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EMBEDDINGS INTO COUNTABLY COMPACT HAUSDORFF SPACES

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We consider the problem of characterization of topological spaces embedded into countably compact Hausdorff topological spaces. We study the separation axioms for subspaces of Hausdorff countably compact topological spaces and construct an example of a regular separable scattered topological space that cannot be embedded into an Urysohn countably compact topological space.

It is well known that a topological space X is homeomorphic to a subspace of a compact Hausdorff space if and only if X is a Tychonoff space.

In the present paper, we discuss the following problem:

Problem 1. *What topological spaces are homeomorphic to subspaces of countably compact Hausdorff spaces?*

A topological space X is:

compact if each open cover of X has a finite subcover;

ω-bounded if each countable set in X has a compact closure in X;

countably compact if each sequence in X has an accumulation point in X;

totally countably compact if each infinite set in X contains an infinite subset with compact closure in X;

ultracompact if each sequence in X has a p-limit for every ultrafilter p on ω;

Lindelöf if each open cover or X has a countable subcover.

These properties are related to each other as follows:

$$\begin{array}{ccccc}
\text{compact} & \Longrightarrow & \omega\text{-bounded} & \Longrightarrow & \text{totally countably compact} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{Lindelöf} & & \text{ultracompact} & \Longrightarrow & \text{countably compact.}
\end{array}$$

Countably compact topological spaces were investigated in [2, 8–12]. The problem of construction of embeddings into ω-bounded or ultracompact spaces was considered in [2] and [1] (see also [7] for the basic information

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about ultracompact spaces). Since the class of countably compact spaces in not closed with respect to the Tychonoff product, there is no possibility to apply the technique of reflections (applied in [1]) for the construction of embeddings into countably compact spaces.

Nevertheless, in the present paper, we establish some properties of subspaces of countably compact Hausdorff spaces and, hence, establish some necessary conditions of embeddability of topological spaces into Hausdorff countably compact spaces. Moreover, we construct an example of regular separable first-countable scattered topological space that cannot be embedded into an Urysohn countably compact topological space.

Let $\mathcal{F}$ be a family of closed subsets of a topological space X. The topological space X is called

$\mathcal{F}$-regular if, for any set $F \in \mathcal{F}$ and a point $x \in X \setminus F$, there exist disjoint open sets $U, V \subset X$ such that $F \subset U$ and $x \in V$.

We recall [6, § 3.6] that the *Wallman extension* $W(X)$ of a topological space X consists of closed ultrafilters, i.e., families $\mathcal{U}$ of closed subsets of X satisfying the following conditions:

$\varnothing \notin \mathcal{U}$;

$A \cap B \in \mathcal{U}$ for any $A, B \in \mathcal{U}$;

a closed set $F \subset X$ belongs to $\mathcal{U}$ if $F \cap U \neq \varnothing$ for every $U \in \mathcal{U}$.

The Wallman extension $W(X)$ of X carries a topology generated by a base formed by the sets

$$\langle U \rangle = \big\{\mathcal{F} \in W(X) \colon \exists F \in \mathcal{F},\ \ F \subset U\big\},$$

where U runs over open subsets of X.

By (the proof of) Theorem 3.6.21 in [6], the Wallman extension $W(X)$ is compact.

If X is a T_1-space, then we can consider a map $j_X \colon X \to W(X)$ by assigning to each $x \in X$ a principal ultrafilter formed by all closed sets $F \subset X$ containing the point x. It is easy to see that the image $j_X(X)$ is dense in $W(X)$. By [6, Theorem 3.6.21], the map $j_X \colon X \to W(X)$ is a topological embedding. Hence, we can identify the T_1-space X with its image $j_X(X)$ in $W(X)$.

In the Wallman extension $W(X)$, we consider a subspace

$$W_\omega X = \bigcup\big\{\overline{j_X(C)} \colon C \subset X,\ \ |C| \le \omega\big\},$$

which is the union of the closures of countable subsets of $j_X(X)$ in $W(X)$. The space $W_\omega X$ is called the *Wallman ω-bounded extension* of X. By Proposition 3.2 from [2], the space $W_\omega X$ is ω-bounded. In [2] (resp., [1]), the Wallman extension was used to construct embeddings of topological spaces into Hausdorff ω-compact (resp., ultracompact) spaces. In what follows, we apply the Wallman extension in Examples 1 and 3.

A topological space X is called

locally countable if each $x \in X$ possesses a countable open neighborhood;

first-countable at a point $x \in X$ if it has a countable neighborhood base at x;

of countable pseudocharacter at a point $x \in X$ if $\{x\} = \bigcap \mathcal{U}$ for a countable family $\mathcal{U}$ of open sets in X;

Fréchet–Urysohn at a point $x \in X$ if, for each subset A of X with $x \in \overline{A}$, there exists a sequence $\{a_n\}_{n\in\omega} \subset A$ that converges to x;

regular at a point $x \in X$ if any neighborhood of x contains a closed neighborhood of x;

completely regular at a point $x \in X$ if, for any neighborhood $U \subset X$ of x, there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f(X \setminus U) \subset \{0\}$.

A topological space X is *first-countable* (resp., *Fréchet–Urysohn*, *regular*, *completely regular*, *or of countable pseudocharacter*) if X has the corresponding property at every point $x \in X$.

Theorem 1. *Let X be a subspace of a countably compact Hausdorff space Y. If X is first-countable at a point $x \in X$, then X is regular at the point x.*

Proof. We fix a countable neighborhood base $\{U_n\}_{n\in\mathbb{N}}$ at x and assume that X is not regular at x. Consequently, there exists a neighborhood U_0 of x such that $\overline{V} \not\subset U_0$ for any neighborhood V of x. Replacing each basic neighborhood U_n with $\bigcap_{k\le n} U_k$, we can assume that $U_n \subset U_{n-1}$ for every $n \in \mathbb{N}$. The choice of the neighborhood U_0 ensures that, for every $n \in \mathbb{N}$, the set $\overline{U}_n \setminus U_0$ contains a point x_n. Since the space Y is countably compact and Hausdorff, the sequence $(x_n)_{n\in\omega}$ has an accumulation point $y \in Y$. Since

$$U_0 \cap \{x_n\}_{n\in\omega} = \varnothing,$$

the point y does not coincide with x. Further, since Y is Hausdorff, there exists a neighborhood $V \subset Y$ of x such that $y \notin \overline{V}$. Finally, we find $n \in \omega$ such that $U_n \subset V$ and observe that $O_y := Y \setminus \overline{V}$ is a neighborhood of y for which

$$O_y \cap \{x_i \colon i \in \omega\} \subset \{x_i\}_{i<n},$$

which means that y is not an accumulating point of the sequence $(x_i)_{i\in\omega}$.

Remark 1. Example 6.1 from [2] shows that, in Theorem 1, the regularity of X at the point x cannot be improved up to the complete regularity at x.

Corollary 1. *Let X be a subspace of a countably compact Hausdorff space Y. If X is first-countable, then X is regular.*

The following example shows that Theorem 1 cannot be generalized over the Fréchet–Urysohn spaces with countable pseudocharacter.

Example 1. There exists a Hausdorff space X such that:

(1) X is locally countable and, hence, has a countable pseudocharacter;

(2) X is separable and Fréchet–Urysohn;

(3) X is not regular;

(4) X is a subspace of a totally countably compact Hausdorff space.

Proof. We choose any point $\infty \notin \omega\times\omega$ and consider a space $Y = \{\infty\} \cup (\omega\times\omega)$ endowed with a topology formed by the sets $U \subset Y$ such that if $\infty \in U$, then, for every $n \in \omega$, the complement $(\{n\}\times\omega) \setminus U$ is finite. The definition of this topology ensures that Y is Fréchet–Urysohn at the unique nonisolated point ∞ of Y.

Let $\mathcal{F}$ be the family of closed infinite subsets of Y that do not contain the point ∞. The definition of the topology on Y implies that, for any $F \in \mathcal{F}$ and $n \in \omega$, the intersection $(\{n\}\times\omega)\cap F$ is finite. By the Kuratowski–Zorn lemma, the family $\mathcal{F}$ contains a maximal almost disjoint subfamily $\mathcal{A} \subset \mathcal{F}$. The maximality of $\mathcal{A}$ guarantees that every set $F \in \mathcal{F}$ has an infinite intersection with some set $A \in \mathcal{A}$.

Consider the space $X = Y \cup \mathcal{A}$ endowed with the topology formed by the sets $U \subset X$ such that $U \cap Y$ is open in Y and, for any $A \in \mathcal{A} \cap U$, the set $A \setminus U \subset \omega \times \omega$ is finite.

We claim that the space X has properties (1)–(4). The definition of the topology of X implies that X is separable, Hausdorff, and locally countable, which implies that X has a countable pseudocharacter. Moreover, X is first-countable at all points except ∞. At the point ∞, the space X is Fréchet–Urysohn (because its open subspace Y is Fréchet–Urysohn at ∞).

The maximality of the maximal almost disjoint family $\mathcal{A}$ guarantees that every neighborhood $U \subset Y \subset X$ of ∞ has an infinite intersection with some set $A \in \mathcal{A}$, which implies that $A \in \overline{U}$ and, hence, $\overline{U} \not\subset Y$. This means that X is not regular (at ∞).

In the Wallman extension $W(X)$ of the space X, we consider a subspace

$$Z := X \cup W_\omega \mathcal{A} = Y \cup W_\omega \mathcal{A}.$$

We claim that the space Z is Hausdorff and totally countably compact. To prove that Z is Hausdorff, we take two distinct ultrafilters $a, b \in Z$. If the ultrafilters a and b are principal, then, in view of the fact that X is Hausdorff, they have disjoint neighborhoods in $W(X)$ and, hence, in Z. We now assume that one of the ultrafilters (a or b) is principal and the other is not principal. Without lose of generality, we can assume that a is principal and b is not principal. If $a \neq \infty$, then we can use the regularity of the space X at a and prove that a and b have disjoint neighborhoods in $W(X) \supset Z$. Hence, we can assume that $a = \infty$. It follows from $b \in Z = X \cup W_\omega \mathcal{A}$ that the ultrafilter b contains a countable set $\{A_n\}_{n\in\omega} \subset \mathcal{A}$. We consider a set

$$V = \bigcup_{n\in\omega} \Big(\{A_n\} \cup A_n \setminus \bigcup_{k\le n} \{k\} \times \omega \Big)$$

and note that V has a finite intersection with every set $\{k\} \times \omega$, which implies that $Y \setminus V$ is a neighborhood of ∞. Then $\langle Y \setminus V \rangle$ and $\langle V \rangle$ are disjoint open neighborhoods of $a = \infty$ and b in $W(X)$.

Finally, we assume that both ultrafilters a and b are not principal. Since $a, b \in W_\omega \mathcal{A}$ are distinct, there are disjoint countable sets $\{A_n\}_{n\in\omega}, \{B_n\}_{n\in\omega} \subset \mathcal{A}$ such that $\{A_n\}_{n\in\omega} \in a$ and $\{B_n\}_{n\in\omega} \in b$. Note that the sets

$$V = \bigcup_{n\in\omega} \Big(\{A_n\} \cup A_n \setminus \bigcup_{k\le n} B_k \Big) \qquad \text{and} \qquad W = \bigcup_{n\in\omega} \Big(\{B_n\} \cup B_n \setminus \bigcup_{k\le n} A_k \Big)$$

are disjoint and open in X. Thus, $\langle V \rangle$ and $\langle W \rangle$ are disjoint open neighborhoods of the ultrafilters a and b in $W(X)$, respectively.

To see that Z is totally countably compact, we take an arbitrary infinite set $I \subset Z$. It is necessary to find an infinite set $J \subset I$ with compact closure $\overline{J}$ in Z. Without loss of generality, we can assume that I is countable and $\infty \notin I$. If $J = I \cap W_\omega \mathcal{A}$ is infinite, then $\overline{J}$ is compact by the ω-boundedness of $W_\omega \mathcal{A}$ (see [2]). If $I \cap W_\omega \mathcal{A}$ is finite, then

$$I \cap Z \setminus W_\omega \mathcal{A} = I \cap Y = I \cap (\omega \times \omega)$$

is infinite. If, for some $n \in \omega$, the set $J_n = I \cap (\{n\} \times \omega)$ is infinite, then $\overline{J}_n = J_n \cup \{\infty\}$ is compact by the definition of topology in the space Y. If, for every $n \in \omega$, the set $I \cap (\{n\} \times \omega)$ is finite, then $I \cap (\omega \times \omega) \in \mathcal{F}$ and, by the maximality of the family $\mathcal{A}$, for some set $A \in \mathcal{A}$, the intersection $J = A \cap I$ is infinite and, hence, $\overline{J} = J \cup \{A\}$ is compact.

A topological space X is called *locally countably compact* if, for each $x \in X$, there exists an open neighborhood U of x such that $\overline{U}$ is countably compact.

Theorem 2. *A first-countable topological space X can be embedded as an open subspace into a Hausdorff countably compact topological space Y if and only if X is locally countably compact.*

Proof. Assume that a first-countable topological space X is an open subspace of a countably compact topological space Y and that X is not locally countably compact. Then there exists $x \in X$ such that, for each open neighborhood U of x, the closure of U in X is not countably compact. We fix any countable base $\{U_n\}_{n\in\omega}$ at the point x such that $U_n \subset U_m$, whenever $n > m$. Then there exists a family $\{A_n\}_{n\in\omega}$ of closed discrete subsets of X such that $A_n \subset U_n$ for each $n \in \omega$. Since Y is countably compact for each $n \in \omega$, the set A_n has an accumulation point $y_n \in Y$. Since A_n is closed in X, we have $y_n \in Y \setminus X$, $n \in \omega$. By using the countable compactness of Y once again, we can find an accumulation point z of the set $\{y_n\}_{n\in\omega}$. Since X is open in Y, we coincide that $z \in Y \setminus X$. It is easy to see that $z \in \overline{U}_n$ for all $n \in \omega$, which contradicts the Hausdorffness of Y.

Let X be a locally countably compact topological space. We set $Y = X \cup \{\infty\}$, where $\infty \notin X$. Let τ be a topology on Y, which satisfies the following conditions:

X is an open subspace of Y;

if $\infty \in U \in \tau$, then $X \setminus U$ is closed and countably compact.

It is easy to check that the space Y is Hausdorff and countably compact.

The following example shows that Theorem 2 does not hold for the topological spaces of character ω_1.

Example 2. By $[0, \omega_1]$ we denote the ordinal $\omega_1 + 1$ endowed with the order topology. Further, by X we denote a subspace $\{\omega_1\} \cup \{\alpha \in \omega_1 \mid \alpha \text{ is isolated in } [0, \omega_1]\}$ of $[0, \omega_1]$. Obviously, X is not locally countably compact (at the point ω_1) and the character of X equals ω_1. Nevertheless, X can be embedded as an open subspace into a Hausdorff countably compact space Y. Let Y be the set $\omega_1 + 1$ endowed with the topology τ satisfying the following conditions:

X is open in Y;

if $\alpha \in U \in \tau$, then there exists an ordinal $\beta \leq \alpha$ such that $\{\gamma \mid \beta < \gamma \leq \alpha\} \subset U$.

Note that $Y \setminus \{\omega_1\}$ is homeomorphic to ω_1 endowed with the order topology. At this point, it is easy to see that Y is countably compact.

A topological space X is called *weakly ∞-regular* if, for any infinite closed subset $F \subset X$ and a point $x \in X \setminus F$, there exist disjoint open sets $V, U \subset X$ such that $x \in V$ and $U \cap F$ is infinite.

Proposition 1. *Every subspace X of a Hausdorff countably compact space Y is weakly ∞-regular.*

Proof. Given an infinite closed subset $F \subset X$ and a point $x \in X \setminus F$, we consider the closure $\overline{F}$ of F in Y and observe that $x \notin \overline{F}$. By the countable compactness of Y, the infinite set F has an accumulation point $y \in \overline{F}$. Since Y is Hausdorff, there are two disjoint open sets $V, U \subset Y$ such that $x \in V$ and $y \in U$. Since y is an accumulation point of the set F, the intersection $F \cap U$ is infinite. Then $V \cap X$ and $U \cap X$ are two disjoint open sets in X such that $x \in V \cap X$ and $F \cap U \cap X$ is infinite, which means that the space X is weakly ∞-regular.

A subset D of a topological space X is called:

> *strictly discrete* if each point $x \in D$ has a neighborhood $O_x \subset X$ such that the family $(O_x)_{x \in D}$ is disjoint in a sense that $O_x \cap O_y = \varnothing$ for any distinct points $x, y \in D$;
>
> *strongly discrete* if each point $x \in D$ has a neighborhood $O_x \subset X$ such that the family $(O_x)_{x \in D}$ is disjoint and locally finite in X.

It is clear that, for every subset $D \subset X$, we have the following implications:

$$\text{strongly discrete} \Rightarrow \text{strictly discrete} \Rightarrow \text{discrete}.$$

Theorem 3. *Let X be a subspace of a countably compact Hausdorff space Y. Then every infinite subset $I \subset X$ contains an infinite subset $D \subset I$, which is strictly discrete in X.*

Proof. By the countable compactness of Y, the set I has an accumulation point $y \in Y$. We choose any point $x_0 \in I \setminus \{y\}$. By using the Hausdorffness of Y, we find disjoint open neighborhoods V_0 and U_0 of the points x_0 and y, respectively. Further, we choose any point $y_1 \in U_0 \cap I \setminus \{y\}$ and, in view of the Hausdorffness of Y, select open disjoint neighborhoods $V_1 \subset U_0$ and $U_1 \subset U_0$ of the points x_1 and y, respectively. Proceeding by induction, we construct a sequence $(x_n)_{n \in \omega}$ of points of X and sequences $(V_n)_{n \in \omega}$ and $(U_n)_{n \in \omega}$ of open sets in Y such that, for every $n \in \mathbb{N}$, the following conditions are satisfied:

(1) $x_n \in V_n \subset U_{n-1}$;

(2) $y \in U_n \subset U_{n-1}$;

(3) $V_n \cap U_n = \varnothing$.

The inductive conditions imply that the sets V_n, $n \in \omega$, are pairwise disjoint, which means that the set $D = \{x_n\}_{n \in \omega} \subset I$ is strictly discrete in X.

Theorem 4. *Let X be a Lindelöf subspace of a countably compact Hausdorff space Y. Then each infinite closed discrete subset $I \subset X$ contains an infinite subset $D \subset I$, which is strongly discrete in X.*

Proof. By the countable compactness of Y, the set I has an accumulation point $y \in Y$. Since I is closed and discrete in X, the point y does not belong to the space X. Further, since Y is Hausdorff, for every $x \in X$, there are disjoint open sets $V_x, W_x \subset Y$ such that $x \in V_x$ and $y \in W_x$. Moreover, since the space X is Lindelöf, the open cover $\{V_x \colon x \in X\}$ has a countable subcover $\{V_{x_n}\}_{n \in \omega}$. For every $n \in \omega$, we consider an open neighborhood $W_n = \bigcap_{k \le n} W_{x_k}$ of y.

We choose an arbitrary point $y_0 \in I \setminus \{y\}$ and, by using the Hausdorffness of Y, find disjoint open neighborhoods V_0 and $U_0 \subset W_0$ of the points y_0 and y, respectively. We also choose an arbitrary point

$$y_1 \in U_0 \cap W_1 \cap I \setminus \{y\}$$

and, by using the Hausdorffness of Y, choose open disjoint neighborhoods $V_1 \subset U_0$ and $U_1 \subset U_0 \cap W_1$ of the points y_1 and y, respectively. Proceeding by induction, we can construct a sequence $(y_n)_{n \in \omega}$ of points of X and sequences $(V_n)_{n \in \omega}$ and $(U_n)_{n \in \omega}$ of open sets in Y such that, for every $n \in \mathbb{N}$, the following conditions are satisfied:

(1) $y_n \in V_n \subset U_{n-1} \cap W_n$;

(2) $y \in U_n \subset U_{n-1} \cap W_n$;

(3) $V_n \cap U_n = \varnothing$.

The inductive conditions imply that the sets in the family $(V_n)_{n\in\omega}$ are pairwise disjoint, thus witnessing that the set $D = \{y_n\}_{n\in\omega} \subset I$ is strictly discrete in X. To prove that D is strongly discrete, it remains to show that the family $(V_n)_{n\in\omega}$ is locally finite in X. Given any point $x \in X$, we find $n \in \omega$ such that $x \in V_{x_n}$ and observe that, for every $i > n$,

$$V_i \cap V_{x_n} \subset W_i \cap V_{x_n} \subset W_n \cap V_{x_n} = \varnothing.$$

A topological space X is called *$\ddot{\omega}$-regular* if it is $\mathcal{F}$-regular for the family $\mathcal{F}$ of countable closed discrete subsets in X.

Proposition 2. *Each countable closed discrete subset D of a (Lindelöf) $\ddot{\omega}$-regular T_1-space X is strictly discrete (strongly discrete) in X.*

Proof. The space X is Hausdorff being an $\ddot{\omega}$-regular T_1-space. If the subset $D \subset X$ is finite, then D is strongly discrete because X is Hausdorff. Hence, we assume that D is infinite and, therefore, $D = \{z_n\}_{n\in\omega}$ for some pairwise distinct points z_n. By the $\ddot{\omega}$-regularity there are two disjoint open sets $V_0, W_0 \subset X$ such that $z_0 \in V_0$ and $\{z_n\}_{n\geq 1} \subset W_0$.

Proceeding by induction, we can construct sequences of open sets $(V_n)_{n\in\omega}$ and $(W_n)_{n\in\omega}$ in X such that, for every $n \in \omega$, the following conditions are satisfied:

$z_n \in V_n \subset W_{n-1}$;

$\{z_k\}_{k>n} \subset W_n \subset W_{n-1}$;

$V_n \cap W_n = \varnothing$.

These conditions imply that the family $(V_n)_{n\in\omega}$ is disjoint, thus witnessing that the set D is strictly discrete in X.

We now assume that the space X is Lindelöf and

$$V = \bigcup_{n\in\omega} V_n.$$

By the $\ddot{\omega}$-regularity of X, each point $x \in X \setminus V$ has a neighborhood $O_x \subset X$ whose closure $\overline{O}_x$ does not intersect a closed discrete subset D of X. Since X is Lindelöf, there exists a countable set $\{x_n\}_{n\in\omega} \subset X \setminus V$ such that

$$X = V \cup \bigcup_{n\in\omega} O_{x_n}.$$

For every $n \in \omega$, we consider an open neighborhood

$$U_n := V_n \setminus \bigcup_{k\leq n} \overline{O}_{x_k}$$

of z_n and observe that the family $(U_n)_{n\in\omega}$ is disjoint and locally finite in X, which means that the set D is strongly discrete in X.

The following proposition shows that the property described in Theorem 3 remains true for $\ddot{\omega}$-regular spaces.

Proposition 3. *Every infinite subset I of an $\ddot{\omega}$-regular T_1-space X contains an infinite subset $D \subset I$, which is strictly discrete in X.*

Proof. If I has an accumulation point in X, then a strictly discrete infinite subset can be constructed by repeating the argument used in the proof of Theorem 3. Thus, we assume that I has no accumulation points in X and, hence, I is closed and discrete in X. Replacing I by a countable infinite subset of I, we can assume that I is countable. By Proposition 2, the set I is strictly discrete in X.

A topological space X is called *superconnected* [3] if, for any nonempty open sets $U_1, \dots, U_n$, the intersection $\overline{U}_1 \cap \dots \cap \overline{U}_n$ is not empty. It is clear that a superconnected space containing more than one point is not regular. An example of a superconnected second-countable Hausdorff space can be found in [3].

Proposition 4. *Any first-countable superconnected Hausdorff space X with $|X| > 1$ contains an infinite set $I \subset X$ such that every infinite subset $D \subset I$ is not strictly discrete in X.*

Proof. For every point $x \in X$, we fix a countable neighborhood base $\{U_{x,n}\}_{n\in\omega}$ at x such that $U_{x,n+1} \subset U_{x,n}$ for every $n \in \omega$.

Choose any two distinct points $x_0, x_1 \in X$ and, for every $n \geq 2$, choose a point $x_n \in \bigcap_{k<n} \overline{U}_{x_k,n}$. We claim that the set $I = \{x_n\}_{n\in\omega}$ is infinite. In the opposite case, we use Hausdorffness and find a neighborhood V of x_0 such that $\overline{V} \cap I = \{x_0\}$. Further, we find $m \in \omega$ such that $U_{x_0,m} \subset V$ and $x_0 \notin \overline{U}_{x_1,m}$. Note that

$$x_m \in I \cap \overline{U}_{x_0,m} \cap \overline{U}_{x_1,m} = \varnothing,$$

which is a desired contradiction showing that the set I is infinite.

Further, we show that any infinite subset $D \subset I$ is not strictly discrete in X. To get a contradiction, we assume that D is strictly discrete. Then each point $x \in D$ has a neighborhood $O_x \subset X$ such that the family $(O_x)_{x\in D}$ is disjoint. We choose any point $x_k \in D$ and find $m \in \omega$ such that $U_{x_k,m} \subset O_{x_k}$. Replacing m by a larger number, we can assume that $m > k$ and $x_m \in D$. Since $x_m \in \overline{U}_{x_k,m} \subset \overline{O}_{x_k}$, the intersection $O_{x_m} \cap O_{x,k}$ is nonempty, which contradicts the choice of the neighborhoods O_x, $x \in D$.

We now establish one property of the subspaces of functionally Hausdorff countably compact spaces. We recall that a topological space X is *functionally Hausdorff* if, for any distinct points $x, y \in X$, there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(x) = 1$.

A subset U of a topological space X is called *functionally open* if $U = f^{-1}(V)$ for some continuous function $f\colon X \to \mathbb{R}$ and some open set $V \subset \mathbb{R}$.

A subset K of a topological space X is called *functionally compact* if each open cover of K by functionally open subsets of X has a finite subcover.

Proposition 5. *If X is a subspace of a functionally Hausdorff countably compact space Y, then no infinite closed discrete subspace $D \subset X$ is contained in a functionally compact subset of X.*

Proof. To get a contradiction, we assume that D is contained in a functionally compact subset K of X. By the countable compactness of Y, the set D has an accumulation point $y \in Y$. Since D is closed and discrete in X, the point y does not belong to X and, hence, $y \notin K$. Since Y is functionally Hausdorff, for every $x \in K$, there exists a continuous function $f_x\colon Y \to [0,1]$ such that

$$f_x(x) = 0 \qquad \text{and} \qquad f_x(y) = 1.$$

By the functional compactness of K, the cover $\left\{f_x^{-1}\left(\left[0,\frac{1}{2}\right)\right)\colon x\in K\right\}$ contains a finite subcover

$$\left\{f_x^{-1}\left(\left[0,\frac{1}{2}\right)\right)\colon x\in E\right\},$$

where E is a finite subset of K. Then

$$D\subset K\subset f^{-1}\left(\left[0,\frac{1}{2}\right)\right)$$

for the continuous function $f=\max_{x\in E}f_x\colon Y\to[0,1]$ and $f^{-1}\left(\left(\frac{1}{2},1\right]\right)$ is a neighborhood of y, which is disjoint with the set D. However, this is impossible as y is an accumulation point of D.

Finally, we construct an example of a regular separable first-countable scattered space that can be embedded into a Hausdorff countably compact space but cannot be embedded into Urysohn countably compact spaces.

Example 3. There exists a topological space X such that:

(1) X is regular, separable, and first-countable;

(2) X cannot be embedded as an open subspace into a Hausdorff countably compact space;

(3) X cannot be embedded into an Urysohn countably compact space;

(4) X can be embedded into an Hausdorff totally countably compact space.

Proof. In the construction of the space X, we use almost disjoint dominating subsets of ω^ω. We recall [5] that a subset $D\subset\omega^\omega$ is called *dominating* if, for any $x\in\omega^\omega$, there exists $y\in D$ such that $x\le^* y$, which means that $x(n)\le y(n)$ for all but finitely many numbers $n\in\omega$. By $\mathfrak{d}$ we denote the smallest cardinality of a dominating subset $D\subset\omega^\omega$. It is clear that $\omega_1\le\mathfrak{d}\le\mathfrak{c}$.

We say that a family of functions $D\subset\omega^\omega$ is *almost disjoint* if, for any distinct $x,y\in D$, the intersection $x\cap y$ is finite. Here, we identify a function $x\in\omega^\omega$ with its graph $\{(n,x(n))\colon n\in\omega\}$ and, hence, identify the set of functions ω^ω with a subset of the family $[\omega\times\omega]^\omega$ of all infinite subsets of $\omega\times\omega$.

Claim 1. *There exists an almost disjoint dominating subset $D\subset\omega^\omega$ of cardinality $|D|=\mathfrak{d}$.*

Proof. By the definition of $\mathfrak{d}$, there exists a dominating family $\{x_\alpha\}_{\alpha\in\mathfrak{d}}\subset\omega^\omega$. It is well known that $[\omega]^\omega$ contains an almost disjoint family $\{A_\alpha\}_{\alpha\in\mathfrak{c}}$ of cardinality continuum. For every $\alpha<\mathfrak{d}$, we choose a strictly increasing function $y_\alpha\colon\omega\to A_\alpha$ such that $x_\alpha\le y_\alpha$. Then the set $D=\{y_\alpha\}_{\alpha\in\mathfrak{d}}$ is dominating and almost disjoint.

By Claim 1, there exists an almost disjoint dominating subset $D\subset\omega^\omega\subset[\omega\times\omega]^\omega$. For every $n\in\omega$, we consider the set $\lambda_n=\{n\}\times\omega$ and note that the family $L=\{\lambda_n\}_{n\in\omega}$ is disjoint and the family $D\cup L\subset[\omega\times\omega]^\omega$ is almost disjoint.

Consider a space

$$Y=(D\cup L)\cup(\omega\times\omega)$$

endowed with the topology formed by the sets $U\subset Y$ such that, for every $y\in(D\cup L)\cap U$, the set $y\setminus U\subset\omega\times\omega$ is finite. Observe that all points from $\omega\times\omega$ are isolated in Y. In view of the almost disjointness of the family $D\cup L$, it can be shown that the space Y is regular, separable, locally countable, scattered, and locally compact.

We choose an arbitrary point $\infty\notin\omega\times Y$ and consider a space $Z=\{\infty\}\cup(\omega\times Y)$ endowed with the topology formed by the sets $W\subset Z$ such that

for every $n\in\omega$, the set $\{y\in Y\colon\ (n,y)\in W\}$ is open in Y

and

if $\infty\in W$, then there exists $n\in\omega$ such that $\bigcup_{m\geq n}\{m\}\times Y\subset W$.

It is easy to see that $Z=\{\infty\}\cup(\omega\times Y)$ is first-countable, separable, scattered, and regular.

Let $\sim$ be the smallest equivalence relation on Z such that

$$(2n,\lambda)\sim(2n+1,\lambda)\qquad\text{and}\qquad(2n+1,d)\sim(2n+2,d)$$

for any $n\in\omega,\ \lambda\in L$ and $d\in D$.

Let X be the quotient space $Z/_\sim$ of Z by the equivalence relation $\sim$. It is easy to see that the equivalence relation $\sim$ has at most two-element equivalence classes and the quotient map $q\colon\ Z\to X$ is closed and, hence, perfect. Applying [6, Theorem 3.7.20], we conclude that the space X is regular. It is easy to see that X is separable, scattered, and first-countable. Note that X is not locally countably compact at the point ∞. Thus, Theorem 2 implies that X cannot be embedded as an open subspace into a Hausdorff countably compact space. It remains to show that X has the properties (3) and (4) from Example 3. This is proved in the following two claims:

Claim 2. *The space X does not admit an embedding into an Urysohn countably compact space.*

Proof. To arrive at a contradiction, we assume that $X=q(Z)$ is a subspace of an Urysohn countably compact space C. By the countable compactness of C, the set $q(\{0\}\times L)\subset X\subset C$ has an accumulation point $c_0\in C$. The point c_0 is distinct from $q(\infty)$, as $q(\infty)$ is not an accumulation point of the set $q(\{0\}\times L)$ in X. Let $l\in\omega$ be the largest number such that c_0 is an accumulation point of the set $q(\{l\}\times L)$ in C.

We now show that the number l is well defined. Indeed, by the Hausdorffness of the space C, there exists a neighborhood $W\subset C$ of $q(\infty)$ such that $c_0\not\subset\overline{W}$. By the definition of the topology of the space Z, there exists $m\in\omega$ such that

$$\bigcup_{k\geq m}\{k\}\times Y\subset q^{-1}(W).$$

Then c_0 is not an accumulation point of the set $\bigcup_{k\geq m}q(\{k\}\times L)$. Hence, the number l is well defined and we have $l<m$.

The definition of the equivalence relation $\sim$ implies that the number l is odd. By the countable compactness of C, the infinite set $q(\{l+1\}\times L)$ has an accumulation point $c_1\in C$. The maximality of l ensures that $c_1\neq c_0$. Since C is Urysohn, the points c_0 and c_1 have open neighborhoods $U_0,U_1\subset C$ with disjoint closures in C.

For every $i\in\{0,1\}$, we consider the set

$$J_i=\big\{n\in\omega\colon\ q(l+i,\lambda_n)\in U_i\big\},$$

which is infinite because c_i is an accumulation point of the set

$$q(\{l+i\}\times L)=\big\{q(l+i,\lambda_n)\colon\ n\in\omega\big\}.$$

For every $n \in J_i$, the open set $q^{-1}(U_i) \subset Z$ contains the pair $(l+i, \lambda_n)$. By the definition of topology at $(l+i, \lambda_n)$, the set

$$(\{l+i\} \times \lambda_n) \setminus q^{-1}(U_i) \subset \{l+i\} \times \{n\} \times \omega$$

is finite and, hence, is contained in the set $\{l+i\} \times \{n\} \times [0, f_i(n)]$ for some number $f_i(n) \in \omega$. By the dominating property of the family D, we can choose a function $f \in D$ such that $f(n) \geq f_i(n)$ for any $i \in \{0,1\}$ and $n \in J_i$. This implies that, for every $i \in \{1,2\}$, the set

$$\{l+i\} \times f \subset \{l+i\} \times (\omega \times \omega)$$

has infinite intersections with the preimage $q^{-1}(U_i)$ and, hence,

$$\{(l+i, f)\} \in \overline{q^{-1}(U_i)} \subset q^{-1}(\overline{U}_i).$$

In view of the fact that the number l is odd, we conclude that

$$q(l, f) = q(l+1, f) \in \overline{U}_0 \cap \overline{U}_1 = \varnothing,$$

which is the desired contradiction. This completes the proof of the claim.

Claim 3. *The space X admits an embedding into a Hausdorff totally countably compact space.*

Proof. By using the Kuratowski–Zorn lemma, we can enlarge an almost disjoint family $D \cup L$ to a maximal almost disjoint family $M \subset [\omega \times \omega]^{\omega}$. Consider a space $Y_M = M \cup (\omega \times \omega)$ endowed with the topology formed by the sets $U \subset Y_M$ such that, for every $y \in M \cap U$, the set $y \setminus U \subset \omega \times \omega$ is finite. This implies that Y_M is a regular locally compact first-countable space containing Y as an open dense subspace. The maximality of M implies that each sequence in $\omega \times \omega$ contains a subsequence that converges to some point of the space Y_M. This property implies that the subspace $\tilde{Y} := (W_\omega M) \cup (\omega \times \omega)$ of the Wallman extension $W(Y_M)$ is totally countably compact. Repeating the argument from Example 1, we can show that the space $\tilde{Y}$ is Hausdorff.

Let $\tilde{Z} = \{\infty\} \cup (\omega \times \tilde{Y})$, where $\infty \notin \omega \times \tilde{Y}$. The space $\tilde{Z}$ is endowed with the topology consisting of the sets $W \subset \tilde{Z}$ such that

for every $n \in \omega$, the set $\{y \in \tilde{Y} : (n, y) \in W\}$ is open in $\tilde{Y}$

and

if $\infty \in W$, then there exists $n \in \omega$ such that $\bigcup_{m \geq n} \{m\} \times \tilde{Y} \subset W$.

In view of the fact that the space $\tilde{Y}$ is Hausdorff and totally countably compact, we can prove that the same is true for the space $\tilde{Z}$.

Let $\sim$ be the smallest equivalence relation on $\tilde{Z}$ such that

$$(2n, \lambda) \sim (2n+1, \lambda) \qquad \text{and} \qquad (2n+1, d) \sim (2n+2, d)$$

for any $n \in \omega$, $\lambda \in W_\omega L$, and $d \in W_\omega D$.

Let $\tilde{X}$ be the quotient space $\tilde{Z}/_\sim$ of $\tilde{Z}$ by the equivalence relation $\sim$. It is easy to see that the space $\tilde{X}$ is Hausdorff totally countably compact and contains the space X as a dense subspace.

However, we do not know the answer for the following intriguing problem (from Lviv Scottish Book [4]):

Problem 2. *Is it true that each (scattered, functionally Hausdorff) regular topological space can be embedded into a Hausdorff countably compact topological space?*

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