $q_2)$ such that

$$\lim_{i\to\infty} \lambda_{bi} = \lambda_{1s} \simeq 1.401155189, \quad i = 1, 2, 3, \ldots;$$

λ_{1s} is the *Myrberg's accumulation point* of this cascade (called *spectrum* in [34]).

(b) The number N_k of all possible cycles with the same period k and the number $N_\lambda(k)$ of bifurcation values leading to these cycles increase very quickly with k : $N_3 = 2$, $N_\lambda(3) = 1$, $N_4 = 3$, $N_\lambda(4) = 2$, $N_5 = 6$, $N_\lambda(5) = 3$, $N_6 = 9$, $N_\lambda(6) = 5, \ldots$, $N_{10} = 99$, $N_\lambda(10) = 51, \ldots$, $N_{20} = 52{,}377$, $N_\lambda(20) = 26{,}214, \ldots$, $N_{30} = 35{,}790{,}267$, $N_\lambda(30) = 17{,}895{,}679, \ldots$, and $N_{37} = 3{,}714{,}566{,}310$ (for the relations giving N_k and $N_\lambda(k)$, cf. [23, p. 93–97]). Cycles with the same period k differ from each other by the *permutation* (cyclic transfer) of their points by successive iterations by T. Each k-cycle can be identified by the symbolism $(k; j)$, where j is an index characterizing this permutation. In what follows, j is simply called *"permutation"* instead of *"permutation of cycle points via k iterations"*. Let $(k; j)$, $k \ne 2$, be one of these cycles. Hence, each period-k cycle can be identified by the symbolism $(k; j)$; j is the index specifying this *permutation*. This permits the distinction between cycles of the same period k. Nevertheless, *this symbolism imposes a convention*: $k \ne 2$, $2^1 \ne 2$, $2^2 \ne 4, \ldots$. Thus, a cycle $(2^1; j)$ has a sense but not $(2; j)$. Two cycles $(2^2; j)$ and $(4; j)$ have different properties.

A $(k; j)$ cycle, $k \ne 2$, can be generated from two basic bifurcations: either a *fold one* $(S = +1)$ or a *flip one* $(S = -1)$. The *fold bifurcation* generates two *basic cycles* from $\lambda = \lambda^j_{(k)_0}$: an *unstable* cycle $(k; j)_{S>1}$ and a *stable* cycle $(k;j)_{S<1}$, as far as $-1 < S < 1$. With increasing the values of λ, a cascade of *flip bifurcations* is created from the cycle $(k;j)_{S<1}$ giving rise to a sequence of $(k2^i;j,p_i)_{S<1}$ cycles with an accumulation value λ_{ks}^j as $i \to \infty$ for $\lambda_{1s} < \lambda_{ks}^j < 2$. Here, p_i is the *permutation* related to the period 2^i cycle generated in the interval ω_1. Similarly, Myrberg also calls *"spectrum"* the following parameter interval: $\omega_k^j = \left[\lambda_{(k)_0}^j; \lambda_{ks}^j\right]$, $k = 1, 3, 4, \ldots$. The interval ω_k^j is made up of parameter intervals corresponding to stable cycles of the period $k2^i$, $i = 0, 1, 2, \ldots$. In ω_k^j, the *flip bifurcation* of a $(k2^{m-1};j,p_{m-1})$ cycle is denoted by λ_{kbm}^j, $m = 1, 2, \ldots$. The cycle symbolisms $(k;j)$ and $(k2^i;j,p_i)$ are related to what is called a *nonembedded representation* in [23] and [24]. This symbolism has the interest of identifying precisely every cycle. For a given value $\lambda = \lambda_g$ of λ, the *embedded boxes* organization permits the identification of all the cycles born for $\lambda < \lambda_g$.

For $\lambda > \lambda_1^* = 2$, $[q_1^{-1}, q_1] \subset \overline{CC_1}$, *the only attractor is the point at infinity, and no other bifurcation takes place*. The derived set E' (without the point at infinity) constitutes the nonwandering set $E' \subset [q_1^{-1}, q_1]$. *The map T generates all possible cycles, which are real and unstable*, and E' is a *totally disconnected* invariant *Cantor set*. This set, which constitutes the basin boundary of the fixed point at infinity, is everywhere disconnected.

The situation equivalent to that of λ_1^* (but now with an *absorbing set* inside $\overline{CC_1}$) is met for $\lambda = \lambda_k^{*j}$ from each $(k;j)$, $k = 1, 3, 4, \ldots$, cycle with multiplier $S > 1$ (generated from a *fold bifurcation*). In this case, λ_k^{*j} is the least λ-value such that the critical points $C_k = T^k(C)$, $C_{k+1}, \ldots, C_{2k-1}$ merge into the k points of the $(k;j)$ cycle with $S > 1$. Considering the map T^k, for k intervals bounded by critical points of the well-defined rank, the value λ_k^{*j} qualitatively reproduces the situation of T with $\lambda = \lambda_1^*$. In the case $\lambda = \lambda_k^{*j}$, the map T gives rise to k nonconnected intervals $\overline{CC_k}$, $\overline{C_1C_{k+1}}, \ldots, \overline{C_{k-1}C_{2k-1}}$, constituting a *k-cyclic chaotic segment* denoted by CH_k^j, which attracts almost all points of $]q_1^{-1}, q_1[\backslash CH_k^j$.

(c) The permutation (cyclic transfer) of one of the points of a $(k;j)$ cycle via k successive iterations by T can be defined either in a *binary form* (*Myrberg's rotation sequence*, the one adopted in this text), or in a decimal form (*decimal rotation sequence*, see [23, p. 135–148]). Each rotation sequence is associated with a well-defined index $j = 1, 2, \ldots, N_\lambda(k)$. These *rotation sequences* are ordered according to *Myrberg's ordering law* [23, 34] and the index j specifies the place of any cycle in this ordering. A necessary and sufficient condition for a permutation of k integers to be that of a cycle generated by a unimodal map was given in [23, p. 136–138].

3.2. General Organization of the Bifurcations. The organization of the bifurcations generated by (1) and described in this section concerns the whole family of unimodal maps T (i.e., $Z_0 - Z_2$ maps) with negative Schwarzian derivative, which are topologically conjugated with (1) in some correctly chosen parameter variation. From the fold bifurcation $\lambda_{(1)_0}$ and the *nonclassical bifurcation* λ_1^* (Fig. 1a with $C_1 \equiv q_1$, $C \equiv q_1^{-1}$, and $T^{-1}(q_1) = q_1 \cup q_1^{-1}$), the study started from simple *"geometric considerations"* by drawing a curve T^k in the (x_n, x_{n+1}) plane (by the classical Koenigs–Lemeray graphical method). Indeed, the *T^k extrema are easily obtained from the successive images and preimages of the minimum of T $(x = 0)$*. Then the k arcs of the "oscillating" curve T^k, each one with only one extremum, locally reproduce the situation of T with respect to the first bisectrix $x_n = x_{n+1}$ (cf. Figs. 1b, c for $k = 3$). The *overall box* contains all the possible bifurcations generated by (1), this box is the closed parameter interval $\Omega_1 = \left[\lambda_{(1)_0}, \lambda_1^*\right]$. This *overall box* contains intervals reproducing the Ω_1 properties with a configuration of the *"Russian dolls"* type. Out of Ω_1, no bifurcation occurs. Taking into account the Myrberg spectrum ω_1 related to the fixed point q_2 $(S < 1)$, the box Ω_1 (see Fig. 2a) is defined by

$$\Omega_1 = [\lambda_{(1)_0}, \lambda_1^*] = \omega_1 \cup \Delta_1, \qquad \Delta_1 =]\lambda_{1s}, \lambda_1^*].$$

The description of *box-within-a-box organization* implies a specific symbolism, where the symbol "2^i" is not used for *cycles of even period* born from a *fold bifurcation* or from a *flip bifurcation* related to a *basic cycle* appearing out of ω_1. Hence, considering the cycle $(2^i;p_i)$ generated inside the *spectrum* ω_1, *in this sense, a period 4*

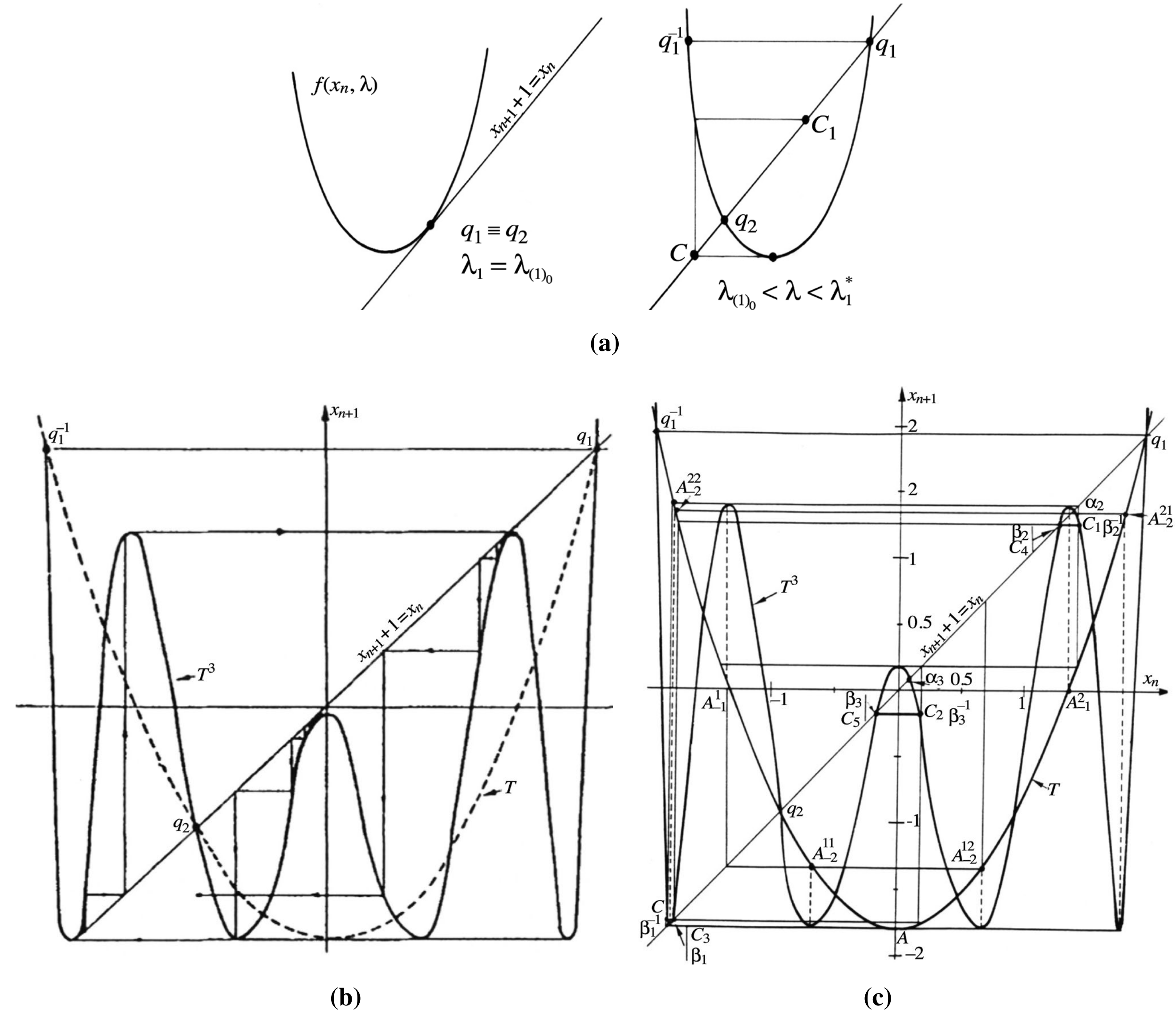


Fig. 1. Map T: $x_{n+1} = x_n^2 - \lambda$: (a) the two basic bifurcations defining the box $\Omega_1 = [\lambda_{(1)_0}, \lambda_1^*]$ if $\lambda = \lambda_1^*$, $C \equiv q_1^{-1}$ and $C_1 \equiv q_1$; (b) and (c) the two basic bifurcations $\lambda_{(k)_0}^j$ and λ_k^{*j} defining the box $\Omega_k^j = [\lambda_{(k)_0}^j, \lambda_k^{*j}]$, $k = 3$, $j = 1$ (only one period 3 attractor exists); (c) $\lambda = \lambda_3^* \simeq 1.790327493$, representation of T^r, $r = 3$, defining the period 3 *"segment S"* (chaotic attractor) $\overline{\beta_1\beta_1^{-1}}$, $\overline{\beta_2\beta_2^{-1}}$, $\overline{\beta_3\beta_3^{-1}}$, β_1, β_2, and β_3 are the points of the period 3 unstable cycle with multiplier (eigenvalue) $S > 1$; β_1^{-1}, β_2^{-1}, and β_3^{-1} are the rank-one preimages (by T^{-3}) of these points. The points α_1, α_2, and α_3 are points of the period 3 unstable cycle with multiplier (eigenvalue) $S < 1$; here, $S < -1$.

cycle is not a period 2^2 cycle, i.e., $2^2 \neq 4$, $2^3 \neq 8, \ldots$; stable cycles different from $(2^i; p_i)$ may appear only for $\lambda \in \Delta_1$. The interval $\lambda < \lambda_{(1)_0} = -1/4$ corresponds to the *absence of fixed points (except the point at infinity) or cycles; here every orbit is divergent*. For $\lambda > \lambda_1^* = 2$, *all the possible period* k *cycles were created. They are unstable*, and the map has the properties indicated in Section 2.

Two basic cycles $(k; j)$, $k = 3, 4, \ldots$, issued from the same *fold bifurcation* $\lambda_{(k)_0}^j$, one with $S > 1$ and the other with $S < 1$, generate a parameter interval, provisionally denoted by $\widehat{\Omega}_k$ and having the same behavior as Ω_1, $\widehat{\Omega}_k \subset \Delta_1$. The box $\widehat{\Omega}_k$ is denoted by Ω_k^j if k *is a prime number and if it is not contained in another interval* $\widehat{\Omega}_{k'}$; k *being a multiple* of k'. Then Ω_k^j defined from $\lambda_{(k)_0}^j$ and λ_k^{*j} (see Figs.1(b, c)) is called a *rank-one box* or

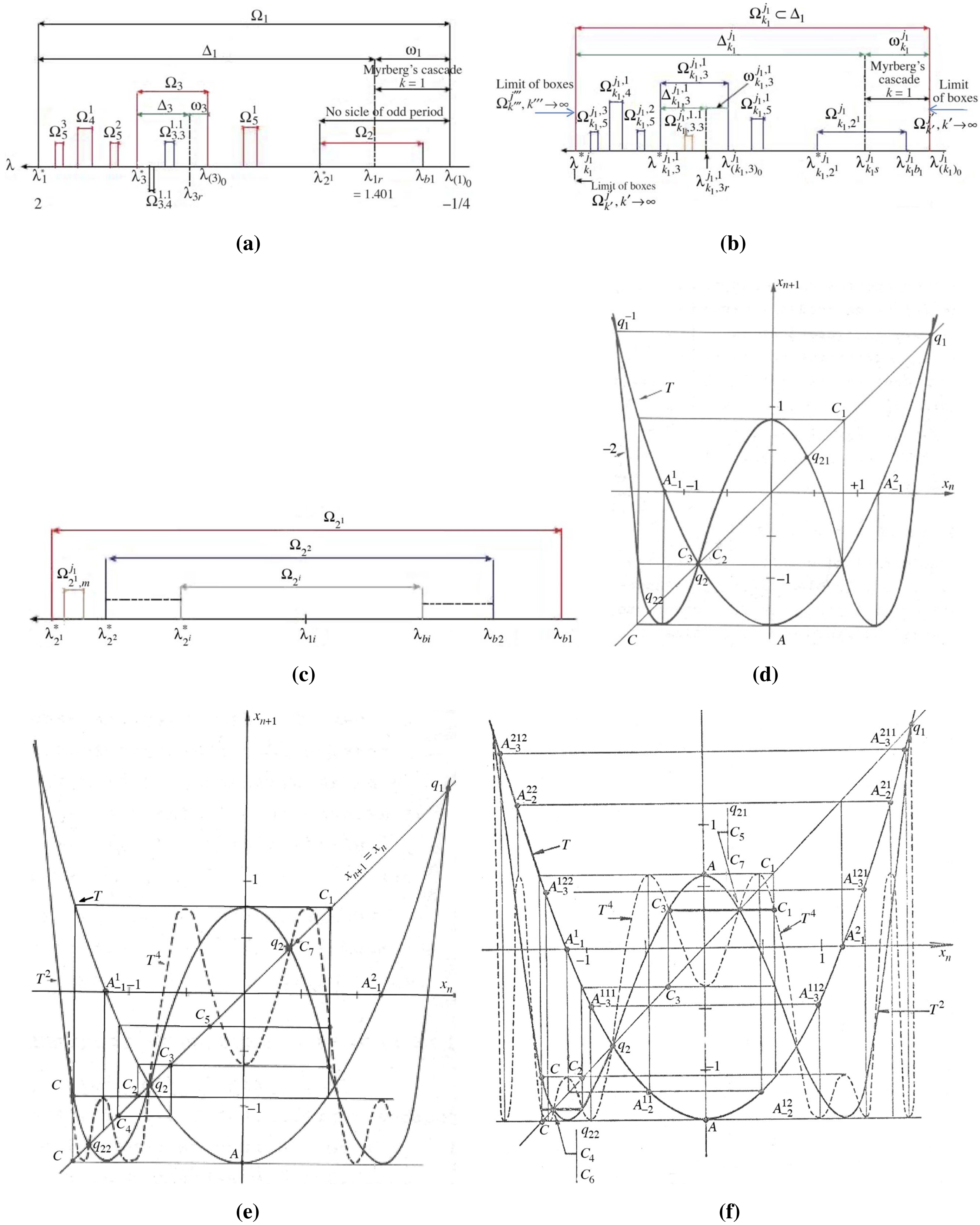


Fig. 2. (a) Embedded boxes organization, global view; (b) enlargement of the $\Omega_{k_1}^{j_1}$ part; (c) *second kind box* Ω_{2^1}; (d) $\lambda = \lambda^*_{2^1}$; curve T^{2^1}; (e) $\lambda^*_{2^1} < \lambda < \lambda^*_{2^2}$; curves T^{2^1} and T^{2^2} (dotted); (f) $\lambda = \lambda^*_{2^2}$; curves T^{2^1} and T^{2^2} (dotted).

a $(k;j)$ *box (nonembedded representation)* with

$$\Omega_k^j = \left[\lambda_{(k)_0}^j, \lambda_k^{*j}\right], \quad \Omega_k^j = \omega_k^j \cup \Delta_k^j \subset \Delta_1, \quad \Delta_k^j = \left]\lambda_{ks}^j, \lambda_k^{*j}\right].$$

The index k is the *basic period* (or *rank-one basic period*) of all the cycles generated in the box Ω_k^j and j is the *basic permutation* (or rank-one basic permutation) of the points of these cycles. The interval ω_k^j is the *spectrum* $(k;j)$, which includes the *Myrberg cascade* (or the *period doubling cascade of flip bifurcations*) starting from the basic $(k;j)$-cycle of the box with $S \leq -1$. Considering T^k, the box Ω_k^j reproduces all bifurcations contained in the box Ω_1 in the same order (self-similarity property) for a set of the cycles (of the map T) having periods multiples of k (but not all possible cycles with these periods). Let $\Omega_{k_1}^{j_1}$ be one of these boxes. Inside $\Omega_{k_1}^{j_1}$, it is possible to define *rank-two* boxes

$$\Omega_{k_1.k_2}^{j_1,j_2} = \left[\lambda_{(k_1.k_2)_0}^{j_1,j_2}, \lambda_{k_1.k_2}^{*j_1,j_2}\right] \subset \Delta_{k_1}^{j_1},$$

related to two $(k_1.k_2; j_1, j_2)$ basic cycles reproducing, for $(T^{k_1})^{k_2}$, in the same order, all bifurcations of the box $\Omega_{k_1}^{j_1}$ and, hence, the bifurcations of Ω_1:

$$\Omega_{k_1.k_2}^{j_1,j_2} = \left[\lambda_{(k_1.k_2)_0}^{j_1,j_2}, \lambda_{k_1.k_2}^{*j_1,j_2}\right] = \omega_{k_1.k_2}^{j_1,j_2} \cup \Delta_{k_1.k_2}^{j_1,j_2} \subset \Delta_{k_1}^{j_1}, \qquad \Delta_{k_1.k_2}^{j_1,j_2} = \left]\lambda_{(k_1.k_2)s}^{j_1,j_2}, \lambda_{k_1.k_2}^{*j_1,j_2}\right].$$

All the cycles generated inside $\Omega_{k_1.k_2}^{j_1,j_2}$ have a *rank-one basic period* k_1, a *rank-one basic permutation* j_1, a *rank-two basic period* $k_1 k_2$, and a *rank-two basic permutation* (j_1, j_2). Similarly, from a couple of basic cycles $(k_1 \ldots k_a;\ j_1, \ldots, j_a)$, one with $S < 1$ and the other with $S > 1$, *rank-a boxes* embedded into a rank-$(a-1)$ box are defined as

$$\Omega_{k_1 \ldots k_a}^{j_1,\ldots,j_a} = \left[\lambda_{(k_1 \ldots k_a)_0}^{j_1,\ldots,j_a}, \lambda_{k_1 \ldots k_a}^{*j_1,\ldots,j_a}\right] = \omega_{k_1 \ldots k_a}^{j_1,\ldots,j_a} \cup \Delta_{k_1 \ldots k_a}^{j_1,\ldots,j_a} \subset \Delta_{k_1 \ldots k_{a-1}}^{j_1,\ldots,j_{a-1}}$$

with cycles having *rank-p basic periods*, $p = 1, \ldots, a$, and rank-p *basic permutations*. Moreover,

$$\Omega_{k_1 \ldots k_a}^{j_1,\ldots,j_a} \subset \Omega_{k_1 \ldots k_{a-1}}^{j_1,\ldots,j_{a-1}}, \quad a = 1, 2, \ldots.$$

The boundary parameter $\lambda_{k_1 \ldots k_a}^{*j_1,\ldots,j_a}$ of each of these boxes $(a = 1, 2, \ldots)$ corresponds to the merging of well-defined critical points with an unstable basic cycle with a multiplier $S > 1$. The boxes $\Omega_{k_1}^{j_1}, \ldots, \Omega_{k_1 \ldots k_a}^{j_1,\ldots,j_a} \ldots$ are called *boxes of the first kind*. The representation of these boxes is given in Fig. 2(a) with the enlargement in Fig. 2b.

The structure of *boxes of the second kind* (see Fig. 2c) can be defined as follows: Consider another type of bifurcation for $\lambda = \lambda_{2^i}^*$, $i = 1, 2, 3, \ldots$, now defined from an unstable cycle with $S < -1$ born from a *flip bifurcation*. The first and largest box is $\Omega_{2^1} \equiv \left[\lambda_{b1}, \lambda_{2^1}^*\right] \subset \Omega_1$, $\lambda_{2^1}^*$, $k = 1$, corresponding to $C_2 \equiv q_2$ $(S < -1)$. Similarly, we have boxes $\Omega_{2^m} \equiv \left[\lambda_{bm}, \lambda_{2^m}^*\right]$, $\lambda_{2^m}^*$, $k = 1$, corresponding to the critical points merging with the period 2^{m-1} cycle $(S < -1)$ from the rank 2^{m+1},

$$\Omega_{2^m} \subset \Omega_{2^{m-1}} \subset \ldots \subset \Omega_{2^1} \subset \Omega_1.$$

The values of $\lambda_{2^i}^*$, $i = 1, 2$, are defined from the curves T^{2^i} given by Figs. 2d – f (see also [23, p. 154, 155]). The first numerical values of $\lambda_{2^m}^*$ are $\lambda_{2^1}^* = 1.543689013$, $\lambda_{2^2}^* = 1.430357$, and $\lambda_{2^3}^* = 1.407405119$. More $\lambda_{2^m}^*$ values until $\lambda_{2^{12}}^* = 1.401155200$ (see Table II in [11, p. 119]) give an idea of $\lambda_{2^m}^*$ convergence rate toward $\lambda_{1s} = 1.401155189$ (also the limit of period doubling bifurcations). Moreover, each interval $\left[\lambda^*_{2^m}, \lambda^*_{2^{m+1}}\right] \subset \Omega_{2^m}$ contains boxes, self-similar with Ω_1 (cf. Fig. 2(c) with $m = 1$). For $\lambda = \lambda^*_{2^m}$, the map T gives rise to *m nonconnected intervals constituting an m-cyclic chaotic segment* denoted by $CH^{p_m}_{2^m}$.

Consider now a box $\Omega^{j_1}_{k_1}$. Bifurcations $\lambda = \lambda^{*j_1,p_m}_{k_1 2^m}$, $m = 1, 2, \ldots,$ $k_1 = 1, 3, 4, \ldots,$ can be defined in an equivalent way. They are characterized by the fact that the *critical points* from the rank $k_1 2^{m+1}$ merge into the unstable period $k_1 2^{m-1}$ cycle $(S < -1)$. From the *flip bifurcation* $\lambda = \lambda^{j_1}_{k_1 bm}$ generating the stable cycle $(k_1 2^m; j_1, p_m)$, the interval

$$\lambda^{j_1}_{k_1 bm} \leq \lambda \leq \lambda^{*j_1,p_m}_{k_1 2^m}$$

defines a *box of the second kind* denoted by $\Omega^{j_1}_{k_1 2^m} \subset \Omega^{j_1}_{k_1}$. As $m \to \infty$, the two boundaries of $\Omega^{j_1}_{k_1 2^m}$ tend toward $\lambda^{j_1}_{k_1 s}$ (the common limit of $\lambda^{*j_1,p_m}_{k_1 2^m}$, $k_1 \to \infty$, and the period doubling $\lambda^{j_1}_{k_1 bm}$, $m \to \infty$, with $\lambda^{*j_1}_{k_1 2^m} > \lambda^{j_1}_{k_1 s}$). For $\lambda = \lambda^{*j_1}_{k_1 2^m}$, the map T gives rise to $k_1 m$ nonconnected intervals constituting a *km-cyclic chaotic segment* denoted by $CH^{j_1,p_m}_{k_1 2^m}$. *For the box-within-a-box symbolism*, we note that a period $2^m k_1$ cycle is not a $k_1 2^m$ cycle because they are not generated in the same box.

3.3. Some Properties. Considering map (1) and *increasing the values of the parameter λ*: *the multiplier S of a cycle $(k; j)_{S>1}$ increases* and *the multiplier S of a cycle $(k; j)_{S<-1}$ decreases. Hence, these cycles become more and more unstable, and cannot disappear by bifurcation.* The following properties result from the fractal embedded organization:

(a) For $\lambda = \lambda_{1s} + \varepsilon$, $\varepsilon > 0$, $\varepsilon \to 0$, the map T generates *infinitely many unstable period 2^i cycles.*

(b) Let $[k, j]$ *(nonembedded representation)*, $k = 1, 3, 4, \ldots,$ be the *basic cycle of the box Ω^j_k* with the multiplier $S < 1$. For $\lambda \geq \lambda^j_{ks}$, the spectrum ω^j_k generates an invariant set $Cs[k, j]$ made up of all the unstable $(k2^i, j, p_i)$-cycles, $i = 0, 1, 2, \ldots,$ with multiplier $S < -1$, and their limit sets.

(c) Let $(k_1; j_1)$ *(nonembedded representation)* be the *basic cycle* $(S < 1)$ of the *rank-one box* $\Omega^{j_1}_{k_1}$. Let $(k_1 k_2 \ldots k_a;\ j_1, j_2, \ldots, j_a)$ be the period k cycle, $k = k_1 k_2 \ldots k_a$ (associated with the permutations $j_1, j_2, \ldots, j_a$). For $\lambda \geq \lambda^{*j_1}_{k_1}$, the box $\Omega^{j_1}_{k_1}$ generates an *invariant set* (in the Pulkin sense, cf. Section 2) $Cs[k_1, j_1]$. This set is made up of infinitely many sets, $Cs[k_1 k_2; j_1, j_2], \ldots, Cs[k_1 k_2 \ldots k_a; j_1, j_2, \ldots, j_a], \ldots,$ $a = 1, 2, \ldots, \infty$, generated from the infinitely many boxes embedded into $\Omega^{j_1}_{k_1}$.

(d) For $\lambda \geq \lambda^{*j_1}_{k_1}$, the map T (thus, not only the box $\Omega^{j_1}_{k_1}$ as in (c)) generates infinitely many invariant sets related to the infinitely many boxes created for $\lambda < \lambda^{*j_1}_{k_1}$.

(e) For $\lambda < \lambda^{j_1}_{(k_1)s}$, the map T generates infinitely many invariant sets related to the infinitely many boxes created for $\lambda < \lambda^{j_1}_{(k_1)_0}$.

(f) For $\lambda \geq \lambda^*_1$, T generates all possible cycles (which are unstable) and their limit sets created from the infinitely many boxes embedded into the *overall box* Ω_1, and *all belong to an invariant set included in the interval* $[q_1^{-1}, q_1]$.

For any $\lambda \geq \lambda_{1s}$, we denote by Λ^*_λ the *fractal invariant set* belonging to $\left[q_1^{-1}, q_1\right]$, which includes all the unstable cycles and their limit sets created for the values of the parameter smaller than λ (whose bifurcation organization is defined and represented in Fig. 2), this together with all preimages and limit points of Λ^*_λ. If $\lambda_{1s} \leq \lambda < \lambda^*_1$, from any initial point $x_0 \in \left[q_1^{-1}, q_1\right] \backslash \Lambda^*_\lambda$, after $N(x_0)$ iterations ($N(x_0)$ depending on the initial point), the iterated points orbit enters an ε-neighborhood ($\varepsilon > 0$ is sufficiently small) of the *unique stable set* existing in $\overline{CC_1}$. If the point x_0 is sufficiently close to Λ_λ^*, then $N(x_0)$ *may be quite high and the orbit of the iterated points has the form of a chaotic transient* toward the *unique stable set* existing in $\overline{CC_1}$. For $\lambda > \lambda_1^*$ and $x_0 \in \left]q_1^{-1}, q_1\right[\backslash\Lambda_\lambda^*$, we denote by $N(x_0)$ the number of iterated points occurring in the interval $\left]q_1^{-1}, q_1\right[$. After $N > N(x_0)$ iterations, the iterated point is mapped outside $\left]q_1^{-1}, q_1\right[$. Then the *orbit diverges tending toward infinity*; first, with a *chaotic transient*.

3.4. Limit Sets of the Boxes and Resulting Properties. Adapting Fatou's results [5, 6] to the case of a real variable, we can deduce the following properties [23 p. 156–160]:

(i) When T has a stable cycle ($|S| < 1$), a point of the critical set E_c, or its derived set E'_c, does not belong to the set E of unstable cycles ($|S| > 1$), or to its derived set E'.

(ii) When $E \cup E'$ contains points of $E_c \cup E'_c$, then some bifurcation occurs, giving either a neutral cycle with $|S| = 1$ or some chaotic stable set, say, for $\lambda = \widehat{\lambda}$, that is T has either a critical attractor A_{cr} or k-cyclic chaotic segments ($k \geq 1$) in the interval $\overline{CC_1}$ (for $k = 1$, the chaotic interval is bounded by the critical points C and C_1). Hence, e.g., λ_k^{*j} (or any closure of a box of the first kind), $\lambda_{k2^i}^{*j}$ (or any closure of a box of the second kind), and λ_{ks}^j (Myrberg's limit point of the sequence of flip bifurcations) are particular values of $\widehat{\lambda}$. For λ_k^{*j}, k points of E_c and their increasing rank images merge into k points of E. For $\lambda_{k2^i}^{*j}$, $k2^i$ points of E_c and their increasing rank images merge into $k2^i$ points of E. For λ_{ks}^j, the whole set E'_c coincides with the critical attractor A_{cr} and belongs to E'.

This first set of properties concerns different types of limit sets of rank-one boxes Ω_r^h, $r = 3, 4, \ldots$, sequences (more details can be found in [23, p. 156–160 and 166–174]). Hence,

(a) Consider a rank-one box of the first kind

$$\Omega_k^j = \left[\lambda_{(k)_0}^j, \lambda_k^{*j}\right], \qquad k = 3, 4, \ldots,$$

and its boundaries. For $\lambda < \lambda_{(k)_0}^j$, *the parameter value* $\lambda_{(k)_0}^j$ (for which the set E'_c consists in the $(k; j)$ cycle) *is a limit point of rank-one boxes of the first kind* $\Omega_{k'}^{j'}$ as $k' \to \infty$. *For* $\lambda > \lambda_k^{*j}$, *the value* λ_k^{*j} *is a limit point of rank-one boxes* $\Omega_{k''}^{j''}$ as $k'' \to \infty$. *For* $\lambda < \lambda_k^{*j}$, $\lambda_k^{*j} \subset \Omega_k^j$, the value λ_k^{*j} is such that E_c includes the unstable $(k; j)_{S>1}$ cycle (i.e., C is either periodic or preperiodic), the set E_c does not contain accumulation points.

(b) Inside each Ω_k^j box with $\lambda > \lambda_{ks}^j$, any bifurcation value λ_{ks}^j is a limit point of the values $\lambda_{k2^i}^{*j}$ as $i \to \infty$ and, for $\lambda < \lambda_{ks}^j$, λ_{ks}^j is a limit point of the sequence of flip bifurcations generated in the interval ω_k^j. The value λ_{ks}^j is such that the whole set E'_c coincides with the critical attractor A_{cr} (i.e., the invariant set $\Lambda_\lambda^* \subset [q_1^{-1}, q_1]$).

(c) *Parameter values of the type* $\widehat{\lambda}$, denoted by $\widetilde{\lambda}$, *exist as limits of the boxes* Ω_r^h but without belonging to the box boundary. For example, $\widetilde{\lambda} \simeq 1.89291098791$ for which $q_2 \equiv C_3$ (and similar values exist for each $k \geq 3$ at which $q_2 \equiv C_k$ [22, 23]). Then $\overline{CC_1}$ is an *absorbing chaotic segment giving rise to a nonclassical invariant measure* (cf. [2], [23, p. 156–160 and 166–174]). Moreover, for these particular bifurcation values (for which the stable set of the map is either a chaotic interval or cyclic chaotic intervals), the set E_c includes an unstable cycle (i.e., C is either periodic or preperiodic; the set E_c does not contain accumulation points).

(d) Due to the property of self-similarity, (a)–(c) also recur for embedded rank-a boxes, $a > 1$, with adaptations related to their rank, e.g., in this case, $\overline{CC_1}$ contains some cyclic chaotic segment giving rise to a nonclassical invariant measure.

Note that, for $\lambda = \lambda_k^{*j}$, the cyclic chaotic segment CH_k^j made up of the k segments $\overline{CC_k}$, $\overline{C_1C_{k+1}}, \ldots,$ $\overline{C_{k-1}C_{2k-1}}$, contains all unstable cycles created inside the Ω_k^j box and their limit sets. Its complementary part $\overline{CC_1} \backslash CH_k^j$ inside $\overline{CC_1}$ contains all unstable cycles created for $\lambda < \lambda_{(k)0}^j$. A value $\widetilde{\lambda}$, limit of the boxes Ω_r^h, is such that $\overline{CC_1}$ contains all cycles created for $\lambda < \widetilde{\lambda}$, except q_1.

3.5. General Occurrence of the Embedded Boxes Organization. The embedded boxes organization generated by the Myrberg map T (1) also occurs for some other types of unimodal maps. In particular, in the case of the general form of quadratic map $y' = ay^2 + 2by + c$, a linear change of variable $y = \alpha x + \beta$ leads to (1) with $\lambda = b^2 - ac - b$, $a\alpha = 1$, and $a\beta = -b$. Moreover, particular classes of *bimodal maps* (maps with two extrema, i.e., $Z_1 - Z_3 - Z_1$ maps) locally create a bifurcations organization related to each of the two possible attractors (see [11, p. 401–418]). The case of maps T, $x_{n+1} = f(x_n, \lambda)$, with only one extremum but with two attractors, e.g., a stable fixed point *A*, and the *absorbing segment* $\overline{CC_1}$ (Section 3.1a) is dealt from the simple Fig. 6 in [21] or [23, p. 75]. Here, the variation of the parameter λ may lead to the notions called *fuzzy basin boundary*, or *chaotic transient*. The notion of *embedded boxes* was extended to the case of the *Hénon map* (a $\mathrm{Dim}\,2$ *invertible map* that depends on two parameters). This situation gives rise to parameter regions (a, b) overlapping with two *boxes* for which two, or more, attractors may coexist for a point (a, b) showing the *"foliated"* nature of the parameter plane made up of *embedded boxes* located in different sheets of this plane. Lorenz [16] quoted these parameter structures [2, 17–19] giving rise to the possibility of sheets communication via local plane structures called *cross-road area*, *spring-area*, and *saddle-area* (see [4, 28], Chapter 6 in [23], and the references therein devoted to this topic).

4. Intermittency and the Embedded Boxes Structure

This section is a shortened version of [29]. The word *intermittency* was introduced in the famous paper *"Intermittent Transition to Turbulence in Dissipative Dynamical Systems"* [35] from observation of an intermittent transition to turbulence in convective fluids. Under the less attractive name *"cycle in average value"* (*"cycle en valeur moyenne"* in French), the *intermittency phenomenon* was the topic of the first part of the French paper *"Sur la Double Interprétation, Déterministe et Statistique, de Certaines Bifurcations Complexes"* [20]. This paper was followed by more details given in Section 2.8.2 of the book [11] (in French), and Section 4.5.4 of the book [23] (in English). The basic approach of the *intermittency problem* is different from the approach adopted in [35]. It directly considers the properties induced by the quadratic map [19].

Figure 1(b) (above Section 2) illustrates this situation in the case where $\lambda = \lambda_{(3)_0} - \varepsilon$, $\varepsilon > 0$ is sufficiently small; the situation is related to a *period 3 "cycle in average value"* [20, 23]. For $\lambda = \lambda_{(3)_0}$, the *period 3 "cycle in average value"* merges into the *period 3 cycle with the multiplier* (eigenvalue) $S = +1$. For sufficiently small $\varepsilon > 0$ (T^3 is the nearest to $x_{n+1} = x_n$), the stable cycle $(k; j)$ is *"announced"* by k intervals of iterated points with *a higher density of these points with respect to the others*. In [20, 23], these regions are called *period 3 "cycle in average value."* This property appears from the Koenigs–Lemeray graphical representation of the map T^k in the $(x_n; x_{n+1})$ plane (see Fig. 1 in Subsection 3.2) for $k = 3$. Then k arcs of T^k with parabolic shapes are close to the first bisectrix $x_{n+1} = x_n$, before their tangency is reached for $\varepsilon = 0$ *(fold bifurcation)*. The notion of *cycle in average value* [20] was introduced as one of the four accumulation points (called *singular* by Myrberg) of boxes generated by the quadratic Myrberg $x' = x^2 - \lambda$. In this sense, the *fold boundary* $\lambda_{(k)_0}^j$ of a box Ω_k^j is *"singular."* When $\lambda = \lambda_{(k)_0}^j - \varepsilon$, $\varepsilon \to 0$, $\varepsilon > 0$ is sufficiently small, $\lambda_{(k)_0}^j$ is the limit of a subset of an infinite sequence of $\Omega_{k'}^{j'}$ boxes with *fold bifurcations* $\lambda_{(k')_0}^{j'} < \lambda_{(k)_0}^{j}$, $\Omega_{k'}^{j'} < \lambda_{(k)_0}^{j}$, and

$$\lim \lambda_{(k')_0 \varepsilon \to 0}^{j'} = \lim \Omega_{k' \varepsilon \to 0}^{j'} = \lambda_{(k)_0}^{j}$$

(cf. above Figs. 2(a, b) in Subsection 3.2).

Considering the above property ("$[\mathrm{r}'] < [\mathrm{r}]$ *is equivalent to* $\lambda' < \lambda$ for $f(x,\lambda) = x^2 - \lambda$"), $\lambda_{(k)_0}^{j}$ satisfies the following *periodic radical series* (see Section 3.3.9 in [23]):

$$\lambda_{(k)_0}^{j} = [\mathrm{p}_0](\lambda_{(k)_0}^{j}), \qquad [\mathrm{p}_0] \equiv [\mathrm{p}_{0k}^{j}] \equiv (B_0)^\infty \equiv (m_1^+ n_1^- \ldots m_u^+ n_u^-)^\infty,$$

$$m_1^+ + n_1^- + \ldots + m_u^+ + n_u^- = k.$$

Here, p_0 (made up of k signs $\pm$) is related not to a cycle with a multiplier $S = 0$, as for (3), *but to the fold bifurcation* $\lambda = \lambda_{(k)_0}^{j}$. The first sign of (B_0) is the sign of the slope of $f(x,\lambda)$ at the rank-2 critical point

$$C_1 = T(C) = T^2(x_e = 0).$$

The following signs corresponds to the slopes at $C_2, C_3, \ldots, C_k$. If $[\mathrm{r}]$ is the *rotation sequence* of the cycle $(k;j)$, then $(B_0) = ([\mathrm{r}] + -)$ if the number of signs $-$ in $[\mathrm{r}]$ is odd and $(B_0) = ([\mathrm{r}] - -)$ if this number is either even or zero. The sequence $[\mathrm{r}_{0k}^{j}]$ is the limit (see Section 4 in [23]) of rotation sequences $(B_0)^m$, $m = 1, 2, \ldots$, of period k' cycles, $k' = 2 + mk$, each of which is a "basic" cycle of a well-defined box $\Omega_{k'}^{j'}$. Examples of *periodic radical series* are as follows: cycle $k = 3$ $(\lambda_{(3)_0}^{j} = 7/4)$, $[\mathrm{r}_{03}^{1}] = (+--)^\infty$; cycle $k = 4$, $[\mathrm{r}_{04}^{1}] = (++--)^\infty$; cycle $(5, j = 1)$, $[\mathrm{r}_{05}^{1}] = (+----)^\infty$; cycle $(5, j = 2)$, $[\mathrm{r}_{05}^{2}] = (++-+-)^\infty$, and cycle $(5, j = 3)$, $[\mathrm{r}_{05}^{3}] = (+++--)^\infty$.

If $\lambda = \lambda_{(k)_0}^{j} - \varepsilon$, $\varepsilon \to 0$, $\varepsilon > 0$, then $\lim\, [\mathrm{r}_{k'}^{j'}]_{\varepsilon \to 0} = \big[\mathrm{r}_{0k}^{j}\big]$ corresponds to

$$\lim_{k' \to \infty} \Omega_{k'}^{j'} = \lambda_{(k)_0}^{j}$$

(Fig. 2b) and

$$\lim_{k' \to \infty} \big[\mathrm{r}_{k'}^{j'}\big] = \big[\mathrm{r}_{0k}^{j}\big].$$

In Section 3.4.3 in [23], it is shown that the convergence of $\lambda_{(k')_0}^{j'}$ to $\lambda_{(k)_0}^{j}$ is the slowest one (the ratio $\rho = 1^-$) as compared to the other *singular values*.

Example. $k = 3$ $\big(\lambda_{(3)_0}^{j=1} = 7/4\big)$; the first term of the *periodic radical series* is $(+--)^1$; $k = 5$, $j = 1$, $\lambda_{S=0} \simeq 1.62541$; the first term is $(+--)^2$; $k = 8$, $j = 4$, $\lambda_{S=0} \simeq 1.71108$; the first term is $(+--)^3; \ldots;$ $k = 11$, $j = 9$, $\lambda_{S=0} \simeq 1.73200$; the first term is $(+--)^4, \ldots.$

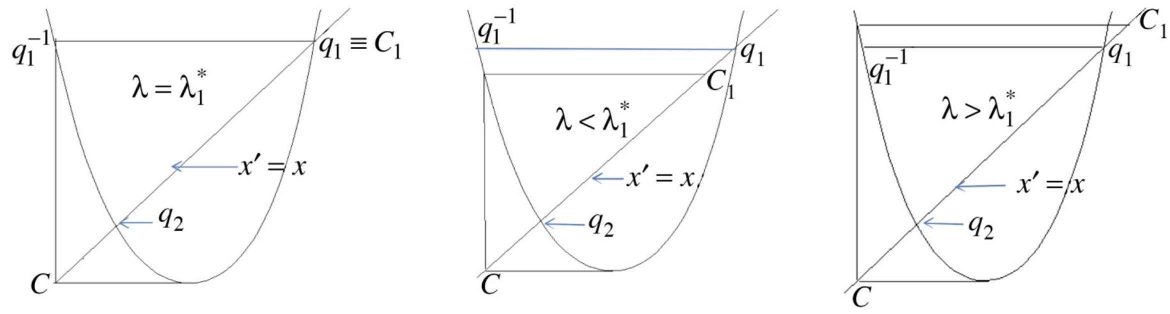
5. Attractors in Crisis and the *Embedded Boxes* Structure

This section is a shortened version of [30] and deals with a bifurcation resulting from the contact of two singularities of different nature: a *chaotic attractor* and *an unstable periodic orbit*. This topic was the purpose of two fundamentally different approaches in [8] and [20], when the attractor is generated by a $\mathrm{Dim}\,1$ unimodal sufficiently smooth map (here, we essentially use a quadratic map). The two above-mentioned approaches were independently considered by the authors without knowledge of their reciprocal contributions. The more recent paper (1982) has called *crisis* of this bifurcation. In the setting of its approach (from the physical point of view), the cited paper is an outstanding publication, which made known this interesting nonlinear dynamical phenomenon leading to many other articles from the same authors and also from many others. The older paper (1976) was published in French.

5.1. The Grebogi, et al. [8] Contribution. Within the scope of sudden qualitative changes in chaotic behaviors, the word *crisis* was introduced in the famous paper *"Chaotic Attractors in Crisis"* [8]. These qualitative changes were investigated *"for the parameter values at which the attractor collides with an unstable periodic orbit."* This type of behavior appeared from the consideration of *bifurcation diagrams* generated by a Dim1 quadratic map T written in the form $x_{n+1} = C - x_n^2$. The study was completed by the formulation of a law (see [8, p. 1509–1510]) related to the dynamical behaviors of orbits followed by the numerical analysis. Two types of *crisis* were considered, namely, the *boundary crisis*, which results from a contact of the attractor with its basin boundary and the *interior crisis* when the size of the chaotic attractor suddenly increases (the attractor is in contact with an unstable periodic orbit located inside the attractor basin). The study carried out by Grebogi, et al. [8] was specially oriented toward the period 3 cycle at $C = C_{*3} \simeq 1.79$ and a period 5 orbit (not defined) at $C = C_{*5}$. At the "tangent bifurcation" *(fold bifurcation)* $k = 3\colon\ C = 7/4$. The paper adds the following results: *"For a range of C smaller than a certain critical value C_{*3} chaos occurs in three different bands but, as C increases and passes through $C_{*3} \simeq 1.79$, the indicated three chaotic regions suddenly widen to form a single band. Furthermore, this precisely coincides with the intersection of the unstable period three orbit created at the original tangent bifurcation with the chaotic region."*

The paper [8] was followed by several papers dealing with certain extensions of the notion of *crisis* and complementary results (for details, see [30]). From the physical point of view, within the scope of investigation of the physical phenomena, it is reasonable to consider as *chaotic* the association of a long chaotic transient with an orbit whose period tends to infinity. This point explains what can be called a *macroscopic* analysis of the phenomenon. Based on the classical methods of chaotic dynamics associated with the formulation of laws and fine numerical studies, the approach of paper [8] served as a source of other outstanding English publications by the authors of the cited 1982 paper aimed at clarifying the *crisis* behavior. For its part, Mira [20] also proposed a *macroscopic* study based on changes in the orbits density in passing through the crisis parameter. Nevertheless, with regard for a more thorough study performed in the 1982 paper and subsequent papers, the "macroscopic" study remains a coarse analysis of the phenomenon without formulation of the laws of dynamical behavior.

5.2. Contributions of the French Texts [20] and the Book [11]). This section is a shortened version of [30]. According to a different approach, the second part of a French paper *"Sur la Double Interprétation, Déterministe et Statistique, de Certaines Bifurcations Complexes"* [20] implicitly deals with the topic of [8] from the notion of *embedded bifurcations boxes*. This is also the case of the book [11] (in French) and the book [23, p. 162–174] (in English), which develop some aspects of the topic. Instead of the 1982 term *"attractor in crisis,"* the second part of the 1976 French paper uses a less attractive name *"segment cyclique stochastique"* or, in abbreviated form, *"segment cyclique S"* (in English, this is *"periodic stochastic segment"* or *"periodic segment S"*). These terms were also used one year earlier in [10, 18, 19]. The chaotic attractor *(segment S)* is defined by its boundaries *(critical points)*. Then, according to Sharkovsky (1969), these *critical points* are *homoclinic* to the unstable cycle (this situation was called *snap-back repeller* by Marotto in 1978). This periodic segment is *fully invariant* [36] under T^k. *Such attractors are bounded by rank-$(n+1)$ critical points* (in the Julia–Fatou sense) $C_n = T^{n+1}(x_e)$, $n = 0, 1, 2, \dots$, $C_0 \equiv C = T(x_e)$, where x_e is the abscissa of the T extremum (in the Julia–Fatou sense, x_e is never a *critical point*, despite a quasigeneral misuse since 1975). When C_n merges into a point of a $(k; j)$ unstable



(a) **(b)** **(c)**

Fig. 3. $x' = x^2 - \lambda$; (a) $\lambda = \lambda_1^* = 2,\ C_1 \equiv q_1$; (b) $\lambda < \lambda_1^*,\ \overline{CC_1} \subset \overline{q_1^{-1}q_1},\ \overline{CC_1}$ is an *absorbing segment* (with $q_1^{-1} \cup q_1$ as the *basin boundary*); (c) $\lambda > \lambda_1^*,\ \overline{CC_1} \notin \overline{q_1^{-1}q_1}$.

cycle, we have a *nonclassical bifurcation (merging, or contact, or collision of singular sets of different nature)* denoted by $\lambda = \lambda_k^{*j}$. *This bifurcation was called crisis in the 1982 paper.*

If we consider $x' = f(x,\lambda) = x^2 - \lambda$ [map (1); see Section 2 and Subsection 3.1 of this text], then the parameter λ_k^{*j} is related to the parameter $C_{*3} \simeq 1.79$ in [8]. Hence, each parameter $\lambda_k^{*j},\ k = 1,3,4,\ldots,$ corresponds to a well-defined $(k;j)$ *attractor in crisis*, $k = 1,3,4,\ldots$. As for this topic, the book [11] gives two tables (see pp. 118–119) of $(k;j)$ cycles. The last column in p. 118 is related to the values of λ_k^{*j}: $\lambda_1^* = 2$ and $\lambda_3^{1*} \simeq 1.79032749$ (see above Fig. 1(c)); paper [8] gives $C_{*3} \simeq 1.79$. I. Gumowski and C. Mira [11] gave more results. Among them $\lambda_4^{1*} \simeq 1.942762011$. For $k = 5$, *the only three possible parameter values* are $\lambda_5^{1*} \simeq 1.633358704,\ \lambda_5^{2*} \simeq 1.862361091$, and $\lambda_5^{3*} \simeq 1.985540378,\ldots$. In addition, see [11, p. 119], we have the numerical values of $\lambda_{2^i}^*,\ i = 1,2,\ldots,11,12$, and $\lambda_{1_s} \simeq 1.401155189$, which is also the accumulation value of period doubling bifurcations from the fixed point q_2 (see Section 2 and Fig. 2). The bifurcation $C = C_{*3}$ (defined, in [8], as the *" intersection of the unstable period three orbit created at the original tangent bifurcation with the chaotic region"*) is the bifurcation obtained for $\lambda = \lambda_3^*$. Several values of λ_k^{j*} are given in the tables presented in [11, p. 118–119]. For $k = 3$ $(C = C_{*3}$ in [8]), what Grebogi calls *"chaotic region"* is made up of a *"period 3 segment S"*: $\overline{CC_3} \cup \overline{C_4C_1} \cup \overline{C_5C_2}$ bounded by *critical points* [11]. This occurs when the points $\beta_i,\ i = 1,2,3,$ of the unstable period 3 cycle (*multiplier*, i.e., eigenvalue, $S > 1$) are such that β_1 merges into C_3, β_2 merges into C_4, and β_3 merges into C_5 (Section 3.2; Fig. 1). The basin boundary of the *period 3 segment S* consists of a fixed point q_1 and its rank-one preimage q_1^{-1} (or the *antecedent* of q_1, $T^{-1}q_1 = q_1 \cup q_1^{-1}$, and T^{-1} is the inverse of T).

In Fig. 3, we present the behavior changes in passing through the crisis parameter λ_1^* :

(a) $\lambda = \lambda_1^* = 2,\ C_1 \equiv q_1$, and the segment S $\overline{CC_1}$ contains a *fractal set formed by all period k unstable cycles*, $k = 2,3,\ldots,$ *with their accumulation points of increasing classes p* (in Pulkin's sense; see Section 3.1).

(b) $\lambda < \lambda_1^*,\ \overline{CC_1} \subset \overline{q_1^{-1}q_1}$, and $\overline{CC_1}$ is an *absorbing segment* (with $q_1^{-1} \cup q_1$ as the *basin boundary*); if $\lambda_1^* > \lambda > \lambda_{1s}$, *then* $\overline{CC_1}$ *contains an attracting set and a fractal set of period k unstable cycles*, $k = 2,3,\ldots,\ k \to \infty$, with their limit points of *increasing classes p*, which leads to a *chaotic transient* toward the attracting set when $x_0 \in \left]q_1^{-1}q_1\right[$.

(c) $\lambda > \lambda_1^*,\ \overline{CC_1} \notin \overline{q_1^{-1}q_1}$, and *contains a fractal set of period k unstable cycles,* $k = 2,3,\ldots,$ *with their accumulation points of the increasing class p leading to a chaotic transient toward the point at infinity* when $x_0 \in \left]q_1^{-1}q_1\right[$; we have equivalent situations for the parameter variations around $\lambda = \lambda_{(2^i)}^*$, $\lambda = \lambda_k^{*j}$, and $\lambda = \lambda_{k2^i}^{*j}$, which lead to sudden and strong *qualitative changes in the map dynamics*;

this can be explained by the fact that, on both sides, $\lambda^*_{(2^i)}$, λ^{*j}_k, and $\lambda^{*j}_{k2^i}$, are the limits of infinitely many boxes $\Omega^{\overline{j}}_{\overline{k}}$, $\overline{k} \to \infty$, $\overline{k} = k'$, or k'', or k''' (cf. Section 3, Fig. 2).

Equivalent situations also occur for the boundaries of *rank-a boxes* $\Omega^{j_1,\dots,j_a}_{k_1\dots k_a} \subset \Omega^{j_1,\dots,j_{a-1}}_{k_1\dots k_{a-1}}$ [rank-$(a-1)$ box; see Figs. 2(a, b)] with self-similarity properties defined by using an *"embedded"* cycle symbolism $(k_1, k_2, \dots, k_a; j_1, j_2, \dots, j_a)$ related to a "basic" period k cycle, $k = k_1 k_2 \dots k_a$ and a *rotation sequence j decomposable* into a rotation sequences $j = j_1, \dots, j_a$ (see Section 4.3 in the book [23]).

Another class of *"nonclassical bifurcations"* $\widetilde{\lambda} \neq \lambda^{*j}_k$, as contacts of critical points with an unstable cycle, are such that $\widetilde{\lambda}$ is an accumulation of boxes on both sides but without being a box boundary (the situation of Fig. 4.12 in the book [23, p. 169]). Here, the critical point C_3 merges into the fixed point q_2 with multiplier $S < 1$ ($\lambda = \widetilde{\lambda} \simeq 1.89291098791$; see Fig. 6 in [22] and Fig. 4.12 in [23]). Each side of λ^{*j}_k, $\lambda^*_{2^i}$, and $\widetilde{\lambda}$ is a limit of an infinite sequence of boxes $\Omega^{\overline{j}}_{\overline{k}}$ defined from the cycles $(\overline{k}, \overline{j})$ of very high periods *(nonembedded representation)* generated from a *fold bifurcation* $\lambda^{\overline{j}}_{(\overline{k})_0}$. See Fig. 2(b) in Section 2 with λ^{*j}_k : $\overline{k} \to \infty$ and $\overline{k} = k''$ if $\lambda = \lambda^{*j}_k - \varepsilon$ or $\overline{k} = k'''$ if $\lambda = \lambda^{*j}_k + \varepsilon$.

Remark. Let ρ be the convergence ratio of a subset of boxes toward the λ-accumulation values λ^*_1, $\lambda^*_{2^1}$, λ^{*j}_k, $\widehat{\lambda}$, and $\widetilde{\lambda}$ (see Section 3.4 in [23]). It is given by the formula $\rho = 1/|S|$, where S is the multiplier (eigenvalue, $|S| > 1$) of the unstable cycle from which these *singular parameter values* are determined.

6. Conclusions

The *bifurcations embedded boxes organization* permits a precise identification of each period k cycle generated by a unimodal smooth map. This is made via a symbolism (k, j) and its *embedded forms*, where *j* allows us to distinguish the cycles with the same period but with different exchanges of their points. In presence of *chaotic behaviors*, this organization leads to a kind of *"microscopic"* study of the situation. This is the case of *intermittency* and *attractors in crisis*, which give rise to *long chaotic transients toward orbits with a period tending toward infinity*. From the physical point of view, within the scope of investigation of physical phenomena, it is reasonable to consider as *chaotic* the association of a *long chaotic transient toward an attractor*. In this sense, studies of this kind constitute what may be called a *macroscopic* view of the phenomenon with formulation of adapted dynamical laws in the presence of changes in the dynamical behavior, as shown in the fine papers [35] and [8].

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