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ON THE POLYCONVOLUTION WITH WEIGHT FUNCTION $\gamma(y) = \cos y$ FOR HARTLEY INTEGRAL TRANSFORMS $\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_1$ AND INTEGRAL EQUATIONS

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We construct and study a new polyconvolution with weight function $\gamma(y) = \cos y$ for Hartley integral transforms $\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_1$ and apply it to the solution of integral equations and a system of integral equations of polyconvolution type.

1. Introduction

In 1997, Kakichev [7] proposed the polyconvolution for $n+1$ arbitrary integral transforms $K, K_1, K_2, \ldots, K_n$ with a weight function $\gamma(y)$ of functions $f_1, f_2, \ldots, f_n$ satisfying the following factorization identity:

$$K\left[\overset{\gamma}{*}(f_1, f_2, \ldots, f_n)\right](y) = \gamma(y)\,(K_1 f_1)(y)\,(K_2 f_2)(y)\ldots(K_n f_n)(y).$$

In recent years, there were some polyconvolutions [13, 14] related to the Hartley integral transforms and some differential integral transforms. At the same time, there were some polyconvolutions [10, 11] related only to the Hartley integral transforms and some differential integral transforms.

In the present paper, we construct and study a new polyconvolution with weight function $\gamma(y) = \cos y$ related to the Hartley integral transforms $\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_1$. We apply this polyconvolution to solve some nonstandard integral equations and system of integral equations. We realize that, for these integral equations, the possibility of representation of their solutions in a closed form is an interesting open problem [4, 8, 9].

In this section, we recall some known convolutions and generalized convolutions. The Hartley integral transform $\mathcal{H}_1,\ \mathcal{H}_2$ was introduced in [3]:

$$(\mathcal{H}f)_{\left\{ \substack{1 \\ 2} \right\}}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x)\,\mathrm{cas}\,(\pm xy)dy, \quad y \in \mathbb{R}.$$

Here, $\mathrm{cas}\,(\pm\theta) = \cos\theta \pm \sin\theta$. The convolution for the Hartley integral transform $\mathcal{H}_1$ [5, 6, 12], i.e.,

$$\left(f \underset{\mathcal{H}_1}{*} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} f(u)\left[g(x+u) + g(x-u) + g(u-x) - f(-x-u)\right]du, \tag{1.1}$$

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satisfies the factorization identity

$$\mathcal{H}_1\left(f \underset{\mathcal{H}_1}{*} g\right)(y) = \left(\mathcal{H}_1 f\right)(y)(\mathcal{H}_1 g)(y).$$

The Hartley integral transform $\mathcal{H}_1,\ \mathcal{H}_1,\ \mathcal{H}_2$ was introduced in [3]

$$\left(f \underset{\mathcal{H}_1,\mathcal{H}_1,\mathcal{H}_2}{*} g\right)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \left[f\left(x+y\right) + f\left(x-y\right) - f\left(-x+y\right) + f(-x-y)\right] g(y)dy$$

and satisfies the factorization identity

$$\mathcal{H}_1\left(f \underset{\mathcal{H}_1,\mathcal{H}_1,\mathcal{H}_2}{*} g\right)(y) = \left(\mathcal{H}_1 f\right)(y)(\mathcal{H}_2 g)(y).$$

2. Polyconvolution with Weight Function $\gamma(y) = \cos y$ for the Hartley Integral Transforms $\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_1$

Definition 2.1. *The polyconvolution with weight function $\gamma(y) = \cos y$ for Hartley integral transforms of the functions $f,\ g$ and h is defined as follows:*

$$[\overset{\gamma}{*}(f,g,h)](x) = \frac{1}{8\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} [f(x+v+w+1) + f(x+v+w-1)$$

$$+ f(x-v-w+1) + f(x-v-w-1)]g(v)h(w)dvdw, \quad x \in \mathbb{R}. \tag{2.1}$$

Theorem 2.1. *Let $f, g,$ and h be functions in $L(\mathbb{R})$. Then the polyconvolution with weight function $\gamma(y) = \cos y$ (2.1) for the Hartley integral transforms of the functions $f,\ g,$ and h belongs to $L(\mathbb{R})$ and the following factorization identity is true:*

$$\mathcal{H}_1\left[\overset{\gamma}{*}(f,g,h)\right](y) = \cos y(\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y) \quad \forall y \in \mathbb{R}. \tag{2.2}$$

Proof. First, we prove that $[\overset{\gamma}{*}(f,g,h)](x) \in L(\mathbb{R})$. Indeed, we have

$$\int\limits_{-\infty}^{\infty} \left|\overset{\gamma}{*}(f,g,h)(x)\right| dx$$

$$\leq \frac{1}{8\pi} \int\limits_{-\infty}^{\infty} |g(v)|\, dv \int\limits_{-\infty}^{\infty} |h(w)|\, dw \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} [|f(x+v+w+1)|$$

$$+ |f(x+v+w-1)| + |f(x-v-w+1)| + |f(x-v-w-1)|]dx.$$

It is easy to see that

$$\int_{-\infty}^{\infty} \Big[|f(x+v+w+1)| + |f(x+v+w-1)| + |f(x-v-w+1)| + |f(x-v-w-1)|\Big] dx = 4\int_{-\infty}^{\infty} |f(u)| du.$$

For this reason, we obtain

$$\int_{-\infty}^{\infty} \left|\overset{\gamma}{*}(f,g,h)(x)\right| dx \le \frac{1}{2\pi}\int_{-\infty}^{\infty} |g(v)|\, dv \int_{-\infty}^{\infty} |h(w)|\, dw \int_{-\infty}^{\infty} |f(u)|\, du < +\infty.$$

Hence, $\overset{\gamma}{*}(f,g,h)(x)$ belongs to $L(\mathbb{R})$. We now prove the factorization identity (2.2). Since

$$2\pi\sqrt{2\pi}\cos y(\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos y \,\mathrm{cas}\,(yu)\,\mathrm{cas}\,(-yv)\,\mathrm{cas}\,(yw)\, f(u)g(v)h(w)dudvdw,$$

in view of the trigonometric identity, we get

$$\cos y\,\mathrm{cas}\,(yu)\,\mathrm{cas}\,(yv)\,\mathrm{cas}\,(yw) = \frac{1}{4}\Big[\mathrm{cas}\, y\,(u+v+w+1) + \mathrm{cas}\, y\,(u+v+w-1) + \mathrm{cas}\, y\,(u-v-w+1) + \mathrm{cas}\, y\,(u-v-w-1)\Big].$$

Thus,

$$\cos y(\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)$$

$$= \frac{1}{8\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[\mathrm{cas}\, y\,(u+v+w+1) + \mathrm{cas}\, y\,(u+v+w-1) + \mathrm{cas}\, y\,(u-v-w+1) + \mathrm{cas}\, y\,(u-v-w-1)\Big] f(u)g(v)h(w)dudvdw$$

$$= \frac{1}{8\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{cas}\,(yt)\Big[f(t-v-w-1) + f(t-v-w+1)$$

$$+ f(t+v+w-1) + f(t+v+w+1)\Big] g(v)h(w)dtdvdw$$

$$= \mathcal{H}_1\left[\overset{\gamma}{*}(f,g,h)\right](y) \quad \forall y \in \mathbb{R}.$$

Theorem 2.1 is proved.

Corollary 2.1. *In the space $L(\mathbb{R})$, polyconvolution (2.1) satisfies the following equality:*

$$\left[\overset{\gamma}{*}(f,g,h)\right](x) = \left[\overset{\gamma}{*}(h,g,f)(x)\right].$$

Proof. From the factorization identity (2.2), we obtain

$$\mathcal{H}\left[\overset{\gamma}{*}(f,g,h)\right](y) = (\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)$$

$$= (\mathcal{H}_1 h)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 f)(y) = \mathcal{H}\left[\overset{\gamma}{*}(h,g,f)\right](y).$$

Therefore,

$$[\overset{\gamma}{*}(f,g,h)](x) = [\overset{\gamma}{*}(h,g,f)(x)].$$

Theorem 2.2. *If $f, g,$ and h belong to $L(\mathbb{R})$, then the following inequality holds:*

$$\left\|\overset{\gamma}{*}(f,g,h)\right\| \le \|f\|\,\|g\|\,\|h\|.$$

Proof. From the proof of Theorem 2.1, we get

$$\int_{-\infty}^{\infty}\left|\overset{\gamma}{*}(f,g,h)(x)\right|dx \le \frac{1}{2\pi}\int_{-\infty}^{\infty}|f(t)|\,dt\int_{-\infty}^{\infty}|g(v)|\,dv\int_{-\infty}^{\infty}|h(w)|\,dw$$

$$= \frac{1}{\sqrt[3]{2\pi}}\int_{-\infty}^{\infty}|f(t)|\,dt\frac{1}{\sqrt[3]{2\pi}}\int_{-\infty}^{\infty}|g(v)|\,dv\frac{1}{\sqrt[3]{2\pi}}\int_{-\infty}^{\infty}|h(w)|\,dw.$$

Hence,

$$\left\|\overset{\gamma}{*}(f,g,h)\right\| \le \|f\|\,\|g\|\,\|h\|\,.$$

Theorem 2.2 is proved.

Theorem 2.3. *Let $g \in L_p(\mathbb{R}),\ h \in L_q(\mathbb{R}),$ and $f \in L_r(\mathbb{R})$ be such that*

$$p,\ q,\ r > 1 \quad \textit{and} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2.$$

Then the following inequality holds:

$$\left\|\overset{\gamma}{*}(f,g,h)\right\| \le \frac{1}{2\pi}\,\|g\|^p_{L_p(\mathbb{R}}\,\|h\|^q_{L_q(\mathbb{R}}\,\|f\|^r_{L_r(\mathbb{R}}\,.$$

Proof. By using (2.1), we get the following estimation:

$$\left|\overset{\gamma}{*}(f,g,h)(x)\right| \leq \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)|\ |h(w)|\ |f(x+v+w+1)|\,dvdw$$

$$+\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)|\ |h(w)|\ |f(x+v+w-1)|\,dvdw$$

$$+\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)|\ |h(w)|\ |f(x-v-w+1)|\,dvdw$$

$$+\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)|\ |h(w)|\ |f(x-v-w-1)|\,dvdw. \tag{2.3}$$

Let $I_1, I_2, \ldots, I_4$ be the corresponding integral terms in this expression. Without loss of generality, we consider

$$I_1 = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(v)|\ |h(w)|\ |f(x+v+w+1)|\,dvdw, \quad x\in\mathbb{R}.$$

Let $p_1,\ q_1,$ and r_1 be the conjugate exponentials for $p,\ q,$ and r and let

$$A_1(u,v) = |h(w)|^{q/p_1}|f(x+v+w+1)|^{q/p_1} \in L_{p_1}(\mathbb{R}^2),$$

$$A_2(u,v) = |f(x+v+w+1)|^{r/q_1}|g(v)|^{p/q_1} \in L_{q_1}(\mathbb{R}^2),$$

$$A_3(u,v) = |g(v)|^{p/r_1}|h(w)|^{q/r_1} \in L_{r_1}(\mathbb{R}^2).$$

We see that

$$A_1.A_2.A_3 = |g(v)|\,|h(w)|\,|f(x+v+w+1)|.$$

By the definition of norm in the space $L_{p_1}(\mathbb{R})^2$, with the help of the Fubini theorem, we get

$$\|A_1\|^{p_1}_{L_{p_1}(\mathbb{R}^2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{|h(w)|^{q/p_1}|f(x+v+w+1)|^{r/p_1}\right\}^{p_1}$$

$$= \int_{-\infty}^{\infty}|h(w)|^q\left(\int_{-\infty}^{\infty}|f(x+v+w+1)|^r dv\right)dw$$

$$= \int_{-\infty}^{\infty}|h(w)|^q\|f\|^r_{L_r(\mathbb{R})}dw = \|h\|^q_{L_q(\mathbb{R})}\|f\|^r_{L_r(\mathbb{R})}.$$

Similarly, we have

$$\|A_2\|^{q_1}_{L_{q_1}(\mathbb{R}^2)} = \|f\|^r_{L_r(\mathbb{R})}\|g\|^p_{L_p(\mathbb{R})},$$
$$\|A_3\|^{r_1}_{L_{r_1}(\mathbb{R}^2)} = \|g\|^p_{L_p(\mathbb{R})}\|h\|^q_{L_q(\mathbb{R})}. \tag{2.4}$$

From the hypothesis $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, it follows that $\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} = 1$. By using the Hölder inequality and (2.4), we obtain the following estimation:

$$I_1 = \frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_1A_2A_3 dvdw$$
$$\leq \frac{1}{8\pi}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_1^{p_1} dvdw\right)^{\frac{1}{p_1}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_2^{q_1} dvdw\right)^{\frac{1}{q_1}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_3^{r_1} dvdw\right)^{\frac{1}{r_1}}$$
$$= \frac{1}{8\pi}\|A_1\|^{p_1}_{L_{p_1}(\mathbb{R}^2)}\|A_2\|^{q_1}_{L_{q_1}(\mathbb{R}^2)}\|A_3\|^{r_1}_{L_{r_1}(\mathbb{R}^2)}$$
$$= \frac{1}{8\pi}\|g\|^p_{L_p(\mathbb{R})}\|h\|^q_{L_q(\mathbb{R})}\|f\|^r_{L_r(\mathbb{R})}. \tag{2.5}$$

In the same way, we get the following estimations for I_2, I_3, and I_4:

$$I_k \leq \frac{1}{8\pi}\|g\|^p_{L_p(\mathbb{R})}\|h\|^q_{L_q(\mathbb{R})}\|f\|^r_{L_r(\mathbb{R})} \tag{2.6}$$

for all $k = 2, 3, 4$. Furthermore, it follows from (2.3)–(2.6) that

$$\left\|\overset{\gamma}{*}(f,g,h)\right\| \leq \frac{1}{2\pi}\,\|g\|^p_{L_p(\mathbb{R}}\,\|h\|^q_{L_q(\mathbb{R}}\,\|f\|^r_{L_r(\mathbb{R}}\,.$$

Theorem 2.3 is proved.

Theorem 2.4 (Titchmarsh-type theorem). *Let* $f, g, h \in L(\mathbb{R})$. *If* $\overset{\gamma}{*}(f,g,h)(x) \equiv 0$ *for all* $x \in \mathbb{R}$, *then either* $f(x) = 0$, *or* $g(x) = 0$, *or* $h(x) = 0$ *for all* $x \in \mathbb{R}$.

Proof. The hypothesis $\overset{\gamma}{*}(f,g,h)(x) \equiv 0$ implies that $\mathcal{H}_1[\overset{\gamma}{*}(f,g,h)](y) = 0$ $\forall y \in \mathbb{R}$. By virtue of Theorem 2.1, we get

$$\cos y(\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y) = 0 \quad \forall y \in \mathbb{R}. \tag{2.7}$$

As $(\mathcal{H}_1 f)(y)$, $(\mathcal{H}_2 g)(y)$, and $(\mathcal{H}_1 h)(y)$ are analytic for all $y \in \mathbb{R}$, relation (2.7) implies that either $(\mathcal{H}_1 f) = 0$ $\forall y \in \mathbb{R}$, or $(\mathcal{H}_2 g) = 0$ $\forall y \in \mathbb{R}$, or $(\mathcal{H}_1 h) = 0$ $\forall y \in \mathbb{R}$.

Therefore, we have either $f(x) = 0$ $\forall x \in \mathbb{R}$, or $g(x) = 0$ $\forall x \in \mathbb{R}$, or $h(x) = 0$ $\forall x \in \mathbb{R}$.

Theorem 2.4 is proved.

3. Application to Solving an Integral Equation and a System of Integral Equations of the Polyconvolution Type

3.1. A Single Integral Equation. In this section, we apply the obtained result to solve an integral equation of the polyconvolution type. To deal with this equation, we prove the existence of solution and present it in the closed form. We examine the following integral equation:

$$f(x)+\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[f(x+v+w+1)+f(x+v+w-1)$$

$$+f(x-v-w+1)+f(x-v-w-1)]g(v)h(w)dvdw=k(x)\quad\forall y\in\mathbb{R}. \tag{3.1}$$

Here, $g,\ h,$ and k are functions of $L(\mathbb{R})$ and f is an unknown function.

Theorem 3.1. *Let $k,g,h\in L(\mathbb{R})$ be given. Equation (3.1) has a unique solution*

$$f(x)=k(x)-\left(k\underset{\mathcal{H}}{*}l\right)(x)$$

in $L(\mathbb{R})$ if

$$1+\cos y(\mathcal{H}_2g)(y)(\mathcal{H}_1h)(y)\neq 0\quad\forall y\in\mathbb{R}.$$

Here, $l\in L(\mathbb{R})$. Moreover, it is determined by the equation

$$(\mathcal{H}l)(y)=\frac{\cos y(\mathcal{H}_2g)(y)(\mathcal{H}_1h)(y)}{1+\cos y(\mathcal{H}_2g)(y)(\mathcal{H}_1h)(y)}.$$

Proof. Equation (3.1) can be rewritten in the form

$$f(x)+\left[\overset{\gamma}{*}g\,(f,g,h)\right](x)=k(x).$$

In view of Theorem 2.1, we obtain

$$(\mathcal{H}_1f)(y)+\cos y(\mathcal{H}_1f)(y)(\mathcal{H}_2g)(y)(\mathcal{H}_1h)(y)=(\mathcal{H}_1k)(y)\quad\forall y\in\mathbb{R}.$$

This yields

$$(\mathcal{H}_1f)(y)\left[1+\cos y(\mathcal{H}_2g)(y)(\mathcal{H}_1h)(y)\right]=(\mathcal{H}_1k)(y).$$

Under the condition

$$1+\cos y(\mathcal{H}_2g)(y)(\mathcal{H}_1h)(y)\neq 0\quad\forall y\in\mathbb{R},$$

we get

$$(\mathcal{H}_1f)(y)=(\mathcal{H}_1k)(y)\left[1-\frac{\cos y(\mathcal{H}_2g)(y)(\mathcal{H}_1h)(y)}{1+\cos y(\mathcal{H}_2g)(y)(\mathcal{H}_1h)(y)}\right].$$

Therefore, by the Wiener–Levy theorem [1, 12], there exists a function $l \in L(\mathbb{R})$ such that

$$(\mathcal{H}_1 l)(y) = \frac{\cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)}{1 + \cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)}.$$

Hence,

$$(\mathcal{H}_1 f)(y) = (\mathcal{H}_1 k)(y) - (\mathcal{H}_1 k)(y)(\mathcal{H}_1 l)(y)$$

$$= (\mathcal{H}_1 k)(y) - \mathcal{H}_1 \left(k \underset{\mathcal{H}}{*} l \right)(y).$$

Therefore,

$$f(x) = k(x) - \left(k \underset{\mathcal{H}_1}{*} l \right)(x).$$

Theorem 3.1 is proved.

3.2. A System of Two Integral Equations of the Polyconvolution Type.

$$\begin{aligned} f(x) + \frac{1}{8\pi} \int_{-\infty}^{\infty} [f(x+v+w+1) + f(x+v+w-1) \\ + f(x-v-w+1) + f(x-v-w-1)]\varphi(v)\psi(w)dvdw = h(x), \\ \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\left[p(x+v) + p(x-v) + p(-x+v) - p(-x-v)\right] dv + g(x) = k(x), \quad x \in \mathbb{R}. \end{aligned} \tag{3.2}$$

Here, $\varphi,\ \psi,\ p,\ h,$ and k are given functions in $L(\mathbb{R})$ and f and g are the unknown functions.

Theorem 3.2. *Assume that*

$$1 - \mathcal{H}_1 \left[\overset{\gamma}{*} (p, \varphi, \psi) \right](y) \neq 0 \quad \forall y \in \mathbb{R}$$

and there exists a unique solution of (3.2) in $L(\mathbb{R})$ *defined as follows:*

$$f(x) = h(x) + \left(h \underset{\mathcal{H}_1}{*} l \right)(x) - \left[\overset{\gamma}{*} (k, \varphi, \psi) \right](x) - \left\{ \left[\overset{\gamma}{*} (k, \varphi, \psi) \right] \underset{\mathcal{H}_1}{*} l \right\}(x),$$

$$g(x) = k(x) + \left(k \underset{\mathcal{H}_1}{*} l \right)(x) - \left(h \underset{\mathcal{H}_1}{*} p \right)(x) - \left[\left(h \underset{\mathcal{H}_1}{*} p \right) \underset{\mathcal{H}_1}{*} l \right](x).$$

Here, $l \in L(\mathbb{R})$ *and defined by the equations*

$$(\mathcal{H}_1 l)(y) = \frac{\mathcal{H}_1 \left[\overset{\gamma}{*}(p, \varphi, \psi) \right](y)}{1 - \mathcal{H}_1 \left[\overset{\gamma}{*}(p, \varphi, \psi) \right](y)}.$$

Proof. System (3.2) can be rewritten in the form

$$f(x) + \left[\overset{\gamma}{*}(g, \varphi, \psi)\right](x) = h(x),$$

$$\left(f \underset{\mathcal{H}_1}{*} p\right)(x) + g(x) = k(x), \quad x \in \mathbb{R}.$$

By using the factorization property of polyconvolution (2.1) and convolution (1.1), we obtain the following linear system of algebraic equations for $(\mathcal{H}_1 f)(y)$ and $(\mathcal{H}_1 g)(y)$:

$$(\mathcal{H}_1 f)(y) + \cos y (\mathcal{H}_1 g)(y) \left(\mathcal{H}_2 \varphi\right)(y) \left(\mathcal{H}_1 \psi\right)(y) = (\mathcal{H}_1 h)(y),$$

$$(\mathcal{H}_1 f)(y)(\mathcal{H}_1 p)(y) + (\mathcal{H}_1 g)(y) = (\mathcal{H}_1 k)(y), \quad y \in \mathbb{R}.$$

We now find the determinants of the system

$$\Delta = \begin{vmatrix} 1 & \cos y (\mathcal{H}_2 \varphi)(y)(\mathcal{H}_1 \psi)(y) \\ (\mathcal{H}_1 p)(y) & 1 \end{vmatrix}$$

$$= 1 - \cos y (\mathcal{H}_1 p)(y)(\mathcal{H}_2 \varphi)(y)(\mathcal{H}_1 \psi)(y) = 1 - \mathcal{H}_1 \left[\overset{\gamma}{*}(p, \varphi, \psi)\right](y),$$

$$\Delta_1 = \begin{vmatrix} (\mathcal{H}_1 h)(y) & \cos y (\mathcal{H}_2 \varphi)(y)(\mathcal{H}_1 \psi)(y) \\ (\mathcal{H}_1 k)(y) & 1 \end{vmatrix}$$

$$= (\mathcal{H}_1 h)(y) - \mathcal{H}_1 \left[\overset{\gamma}{*}(k, \varphi, \psi)\right](y),$$

$$\Delta_2 = \begin{vmatrix} 1 & (\mathcal{H}_1 h)(y) \\ (\mathcal{H}_1 p)(y) & (\mathcal{H}_1 k)(y) \end{vmatrix} = (\mathcal{H}_1 k)(y) - \mathcal{H}_1 \left(h \underset{\mathcal{H}_1}{*} p\right)(y).$$

Since

$$1 - \mathcal{H}_1 \left[\overset{\gamma}{*}(p, \varphi, \psi)\right](y) \neq 0 \quad \forall y \in \mathbb{R},$$

we find

$$(\mathcal{H}_1 f)(y) = \left\{(\mathcal{H}_1 h)(y) - \mathcal{H}_1 \left[\overset{\gamma}{*}(k, \varphi, \psi)\right](y)\right\} \frac{1}{1 - \mathcal{H}_1 \left[\overset{\gamma}{*}(p, \varphi, \psi)\right](y)}$$

$$= \left\{(\mathcal{H}_1 h)(y) - \mathcal{H}_1 \left[\overset{\gamma}{*}(k, \varphi, \psi)\right](y)\right\} \left\{1 + \frac{\mathcal{H}_1 \left[\overset{\gamma}{*}(p, \varphi, \psi)\right](y)}{1 - \mathcal{H}_1 \left[\overset{\gamma}{*}(p, \varphi, \psi)\right](y)}\right\}.$$

Furthermore, according to the Wiener–Levy theorem [1, 12], there exists a function $l \in L(\mathbb{R})$ such that

$$(\mathcal{H}_1 l)(y) = \frac{\mathcal{H}_1 \left[\overset{\gamma}{*}(p, \varphi, \psi)\right](y)}{1 - \mathcal{H}_1 \left[\overset{\gamma}{*}(p, \varphi, \psi)\right](y)}.$$

This yields

$$(\mathcal{H}_1 f)(y) = \left\{(\mathcal{H}_1 h)(y) - \mathcal{H}_1 \left[\overset{\gamma}{*}(k, \varphi, \psi)\right](y)\right\}\{1 + (\mathcal{H}_1 l)(y)\}$$

$$= (\mathcal{H}_1 h)(y) + \mathcal{H}_1 \left(h \underset{\mathcal{H}_1}{*} l\right)(y) - \mathcal{H}_1 \left[\overset{\gamma}{*}(k, \varphi, \psi)\right](y)$$

$$- \mathcal{H}_1 \left\{\left[\overset{\gamma}{*}(k, \varphi, \psi)\right] \underset{\mathcal{H}_1}{*} l\right\}(y).$$

Hence,

$$f(x) = h(x) + \left(h \underset{\mathcal{H}_1}{*} l\right)(x) - \left[\overset{\gamma}{*}(k, \varphi, \psi)\right](x) - \left\{\left[\overset{\gamma}{*}(k, \varphi, \psi)\right] \underset{\mathcal{H}_1}{*} l\right\}(x) \in L(\mathbb{R}).$$

In the same way, we obtain

$$(\mathcal{H}_1 g)(y) = \left\{(\mathcal{H}_1 k)(y) - \mathcal{H}_1 \left(h \underset{\mathcal{H}_1}{*} p\right)(y)\right\}\{1 + (\mathcal{H}_1 l)(y)\}$$

$$= (\mathcal{H}_1 k)(y) + \mathcal{H}_1 \left(k \underset{\mathcal{H}_1}{*} l\right)(y) - \mathcal{H}_1 \left(h \underset{\mathcal{H}_1}{*} p\right)(y) - \mathcal{H}_1 \left[\left(h \underset{\mathcal{H}_1}{*} p\right) \underset{\mathcal{H}_1}{*} l\right](y).$$

Thus, we can write

$$g(x) = k(x) + \left(k \underset{\mathcal{H}_1}{*} l\right)(x) - \left(h \underset{\mathcal{H}_1}{*} p\right)(x) - \left[\left(h \underset{\mathcal{H}_1}{*} p\right) \underset{\mathcal{H}_1}{*} l\right] \quad (x) \in L(\mathbb{R}).$$

Theorem 3.2 is proved.

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