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PERIODIC AND WEAKLY PERIODIC GROUND STATES CORRESPONDING TO THE SUBGROUPS OF INDEX THREE FOR THE ISING MODEL ON THE CAYLEY TREE OF ORDER THREE

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We determine periodic and weakly periodic ground states with subgroups of index three for the Ising model on the Cayley tree of order three.

1. Introduction

The Ising model with two values of spin ± 1 was considered in [11, 14]. It became extensively investigated since the 1990's and later (see, e.g., [1–7, 10, 12]).

Each Gibbs measure is associated with a single phase of a physical system. The existence of two or more Gibbs measures corresponds to the existence of phase transitions. One of fundamental problems is to describe the extreme Gibbs measures corresponding to a given Hamiltonian. It is known that the phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measure, see [13, 17]. Thus, we naturally arrive at the problem of description of periodic and weakly periodic ground states. For the Ising model with competing interactions on the Cayley tree, the translation-invariant and periodic ground states corresponding to normal subgroups with even indices were studied in [1, 16]. As usual, periodic ground states are simpler and more interesting. On the other hand, it is necessary to find weakly periodic ground states for some parameters for which periodic ground states do not exist.

Main concepts and notation for the case of weakly periodic ground states were introduced in [18]. For the Ising model with competing interactions, weakly periodic ground states corresponding to normal subgroups of indices two and four were described in [18, 20]. For the Potts model, these states were studied for normal subgroups of index 2 in [21, 22]. Moreover, for the Potts model, periodic and weakly periodic ground states were studied for the normal subgroups of index 4 in [23].

A full description of the normal subgroups of indices $2i,\ i = \overline{1,5}$, for the group representation of the Cayley tree was given in [8, 9, 19]. In addition, the existence of all subgroups of finite index for the group was proved and a full description of (not normal) subgroups of index 3 was given in [15]. Note that there are some papers devoted to periodic and weakly periodic ground states for normal groups of finite index. In the present paper, for the first time, we study periodic and weakly periodic ground states for (not normal) subgroups of index 3. Note that periodic and weakly periodic ground states depend on the subgroups (in particular, normal subgroups). Moreover, the invariance properties are not true for the (not normal) subgroup. Note that this problem is more difficult than to study periodic and weakly periodic ground states constructed by the normal subgroups. Thus, it is naturally of interest to study the subgroups of index 3.

The present paper is organized as follows. In Section 2, we recall main definitions and known facts. In Section 3, we describe periodic and weakly periodic ground states.

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2. Main Definitions and Known Facts

The Cayley Tree. The Cayley tree Γ^k (see [2]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles with exactly $k+1$ edges originating from each its vertex. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighboring vertices*, and we write $l = \langle x, y\rangle$. The distance on this tree is defined as the number of pairs of nearest neighbors in the minimal path between the vertices x and y (where the path is the collection of pairs of nearest neighbors in which two consecutive pairs share at least a given vertex) and denoted by $d(x, y)$.

For a fixed $x^0 \in V$ (as usual, x^0 is called the root of the tree), we set

$$W_n = \{x \in V \mid d(x, x^0) = n\},$$

$$V_n = \{x \in V \mid d(x, x^0) \leq n\}, \qquad L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\}.$$

We write $x < y$ if the path from x^0 to y goes through x and $|x| = d(x, x^0),\ x \in V$.

It is known (see [6]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and a group G_k of the free products of $k+1$ cyclic groups $\{e, a_i\},\ i = 1, \dots, k+1$, of the second order (i.e., $a_i^2 = e,\ a_i \neq e$) with generators $a_1, a_2, \dots, a_{k+1}$.

Let $S(x)$ be the set of "direct successors" of $x \in G_k$, i.e.,

$$S(x) = \{y \in W_{n+1} \mid d(y, x) = 1\}, \quad x \in W_n.$$

In addition, $S_1(x)$ is the set of all nearest-neighboring vertices of $x \in G_k$, i.e.,

$$S_1(x) = \{y \in G_k\colon \langle x, y\rangle\} \qquad \text{and} \qquad \{x_\downarrow\} = S_1(x) \setminus S(x).$$

The Ising Model. First, we present the main definitions and facts about the Ising model. We consider models in which the spin takes values from the set $\Phi = \{-1, 1\}$. For $A \subseteq V$ a spin *configuration* σ_A on A is defined as a function $x \in A \to \sigma_A(x) \in \Phi$; the set of all configurations is denoted by $\Omega_A = \Phi^A$. We set

$$\Omega = \Omega_V, \quad \sigma = \sigma_V, \quad \text{and} \quad -\sigma_A = \{-\sigma_A(x),\ x \in A\}.$$

We define a *periodic configuration* as a configuration $\sigma \in \Omega$ invariant under the cosets of a subgroup $G_k^* \subset G_k$ of finite index. More precisely, a configuration $\sigma \in \Omega$ is called G_k^*-periodic if $\sigma(yx) = \sigma(x)$ for any $x \in G_k$ and $y \in G_k^*$.

The index of a subgroup is called the *period of the corresponding periodic configuration*. A configuration invariant with respect to all cosets is called *translation-invariant*.

Let $G_k/G_k^* = \{H_1, \dots, H_r\}$ be a family of cosets, where G_k^* is a subgroup of index $r \geq 1$. A configuration $\sigma(x),\ x \in V$, is called G_k^*-*weakly periodic* if $\sigma(x) = \sigma_{ij}$ for $x \in H_i,\ x_\downarrow \in H_j\ \forall x \in G_k$.

The Ising model with competing interactions has the form

$$H(\sigma) = J_1 \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y) + J_2 \sum_{\substack{x,y \in V:\\ d(x,y)=2}} \sigma(x)\sigma(y), \tag{1}$$

where $J_1, J_2 \in R$ and $\sigma \in \Omega$.

For a pair of configurations σ and φ that coincide almost everywhere, i.e., everywhere except finitely many positions, we consider a relative Hamiltonian $H(\sigma, \varphi)$, i.e., the difference between the energies of configurations σ and φ, which has the form

$$H(\sigma, \varphi) = J_1 \sum_{\langle x,y\rangle \in L} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)) + J_2 \sum_{\substack{x,y\in V:\\ d(x,y)=2}} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)),$$

where $J = (J_1, J_2) \in R^2$ is an arbitrary fixed parameter.

Let M be the set of unit balls with vertices in V. The restriction of a configuration σ to the ball $b \in M$ is called a *bounded configuration* σ_b.

The energy of a ball b for the configuration σ is defined by

$$U(\sigma_b) \equiv U(\sigma_b, J) = \frac{1}{2} J_1 \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y) + J_2 \sum_{d(x,y)=2} \sigma(x)\sigma(y), \quad x, y \in b,$$

where $J = (J_1, J_2) \in R^2$.

We say that two bounded configurations σ_b and $\sigma'_{b'}$ belong to the same class if $U(\sigma_b) = U(\sigma'_{b'})$, and we write $\sigma'_{b'} \sim \sigma_b$.

Let A be a set. Then $| A |$ is the cardinality of A.

Lemma 1 [1].

1. *For any configuration* σ_b, *the following relation is true:*

$$U(\sigma_b) \in \{U_0, U_1, \dots, U_{k+1}\},$$

where

$$U_i = \left(\frac{k+1}{2} - i\right) J_1 + \left(\frac{k(k+1)}{2} + 2i(i-k-1)\right) J_2, \quad i = 0, 1, \dots, k+1.$$

2. *Let* $\mathcal{C}_i = \Omega_i \cup \Omega_i^-$, $i = 0, \dots, k+1$, *where*

$$\Omega_i = \big\{\sigma_b \colon \sigma_b(c_b) = +1, \ \big|\{x \in b \setminus \{c_b\} \colon \sigma_b(x) = -1\}\big| = i\big\},$$

$$\Omega_i^- = \big\{-\sigma_b = \{-\sigma_b(x), x \in b\} \colon \sigma_b \in \Omega_i\big\},$$

and c_b *is the center of the ball* b. *Then, for* $\sigma_b \in \mathcal{C}_i$, *the following equality is true:* $U(\sigma_b) = U_i$.

3. *The class* $\mathcal{C}_i$ *contains* $\dfrac{2(k+1)!}{i!(k-i+1)!}$ *configurations.*

Definition 1. *A configuration* φ *is called a ground state for Hamiltonian (1) if it satisfies the condition*

$$U(\varphi_b) = \min\{U_0, U_1, \dots, U_{k+1}\} \quad \textit{for any} \quad b \in M.$$

Denote

$$U_i(J) = U(\sigma_b, J) \quad \text{if} \quad \sigma_b \in \mathcal{C}_i, \quad i = 0, 1, \ldots, k+1.$$

The quantity $U_i(J)$ is a linear function of the parameter $J \in R^2$. For any fixed $m = 0, 1, \ldots, k+1,$ we denote

$$A_m = \{J \in R^2 \colon U_m(J) = \min\{U_0(J), U_1(J), \ldots, U_{k+1}(J)\}\}. \tag{2}$$

It is easy to see that

$$A_0 = \{J \in R^2 \colon J_1 \leq 0, \ J_1 + 2kJ_2 \leq 0\},$$

$$A_m = \left\{J \in R^2 \colon J_2 \geq 0, \ 2(2m-k-2)J_2 \leq J_1 \leq 2(2m-k)J_2\right\}, \quad m = 1, 2, \ldots, k,$$

$$A_{k+1} = \{J \in R^2 \colon J_1 \geq 0, \ J_1 - 2kJ_2 \geq 0\},$$

and

$$R^2 = \bigcup_{i=0}^{k+1} A_i.$$

3. Periodic and Weakly Periodic Ground States

In this section, we study periodic and weakly periodic ground states. It is known that ground states depend on the choice of subgroups for a given Hamiltonian. For this reason, we now explain how to choose a subgroup with index 3 of the group G_k.

Let G_k be a free product of $k+1$ cyclic groups of the second order with generators $a_1, a_2, \ldots, a_{k+1}$, respectively. Then it follows from Theorem 1 in [16] that:

the group G_k does not have normal subgroups of odd index ($\neq 1$);

the group G_k has normal subgroups of an arbitrary even index.

We now present a construction of subgroups of index 3 for the group G_k (for more detail, see [15]).

Let $N_k = \{1, 2, \ldots, k+1\}$, let $B_0 \subset N_k, \ 0 \leq |B_0| \leq k-1,$ and let (B_1, B_2) be a partition of the set $N_k \setminus B_0$. Also let m_j be a minimal element of $B_j, \ j \in \{1, 2\}$. Thus, we consider a homomorphism

$$u_{B_1B_2} \colon \langle e, a_1, a_2, \ldots, a_{k+1}\rangle \to \langle e, a_{m_1}, a_{m_2}\rangle,$$

(where e is the identity element) given by

$$u_{B_1B_2}(x) = \begin{cases} e, & \text{if} \quad x = a_i, \quad i \in N_k \backslash (B_1 \cup B_2), \\ a_{m_j}, & \text{if} \quad x = a_i, \quad i \in B_j, \quad j = 1, 2. \end{cases} \tag{3}$$

Let $l(x)$ be the length of x. For $1 \leq q \leq s,$ we define

$$\gamma_s \colon \langle e, a_{m_1}, a_{m_2}\rangle \to \{e, a_{m_1}, a_{m_2}\}$$

by the formula

$$\gamma_s(x)=\begin{cases} e, \quad \text{if} \quad x=e,\\ a_{m_1}a_{m_2}a_{m_1}\dots a_{m_j}, \quad \text{if} \quad x\in\Big\{\underbrace{a_{m_1}a_{m_2}a_{m_1}\dots a_{m_j}}_{q},\ \underbrace{a_{m_2}a_{m_1}a_{m_2}\dots a_{m_{3-j}}}_{2s+1-q}\Big\},\\ a_{m_2}a_{m_1}a_{m_2}\dots a_{m_j}, \quad \text{if} \quad x\in\Big\{\underbrace{a_{m_2}a_{m_1}a_{m_2}\dots a_{m_j}}_{q},\ \underbrace{a_{m_1}a_{m_2}a_{m_1}\dots a_{m_{3-j}}}_{2s+1-q}\Big\},\\ \gamma_s\Big(a_{m_j}\dots\gamma_s(\underbrace{a_{m_j}a_{m_{3-j}}\dots a_{m_{3-j}}}_{2s})\Big), \quad \text{if} \quad x=a_{m_j}a_{m_{3-j}}\dots a_{m_{3-j}}, \quad l(x)>2s,\\ \gamma_s\Big(a_{m_j}\dots\gamma_s(\underbrace{a_{m_{3-j}}a_{m_j}\dots a_{m_j}}_{2s})\Big), \quad \text{if} \quad x=a_{m_j}a_{m_{3-j}}\dots a_{m_{3-j}}, \quad l(x)>2s. \end{cases} \tag{4}$$

We denote

$$\Im^s_{B_1B_2}(G_k)=\{x\in G_k \mid \gamma_s(u_{B_1B_2}(x))=e\}.$$

Lemma 2 [15]. *Let (B_1,B_2) be a partition of the set $N_k\setminus B_0,\ 0\le|B_0|\le k-1$. Then $x\in\Im^s_{B_1B_2}(G_k)$ if and only if the number $l(u_{B_1B_2}(x))$ is divisible by $2s+1$.*

Proposition 1 [15]. *For the group G_k the following equality holds:*

$$\{K\mid K\ \textit{is a subgroup of}\ G_k\ \textit{of index}\ 3\}=\left\{\Im^1_{B_1B_2}\mid B_1,B_2\ \textit{is a partition of}\ N_k\setminus B_0\right\}.$$

We consider periodic and weakly periodic ground states on the Cayley tree of order three, i.e., $k=3$. Further, we consider all cases of subgroups of index 3 of the group G_3.

1. Let $B_0=\{3,4\}$ and $B_d=\{d\},\ d\in\{1,2\}$, i.e., $m_i=i, i\in\{1,2\}$. We now consider the homomorphisms $u^{(1)}_{B_1B_2}:\langle e,a_1,a_2,a_3,a_4\rangle\to\langle e,a_1,a_2\rangle$ (3) and $\gamma^{(1)}:\langle e,a_1,a_2\rangle\to\{e,a_1,a_2\}$ (4):

$$u^{(1)}_{B_1B_2}(x)=\begin{cases} e, & \text{if} \quad x\in\{e,a_3,a_4\},\\ a_i & \text{if} \quad x=a_i, \quad i=1,2, \end{cases}$$

$$\gamma^{(1)}(x)=\begin{cases} e, & \text{if} \quad x=e,\\ a_1, & \text{if} \quad x\in\{a_1,a_2a_1\},\\ a_2, & \text{if} \quad x\in\{a_2,a_1a_2\},\\ \gamma^{(1)}(a_ia_{3-i}\dots\gamma^{(1)}(a_ia_{3-i})), & \text{if} \quad x=a_ia_{3-i}\dots a_{3-i}, \quad l(x)\ge3, \quad i=1,2,\\ \gamma^{(1)}(a_ia_{3-i}\dots\gamma^{(1)}(a_{3-i}a_i)), & \text{if} \quad x=a_ia_{3-i}\dots a_i, \quad l(x)\ge3, \quad i=1,2. \end{cases}$$

Let $H_1^{(1)} := \Im^1_{B_1B_2}(G_3)$. Then

$$H_1^{(1)} = \left\{x \in G_3 \mid \gamma^{(1)}(u^{(1)}_{B_1B_2}(x)) = e\right\}.$$

Since $H_1^{(1)}$ is a subgroup of index 3 of the group G_3, we define a family of cosets:

$$G_3/H_1^{(1)} = \left\{H_1^{(1)}, H_2^{(1)}, H_3^{(1)}\right\},$$

where

$$H_2^{(1)} = \left\{x \in G_3 \mid \gamma^{(1)}(u^{(1)}_{B_1B_2}(x)) = a_1\right\} \qquad \text{and} \qquad H_3^{(1)} = \left\{x \in G_3 \mid \gamma^{(1)}(u^{(1)}_{B_1B_2}(x)) = a_2\right\}.$$

2. Let $B_0 = \{1\}$, $B_1 = \{2,3\}$, and $B_2 = \{4\}$, i.e., $m_1 = 2$ and $m_2 = 4$. We now consider the homomorphisms

$$u^{(2)}_{B_1B_2}\colon \langle e, a_1, a_2, a_3, a_4\rangle \to \langle e, a_2, a_4\rangle \ (3) \qquad \text{and} \qquad \gamma^{(2)}\colon \langle e, a_2, a_4\rangle \to \{e, a_2, a_4\} \ (4):$$

$$u^{(2)}_{B_1B_2}(x) = \begin{cases} e, & \text{if} \quad x \in \{e, a_1\}, \\ a_2, & \text{if} \quad x \in \{a_2, a_3\}, \\ a_4, & \text{if} \quad x = a_4, \end{cases}$$

$$\gamma^{(2)}(x) = \begin{cases} e, & \text{if} \quad x = e, \\ a_2, & \text{if} \quad x \in \{a_2, a_4a_2\}, \\ a_4, & \text{if} \quad x \in \{a_4, a_2a_4\}, \\ \gamma^{(2)}\big(a_ia_{6-i}\dots\gamma^{(2)}(a_ia_{6-i})\big), & \text{if} \quad x = a_ia_{6-i}\dots a_{6-i}, \quad l(x) \geq 3, \quad i \in \{2;4\}, \\ \gamma^{(2)}\big(a_ia_{6-i}\dots\gamma^{(2)}(a_{6-i}a_i)\big), & \text{if} \quad x = a_ia_{6-i}\dots a_i, \quad l(x) \geq 3, \quad i \in \{2;4\}. \end{cases}$$

Let $H_1^{(2)} := \Im^1_{B_1B_2}(G_3)$. Then

$$H_1^{(2)} = \left\{x \in G_3 \mid \gamma^{(2)}\left(u^{(2)}_{B_1B_2}(x)\right) = e\right\}.$$

Since $H_1^{(2)}$ is a subgroup of index 3 of the group G_3, we define a family of cosets:

$$G_3/H_1^{(2)} = \left\{H_1^{(2)}, H_2^{(2)}, H_3^{(2)}\right\},$$

where

$$H_2^{(2)} = \left\{x \in G_3 \mid \gamma^{(2)}\left(u^{(2)}_{B_1B_2}(x)\right) = a_2\right\} \qquad \text{and} \qquad H_3^{(2)} = \left\{x \in G_3 \mid \gamma^{(2)}\left(u^{(2)}_{B_1B_2}(x)\right) = a_4\right\}.$$

3. Let $B_0=\{\varnothing\},\ B_1=\{1\}$, and $B_2=\{2,3,4\}$, i.e., $m_1=1$ and $m_2=2$. We now consider homomorphisms

$$u^{(3)}_{B_1B_2}\colon \langle e,a_1,a_2,a_3,a_4\rangle \to \langle e,a_1,a_2\rangle \ (3) \qquad \text{and} \qquad \gamma^{(3)}\colon\langle e,a_1,a_2\rangle\to\{e,a_1,a_2\}\ (4):$$

$$u^{(3)}_{B_1B_2}(x)=\begin{cases} e, & \text{if} \quad x=e,\\ a_1, & \text{if} \quad x=a_1,\\ a_2, & \text{if} \quad x=a_i, \quad i=\overline{2,4}, \end{cases}$$

$$\gamma^{(3)}(x)=\begin{cases} e, & \text{if} \quad x=e,\\ a_1, & \text{if} \quad x\in\{a_1,a_2a_1\},\\ a_2, & \text{if} \quad x\in\{a_2,a_1a_2\},\\ \gamma^{(3)}(a_ia_{3-i}\ldots\gamma^{(3)}(a_ia_{3-i})), & \text{if} \quad x=a_ia_{3-i}\ldots a_{3-i}, \quad l(x)\geq 3, \quad i\in\{1;2\},\\ \gamma^{(3)}(a_ia_{3-i}\ldots\gamma^{(3)}(a_{3-i}a_i)), & \text{if} \quad x=a_ia_{3-i}\ldots a_i, \quad l(x)\geq 3, \quad i\in\{1;2\}. \end{cases}$$

Let $H^{(3)}_1:=\Im^1_{B_1B_2}(G_3)$. Then

$$H^{(3)}_1=\left\{x\in G_3 \mid \gamma^{(3)}\left(u^{(3)}_{B_1B_2}(x)\right)=e\right\}.$$

Since $H^{(3)}_1$ is a subgroup of index 3 of the group G_3, we define a family of cosets:

$$G_3/H^{(3)}_1=\left\{H^{(3)}_1,H^{(3)}_2,H^{(3)}_3\right\},$$

where

$$H^{(3)}_2=\left\{x\in G_3 \mid \gamma^{(3)}\left(u^{(3)}_{B_1B_2}(x)\right)=a_1\right\} \qquad \text{and} \qquad H^{(3)}_3=\left\{x\in G_3 \mid \gamma^{(3)}\left(u^{(3)}_{B_1B_2}(x)\right)=a_2\right\}.$$

4. Let $B_0=\{\varnothing\},\ B_1=\{1,2\}$, and $B_2=\{3,4\}$, i.e., $m_1=1$ and $m_2=3$. We consider the homomorphisms

$$u^{(4)}_{B_1B_2}\colon \langle e,a_1,a_2,a_3,a_4\rangle \to \langle e,a_1,a_3\rangle \ (3) \qquad \text{and} \qquad \gamma^{(4)}\colon\langle e,a_1,a_3\rangle\to\{e,a_1,a_3\}\ (4):$$

$$u^{(4)}_{B_1B_2}(x)=\begin{cases} e, & \text{if} \quad x=e,\\ a_1, & \text{if} \quad x=a_i, \quad i=1,2,\\ a_3, & \text{if} \quad x=a_i, \quad i=3,4, \end{cases}$$

$$\gamma^{(4)}(x)=\begin{cases} e, & \text{if} \quad x=e, \\ a_1, & \text{if} \quad x\in\{a_1,a_3a_1\}, \\ a_3, & \text{if} \quad x\in\{a_3,a_1a_3\}, \\ \gamma^{(4)}\big(a_ia_{4-i}\dots\gamma^{(4)}(a_ia_{4-i})\big), & \text{if} \quad x=a_ia_{4-i}\dots a_{4-i}, \quad l(x)\geq 3, \quad i\in\{1;3\}, \\ \gamma^{(4)}\big(a_ia_{4-i}\dots\gamma^{(4)}(a_{4-i}a_i)\big), & \text{if} \quad x=a_ia_{4-i}\dots a_i, \quad l(x)\geq 3, \quad i\in\{1;3\}. \end{cases}$$

Let $H_1^{(4)}:=\Im^1_{B_1B_2}(G_3)$. Then

$$H_1^{(4)}=\left\{x\in G_3\mid \gamma^{(4)}\left(u^{(4)}_{B_1B_2}(x)\right)=e\right\}.$$

Since $H_1^{(4)}$ is a subgroup of index 3 of the group G_3, we define a family of cosets:

$$G_3/H_1^{(4)}=\{H_1^{(4)},H_2^{(4)},H_3^{(4)}\},$$

where

$$H_2^{(4)}=\left\{x\in G_3\mid \gamma^{(4)}\left(u^{(4)}_{B_1B_2}(x)\right)=a_1\right\} \qquad \text{and} \qquad H_3^{(4)}=\left\{x\in G_3\mid \gamma^{(4)}\left(u^{(4)}_{B_1B_2}(x)\right)=a_3\right\}.$$

The $H_1^{(j)}$-periodic configurations have the following form:

$$\sigma(x)=\begin{cases} \sigma_1, & x\in H_1^{(j)}, \\ \sigma_2, & x\in H_2^{(j)}, \\ \sigma_3, & x\in H_3^{(j)}, \end{cases}$$

where $\sigma_i\in\Phi,\ i\in\{1,2,3\},\ j=\overline{1,4}$.

Note that if $\sigma_1=\sigma_2=\sigma_3$, then this configuration is *translation-invariant*; for the full details about this configuration, see [16].

Theorem 1. *Let $k=3$.*

1. *If $(J_1,J_2)\in A_1\cap A_2$, then there exist six $H_1^{(1)}$-periodic (with the exception of translation-invariant) ground states corresponding to the following configurations:*

$$\sigma(x)=\pm\begin{cases} \sigma_1, & \textit{if} \quad x\in H_1^{(1)}, \\ \sigma_2, & \textit{if} \quad x\in H_2^{(1)}, \\ \sigma_3, & \textit{if} \quad x\in H_3^{(1)}, \end{cases}$$

where $(\sigma_1,\sigma_2,\sigma_3)\in\{(-1,1,1),(1,-1,1),(1,1,-1)\}$.

2. *If* $(J_1, J_2) \in R^2 \backslash (A_1 \cap A_2)$, *then there exist non-*$H_1^{(1)}$*-periodic (with the exception of translation-invariant) ground states.*

Proof. Let $(\sigma_1, \sigma_2, \sigma_3) = (-1, 1, 1)$. Consider the following configuration:

$$\varphi_1(x) = \begin{cases} -1, & \text{if} \quad x \in H_1^{(1)}, \\ 1, & \text{if} \quad x \in H_2^{(1)}, \\ 1, & \text{if} \quad x \in H_3^{(1)}. \end{cases}$$

Denote

$$A_- = \{x \in S_1(c_b) \colon \varphi_b(x) = -1\}, \quad A_+ = \{x \in S_1(c_b) \colon \varphi_b(x) = +1\}, \quad \text{and} \quad \varphi_{i,b} = (\varphi_i)_b$$

for any i. If $c_b \in H_1^{(1)}$, then

$$\varphi_1(c_b) = -1, \quad |A_-| = 2, \quad \text{and} \quad |A_+| = 2,$$

which implies that $\varphi_{1,b} \in C_2$. For the case, $c_b \in H_2^{(1)}$, we get

$$\varphi_1(c_b) = 1, \quad |A_-| = 1, \quad \text{and} \quad |A_+| = 3,$$

which implies that $\varphi_{1,b} \in C_1$. Finally, if $c_b \in H_3^{(1)}$, then

$$\varphi_1(c_b) = 1, \quad |A_-| = 1, \quad \text{and} \quad |A_+| = 3$$

which implies that $\varphi_{1,b} \in C_1$. Hence, for any $b \in M$, we find $\varphi_{1,b} \in C_1 \cup C_2$.

It follows from (2) that

$$A_1 \cap A_2 = \left\{(J_1, J_2) \colon J_2 = -\frac{1}{2}J_1, \ J_1 \leq 0\right\}.$$

By Lemma 1, we conclude that the periodic configuration φ_1 is an $H_1^{(1)}$-periodic ground state on the set $A_1 \cap A_2$. Note that, for any $b \in M$, we have $\varphi_{1,b} \sim -\varphi_{1,b}$, i.e., $-\varphi_{1,b} \in C_1 \cup C_2$ for all $b \in M$. Consequently, the periodic configuration $-\varphi_1$ is an $H_1^{(1)}$-periodic ground state on the set $A_1 \cap A_2$.

Similar arguments can be also applied to the periodic configurations $\pm\varphi_2$ and $\pm\varphi_3$ corresponding to

$$(\sigma_1, \sigma_2, \sigma_3) \in \{(1, -1, 1), (1, 1, -1)\}.$$

Note that there exist nonperiodic (not translation-invariant) configurations not mentioned in Assertion 1. As above, we prove that these configurations are ground states on the set $A_1 \cap A_2$. Hence, if $(J_1, J_2) \in R^2 \setminus (A_1 \cap A_2)$, then there exist non-$H_1^{(1)}$-periodic ground states (not translation-invariant).

Theorem 1 is proved.

Remark 1. The $H_1^{(1)}$-periodic ground states mentioned in Theorem 1 differ from the periodic ground states described in [1]. In addition, in [1], it was proved that, for fixed $J = (J_1, J_2)$, the maximum number of periodic ground states is equal to four. In our case, it is equal to six.

In [20, 21], for the normal subgroups of indices two and four, the author studied weakly periodic ground states. In [24], we studied H_1-weakly periodic ground states on the Cayley tree of order two. We now study H_1-weakly periodic ground states corresponding to the subgroups of index 3 of the group representation of Cayley tree of order three.

For any element x of G_k, we recall that $x_\downarrow$ on the element satisfying the following condition:

$$x^{-1} \cdot x_\downarrow \in \{a_i \mid i \in N_k\}.$$

Invariance Property. For $B_i = \{m_i\}$ and $x, y \in G_k$, if

$$\gamma(u_{B_1B_2}(x)) = \gamma(u_{B_1B_2}(y)) \quad \text{and} \quad \gamma(u_{B_1B_2}(x_\downarrow)) = \gamma(u_{B_1B_2}(y_\downarrow)),$$

then

$$\langle\langle \gamma(u_{B_1B_2}(xa_i)) \mid xa_i \in S(x)\rangle\rangle = \langle\langle \gamma(u_{B_1B_2}(ya_i)) \mid ya_i \in S(y)\rangle\rangle,$$

where $\langle\langle \ldots \rangle\rangle$ stands for ordered k-tuples (for more details, see [15]).

In [15], one can find a condition imposed on subgroups of the group representation of Cayley tree such that an invariance property is true. Generally speaking, except for the given condition, the invariance property does not hold. The $H_1^{(z)}$-weakly periodic configurations have the following form:

$$\varphi(x) = \begin{cases} a_{11}, & x_\downarrow \in H_1^{(z)} \text{ and } x \in H_1^{(z)}, \\ a_{12}, & x_\downarrow \in H_1^{(z)} \text{ and } x \in H_2^{(z)}, \\ a_{13}, & x_\downarrow \in H_1^{(z)} \text{ and } x \in H_3^{(z)}, \\ a_{21}, & x_\downarrow \in H_2^{(z)} \text{ and } x \in H_1^{(z)}, \\ a_{22}, & x_\downarrow \in H_2^{(z)} \text{ and } x \in H_2^{(z)}, \\ a_{23}, & x_\downarrow \in H_2^{(z)} \text{ and } x \in H_3^{(z)}, \\ a_{31}, & x_\downarrow \in H_3^{(z)} \text{ and } x \in H_1^{(z)}, \\ a_{32}, & x_\downarrow \in H_3^{(z)} \text{ and } x \in H_2^{(z)}, \\ a_{33}, & x_\downarrow \in H_3^{(z)} \text{ and } x \in H_3^{(z)}, \end{cases}$$

where $a_{ij} \in \Phi,\ i, j \in \{1, 2, 3\},\ z = \overline{1, 2}$. For the sake of convenience, we write

$$\varphi(x) = (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33})$$

for this weakly periodic configuration φ.

Theorem 2. *Let* $k = 3$.

1. *There are no* $H_1^{(1)}$*-weakly periodic ground states (with the exception of translation-invariant and periodic).*

2. *There are no* $H_1^{(l)}$*-periodic, where* $l = 2, 3,$ *and weakly periodic ground states (with the exception of translation-invariant).*

3. *There are no* $H_1^{(4)}$*-periodic ground states (with the exception of translation-invariant).*

Proof. 1. We now prove part 1 of Theorem 2.
Consider $\varphi_1 = (-1, -1, 1, -1, 1, 1, -1, 1, 1)$.

1.1. Assume that $c_b \in H_1^{(1)}$. Then all possible cases are as follows:

(a) if $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = -1,\ |A_-| = 3,\ |A_+| = 1$, and $\varphi_{1,b} \in C_1$;
(b) $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$; this case is impossible;
(c) $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$; this case is impossible;
(d) if $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = -1,\ |A_-| = 2,\ |A_+| = 2$, and $\varphi_{1,b} \in C_2$;
(e) if $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = -1,\ |A_-| = 3,\ |A_+| = 1$, and $\varphi_{1,b} \in C_1$,
(f) $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$; this case is impossible.

1.2. Let $c_b \in H_2^{(1)}$, then all possible cases are as follows:

(a) if $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = -1,\ |A_-| = 1,\ |A_+| = 3$, and $\varphi_{1,b} \in C_3$;
(b) $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, which is impossible;
(c) if $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = 1,\ |A_-| = 2,\ |A_+| = 2$, and $\varphi_{1,b} \in C_2$;
(d) if $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1,\ |A_-| = 1,\ |A_+| = 3$, and $\varphi_{1,b} \in C_1$;
(e) if $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1,\ |A_-| = 1,\ |A_+| = 3$, and $\varphi_{1,b} \in C_1$;
(f) $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$; this is impossible.

1.3. If $c_b \in H_3^{(1)}$, then all possible cases are as follows:

(a) if $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = 1,\ |A_-| = 1,\ |A_+| = 3$, and $\varphi_{1,b} \in C_1$;
(b) $c_{b\downarrow} \in H_1^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, which is impossible;
(c) if $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, then $\varphi_{1,b}(c_b) = 1,\ |A_-| = 2,\ |A_+| = 2$, and $\varphi_{1,b} \in C_2$;
(d) if $c_{b\downarrow} \in H_2^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1,\ |A_-| = 1,\ |A_+| = 3$, and $\varphi_{1,b} \in C_1$;
(e) if $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = 1$, then $\varphi_{1,b}(c_b) = 1,\ |A_-| = 1,\ |A_+| = 3$, and $\varphi_{1,b} \in C_1$;
(f) $c_{b\downarrow} \in H_3^{(1)}$ and $\varphi_{1,b}(c_{b\downarrow}) = -1$, this case is impossible.

Hence, we have proved that $\varphi_{1,b} \in C_1 \cup C_2 \cup C_3$ for all $b \in M$.
It follows from (2) that

$$A_1 \cap A_2 \cap A_3 = \{(J_1, J_2) \in R^2 :\ J_1 = J_2 = 0\}.$$

This implies that the configuration φ_1 is a non-$H_1^{(1)}$-weakly periodic ground state. The same conclusion can be made for the remaining configurations. The remaining part of the proof is performed by analogy with the proof of part 1 of Theorem 2 and Theorem 1.

Theorem 2 is proved.

Remark 2. In [24], for $k = 2,$ the authors found both periodic (not translation-invariant) and weakly periodic (not translation-invariant and nonperiodic) ground states.

Remark 3. The $H_1^{(l)}$-subgroups with $l = 2, 3$ do not possess the invariance property.

The $H_1^{(m)}$-weakly periodic configurations have the following form:

$$\varphi(x) = \begin{cases} a_{12}, & x_\downarrow \in H_1^{(m)} \quad \text{and} \quad x \in H_2^{(m)}, \\ a_{13}, & x_\downarrow \in H_1^{(m)} \quad \text{and} \quad x \in H_3^{(m)}, \\ a_{21}, & x_\downarrow \in H_2^{(m)} \quad \text{and} \quad x \in H_1^{(m)}, \\ a_{23}, & x_\downarrow \in H_2^{(m)} \quad \text{and} \quad x \in H_3^{(m)}, \\ a_{31}, & x_\downarrow \in H_3^{(m)} \quad \text{and} \quad x \in H_1^{(m)}, \\ a_{32}, & x_\downarrow \in H_3^{(m)} \quad \text{and} \quad x \in H_2^{(m)}, \end{cases}$$

where $a_{ij} \in \Phi,\ i, j \in \{1, 2, 3\},\ m = \overline{3, 4}.$

In what follows, we write $\varphi(x) = (a_{12}, a_{13}, a_{21}, a_{23}, a_{31}, a_{32})$ for this weakly periodic configuration φ.

Theorem 3. *Let $k = 3$. Then the following assertions hold:*

1(a). There exist exactly six $H_1^{(4)}$-weakly periodic ground states on $\left\{J_2 = \frac{1}{2}J_1,\ J_1 \geq 0\right\}$ that are nonperiodic and have the form $\varphi_{1,2} = \pm(i, j, i, j, i, j),\ \varphi_{3,4} = \pm(i, j, i, j, j, i)$, and $\varphi_{5,6} = \pm(i, j, j, i, j, i)$, where $i \neq j,\ i, j \in \Phi$.

1(b). There are exactly two $H_1^{(4)}$-weakly periodic ground states on $\left\{J_2 = -\frac{1}{2}J_1,\ J_1 \leq 0\right\}$ that are nonperiodic and have the form $\varphi_{7,8} = \pm(i, j, j, i, i, j)$, where $i \neq j,\ i, j \in \Phi$.

2. If $(J_1, J_2) \in R^2 \setminus ((A_1 \cap A_2) \cup (A_2 \cap A_3))$, then there exist non-$H_1^{(4)}$-weakly periodic ground states (with the exception of translation-invariant).

Proof. The proof is performed by using the same method as in the proof of part 1 of Theorem 2.

Remark 4. The results of Theorems 1 and 3 do not depend on the choice of elements of N_k; however, they depend only on the power of partition sets of N_k.

Remark 5. The weakly periodic ground states obtained in Theorem 3 differ from the weakly periodic ground states determined in [20].

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