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ON THE ASYMPTOTICS OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS

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Consider a one-dimensional stochastic differential equation with jumps

$$dX(t) = a(X(t))dt + \sum_{k=1}^{m} b_k(X(t-))dZ_k(t),$$

where $Z_k,\ k \in \{1, 2, \ldots, m\}$, are independent centered Lévy processes with finite second moments. We prove that if the coefficient $a(x)$ has a certain power asymptotics as $x \to \infty$ and the coefficients b_k, $k \in \{1, 2, \ldots, m\}$, satisfy certain growth condition, then the solution $X(t)$ has the same asymptotics as the solution of the ordinary differential equation $dx(t) = a(x(t))dt$ as $t \to \infty$ a.s.

1. Introduction

As a rule, the researchers consider two types of behavior of the solutions of stochastic differential equations as $t \to \infty$, namely, tending to infinity and recurrence. In the present paper, we assume that the solution of a stochastic differential equation tends to infinity and study its exact asymptotics.

For the first time, this problem was considered by Gikhman and Skorokhod in [3] for the one-dimensional stochastic differential equation

$$dX(t) = a(X(t))dt + b(X(t))dW(t), \tag{1}$$

where W is a one-dimensional Wiener process. In particular, they established sufficient conditions for $X(t) \to +\infty$ as $t \to \infty$ and $X(t) \sim x(t)$ as $t \to \infty$ almost surely, where x is a solution of the ordinary differential equation

$$dx(t) = a(x(t))dt. \tag{2}$$

Later, this problem was investigated in [4]. Some types of nonautonomous stochastic differential equations were studied in [1]. Stochastic differential equations with non-Gaussian noise were investigated in [9, 10].

In the monograph [2], the author studied the problems of tending to infinity and recurrence for the solutions to the system of linear stochastic differential equations and the behavior of the polar angle of solution to a two-dimensional stochastic differential equation. In [12], the asymptotic behavior of multidimensional stochastic differential equations was compared with the behavior of linear ordinary differential equations. In [13], we considered a multidimensional stochastic differential equation of the form (1) and studied the behavior of its solution as $t \to \infty$ a.s., namely, the conditions of tending to infinity for the modulus of solution, stabilization of the angle $X(t)/|X(t)|$, and asymptotics of the modulus of solution. Similar problems for the additive Lévy noise were studied in [11].

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The problem of asymptotic behavior of the stochastic differential equations with multiplicative Lévy noise was not investigated in the literature. In the present paper, we consider a stochastic differential equation with jumps of the form

$$dX(t) = a(X(t))dt + \sum_{k=1}^{m} b_k(X(t-))dZ_k(t),$$

where $Z_k,\ k \in \{1, 2, \ldots, m\}$, are independent centered Lévy processes with finite second moment. We prove that if the coefficient $a(t)$ has a certain power asymptotics as $t \to \infty$ and the coefficients $b_k,\ k \in \{1, 2, \ldots, m\}$, satisfy certain growth conditions, then the solution $X(t)$ a.s. has the same asymptotics as $t \to \infty$ as the solution of the ordinary differential equation (2).

The present paper is organized as follows: In Sec. 2, we prove lemmas on the asymptotic behavior of stochastic integrals with respect to a Wiener process and with respect to a compensated Poisson measure. In Sec. 3, we establish two main results on the asymptotic behavior of solutions to stochastic differential equations with jumps, namely, Theorem 1 in which the drift coefficient is equivalent to a positive constant and Theorem 3 in which the drift coefficient is equivalent to a positive power function. In both theorems, certain conditions are imposed on the growth rate of the characteristics of noise. In Sec. 4, we prove the lemma required for the proof of the theorems presented in Sec. 3.

2. Asymptotics of Stochastic Integrals

In the present section, we obtain some additional results on the asymptotics of stochastic integrals with variable upper bound t as $t \to \infty$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, let $W = W(t)$ be an $\mathbb{F}$-Wiener process, and let $N = N(dt, du)$ be an $\mathbb{F}$-Poisson random measure on[1] $\mathbb{R}_+ \times \mathbb{R}$ independent of W with intensity measure $dt \cdot \nu(du)$, where the measure ν is such that

$$\int_{\mathbb{R}} u^2 \nu(du) < \infty \qquad \text{and} \qquad \tilde{N} = \tilde{N}(dt, du) := N(dt, du) - dt \cdot \nu(du).$$

Lemma 1. *Suppose that $M = M(t)$ is a square-integrable martingale. If $\mathbb{E}M^2(t) = O(t^\gamma)$ as $t \to \infty$ for some $\gamma < 2$, then*

$$\frac{M(t)}{t} \to 0 \quad \textit{as} \quad t \to \infty \quad \textit{a.s.}$$

Proof. It follows from the condition that there exists $T \geq 0$ such that

$$\mathbb{E}M^2(t) \leq Ct^\gamma, \quad t \geq T,$$

where $C \geq 0$. Let $\varepsilon > 0$ and let $k \in \mathbb{N}$ be such that $2^{k+1} \geq T$. We estimate the probability

$$\mathbb{P}\left\{ \sup_{2^k \leq t \leq 2^{k+1}} \left| \frac{M(t)}{t} \right| \geq \varepsilon \right\} \leq \mathbb{P}\left\{ \sup_{2^k \leq t \leq 2^{k+1}} \frac{|M(t)|}{2^k} \geq \varepsilon \right\}$$

[1] $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

$$\le \mathbb{P}\left\{\sup_{t\le 2^{k+1}} |M(t)| \ge \varepsilon 2^k\right\}$$

$$\le \frac{\mathbb{E}M^2(2^{k+1})}{(\varepsilon 2^k)^2} \le \frac{C\left(2^{k+1}\right)^\gamma}{\varepsilon^2 2^{2k}} = \frac{C2^\gamma}{\varepsilon^2}\left(2^{\gamma-2}\right)^k \qquad \text{(by the Doob inequality).}$$

For $n \in \mathbb{N}$ such that $2^{n+1} \ge T$, we have

$$\mathbb{P}\left\{\limsup_{t\to\infty}\left|\frac{M(t)}{t}\right| \ge \varepsilon\right\} \le \mathbb{P}\left\{\sup_{t\ge 2^n}\left|\frac{M(t)}{t}\right| \ge \varepsilon\right\}$$

$$\le \sum_{k=n}^{\infty}\mathbb{P}\left\{\sup_{2^k\le t\le 2^{k+1}}\left|\frac{M(t)}{t}\right| \ge \varepsilon\right\}$$

$$\le \frac{C2^\gamma}{\varepsilon^2}\sum_{k=n}^{\infty}\left(2^{\gamma-2}\right)^k.$$

The last series converges to 0 as $n \to \infty$ (because $\gamma - 2 < 0$). Therefore, the probability at the beginning of the chain of inequalities is equal to 0. Since $\varepsilon > 0$ is arbitrary, we find

$$\mathbb{P}\left\{\limsup_{t\to\infty}\left|\frac{M(t)}{t}\right| > 0\right\} = 0 \Longrightarrow \mathbb{P}\left\{\limsup_{t\to\infty}\left|\frac{M(t)}{t}\right| = 0\right\} = 1$$

$$\Longrightarrow \mathbb{P}\left\{\lim_{t\to\infty}\left|\frac{M(t)}{t}\right| = 0\right\} = 1$$

$$\Longrightarrow \mathbb{P}\left\{\lim_{t\to\infty}\frac{M(t)}{t} = 0\right\} = 1.$$

Q.E.D.

Corollary 1. *Let a progressively measurable random process $b = b(t)$ be such that*

$$\mathbb{E}b^2(t) \le C(1 + t^{2\beta}), \quad t \ge 0,$$

for some $C \ge 0$ and $0 \le \beta < \dfrac{1}{2}$. Then

$$\frac{1}{t}\int_0^t b(s)dW(s) \to 0 \quad \textit{as} \quad t \to \infty, \quad \textit{almost surely.}$$

Proof. We set

$$M(t) = \int_0^t b(s)dW(s).$$

Then

$$\mathbb{E}M^2(t) = \mathbb{E}\int_0^t b^2(s)ds \qquad \text{(by the Itô isometry)}$$

$$= \int_0^t \mathbb{E}b^2(s)ds \le \int_0^t C(1+s^{2\beta})ds = O(t^{2\beta+1}), \quad t\to\infty \qquad \text{(by the Fubini theorem).}$$

By using Lemma 1, we complete the proof.

By $\mathcal{P}$ we denote a sigma algebra generated by random fields of the form

$$c(t,u) = \zeta_0 \mathbb{I}_{t=0,u\in U_0} + \sum_{k=1}^{n} \zeta_k \mathbb{I}_{t\in(t_{k-1},t_k],u\in U_k},$$

where $n \in \mathbb{N}$, ζ_0 is an $\mathcal{F}_0$-measurable random variable, ζ_k is an $\mathcal{F}_{t_{k-1}}$-measurable random variable, $k \in \{1,2,\ldots,n\}$, $U_k \in \mathcal{B}(\mathbb{R})$, $k \in \{0,1,2,\ldots,n\}$, and $0 = t_0 < t_1 < \ldots < t_n = \infty$.

Corollary 2. *Let a $\mathcal{P}$-measurable random field $c = c(t,u)$ be such that*

$$\mathbb{E}\int_{\mathbb{R}} c^2(t,u)\nu(du) \le C(1+t^{2\beta}), \quad t \ge 0,$$

for some $C \ge 0$ and $0 \le \beta < \frac{1}{2}$. Then

$$\frac{1}{t}\int_0^t \int_{\mathbb{R}} c(s,u)\tilde{N}(ds,du) \to 0 \quad \textit{as} \quad t\to\infty, \quad \textit{almost surely.}$$

Proof. We set

$$M(t) = \int_0^t \int_{\mathbb{R}} c(s,u)\tilde{N}(ds,du).$$

Then

$$\mathbb{E}M^2(t) = \mathbb{E}\int_0^t \int_{\mathbb{R}} c^2(s,u)\nu(du)ds \qquad \text{(by the Itô isometry)}$$

$$= \int_0^t \mathbb{E}\int_{\mathbb{R}} c^2(s,u)\nu(du)ds \qquad \text{(by the Fubini theorem)}$$

$$\le \int_0^t C(1+s^{2\beta})ds = O(t^{2\beta+1}), \quad t\to\infty.$$

By using Lemma 1, we complete the proof.

3. Asymptotics of the Solutions to Stochastic Equations

Let $W_k,\ k \in \{1, 2, \ldots, m\}$, be a Wiener process and let $\tilde{N}_k$ be a compensated Poisson measure with compensator $dt \cdot \nu_k(du)$, where the measure ν_k is such that

$$\int_{\mathbb{R}} u^2 \nu_k(du) < \infty, \quad k \in \{1, 2, \ldots, l\}.$$

In addition, $W_1, W_2, \ldots, W_m$ and $\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_l$ are independent.

The next theorem establishes the equivalence of solutions of stochastic and ordinary differential equations in the case where the drift coefficient has a positive limit as $t \to \infty$ and the characteristics of noise are not increasing very rapidly. This theorem is an important result, which is used in what follows (see Theorem 3) to establish the power-type character of growth of the solutions to stochastic differential equations with coefficients increasing according to the power law.

Theorem 1. *Suppose that $a = a(t)$ and $b_k = b_k(t),\ k \in \{1, 2, \ldots, m\}$, are progressively measurable*[2] *random processes, $c_k = c_k(t, u),\ k \in \{1, 2, \ldots, l\}$, are $\mathcal{P}$-measurable random fields, and an $\mathbb{F}$-adapted càdlàg*[3] *random process $X = X(t)$ has a stochastic differential*

$$dX(t) = a(t)dt + \sum_{k=1}^{m} b_k(t)dW_k(t) + \sum_{k=1}^{l} \int_{\mathbb{R}} c_k(t, u)\tilde{N}_k(dt, du).$$

Moreover, $\mathbb{E}X^2(0) < \infty$. Assume that:

(A) the random process a is bounded and $a(t) \to A,\ t \to \infty$, a.s., where $A > 0$ is a random variable;

(B) for some $C \geq 0$ and $0 \leq \beta < \frac{1}{2}$,

$$\sum_{k=1}^{m} b_k^2(t) + \sum_{k=1}^{l} \int_{\mathbb{R}} c_k^2(t, u)\nu_k(du) \leq C(1 + |X(t-)|^{2\beta}), \quad t \geq 0. \tag{3}$$

Then $X(t) \sim At,\ t \to \infty$, a.s.

Proof. We rewrite the process X in the integral form:

$$X(t) = X(0) + \int_0^t a(s)ds + \sum_{k=1}^{m} \int_0^t b_k(s)dW_k(s) + \sum_{k=1}^{l} \int_0^t \int_{\mathbb{R}} c_k(s, u)\tilde{N}_k(ds, du).$$

Step 1. We first verify that $\mathbb{E}X^2(t) \leq \tilde{C}(1 + t^2),\ t \geq 0$, for some $\tilde{C} > 0$. By analogy with Lemma 3.3.2 in [6], we can show that condition (3) implies that

$$\sup_{0 \leq t \leq T} \mathbb{E}X^2(t) < \infty, \quad T \geq 0.$$

[2] A random process $a = a(t)$ is called progressively measurable if, for any $t \geq 0$, the restriction of the map a to the set $[0, t] \times \Omega$ is measurable with respect to the sigma algebra $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$.

[3] We say that a random process is càdlàg if its trajectories are right continuous and have left limits with probability 1.

By the Cauchy–Schwarz inequality, we obtain

$$\frac{1}{4}\mathbb{E}X^2(t) \le \mathbb{E}X^2(0) + \mathbb{E}\left(\int_0^t a(s)ds\right)^2$$

$$+ \mathbb{E}\left[\left(\sum_{k=1}^{m}\int_0^t b_k(s)dW_k(s)\right)^2 + \left(\sum_{k=1}^{l}\int_0^t\int_{\mathbb{R}} c_k(s,u)\tilde{N}_k(ds,du)\right)^2\right]$$

$$=: E_1 + E_2(t) + E_3(t).$$

We now estimate the terms on the right-hand side:

$$E_1 = \mathbb{E}X^2(0) < \infty \quad \text{by assumption,}$$

$$E_2(t) = \mathbb{E}\left(\int_0^t a(s)ds\right)^2 \le C_1 t^2 \quad \text{for some } C_1 \ge 0 \text{ because } a \text{ is bounded,}$$

$$E_3(t) = \mathbb{E}\left[\left(\sum_{k=1}^{m}\int_0^t b_k(s)dW_k(s)\right)^2 + \left(\sum_{k=1}^{l}\int_0^t\int_{\mathbb{R}} c_k^2(s,u)\tilde{N}_k(ds,du)\right)^2\right]$$

(because W_i, and W_j and independent and $\tilde{N}_i$ and $\tilde{N}_j$ are independent for $i \neq j$)

$$= \mathbb{E}\left[\sum_{k=1}^{m}\left(\int_0^t b_k(s)dW_k(s)\right)^2 + \sum_{k=1}^{l}\left(\int_0^t\int_{\mathbb{R}} c_k^2(s,u)\tilde{N}_k(ds,du)\right)^2\right]$$

$$= \mathbb{E}\left[\sum_{k=1}^{m}\int_0^t b_k^2(s)ds + \sum_{k=1}^{l}\int_0^t\int_{\mathbb{R}} c_k^2(s,u)\nu(du)ds\right] \qquad \text{(by the Itô isometry)}$$

$$\le C\mathbb{E}\int_0^t \left(1 + |X(s-)|^{2\beta}\right) ds \qquad \text{(by condition (B))}$$

$$= \int_0^t \mathbb{E}\left(C(1 + |X(s-)|^{2\beta})ds\right) \qquad \text{(by the Fubini theorem)}$$

$$\le C\left(t + \int_0^t \left(\mathbb{E}X^2(s-)\right)^{\beta} ds\right) \qquad \text{(by the Jensen inequality).}$$

As a result, we arrive at the estimate

$$\mathbb{E}X^2(t) \le C_2(1+t^2) + C\int_0^t \left(\mathbb{E}X^2(s-)\right)^\beta ds,$$

where[4] $C_2 := C_1 \wedge \mathbb{E}X^2(0)$. By using the Wendroff inequality (see Theorem 7.3 in [8]), which is a generalization of the Grönwall inequality, we get

$$\mathbb{E}X^2(t) \le C_3\left((1-\beta)t + (1+t^2)^{1-\beta}\right)^{\frac{1}{1-\beta}},$$

where $C_3 := (C_2 \wedge C)^{\frac{1}{1-\beta}}$. This yields

$$\mathbb{E}X^2(t) \le \tilde{C}(1+t^2), \quad t \ge 0,$$

where $\tilde{C} \ge 0$.

Step 2. We now determine the asymptotics of the solution $X(t)$ as $t \to \infty$. We divide the stochastic differential equation by $t > 0$. This yields :

$$\begin{aligned}\frac{X(t)}{t} &= \frac{X(0)}{t} + \frac{1}{t}\int_0^t a(s)ds \\ &\quad + \sum_{k=1}^m \frac{1}{t}\int_0^t b_k(s)dW_k(s) + \sum_{k=1}^l \frac{1}{t}\int_0^t\int_{\mathbb{R}} c_k(s,u)\tilde{N}_k(ds,du) \\ &=: T_1(t) + T_2(t) + T_3(t) + T_4(t).\end{aligned}$$

It is necessary to study the convergence of the terms on the right-hand side as $t \to \infty$. Thus, we have

$$T_1(t) = \frac{X(0)}{t} \to 0, \quad t \to \infty.$$

Under the conditions of the theorem, we obtain

$$\lim_{t\to\infty} T_2(t) = \lim_{t\to\infty} \frac{1}{t}\int_0^t a(s)ds = \lim_{t\to\infty} a(t) = A \quad \text{almost surely.}$$

To estimate the term T_3, we note that

$$\mathbb{E}b_k^2(t) \le \mathbb{E}\left(C(1+|X(t-)|^{2\beta})\right) = C\left(1+\mathbb{E}\left(X^2(t-)\right)^\beta\right)$$

[4] $\wedge$ and $\vee$ stand for the operations of taking minimum and maximum, respectively.

$$\le C\left(1+\left(\mathbb{E}X^2(t-)\right)^{\beta}\right) \qquad \text{(by the Jensen inequality)}$$

$$\le C\left(1+\left(\tilde{C}(1+t^2)\right)^{\beta}\right) \le C_4(1+t^{2\beta}), \quad k\in\{1,2,\ldots,m\},$$

where $C_4 \ge 0.$ By Corollary 1, we find

$$T_3(t)=\sum_{k=1}^{m}\frac{1}{t}\int\limits_0^t b_k(s)dW_k(s)\to 0, \quad t\to\infty, \quad \text{almost surely.}$$

By analogy with the previous case, we get

$$\mathbb{E}\int\limits_{\mathbb{R}} c_k^2(t,u)\nu_k(du)\le C_4(1+t^{2\beta}), \quad k\in\{1,2,\ldots,l\}.$$

Hence, by Corollary 2, we obtain

$$T_4(t)=\sum_{k=1}^{l}\frac{1}{t}\int\limits_0^t\int\limits_{\mathbb{R}} c_k(s,u)\tilde{N}_k(ds,du)\to 0, \quad t\to\infty, \quad \text{almost surely.}$$

By using the established convergences, we arrive at the statement of the theorem.

Replacing condition (A) in the previous theorem by the condition

(A$'$) $A_-\le a(t)\le A_+,\ t\ge 0,$ where $A_->0$ and $A_+>0$ are random variables,

we can prove the following statement:

Theorem 2. *Suppose that conditions (A$'$) and (B) are satisfied. Then*

$$\liminf_{t\to\infty} a(t)\le \liminf_{t\to\infty}\frac{X(t)}{t}\le \limsup_{t\to\infty}\frac{X(t)}{t}\le \limsup_{t\to\infty} a(t) \qquad \textit{a.s.}$$

In what follows, we need the following lemma, which is proved in the Appendix (Section 4):

Lemma 2. *Let $\alpha\in(0,1),$ let $f=f(x)$ be a twice continuously differentiable function such that*

$$f(x)=\begin{cases}0, & x\le 0,\\ \dfrac{x^{1-\alpha}}{1-\alpha}, & x\ge 1,\end{cases}$$

and, in addition,

$$f(x)\le\frac{x^{1-\alpha}}{1-\alpha}, \quad 0<x<1;$$

let $c = c(x)$ *be a measurable function such that, for some* $C \geq 0$ *and* $\beta \in \left[0, \dfrac{1+\alpha}{2}\right)$,

$$c^2(x) \leq C\left(1 + |x|^{2\beta}\right), \quad x \in \mathbb{R},$$

and let $\nu = \nu(du)$ *be the measure on* $\mathcal{B}(\mathbb{R})$ *such that*

$$\int\limits_{\mathbb{R}} u^2 \nu(du) < \infty.$$

Then:

(A) $\displaystyle\int_{\mathbb{R}} \big(f(x + c(x)u) - f(x) - f'(x)c(x)u\big)\nu(du) \to 0, \ x \to +\infty;$

(B) $\displaystyle\int_{\mathbb{R}} \big(f(x + c(x)u) - f(x)\big)^2 \nu(du) \leq Cx^{2(\beta-\alpha)}, \ x \geq 1,$ *where* $C \geq 0$ *is a constant.*

The following theorem is the main result of the present paper:

Theorem 3. *Suppose that* X *is a certain (not necessarily unique) solution of the stochastic differential equation*

$$dX(t) = a(X(t))dt + \sum_{k=1}^{q} h_k(X(t-))dZ_k(t), \tag{4}$$

where $a = a(x)$ *and* $h_k = h_k(x)$, $k \in \{1, 2, \ldots, q\}$, *are locally bounded measurable functions,* $Z_k = Z_k(t)$, $k \in \{1, 2, \ldots, q\}$, *are independent centered Lévy processes with finite second moment, and* $\mathbb{E}X^2(0) < \infty$. *Also let* $\alpha \in [0, 1)$. *Assume that:*

(A) $a(x) \sim Ax^{\alpha}$, $x \to +\infty$, *where* $A > 0$ *is a nonrandom constant;*

(B) for some $C \geq 0$ *and* $2\beta \in [0, 1+\alpha)$,

$$\sum_{k=1}^{q} h_k^2(x) \leq C\left(1 + |x|^{2\beta}\right), \quad x \in \mathbb{R}; \tag{5}$$

(C) $X(t) \to +\infty$, $t \to \infty$, *a.s.*

Then

$$X(t) \sim ((1-\alpha)At)^{\frac{1}{1-\alpha}}, \quad t \to \infty, \quad \text{a.s.} \tag{6}$$

Remark. Condition (C) also appears in the works [3, 5]; it is essential and does not follow from conditions (A) and (B). It is possible to prove that condition (C) is satisfied, e.g., under the following conditions:

the coefficients a and h_k, $k \in \{1, 2, \ldots, q\}$, satisfy the Lipschitz condition;

$\lim_{|x|\to\infty} \dfrac{a(x)}{|x|^{\alpha}} > 0;$

for some $k \in \{1, 2, \ldots, q\}$, the following conditions are satisfied:

$\inf_{|x| \le R} |h_k(x)| > 0$ for any $R > 0$,

$Z_k(t),\ t \ge 0$, has a nondegenerate Gaussian component or positive jumps with probability 1.

Proof. The validity of the theorem for $\alpha = 0$ follows from Theorem 1. Further, we assume that $\alpha \in (0, 1)$.

In view of the fact that the processes $Z_k,\ k \in \{1, 2, \ldots, q\}$, are centered and have a finite second moment, by the Lévy–Itô representation, we get

$$dZ_k(t) = \sigma_k dW_k(t) + \int_{\mathbb{R}} u\tilde{N}_k(dt, du), \quad k \in \{1, 2, \ldots, q\},$$

where $\sigma_k \ge 0,\ W_k$ is a Wiener process, $\tilde{N}_k$ is a compensated Poisson measure with compensator $dt \cdot \nu_k(du)$, the measure ν_k is such that

$$\int_{\mathbb{R}} u^2 \nu_k(du) < \infty, \quad k \in \{1, 2, \ldots, q\},$$

and, in addition, $W_1, W_2, \ldots, W_q$ and $\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_q$ are independent. Hence, the stochastic differential equation (4) can be rewritten in the form

$$dX(t) = a(X(t))dt + \sum_{k=1}^{m} \sigma_k h_k(X(t)) dW_k(t) + \sum_{k=1}^{l} \int_{\mathbb{R}} h_k(X(t-)) u\tilde{N}_k(dt, du).$$

To simplify notation, we consider the case $m = l = 1$ (the general case is studied similarly) and denote

$$b := \sigma_1 h_1, \quad c := h_1, \quad W := W_1, \quad \tilde{N} := \tilde{N}_1, \quad \text{and} \quad \nu := \nu_1.$$

Further, we consider an equation

$$dX(t) = a(X(t))dt + b(X(t))dW(t) + \int_{\mathbb{R}} c(X(t-)) u\tilde{N}(dt, du).$$

Note that condition (5) yields the estimate

$$b^2(x) + c^2(x) \le C_0(1 + |x|^{2\beta}), \quad x \in \mathbb{R},$$

where $C_0 \ge 0$ is a constant.

We take the same function $f = f(x)$ as in Lemma 2. Denote $\tilde{X}(t) = f(X(t))$. By the Itô formula with jumps (see Theorem 5.1 in [7]), we get

$$\mathrm{d}\tilde{X}(t) = \tilde{a}(t)dt + \tilde{b}(t)dW(t) + \int_{\mathbb{R}} \tilde{c}(t, u)\tilde{N}(dt, du), \tag{7}$$

where

$$\tilde{a}(t) = a(X(t-))f'(X(t-)) + \frac{1}{2}b^2(X(t-))f''(X(t-))$$

$$+ \int_{\mathbb{R}} \Big(f\big(X(t-) + c(X(t-))u\big) - f(X(t-)) - c(X(t-))uf'(X(t-))\Big)\nu(du)$$

$$=: \tilde{a}_1(t) + \tilde{a}_2(t) + \tilde{a}_3(t),$$

$$\tilde{b}(t) = b(X(t-))f'(X(t-)), \qquad \tilde{c}(t,u) = f\big(X(t-) + c(X(t-))u\big) - f(X(t-)).$$

We now check that the stochastic differential equation (7) satisfies the conditions of Theorem 1.

Note that the coefficient $\tilde{a}$ is bounded. To investigate its asymptotic behavior, we study the behaviors of the terms $\tilde{a}_1(t)$, $\tilde{a}_2(t)$, and $\tilde{a}_3(t)$ separately:

$$\lim_{t\to\infty} \tilde{a}_1(t) = \lim_{t\to\infty} a(X(t))f'(X(t)) = \lim_{t\to\infty} \frac{a(X(t))}{X^\alpha(t)} = \lim_{t\to\infty} \frac{AX^\alpha(t)}{X^\alpha(t)} = A \quad \text{a.s.},$$

$$\lim_{t\to\infty} |\tilde{a}_2(t)| = \lim_{t\to\infty} \left|b^2(X(t))f''(X(t))\right|$$

$$\le \lim_{t\to\infty} \frac{C_1|\alpha|X^{2\beta}(t)}{X^{1+\alpha}(t)} \le C_1 \lim_{t\to\infty} \frac{1}{X^{1+\alpha-2\beta}(t)} = 0 \quad \text{a.s.}$$

because

$$1 + \alpha - 2\beta > 0 \qquad \text{and} \qquad X(t) \to \infty, \quad t \to \infty, \quad \text{a.s.}$$

(here, $C_1 \ge 0$ is such that $b^2(X(t)) \le C_1 X^{2\beta}(t)$ for $X(t) \ge 1$);

$$\lim_{t\to\infty} \tilde{a}_3(t) = \lim_{t\to\infty} \int_{\mathbb{R}} \Big(f\big(X(t-) + c(X(t-))u\big) - f(X(t-))$$

$$- c(X(t-))uf'(X(t-))\Big)\nu(du) = 0, \quad t \to \infty, \quad \text{a.s.}$$

by Assertion (A) of Lemma 2 because $X(t) \to \infty,\ t \to \infty$, a.s.

Thus, $\lim_{t\to\infty} \tilde{a}(t) = A$ a.s.

We now estimate the coefficient $\tilde{b}$. If $X(t-) \ge 1$, then

$$\tilde{b}^2(t) = \big(b(X(t-))f'(X(t-))\big)^2 = b^2(X(t-))\left(f'(X(t-))\right)^2$$

$$\le \frac{C_1 X^{2\beta}(t-)}{X^{2\alpha}(t-)} \le C_1 X^{2(\beta-\alpha)}(t-) = C_1 C_2 \tilde{X}^{\frac{2(\beta-\alpha)}{1-\alpha}}(t-),$$

where

$$C_2 := (1-\alpha)^{\frac{2(\beta-\alpha)}{1-\alpha}}.$$

If $X(t-) < 1$, then $\tilde{b}^2(t)$ is uniformly bounded in t by a certain nonrandom constant $C_3 \geq 0$ because f' is bounded.

Thus,

$$b^2(t) \leq C_4 \left(1 + |\tilde{X}(t-)|^{2\tilde{\beta}}\right), \qquad t \geq 0,$$

where

$$C_4 := (C_1 C_2) \wedge C_3 \qquad \text{and} \qquad \tilde{\beta} := \frac{\beta - \alpha}{1 - \alpha} < 1.$$

We now estimate the coefficient $\tilde{c}$. If $X(t-) \geq 1$, then

$$\int_{\mathbb{R}} \tilde{c}^2(t,u)\nu(du) = \int_{\mathbb{R}} \left(f\big(X(t-) + c(X(t-))u\big) - f(X(t-))\right)^2 \nu(du)$$

$$\leq C_5 X^{2(\beta-\alpha)}(t-) = C_2 C_5 \tilde{X}^{\frac{2(\beta-\alpha)}{1-\alpha}}(t-) \quad \text{(by Assertion (B) of Lemma 2)},$$

where $C_5 \geq 0$. If $X(t-) < 1$, then, by the Taylor formula, we get

$$\Big(f\big(X(t-) + c(X(t-))u\big) - f(X(t-))\Big)^2 = \big(f'(\xi_{X(t-),u})c(X(t-))u\big)^2 \leq C_6 u^2,$$

where $C_6 \geq 0$. Hence, $\int_{\mathbb{R}} \tilde{c}^2(t,u)\nu(du)$ is bounded $\big($here, $\xi_{x,u} \in \big[x \wedge (x + c(x)u), x \vee (x + c(x)u)\big]\big)$.

Therefore,

$$\int_{\mathbb{R}} \tilde{c}^2(t,u)\nu(du) \leq C_7 \left(1 + |\tilde{X}(t-)|^{2\tilde{\beta}}\right),$$

where $C_7 \geq 0$.

Thus, the coefficients $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ of the stochastic differential equation (7) satisfy the conditions of Theorem 1. Hence, $\tilde{X}(t) \sim At$, $t \to \infty$, a.s. Carrying out the change of variables

$$\tilde{X}(t) = \frac{X^{1-\alpha}(t)}{1-\alpha}, \quad X(t) \geq 1,$$

and using condition (C), we establish equivalence (6).

Theorem 3 is proved.

4. Appendix

Proof of Lemma 2. For the sake of simplicity, let $C \geq 0$ be a universal constant that may vary from line to line.

Proof of Assertion (A). Let $x \geq 1$. Under the conditions of the lemma, we have $c^2(x) \leq C x^{2\beta}$. We split the integral as follows:

$$\int_{\mathbb{R}} [\ldots]\nu(du) = \int_{|u|<Kx^{1-\beta}} [\ldots]\nu(du) + \int_{|u|\geq Kx^{1-\beta}} [\ldots]\nu(du),$$

where

$$[\dots] := f(x + c(x)u) - f(x) - f'(x)c(x)u$$

and $K > 0$ is a constant.

First, let $|u| < Kx^{1-\beta}$. By the Taylor formula,

$$f\big(x + c(x)u\big) - f(x) - f'(x)c(x)u = \frac{1}{2}f''(\xi_{x,u})c^2(x)u^2,$$

where $\xi_{x,u} \in \big[x \wedge (x + c(x)u), x \vee (x + c(x)u)\big]$. We have

$$\begin{aligned}
&|\xi_{x,u} - x| \le |c(x)u| \\
&\quad \Longrightarrow (\xi_{x,u} - x)^2 \le (c(x)u)^2 = c^2(x)u^2 \le Cx^{2\beta}u^2 \\
&\quad \Longrightarrow |\xi_{x,u} - x| \le Cx^{\beta}|u|.
\end{aligned}$$

Further, we take K such that

$$Cx^{\beta}|u| \le Cx^{\beta}Kx^{1-\beta} = CKx \le \frac{1}{2}x, \qquad x \ge 1, \quad |u| < Kx^{1-\beta}.$$

Therefore,

$$\frac{1}{2}x \le \xi_{x,u} \le \frac{3}{2}x.$$

Thus, we get

$$\begin{aligned}
\int\limits_{|u|<Kx^{1-\beta}} [\dots]\nu(du) &= \int\limits_{|u|<Kx^{1-\beta}} f''(\xi_{x,u})c^2(x)u^2\nu(du) \\
&\le Cx^{2\beta} \int\limits_{|u|<Kx^{1-\beta}} \frac{u^2}{\xi_{x,u}^{\alpha+1}}\nu(du) \\
&\le Cx^{2\beta}\left(\frac{1}{2}x\right)^{-(\alpha+1)} \int\limits_{\mathbb{R}} u^2\nu(du) \\
&\le \frac{C}{x^{1+\alpha-2\beta}} \to 0, \quad x \to +\infty,
\end{aligned}$$

because

$$1 + \alpha - 2\beta > 0 \qquad \text{and} \qquad \int\limits_{\mathbb{R}} u^2\nu(du) < \infty.$$

We now consider the integral over the set $\{u \in \mathbb{R}\colon |u| \geq Kx^{1-\beta}\}$. We split the analyzed integral as follows:

$$\int\limits_{|u|\geq Kx^{1-\beta}} [\ldots]\nu(du) = \int\limits_{|u|\geq Kx^{1-\beta}} f\left(x + c(x)u\right)\nu(du)$$

$$- f(x) \int\limits_{|u|\geq Kx^{1-\beta}} \nu(du) - f'(x)c(x) \int\limits_{|u|\geq Kx^{1-\beta}} u\nu(du)$$

$$=: I_1(x) - I_2(x) - I_3(x)$$

and estimate each term on the right-hand side separately. Note that

$$f(x) \leq \frac{|x|^{1-\alpha}}{1-\alpha}, \quad x \in \mathbb{R}.$$

Thus, we get

$$I_1(x) = \int\limits_{|u|\geq Kx^{1-\beta}} f\left(x + c(x)u\right)\nu(du) \leq \frac{1}{1-\alpha} \int\limits_{|u|\geq Kx^{1-\beta}} 1 \cdot |x + c(x)u|^{1-\alpha}\,\nu(du)$$

(by the Hölder inequality)

$$\leq \frac{1}{1-\alpha} \left(\int\limits_{|u|\geq Kx^{1-\beta}} 1^{\frac{2}{1+\alpha}}\nu(du) \right)^{\frac{1+\alpha}{2}} \left(\int\limits_{|u|\geq Kx^{1-\beta}} \left(|x + c(x)u|^{1-\alpha}\right)^{\frac{2}{1-\alpha}} \nu(du) \right)^{\frac{1-\alpha}{2}}$$

$$= \frac{1}{1-\alpha} \left(\int\limits_{|u|\geq Kx^{1-\beta}} \nu(du) \right)^{\frac{1+\alpha}{2}} \left(\int\limits_{|u|\geq Kx^{1-\beta}} (x + c(x)u)^2\,\nu(du) \right)^{\frac{1-\alpha}{2}}.$$

We now estimate each integral as follows:

$$\int\limits_{|u|\geq Kx^{1-\beta}} \nu(du) = \int\limits_{u^2\geq K^2x^{2-2\beta}} \nu(du)$$

(by the Chebyshev inequality)

$$\leq \frac{1}{K^2x^{2-2\beta}} \int\limits_{\mathbb{R}} u^2\nu(du) \leq \frac{C}{x^{2-2\beta}}, \tag{8}$$

$$\int\limits_{|u|\geq Kx^{1-\beta}} (x + c(x)u)^2\nu(du)$$ (by the Cauchy–Schwarz inequality)

$$\leq 2x^2 \int\limits_{|u|\geq Kx^{1-\beta}} \nu(du) + 2c^2(x) \int\limits_{|u|\geq Kx^{1-\beta}} u^2\nu(du)$$

(by relation (8))

$$\leq 2x^2 \frac{C}{x^{2-2\beta}} + 2c^2(x) \int\limits_{|u|\geq Kx^{1-\beta}} u^2\nu(du)$$

$$\leq Cx^{2\beta} + Cx^{2\beta} \leq Cx^{2\beta}. \tag{9}$$

Thus, by virtue of relations (8) and (9), we obtain

$$I_1(x) \leq \frac{1}{1-\alpha}\left(\frac{C}{x^{2-2\beta}}\right)^{\frac{1+\alpha}{2}} \left(Cx^{2\beta}\right)^{\frac{1-\alpha}{2}} \leq \frac{C}{x^{1+\alpha-2\beta}} \to 0, \quad x \to +\infty,$$

because $1+\alpha-2\beta > 0$. Further, we get

$$I_2(x) = f(x) \int\limits_{|u|\geq Kx^{1-\beta}} \nu(du)$$

$$\leq \frac{x^{1-\alpha}}{1-\alpha} \int\limits_{|u|\geq Kx^{1-\beta}} \nu(du)$$

$$\leq \frac{x^{1-\alpha}}{1-\alpha}\frac{C}{x^{2-2\beta}} \leq \frac{C}{x^{1+\alpha-2\beta}} \to 0, \quad x \to +\infty \qquad \text{(by relation (8))}$$

because $1+\alpha-2\beta > 0$. Finally, we find

$$|I_3(x)| = \left| f'(x)c(x) \int\limits_{|u|\geq Kx^{1-\beta}} u\nu(du)\right|$$

$$\leq \frac{1}{x^\alpha} Cx^\beta \int\limits_{|u|\geq Kx^{1-\beta}} |u|\nu(du) = \frac{C}{x^{\alpha-\beta}} \int\limits_{|u|\geq Kx^{1-\beta}} |u|\nu(du).$$

We now separately estimate the integral

$$\int\limits_{|u|\geq Kx^{1-\beta}} |u|\nu(du) = \int\limits_{|u|\geq Kx^{1-\beta}} 1\cdot|u|\nu(du)$$

$$\leq \left(\int\limits_{|u|\geq Kx^{1-\beta}} 1^2\nu(du)\right)^{\frac{1}{2}} \left(\int\limits_{|u|\geq Kx^{1-\beta}} |u|^2\nu(du)\right)^{\frac{1}{2}} \qquad \text{(by the Cauchy–Schwarz inequality)}$$

$$\leq \left(\int\limits_{|u|\geq Kx^{1-\beta}} \nu(du) \right)^{\frac{1}{2}} \left(\int\limits_{\mathbb{R}} u^2 \nu(du) \right)^{\frac{1}{2}} \leq C \left(\int\limits_{|u|\geq Kx^{1-\beta}} \nu(du) \right)^{\frac{1}{2}}$$

$$\leq C \left(\frac{C}{x^{2-2\beta}} \right)^{\frac{1}{2}} \leq \frac{C}{x^{1-\beta}} \qquad \text{(by relation (8))}.$$

Thus,

$$|I_3(x)| \leq \frac{C}{x^{\alpha-\beta}} \frac{C}{x^{1-\beta}} \leq \frac{C}{x^{1+\alpha-2\beta}} \to 0, \quad x \to +\infty,$$

because $1 + \alpha - 2\beta > 0$.

Since all three terms $I_1(x)$, $I_2(x)$, and $I_3(x)$ approach zero as $x \to +\infty$, we obtain

$$\int\limits_{|u|\geq Kx^{1-\beta}} [\ldots]\nu(du) \to 0, \quad x \to +\infty.$$

Thus, Assertion (A) of the lemma is proved.

Proof of Assertion (B). Let $x \geq 1$. Under the condition of the lemma, we have

$$c^2(x) \leq Cx^{2\beta}, \quad x \geq 1.$$

We now split the integral as follows:

$$\int\limits_{\mathbb{R}} [\ldots]\nu(du) = \int\limits_{|u|<Kx^{1-\beta}} [\ldots]\nu(du) + \int\limits_{|u|\geq Kx^{1-\beta}} [\ldots]\nu(du),$$

where $[\ldots] := (f(x + c(x)u) - f(x))^2$ and $K > 0$ is a constant.

First, let $|u| < Kx^{1-\beta}$. By the mean-value theorem, we get

$$f(x + c(x)u) - f(x) = f'(\xi_{x,u})c(x)u,$$

where $\xi_{x,u} \in \big[x \wedge (x + c(x)u), x \vee (x + c(x)u)\big]$. As in the proof of Assertion (A), we have

$$\frac{1}{2}x \leq \xi_{x,u} \leq \frac{3}{2}x$$

for sufficiently small K. Therefore,

$$\int\limits_{|u|<Kx^{1-\beta}} [\ldots]\nu(du) = \int\limits_{|u|<Kx^{1-\beta}} \left(f'(\xi_{x,u})\right)^2 c^2(x)u^2\nu(du)$$

$$\leq Cx^{2\beta} \int\limits_{|u|<Kx^{1-\beta}} \frac{u^2}{\xi_{x,u}^{2\alpha}} \nu(du).$$

Since $\alpha > 0,$ we find

$$\int\limits_{|u|<Kx^{1-\beta}} \frac{u^2}{\xi_{x,u}^{2\alpha}}\nu(du) \le \left(\frac{1}{2}x\right)^{-2\alpha} \int\limits_{\mathbb{R}} u^2\nu(du) \le Cx^{-2\alpha}.$$

Thus,

$$\int\limits_{|u|<Kx^{1-\beta}} [\ldots]\nu(du) \le Cx^{2\beta}Cx^{-2\alpha} \le Cx^{2(\beta-\alpha)}.$$

We now consider the integral over the set $\{u \in \mathbb{R} : |u| \ge Kx^{1-\beta}\}$:

$$\frac{1}{2} \int\limits_{|u|\ge Kx^{1-\beta}} [\ldots]\nu(du)$$

(by the Cauchy–Schwarz inequality)

$$\le \int\limits_{|u|\ge Kx^{1-\beta}} f^2\left(x + c(x)u\right)\nu(du) + f^2(x) \int\limits_{|u|\ge Kx^{1-\beta}} \nu(du)$$

$$=: J_1(x) + J_2(x).$$

By using the formulas established in the proof of Assertion (A), we estimate each term on the right-hand side as follows:

$$J_1(x) = \int\limits_{|u|\ge Kx^{1-\beta}} f^2\left(x + c(x)u\right)\nu(du)$$

$$\le \frac{1}{(1-\alpha)^2} \int\limits_{|u|\ge Kx^{1-\beta}} 1 \cdot |x + c(x)u|^{2(1-\alpha)}\nu(du)$$

(by the Hölder inequality)

$$\le \frac{1}{(1-\alpha)^2} \left(\int\limits_{|u|\ge Kx^{1-\beta}} 1^{\frac{1}{\alpha}}\nu(du)\right)^{\alpha} \left(\int\limits_{|u|\ge Kx^{1-\beta}} \left(|x + c(x)u|^{2(1-\alpha)}\right)^{\frac{1}{1-\alpha}}\nu(du)\right)^{1-\alpha}$$

$$= \frac{1}{(1-\alpha)^2} \left(\int\limits_{|u|\ge Kx^{1-\beta}} \nu(du)\right)^{\alpha} \left(\int\limits_{|u|\ge Kx^{1-\beta}} (x + c(x)u)^2\nu(du)\right)^{1-\alpha}$$

$$\le \frac{1}{(1-\alpha)^2} \left(\frac{C}{x^{2-2\beta}}\right)^{\alpha} \left(Cx^{2\beta}\right)^{1-\alpha} \le Cx^{2(\beta-\alpha)} \qquad \text{(by reltions (8) and (9))},$$

$$J_2(x) = f^2(x) \int\limits_{|u| \geq Kx^{1-\beta}} \nu(du) \leq \left(\frac{x^{1-\alpha}}{1-\alpha}\right)^2 \int\limits_{|u| \geq Kx^{1-\beta}} \nu(du)$$

$$\leq \left(\frac{x^{1-\alpha}}{1-\alpha}\right)^2 \frac{C}{x^{2-2\beta}} \leq Cx^{2(\beta-\alpha)} \qquad \text{(by relation (8)).}$$

Both these terms are estimated as $Cx^{2(\beta-\alpha)}$. Hence,

$$\int\limits_{|u| \geq Kx^{1-\beta}} [\ldots]\nu(du) \leq Cx^{2(\beta-\alpha)}.$$

The lemma is proved.

The author states that there is no conflict of interest.

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