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BOJANOV–NAIDENOV PROBLEM FOR DIFFERENTIABLE FUNCTIONS AND THE ERDŐS PROBLEM FOR POLYNOMIALS AND SPLINES

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We solve an extremal problem

$$\|x_{\pm}^{(k)}\|_{L_p[a,b]} \to \sup, \quad k = 0, 1, \ldots, r-1, \quad p > 0,$$

in a class of pairs (x, I) of functions $x \in S_\varphi^k$ such that $\varphi^{(i)}$ are the comparison functions for $x^{(i)}$, $i = 0, 1, \ldots, k,$ and the intervals $I = [a, b]$ satisfy the conditions

$$L(x)_p \le A, \quad \mu\{\operatorname{supp}_{[a,b]} x_{\pm}^{(k)}\} \le \mu,$$

where

$$L(x)_p := \sup\left\{\left(\int_a^b |x(t)|^p dt\right)^{\frac{1}{p}} : \, a, b \in \mathbf{R}, \; |x(t)| > 0, \; t \in (a, b)\right\}.$$

In particular, we solve the same problems on the classes $W_\infty^r(\mathbf{R})$ and on bounded sets of spaces of trigonometric polynomials and splines, as well as the Erdős problem for the positive (negative) parts of polynomials and splines.

1. Introduction

Let $G = \mathbf{R}$ or $G = [\alpha, \beta]$. Consider the spaces $L_p(G),\ 0 < p \le \infty,$ of all Lebesgue-measurable functions $x : G \to \mathbf{R}$ such that $\|x\|_{L_p(G)} < \infty,$ where

$$\|x\|_{L_p(G)} := \begin{cases} \left(\displaystyle\int_G |x(t)|^p dt \right)^{1/p} & \text{for} \quad 0 < p < \infty, \\ \operatorname{vrai\,sup}_{t \in G} |x(t)| & \text{for} \quad p = \infty. \end{cases}$$

For $r \in \mathbf{N}$ and $p, s \in (0, \infty]$, by $L_{p,s}^r$ we denote the space of all functions $x \in L_p(\mathbf{R})$ with locally absolutely continuous derivatives up to the $(r-1)$th order, inclusively, such that, in addition, $x^{(r)} \in L_s(\mathbf{R})$. We write $\|x\|_p$ instead of $\|x\|_{L_p(\mathbf{R})}$ and L_∞^r instead of $L_{\infty,\infty}^r$.

It is known (see, e.g., [1, p. 47]) that the problem of determination of the exact constant C in the Kolmogorov–Nagy-type inequality

$$\left\|x^{(k)}\right\|_q \le C \|x\|_p^\alpha \left\|x^{(r)}\right\|_s^{1-\alpha} \tag{1.1}$$

in the class of functions $x \in L_{p,s}^r$, where

$$\alpha = \frac{r - k + 1/q - 1/s}{r + 1/p - 1/s}, \qquad q, p, s \ge 1,$$

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and the parameters

$$r \in \mathbf{N} \quad \text{and} \quad k \in \mathbf{N_0} := \mathbf{N} \bigcup \{0\}, \quad k < r,$$

satisfy the condition

$$\alpha \le (r-k)/r,$$

is equivalent to the extremal problem

$$\left\| x^{(k)} \right\|_q \to \sup \tag{1.2}$$

on the class of functions $x \in L^r_{p,s}$ with the following restrictions:

$$\left\| x^{(r)} \right\|_s \le A_r \qquad \text{and} \qquad \|x\|_p \le A_0, \tag{1.3}$$

where A_0 and A_r are given positive numbers.

There are numerous works devoted to this class of problems (for the detailed bibliography, see [1–3]). Note that the problem of coincidence of the exact constants in inequalities of type (1.1) for periodic functions and the same inequalities for functions nonperiodic on the axis was investigated in [4]. Despite a great number of works devoted to inequalities of the form (1.1), the exact constant C in this inequality is known for all $r \in \mathbf{N}$ and all $k < r$ only in a few cases. For this reason, it is of interest to analyze the Bojanov–Naidenov modification of problem (1.2) with restrictions (1.3) proposed in [5].

We say that $f \in L^1_\infty$ is a comparison function for $x \in L^1_\infty$ if $\|x_\pm\|_\infty \le \|f_\pm\|_\infty$ and the equality

$$x(\xi) = f(\eta), \quad \xi, \eta \in \mathbf{R},$$

yields the inequality

$$|x'(\xi)| \le |f'(\eta)|$$

provided that the indicated derivatives exist.

We say that an odd 2ω-periodic function $\varphi \in L^1_\infty$ is an S-function if it has the following properties: φ is even with respect to $\omega/2$ and $|\varphi|$ is convex upward on $[0, \omega]$ and strictly monotone on $[0, \omega/2]$. For $k = 0, 1, 2, \ldots$ and an S-function $\varphi \in L^{k+1}_\infty$, by S^k_φ we denote a class of functions $x \in L^{k+1}_\infty$ such that $\varphi^{(i)}$ is a comparison function for $x^{(i)},\ i = 0, 1, \ldots, k.$ As examples of the classes S^k_φ, we can mention the Sobolev classes

$$\left\{ x \in L^r_\infty \colon \left\| x^{(r)} \right\|_\infty \le A_r,\ \|x\|_\infty \le A_0 \right\},$$

bounded subsets of the spaces T_n (of trigonometric polynomials of degree $\le n$), and the spaces $S_{n,r}$ (of splines of order r with defect 1 and nodes at the points $l\pi/n,\ l \in \mathbf{Z}$).

For an arbitrary segment $[\alpha, \beta] \subset \mathbf{R}$, in [5], Bojanov and Naidenov solved the following problem:

$$\int\limits_\alpha^\beta \Phi\big(|x^{(k)}(t)|\big) dt \to \sup, \quad k = 1, 2, \ldots,$$

on the class S^k_φ, where Φ is a continuously differentiable function on $[0, \infty)$ such that $\Phi(t)/t$ is nondecreasing and $\Phi(0) = 0$. As a result, they solved the Erdős problem of characterization of a trigonometric polynomial [6] with fixed uniform norm whose graph has the maximal length on a given segment $[\alpha, \beta] \subset \mathbf{R}$. For continuous splines on the axis, this problem was solved in [7].

By W we denote a class of continuous, nonnegative, and convex functions Φ defined on $[0, \infty)$ and such that $\Phi(0) = 0$. For $p > 0$, we set [8]

$$L(x)_p := \sup \left\{ \left(\int\limits_a^b |x(t)|^p dt \right)^{\frac{1}{p}} : \, a, b \in \mathbf{R}, \; |x(t)| > 0, \; t \in (a, b) \right\}. \tag{1.4}$$

Note that

$$L(x)_\infty = \|x\|_\infty \qquad \text{and} \qquad L(x')_1 \le 2\|x\|_\infty.$$

In [9–11], the Bojanov–Naidenov problem was also solved for $k = 0$, namely,

$$\int\limits_\alpha^\beta \Phi(|x(t)|^p) dt \to \sup, \qquad \Phi \in W, \quad p > 0, \tag{1.5}$$

on a class of functions S^0_φ satisfying the condition $L(x)_p \le L(\varphi)_p$. As a result, we obtained the solution of the problem

$$\int\limits_\alpha^\beta \Phi\big(|x^{(k)}(t)|\big) dt \to \sup, \qquad \Phi \in W, \quad k = 1, 2, \ldots, \tag{1.6}$$

on the classes of functions $x \in S^k_\varphi$.

The Bojanov–Naidenov problem and the Kolmogorov–Nagy-type inequalities for functions with asymmetric restrictions imposed on the higher derivative were studied in [12, 13]. Among other works devoted to the investigation of problems of this kind, we can mention [14, 15].

In the present paper, we solve the problem (Theorem 1)

$$\int\limits_a^b \Phi\big(x^p_\pm(t)\big) dt \to \sup, \qquad \Phi \in W, \quad p > 0, \tag{1.7}$$

on a class of pairs (x, I) of functions $x \in S^0_\varphi$ and segments $I = [a, b]$ such that $L(x)_p \le L(\varphi)_p$ and the following condition is satisfied:

$$\mu\big(\operatorname{supp}_{[a,b]} x_\pm\big) \le \mu, \qquad \mu > 0. \tag{1.8}$$

In addition, we also solve the problem (Theorem 2)

$$\int\limits_a^b \Phi\big(x^{(k)}_\pm(t)\big) dt \to \sup, \qquad \Phi \in W, \quad k = 1, 2, \ldots, \tag{1.9}$$

on a class of pairs (x, I) of functions $x \in S^k_\varphi$ and segments $I = [a, b]$ for which the following condition is satisfied:

$$\mu\Big(\operatorname{supp}_{[a,b]} x^{(k)}_\pm\Big) \le \mu, \quad \mu > 0, \tag{1.10}$$

where

$$\operatorname{supp}_{[a,b]} x := \big\{t \in [a,b] \colon\, |x(t)| > 0\big\}.$$

In particular, problems (1.7) and (1.9) with restrictions (1.8) and (1.10) are solved, respectively, on the classes

$$\Omega_p^r(A_0, A_r) := \big\{x \in L_\infty^r \colon\, \big\|x^{(r)}\big\|_\infty \le A_r,\; L(x)_p \le A_0\big\}$$

(Theorem 3) and on bounded subsets of the spaces T_n and $S_{n,r}$ (Theorems 4 and 5).

In addition, we obtain the solution (Theorem 6) of an analog of the Erdős problem of characterization of a pair (x, I) formed by a polynomial $T \in T_n$ with given uniform norm and a segment I whose support measure $\mu\big(\operatorname{supp}_I T'_\pm\big)$ is bounded by a given number and is such that the total length of arcs of the graph of positive (negative) part of the polynomial $T_\pm$ is maximal on the segment I. A similar problem is solved by the same theorem for splines from the set

$$\tilde{S}_{n,r} := \big\{s(\cdot + \tau) \colon\, s \in S_{n,r},\; \tau \in \mathbf{R}\big\}.$$

2. Auxiliary Statements

Note that if a function $x \in S_\varphi^0$ satisfies the condition $L(x)_p < \infty$ with some $p > 0$, $|x(t)| > 0$ for $t \in (a,b)$, and moreover, $a = -\infty$ or $b = +\infty$, then

$$x(t) \to 0 \quad \text{as} \quad t \to -\infty \quad \text{or} \quad t \to +\infty.$$

In this case, we assume that $x(-\infty) = 0$ or $x(+\infty) = 0$.

For a summable function x on the segment $[a,b]$, by $r(x,t)$ we denote the permutation of the function $|x|$ (see, e.g., [16], Sec. 1.3). Moreover, we set $r(x,t) = 0$ for $t > b - a$.

Lemma 1. *Suppose that φ is a function with period 2ω, $p > 0$, $\Phi \in W$, and a function $x \in S_\varphi^0$ satisfies the condition*

$$L(x)_p \le L(\varphi)_p, \tag{2.1}$$

where the quantity $L(x)_p$ is given by equality (1.4).

If a (finite or infinite) interval $(a_\pm, b_\pm) \subset \mathbf{R}$ and a segment $[A_\pm, B_\pm] \subset \mathbf{R}$ are such that

$$x(a_\pm) = x(b_\pm) = 0, \qquad x_\pm(t) > 0, \quad t \in (a_\pm, b_\pm), \tag{2.2}$$

and

$$\varphi(A_\pm) = \varphi(B_\pm) = 0, \qquad \varphi_\pm(t) > 0, \quad t \in (A_\pm, B_\pm), \tag{2.3}$$

then, for any $\xi > 0$ and any function $\Phi \in W$, the following inequalities are true:

$$\int\limits_{a_\pm}^{a_\pm+\xi} \Phi\big(\overline{x}_\pm^p(t)\big)\,dt \le \int\limits_{A_\pm}^{A_\pm+\xi} \Phi\big(\overline{\varphi}_\pm^p(t)\big)\,dt \tag{2.4}$$

and

$$\int\limits_{b_\pm-\xi}^{b_\pm} \Phi\big(\overline{x}_\pm^p(t)\big)dt \le \int\limits_{B_\pm-\xi}^{B_\pm} \Phi\big(\overline{\varphi}_\pm^p(t)\big)dt, \tag{2.5}$$

where $\overline{x}_\pm$ is the restriction of $x_\pm$ to $(a_\pm, b_\pm)$ and $\overline{\varphi}_\pm$ is the restriction of $\varphi_\pm$ to $[A_\pm, B_\pm]$. Moreover, outside the corresponding intervals, the functions $\overline{x}_\pm$ and $\overline{\varphi}_\pm$ are set equal to zero.

In addition, if

$$b_\pm - a_\pm \le B_\pm - A_\pm, \tag{2.6}$$

then, for any segment $[\alpha_\pm, \beta_\pm] \subset [A_\pm, B_\pm]$ such that

$$\beta_\pm - \alpha_\pm = b_\pm - a_\pm, \tag{2.7}$$

the following inequality is true:

$$\int\limits_{a_\pm}^{b_\pm} \Phi\big(x_\pm^p(t)\big)dt \le \int\limits_{\alpha_\pm}^{\beta_\pm} \Phi\big(\varphi_\pm^p(t)\big)dt, \qquad \Phi \in W. \tag{2.8}$$

Proof. We fix a function x and segments $(a_\pm, b_\pm)$ and $[A_\pm, B_\pm]$ satisfying the conditions of the lemma. We now establish inequality (2.4) [inequality (2.5) is proved similarly].

We first establish the inequality

$$\int\limits_0^\xi r^p(\overline{x}_\pm, t)dt \le \int\limits_0^\xi r^p(\overline{\varphi}_\pm, t)dt, \quad \xi > 0. \tag{2.9}$$

To do this, we first show that the difference

$$\delta_\pm(t) := r(\overline{x}_\pm, t) - r(\overline{\varphi}_\pm, t)$$

changes its sign (from minus to plus) on $[0, \infty)$ at most once. To prove this, we note that

$$\delta_\pm(0) \le \|x_\pm\|_\infty - \|\varphi\|_\infty \le 0 \tag{2.10}$$

because $x \in S_\varphi^0$. In view of this inequality and relations (2.2) and (2.3), for any $z_\pm \in \big[0, \|\overline{x}_\pm\|_{L_\infty[a_\pm, b_\pm]}\big)$, there exist points

$$t_i^\pm \in [a_\pm, b_\pm], \quad i = 1, \ldots, m, \quad m \ge 2,$$

$$y_j^\pm \in [A_\pm, B_\pm], \quad j = 1, 2,$$

such that

$$z_\pm = \overline{x}_\pm(t_i^\pm) = \overline{\varphi}_\pm(y_j^\pm). \tag{2.11}$$

In view of the inclusion $x \in S^0_\varphi$, the following inequality holds for points $t_i^\pm$ and $y_j^\pm$ satisfying relation (2.11):

$$\left|\overline{x}'_\pm(t_i^\pm)\right| \le \left|\overline{\varphi}'_\pm(y_j^\pm)\right|. \tag{2.12}$$

If the points $\theta_1^\pm, \theta_2^\pm > 0$ are chosen such that

$$z_\pm = r(\overline{x}_\pm, \theta_1^\pm) = r(\overline{\varphi}_\pm, \theta_2^\pm),$$

then, by the theorem on the derivative of permutation (see, e.g., [16], Proposition 1.3.2), in view of inequality (2.12), we get

$$\begin{aligned}\left|r'(\overline{x}_\pm, \theta_1^\pm)\right| &= \left[\sum_{i=1}^{m} \left|\overline{x}'_\pm(t_i^\pm)\right|^{-1}\right]^{-1} \\ &\le \left[\sum_{j=1}^{2} \left|\overline{\varphi}'_\pm(y_j^\pm)\right|^{-1}\right]^{-1} = \left|r'(\overline{\varphi}_\pm, \theta_2^\pm)\right|.\end{aligned}$$

By virtue of (2.10), this implies that the difference $\delta^\pm(t) := r(\overline{x}_\pm, t) - r(\overline{\varphi}_\pm, t)$ changes its sign (from minus to plus) on $[0, \infty)$ at most once. The same is also true for the difference

$$\delta_p^\pm(t) := r^p(\overline{x}_\pm, t) - r^p(\overline{\varphi}_\pm, t).$$

Consider an integral

$$I_p^\pm(\xi) := \int_0^\xi \delta_p^\pm(t)dt, \quad \xi \ge 0.$$

It is clear that $I_p^\pm(0) = 0$ and, in view of condition (2.1), for $\xi \ge \max\{b_\pm - a_\pm, B_\pm - A_\pm\}$, we get

$$I_p^\pm(\xi) \le L(x_\pm)_p - L\big(\varphi_\pm\big)_p \le 0.$$

Moreover, the derivative $(I_p^\pm)'(t) = \delta_p^\pm(t)$ changes its sign (from minus to plus) at most once. Thus,

$$I_p^\pm(\xi) \le 0$$

for all $\xi \ge 0$. Inequality (2.9) is true. By the Hardy–Littlewood–Pólya theorem (see, e.g., [16], Theorem 1.3.11), this inequality implies that

$$\int_{a_\pm}^{b_\pm} \Phi\big(x_\pm^p(t)\big)dt \le \int_{A_\pm}^{B_\pm} \Phi\big(\varphi_\pm^p(t)\big)dt, \quad \Phi \in W. \tag{2.13}$$

We now establish inequality (2.4). Passing to the shifts of the functions $\overline{x}$ and $\overline{\varphi}$, we can assume that

$$a_\pm = A_\pm = 0. \tag{2.14}$$

In view of the inclusion $x \in S_\varphi^0$, the difference $\Delta^\pm(t) := \overline{x}_\pm(t) - \overline{\varphi}_\pm(t)$ changes its sign (from minus to plus) on $[0,\infty)$ at most once. Since the functions $f(t) = t^p$ and $\Phi \in W$ are monotonically increasing, the same is also true for the difference

$$\Delta_\Phi^\pm(t) := \Phi\big(\overline{x}_\pm^p(t)\big) - \Phi\big(\overline{\varphi}_\pm^p(t)\big).$$

We set

$$I_\Phi^\pm(\xi) := \int\limits_0^\xi \Delta_\Phi^\pm(t)dt, \quad \xi \ge 0.$$

It is clear that $I_\Phi^\pm(0) = 0.$ Further, by using inequality (2.13) and assumption (2.14), we obtain

$$I_\Phi^\pm(\xi) \le \int\limits_{a_\pm}^{b_\pm} \Phi\big(\overline{x}_\pm^p(t)\big)dt - \int\limits_{A_\pm}^{B_\pm} \Phi\big(\overline{\varphi}_\pm^p(t)\big)dt \le 0$$

for $\xi \ge \max\{b_\pm - a_\pm, B_\pm - A_\pm\}$. In addition, the derivative $(I_\Phi^\pm)'(t) = \Delta_\Phi^\pm(t)$ changes its sign (from minus to plus) on $[0,\infty)$ at most once. Thus,

$$I_\Phi^\pm(\xi) \le 0 \quad \text{for all} \quad \xi \ge 0.$$

In view of assumption (2.14), this is equivalent to inequality (2.4).

It remains to establish inequality (2.8) under conditions (2.6) and (2.7). Assume that the last two conditions are satisfied. Thus, passing, if necessary, to a shift of the function $x,$ we can assume that

$$a_\pm = \alpha_\pm, \qquad b_\pm = \beta_\pm. \tag{2.15}$$

Hence, by using the inclusion $x \in S_\varphi^0$ and condition (2.2), we arrive at the inequality

$$x_\pm(t) \le \varphi_\pm(t), \quad t \in [a_\pm, b_\pm].$$

In view of assumption (2.15), this directly yields inequality (2.8).

Lemma 1 is proved.

In the proof of Lemma 1, we have established inequality (2.13). Thus, the following corollary is true:

Corollary 1. *Under the conditions of Lemma 1, for any function $\Phi \in W,$ the inequality*

$$\int\limits_{a_\pm}^{b_\pm} \Phi\big(x_\pm^p(t)\big)dt \le \int\limits_{A_\pm}^{B_\pm} \Phi\big(\varphi_\pm^p(t)\big)dt = \int\limits_0^{2\omega} \Phi\big(\varphi_\pm^p(t)\big)dt \tag{2.16}$$

is true.

Lemma 2. *Suppose that φ is an S-function with period $2\omega,$ $p > 0,$ $\Phi \in W,$ and $[a,b] \subset \mathbf{R}$. If the function $x \in S_\varphi^0$ satisfies the condition*

$$L(x)_p \le L(\varphi)_p, \tag{2.17}$$

where the quantity $L(x)_p$ *is given by equality (1.4), and one of the requirements*

$$\delta_\pm := \mu\left(\operatorname{supp}_{[a,b]} x_\pm\right) \le \omega, \tag{2.18}$$

then, for any function $\Phi \in W,$ *the following inequality is true:*

$$\int\limits_a^b \Phi\left(x_\pm^p(t)\right) dt \le \int\limits_{m^\pm-\Theta^\pm}^{m^\pm+\Theta^\pm} \Phi\left(\varphi_\pm^p(t)\right) dt. \tag{2.19}$$

Here, $m^\pm$ *are points of local maximum of the functions* $\varphi_\pm$ *and the numbers* $\Theta^\pm > 0$ *are such that*

$$\varphi(m^\pm - \Theta^\pm) = \varphi(m^\pm + \Theta^\pm) \tag{2.20}$$

and, moreover,

$$2\Theta^\pm = \delta_\pm. \tag{2.21}$$

Proof. We fix a function $x \in S_\varphi^0$ and a segment $[a,b]$ satisfying the conditions of Lemma 2. Inequality (2.19) is established for x_+ (for x_-, the proof is similar). Suppose that the segment $[a,b]$ satisfies the corresponding requirement (2.18). Assume that

$$x_+(a) > 0, \qquad x_+(b) > 0 \tag{2.22}$$

[if at least one of these inequalities is not true, then the proof of inequality (2.19) is simplified].

Assume that the function x does not have zeros on (a,b). Since $L(x)_p < \infty$ by condition (2.17), there exists a (finite or infinite) interval (c,d) such that $(a,b) \subset (c,d)$ and, in addition,

$$x_+(c) = x_+(d) = 0, \qquad x_+(t) > 0, \quad t \in (c,d).$$

By $\overline{x}_+$ we denote the restriction of x_+ to (c,d). Moreover, by $\overline{\varphi}_+$ we denote the restriction of φ_+ to $[0,2\omega]$. Applying inequality (2.16) to the interval (c,d), we arrive at the estimate

$$\int\limits_c^d \Phi\big(\overline{x}_+^p(t)\big) dt \le \int\limits_0^{2\omega} \Phi\big(\overline{\varphi}_+^p(t)\big) dt,$$

which can be rewritten in the form

$$\int\limits_0^{d-c} \Phi\big(r^p(\overline{x}_+, t)\big) dt \le \int\limits_0^{2\omega} \Phi\big(r^p(\overline{\varphi}_+, t)\big) dt. \tag{2.23}$$

As in the proof of Lemma 1, we can show that the difference

$$\delta_\Phi(t) := \Phi\big(r^p(\overline{x}_+, t)\big) - \Phi\big(r^p(\overline{\varphi}_+, t)\big)$$

changes its sign (from minus to plus) on $[0, \infty)$ at most once. In view of this fact and inequality (2.23), we get the following inequality:

$$\int\limits_0^\xi \Phi\big(r^p(\overline{x}_+, t)\big)dt \le \int\limits_0^\xi \Phi\big(r^p(\overline{\varphi}_+, t)\big)dt, \quad \xi > 0.$$

It is clear that this inequality also holds if $\overline{x}_+$ is the restriction of x_+ to (a, b). For the same restriction $\overline{x}_+$, we conclude that

$$\int\limits_a^b \Phi\big((\overline{x}_+^p(t)\big)dt = \int\limits_0^{b-a} \Phi\big(r^p(\overline{x}_+, t)\big)dt \le \int\limits_0^{b-a} \Phi\big(r^p(\overline{\varphi}_+, t)\big)dt = \int\limits_{m^+-\Theta^+}^{m^++\Theta^+} \Phi\big(\varphi_+^p(t)\big)dt,$$

where m^+ is a point of local maximum of the spline φ_+ and $\Theta^+ > 0$ satisfies conditions (2.20) and (2.21). Moreover, $\delta_+ = b - a$. Thus, in the case where x does not have zeros on (a, b), inequality (2.19) is true.

We now assume that x has zeros on (a, b). We set

$$a' := \inf\big\{t \in (a, b) \colon\, x_+(t) = 0\big\} \qquad \text{and} \qquad b' := \sup\big\{t \in (a, b) \colon\, x_+(t) = 0\big\}.$$

In view of (2.22), the support $\operatorname{supp}_{[a,b]} x_+$ has the form

$$\operatorname{supp}_{[a,b]} x_+ = (a, a')\bigcup(b', b)\bigcup\bigcup_k (a_k, b_k), \tag{2.24}$$

where $(a_k, b_k) \subset (a', b')$. Moreover,

$$x(a_k) = x(b_k) = 0, \qquad x_+(t) > 0, \quad t \in (a_k, b_k)$$

(the set of these intervals (a_k, b_k) can be empty). In view of relation (2.18), assumption (2.22), and the definitions of the numbers a' and b', we obtain

$$\delta_+ = (a' - a) + (b - b') + \sum_k (b_k - a_k) \le \omega. \tag{2.25}$$

Let A_+ and B_+ be two neighboring zeros of the function φ and, moreover, $\varphi_+(t) > 0$ for $t \in (A_+, B_+)$. In view of (2.17), we have $L(x)_p < \infty$. Hence, there exist (finite or infinite) intervals (α', a') and (b', β') such that

$$x_+(\alpha') = x_+(a') = 0, \qquad x_+(t) > 0, \quad t \in (\alpha', a'),$$

and

$$x_+(b') = x_+(\beta') = 0, \qquad x_+(t) > 0, \quad t \in (b', \beta').$$

Applying inequalities (2.4) and (2.5) to the intervals (α', a') and (b', β') and the segment $[A_+, B_+]$, we find

$$\int_{b'}^{b} \Phi\big(x_+^p(t)\big)dt \le \int_{A_+}^{A_+ + \xi} \Phi\big(\varphi_+^p(t)\big)dt, \quad \xi = b - b', \tag{2.26}$$

and

$$\int_{a}^{a'} \Phi\big(x_+^p(t)\big)dt \le \int_{B_+ - \eta}^{B_+} \Phi\big(\varphi_+^p(t)\big)dt, \quad \eta = a' - a \tag{2.27}$$

[in view of (2.25), $\overline{x}_+$ in inequality (2.4) can be replaced by x_+, whereas $\overline{\varphi}_+$ can be replaced by φ_+]. By virtue of (2.25), there exist mutually disjoint intervals (α_k, β_k) such that

$$(\alpha_k, \beta_k) \subset (A_+ + \xi,\ B_+ - \eta) \quad \text{and} \quad \beta_k - \alpha_k = b_k - a_k.$$

According to relation (2.8), for these intervals, the following inequality is true:

$$\int_{a_k}^{b_k} \Phi\big(x_+^p(t)\big)dt \le \int_{\alpha_k}^{\beta_k} \Phi\big(\varphi_+^p(t)\big)dt. \tag{2.28}$$

By using estimates (2.26)–(2.28) and relation (2.24), we get

$$\int_{a}^{b} \Phi\big(x_+^p(t)\big)dt = \int_{a}^{a'} \Phi\big(x_+^p(t)\big)dt + \int_{b'}^{b} \Phi\big(x_+^p(t)\big)dt + \sum_k \int_{a_k}^{b_k} \Phi\big(x_+^p(t)\big)dt$$

$$\le \int_{A_+}^{A_+ + \xi} \Phi\big(\varphi_+^p(t)\big)dt + \int_{B_+ - \eta}^{B_+} \Phi\big(\varphi_+^p(t)\big)dt + \sum_k \int_{\alpha_k}^{\beta_k} \Phi\big(\varphi_+^p(t)\big)dt.$$

Since $\beta_k - \alpha_k = b_k - a_k$, in view of (2.25), we can write

$$\xi + \eta + \sum_k (\beta_k - \alpha_k) = \delta_+.$$

Thus, the sum of integrals on the right-hand side of the obtained estimate does not exceed

$$\int_0^{\delta_+} r\left(\Phi\big(\varphi_+^p\big), t\right) dt = \int_{m^+ - \Theta^+}^{m^+ + \Theta^+} \Phi\big(\varphi_+^p(t)\big)dt,$$

where m^+ is the point of local maximum of the function φ_+ and $\Theta^+ > 0$ satisfies relations (2.20) and (2.21). Inequality (2.19) is proved.

Lemma 2 is proved.

Corollary 2. *Under the conditions of Lemma 2 and in the case where one of the following assumptions is true:*

$$\mu\big(\operatorname{supp}_{[a,b]} x_\pm\big) \le \omega,$$

the corresponding inequality

$$\int\limits_a^b \Phi\big(x_\pm^p(t)\big)dt \le \int\limits_0^{2\omega} \Phi\big(\varphi_\pm^p(t)\big)dt \tag{2.29}$$

holds.

3. Solving of the Bojanov–Naidenov Problem in the Classes of Functions with Given Comparison Function

Let $p, \omega > 0$ and let φ be an S-function with period 2ω. We set

$$L_\varphi(p,\omega) := \big\{x \in S_\varphi^0 :\, L(x)_p \le L(\varphi)_p\big\}, \tag{3.1}$$

where the quantity $L(x)_p$ is given by equality (1.4). We fix a number $\mu > 0$ and introduce a class $L_\varphi^\pm(p,\omega,\mu)$ of pairs (x, I) of functions x and segments $I = [a,b]$ by the formula

$$L_\varphi^\pm(p,\omega,\mu) := \big\{(x,I) :\, x \in L_\varphi(p,\omega),\ \mu(\operatorname{supp}_I x_\pm) \le \mu\big\}. \tag{3.2}$$

We rewrite the number μ in the form

$$\mu = n\cdot\omega + 2\Theta, \qquad n \in \mathbf{N}\bigcup\{0\}, \quad \Theta \in [0, \omega/2). \tag{3.3}$$

Note that if the numbers $\tau^\pm \in \mathbf{R}$ and the segment $[A,B]$ are such that

$$B - A = 2n\omega + 2\Theta, \tag{3.4}$$

$$\varphi_\pm(A + \Theta + \tau^\pm) = \varphi_\pm(B - \Theta + \tau^\pm) = \|\varphi\|_\infty, \tag{3.5}$$

then $\big(\varphi(\cdot + \tau^\pm), [A,B]\big) \in L_\varphi^\pm(p,\omega,\mu)$.

Theorem 1. *Suppose that $p, \omega,\ \mu > 0$, and φ is an S-function with period 2ω. Then, for any function $\Phi \in W$,*

$$\sup\left\{\int\limits_a^b \Phi\big(x_\pm^p(t)\big)dt :\, (x,[a,b]) \in L_\varphi^\pm(p,\omega,\mu)\right\} = \int\limits_A^B \Phi\big(\varphi_\pm^p(t+\tau^\pm)\big)dt,$$

where the sets $L_\varphi^\pm(p,\omega,\mu)$, the numbers $\tau^\pm$, and the segment $[A,B]$ are given by relations (3.1)–(3.5).

Proof. We fix an arbitrary pair $(x, I) \in L^{\pm}_{\varphi}(p, \omega, \mu)$ formed by a function x and a segment $I = [a, b]$. We prove the theorem for x_+ (for x_-, the proof is similar). To do this, we first establish the inequality

$$\mathcal{I} := \int_a^b \Phi\big(x_+^p(t)\big)dt \le \int_A^B \Phi\big(\varphi_+^p(t+\tau^+)\big)dt := \mathcal{I}(\mu). \tag{3.6}$$

We first consider the case where $\operatorname{supp}_{[a,b]} x_+ = \mu$. Since μ satisfies relation (3.3), the segment $[a, b]$ can be rewritten in the form

$$[a, b] = \bigcup_{k=1}^{n} [\alpha_k, \beta_k] \bigcup [\alpha.\beta],$$

Moreover, the intervals (α_k, β_k) and (α, β) are mutually disjoint and

$$\mu(\operatorname{supp}_{[\alpha_k,\beta_k]} x_+) = \omega, \qquad \mu(\operatorname{supp}_{[\alpha,\beta]} x_+) = 2\Theta.$$

Hence,

$$\int_a^b \Phi\big(x_+^p(t)\big)dt = \sum_{k=1}^{n} \int_{\alpha_k}^{\beta_k} \Phi\big(x_+^p(t)\big)dt + \int_{\alpha}^{\beta} \Phi\big(x_+^p(t)\big)dt.$$

To estimate the integrals on the right-hand side of this equality, we apply inequalities (2.29) and (2.19) and obtain

$$\int_a^b \Phi\big(x_+^p(t)\big)dt \ \le\ n\int_0^{2\omega} \Phi\big(\varphi_+^p(t)\big)dt \ + \int_{m^+-\Theta}^{m^++\Theta} \Phi\big(\varphi_+^p(t)\big)dt \ = \int_A^B \Phi\big(\varphi_+^p(t+\tau^+)\big)dt,$$

where m^+ is the point of local maximum of the function φ, and the last equality in this sequence of relations follows from (3.4). Thus, inequality (3.6) is established in the case where $\operatorname{supp}_{[a,b]} x_+ = \mu$.

Now let

$$\mu_1 := \operatorname{supp}_{[a,b]} x_+ < \mu.$$

Note that the number μ can be uniquely represented in the form (3.3). Hence, the segment $[A, B]$ and the number τ^+ are uniquely (to within a shift) determined by this number. Therefore, the integral $\mathcal{I}(\mu)$ on the right-hand side of (3.6) is uniquely determined by the number μ. Moreover, it is clear that $\mathcal{I}(\mu)$ does not decrease as a function of μ. Hence, repeating the reasoning used in the previous case, we obtain the following estimate for the integral $\mathcal{I}$ on the left-hand side of (3.6):

$$\mathcal{I} \le \mathcal{I}(\mu_1) \le I(\mu).$$

Thus, the proof of inequality (3.6) is completed. Note that, for the pair

$$\big(\varphi(\cdot + \tau^{\pm}), [A, B]\big) \in L^{\pm}_{\varphi}(p, \omega, \mu)$$

formed by a function $x(\cdot) = \varphi(\cdot + \tau^+)$ and a segment $[A, B]$ given by relations (3.3)–(3.5), inequality (3.6) turns into the equality.

Theorem 1 is proved.

Let $k \in \mathbf{N}$, let $\omega > 0$, and let φ be an S-function with period 2ω such that $\varphi \in L_\infty^{k+1}$, $x \in S_\varphi^k$. Thus, $\varphi^{(i)}$ is a comparison function for $x^{(i)}$, $i = 0, 1, \ldots, k$. Therefore,

$$L(x^{(k)})_1 \le 2\|x^{(k-1)}\|_\infty \le 2\|\varphi^{(k-1)}\|_\infty = L(\varphi^{(k)})_1. \tag{3.7}$$

Hence, $x^{(k)} \in S_{\varphi^{(k)}}(1,\omega)$. We fix a number $\mu > 0$ and introduce a class of pairs (x, I) of functions x and segments $I = [a, b]$ by the formula

$$S^{\pm}_{\varphi,k}(\omega,\mu) := \{(x,I)\colon\, x \in S_\varphi^k,\ \mu\big(\operatorname{supp}_I x_\pm^{(k)}\big) \le \mu\}. \tag{3.8}$$

By using these definitions and relation (3.7), we arrive at the implication

$$(x, I) \in S^{\pm}_{\varphi,\,k}(\omega,\mu) \Rightarrow (x^{(k)}, I) \in L^{\pm}_{\varphi^{(k)}}(1,\omega,\mu), \tag{3.9}$$

where the sets $L^{\pm}_\varphi(p,\omega,\mu)$ are determined by using (3.2).

We represent the number μ in the form

$$\mu = n\omega + 2\Theta, \qquad n \in \mathbf{N}\bigcup\{0\}, \quad \Theta \in (0, \omega/2). \tag{3.10}$$

Further, we choose the numbers $\tau_k^{\pm} \in \mathbf{R}$ and the segment $[A, B]$ such that

$$B - A = 2n\omega + 2\Theta, \tag{3.11}$$

$$\varphi_\pm^{(k)}(A + \Theta + \tau_k^\pm) = \varphi_\pm^{(k)}(B - \Theta + \tau_k^\pm) = \|\varphi^{(k)}\|_\infty. \tag{3.12}$$

Then $\big(\varphi(\cdot + \tau^\pm), [A,B]\big) \in S^{\pm}_{\varphi,k}(\omega,\mu)$.

Theorem 2. *Suppose that $k \in \mathbf{N}$, $\omega, \mu > 0$, and φ is an S-function with period 2ω such that $\varphi \in L_\infty^{k+1}$. Then, for any function $\Phi \in W$,*

$$\sup\left\{\int_a^b \Phi\left(x_\pm^{(k)}(t)\right) dt\colon\, (x,I) \in S^{\pm}_{\varphi,k}(\omega,\mu)\right\} = \int_A^B \Phi\left(\varphi_\pm^{(k)}(t + \tau_k^\pm)\right) dt,$$

where the set $S^{\pm}_{\varphi,k}(\omega,\mu)$, the numbers $\tau_k^\pm$, and the segment $[A,B]$ are given by relations (3.8)–(3.12).

Proof. In view of implication (3.9), if $(x,I) \in S^{\pm}_{\varphi,\,k}(\omega,\mu)$, then $(x^{(k)}, I) \in L^{\pm}_{\varphi^{(k)}}(1,\omega,\mu)$, where the set $L^{\pm}_\varphi(p,\omega,\mu)$ is given by (3.2). Thus, by applying Theorem 1 to the class $L^{\pm}_{\varphi^{(k)}}(1,\omega,\mu)$, we arrive at the assertion of Theorem 2.

Theorem 2 is proved.

Setting $\Phi(t) = t^{q/p}$ in Theorem 1 and $\Phi(t) = t^q$ in Theorem 2, we get the following corollary:

Corollary 3. *Let $k \in \mathbf{N}$, let $p, \omega, \mu > 0$, let φ be an S-function with period 2ω, and let $\Phi \in W$. Then, for any $q \ge p$,*

$$\sup\left\{\int_a^b x_\pm^q(t)dt\colon\ (x,I)\in L_\varphi^\pm(p,\omega,\mu)\right\}=\int_A^B \varphi_\pm^q(t+\tau^\pm)dt,$$

where the sets $L_\varphi^\pm(p,\omega,\mu)$, *the numbers* $\tau^\pm$, *and the segment* $[A,B]$ *are given by relations (3.1)–(3.5).*

In addition, if $k\in\mathbf{N},\ \mu>0$, *and* $\varphi\in L_\infty^{k+1}$, *then, for any* $q\ge 1$,

$$\sup\left\{\int_a^b \left(x_\pm^{(k)}(t)\right)^q dt\colon\ (x,I)\in S_{\varphi,\,k}^\pm(\omega,\mu)\right\}=\int_A^B \left(\varphi_\pm^{(k)}(t+\tau_k^\pm)\right)^q dt,$$

where the sets $S_{\varphi,k}^\pm(\omega,\mu)$, *the numbers* $\tau_k^\pm$, *and the segment are given by relations (3.8)–(3.12).*

4. Solving of the Bojanov–Naidenov Problem in Sobolev Classes

By $\varphi_r(t),\ r\in\mathbf{N}$, we denote a shift of the rth 2π-periodic integral with mean value zero over a period of the function $\varphi_0(t)=\operatorname{sgn}\sin t$. This quantity satisfies the condition $\varphi_r(0)=0$. For $\lambda>0$, we set

$$\varphi_{\lambda,r}(t):=\lambda^{-r}\varphi_r(\lambda t).$$

Let $A_r, A_0,\ p>0$. We choose $\lambda>0$ such that

$$A_0=A_rL(\varphi_{\lambda,r})_p, \tag{4.1}$$

where the quantity $L(x)_p$ is given by equality (1.4), and set

$$\varphi(t):=A_r\varphi_{\lambda,r}(t). \tag{4.2}$$

It is clear that φ is an S-function with period $2\pi/\lambda$; moreover,

$$\left\|\varphi^{(r)}\right\|_\infty=A_r \qquad\text{and}\qquad L(\varphi)_p=A_0.$$

Consider a class of functions

$$\Omega_p^r(A_0,A_r):=\left\{x\in L_\infty^r\colon\ \|x^{(r)}\|_\infty\le A_r,\ L(x)_p\le A_0\right\}. \tag{4.3}$$

Lemma 3 [11]. *Suppose that* $r\in\mathbf{N}$ *and* $A_0,A_r,\ p>0$. *Then, for any* $k=0,1,\ldots,r-1$,

$$\Omega_p^r(A_0,A_r)\subset S_\varphi^k,$$

where the function φ *is defined by equality (4.2) and the number* λ *is given by equality (4.1).*

Let $r\in\mathbf{N},\ k=0,1,\ldots,r-1$, and $\mu>0$. Consider the set of pairs (x,I) of functions x and segments $I=[\alpha,\beta]$ given by the formula

$$\Omega_p^{r,k}(A_0,A_r)_\pm:=\left\{(x,I)\colon\ x\in\Omega_p^r(A_0,A_r),\ \mu\big(\operatorname{supp}_I x_\pm^{(k)}\big)\le\mu\right\}. \tag{4.4}$$

We represent the number μ in the form

$$\mu = n\frac{\pi}{\lambda} + 2\Theta, \qquad n \in \mathbf{N}\bigcup\{0\}, \quad \Theta \in (0, \pi/(2\lambda)). \tag{4.5}$$

Further, we choose numbers $\tau^{\pm} \in \mathbf{R}$ and a segment $[A, B]$ such that

$$B - A = 2n\frac{\pi}{\lambda} + 2\Theta, \tag{4.6}$$

$$\big(\varphi_{\lambda,r-k}\big)_{\pm}(A + \Theta + \tau^{\pm}) = \big(\varphi_{\lambda,r-k}\big)_{\pm}(B - \Theta + \tau_k^{\pm}) = \big\|\varphi_{\lambda,r-k}\big\|_{\infty}. \tag{4.7}$$

Thus,

$$\big(\varphi_{\lambda,r}(\cdot + \tau^{\pm}), [A, B]\big) \in \Omega_p^{r,k}(A_0, A_r)_{\pm}.$$

Theorems 1 and 2 and Lemma 3 imply the following statement:

Theorem 3. *Suppose that $r \in \mathbf{N},\ A_0, A_r,\ p > 0$, and $\Phi \in W$. Then*

$$\sup\left\{\int\limits_{\alpha}^{\beta} \Phi\big(x_{\pm}^{p}(t)\big)dt\colon\ \big(x, [\alpha, \beta]\big) \in \Omega_p^{r,0}(A_0, A_r)_{\pm}\right\} = \int\limits_{A}^{B} \Phi\big((A_r\varphi_{\lambda,r})_{\pm}^{p}(t + \tau^{\pm})\big)dt.$$

At the same time, if $k \in \mathbf{N},\ k < r$, then

$$\sup\left\{\int\limits_{\alpha}^{\beta} \Phi(x_{\pm}^{(k)}(t))dt\colon\ \big(x, [\alpha, \beta]\big) \in \Omega_p^{r,k}(A_0, A_r)_{\pm}\right\} = \int\limits_{A}^{B} \Phi\big(A_r(\varphi_{\lambda,r-k})_{\pm}(t + \tau^{\pm})\big)dt,$$

where the classes $\Omega_p^{r,k}(A_0, A_r)_{\pm}$, the numbers λ and $\tau^{\pm}$, and the segment $[A, B]$ are given by (4.3)–(4.7).

Setting $\Phi(t) = t^{q/p},\ q \geq p$, in the first relation of Theorem 3 and $\Phi(t) = t^{q},\ q \geq 1$, in the second relation, we obtain, as in Corollary 3, sharp estimates for the norms $\big\|x_{\pm}^{(k)}\big\|_{L_q[\alpha,\beta]},\ k = 0, 1, \ldots, r-1$, in the classes $\Omega_p^{r,k}(A_0, A_r)_{\pm}$.

Theorem 3 is proved.

5. Solving of the Bojanov–Naidenov Problem in the Spaces of Trigonometric Polynomials

By T_n we denote the space of trigonometric polynomials of degree at most n. For $A_0,\ p > 0$, we set

$$T_n(A_0, p) := \big\{T \in T_n\colon\ L(T)_p \leq A_0 L(\sin n(\cdot))_p\big\},$$

where the quantity $L(x)_p$ is given by equality (1.4).

Lemma 4 [11]. *Suppose that $n \in \mathbf{N}$ and $A_0,\ p > 0$. Then, for any $k = 0, 1, \ldots,$*

$$T_n(A_0, p) \subset S_{\varphi}^{k},$$

where $\varphi(t) = A_0 \sin nt$.

Let $k \in \mathbf{N}\bigcup\{0\}$ and let $\mu > 0$. We introduce a set of pairs (T, I) formed by polynomials T and the segment $I = [\alpha, \beta]$ by the formula

$$T^{\pm}_{n,k}(A_0, p, \mu) := \Big\{(T, I)\colon\ T \in T_n(A_0, p),\ \mu\big(\operatorname{supp}_I T^{(k)}_{\pm}\big) \le \mu\Big\}. \tag{5.1}$$

The number μ is represented in the form

$$\mu = m\frac{\pi}{n} + 2\Theta, \qquad m \in \mathbf{N}\bigcup\{0\}, \quad \Theta \in (0,\ \pi/(2n)). \tag{5.2}$$

Further, we choose the numbers $\tau^{\pm} \in \mathbf{R}$ and the segment $[A, B]$ such that

$$B - A = 2m\frac{\pi}{n} + 2\Theta, \tag{5.3}$$

$$\big(\sin n\big(A + \Theta + \tau^{\pm}\big)\big)_{\pm} = \big(\sin n\big(B - \Theta + \tau^{\pm}\big)\big)_{\pm} = 1. \tag{5.4}$$

By Theorems 1 and 2 and Lemma 4, we obtain the following statement:

Theorem 4. *Suppose that $A_0,\ p,\ \mu > 0$ and $\Phi \in W$. Then*

$$\sup\left\{\int\limits_{\alpha}^{\beta} \Phi(T^{p}_{\pm}(t))dt\colon\ \big(T, [\alpha, \beta]\big) \in T^{\pm}_{n,0}(A_0, p, \mu)\right\} = \int\limits_{A}^{B} \Phi\left(\Big(A_0 \sin n\big(t + \tau^{\pm}\big)\Big)^{p}_{\pm}\right) dt$$

and, for any $k \in \mathbf{N}$,

$$\sup\left\{\int\limits_{\alpha}^{\beta} \Phi(T^{(k)}_{\pm}(t))dt\colon\ \big(T, [\alpha, \beta]\big) \in T^{\pm}_{n,k}(A_0, p, \mu)\right\} = \int\limits_{A}^{B} \Phi\left(n^k A_0\Big(\sin n\big(t + \tau^{\pm}\big)\Big)_{\pm}\right) dt,$$

where the classes $T^{\pm}_{n,k}(A_0, p, \mu)$, the numbers $\tau^{\pm}$, and the segment $[A, B]$ are given by (5.1)–(5.4).

6. Solving of the Bojanov–Naidenov Problem in the Spaces of Splines

By $S_{n,r}$ we denote a space of 2π-periodic polynomial splines of order r with defect 1 and nodes at the points $k\pi/n,\ k \in \mathbf{Z}$. For $A_0,\ p > 0,$ we set

$$S_{n,r}(A_0, p) := \big\{s(\cdot + \tau)\colon\ s \in S_{n,r},\ L(s)_p \le A_0 L(\varphi_{n,r})_p,\ \tau \in \mathbf{R}\big\},$$

where the quantity $L(x)_p$ is given by equality (1.4).

Lemma 5 [11]**.** *Suppose that $r, n \in \mathbf{N}$ and $A_0,\ p > 0$. Then, for any $k = 0, 1, \ldots, r - 1,$*

$$S_{n,r}(A_0, p) \subset S^k_{\varphi},$$

where $\varphi(t) = A_0\varphi_{n,r}(t)$.

Let $r, n \in \mathbf{N},\ k = 0, 1, \dots, r-1,$ and $\mu > 0$. We consider the set of pairs (s, I) of splines s and segments $I = [\alpha, \beta]$ given by the formula

$$S^k_{n,r}(A_0, p, \mu)_\pm := \Big\{(s, I)\colon\ s \in S_{n,r}(A_0, p),\ \mu\Big(\operatorname{supp}_I s^{(k)}_\pm\Big) \le \mu\Big\}. \tag{6.1}$$

We rewrite the number μ in the form

$$\mu = m\frac{\pi}{n} + 2\Theta, \qquad m \in \mathbf{N}\bigcup\{0\}, \quad \Theta \in \big(0, \pi/(2n)\big). \tag{6.2}$$

Further, we choose the numbers $\tau^\pm \in \mathbf{R}$ and the segment $[A, B]$ such that

$$B - A = 2m\frac{\pi}{n} + 2\Theta, \tag{6.3}$$

$$(\varphi_{n,r-k})_\pm\big(A + \Theta + \tau^\pm\big) = \big(\varphi_{n,r-k}\big)_\pm\big(B - \Theta + \tau^\pm\big) = \|\varphi_{n,r-k}\|_\infty. \tag{6.4}$$

By using Theorems 1 and 2 and Lemma 5, we obtain the following statement:

Theorem 5. *Suppose that* $r, n \in \mathbf{N},\ A_0,\ p,\ \mu > 0,$ *and* $\Phi \in W$. *Then*

$$\sup\left\{\int\limits_\alpha^\beta \Phi(s^p_\pm(t))dt\colon\ \big(s, [\alpha, \beta]\big) \in S^0_{n,r}(A_0, p, \mu)_\pm\right\} = \int\limits_A^B \Phi\big((A_0\varphi_{n,r})^p_\pm(t + \tau^\pm)\big)dt$$

and, for any $k = 1, 2, \dots, r-1,$

$$\sup\left\{\int\limits_\alpha^\beta \Phi(s^{(k)}_\pm(t))dt\colon\ s \in S^k_{n,r}(A_0, p, \mu)_\pm\right\} = \int\limits_\alpha^\beta \Phi\big(A_0(\varphi_{n,r-k})_\pm(t + \tau^\pm)|\big)dt,$$

where the classes $S^k_{n,r}(A_0, p, \mu)_\pm$, *the numbers* $\tau^\pm$, *and the segment* $[A, B]$ *are given by (6.1)–(6.4).*

7. Solving of the Erdős Problem in the Spaces of Trigonometric Polynomials and Splines

In [5], Bojanov and Naidenov solved the Erdős problem [6] of characterization of a trigonometric polynomial $T \in T_n$ with fixed uniform norm whose graph has the maximal length on a given segment $[\alpha, \beta] \subset \mathbf{R}$.

In the next theorem, we solve a similar problem of characterization of a pair (T, I) formed by a polynomial $T \in T_n$ with given uniform norm and a segment I whose support measure $\mu\big(\operatorname{supp}_I T'_\pm\big)$ is bounded by a given number and is such that the total length of arcs of the graph of positive (negative) part of the polynomial T on the segment I is maximal. In this theorem, the same problem is solved for splines from the set

$$\tilde{S}_{n,r} := \big\{s(\cdot + \tau)\colon\ s \in S_{n,r},\ \tau \in \mathbf{R}\big\}.$$

It is known that the length of an arc $l[a, b]$ of the graph of a function $x \in L^1[a, b]$ is given by the formula

$$l[a, b] = \int\limits_a^b \sqrt{1 + x'(t)^2}\, dt.$$

It is clear that, for the function $\Phi_0(t) = \sqrt{1+t^2}$, the inclusion $\Phi_0 \in W$ is true. Setting $\Phi = \Phi_0,\ k = 1,$ and $p = \infty$ in Theorems 4 and 5, we obtain the following assertion:

Theorem 6. *Suppose that $n \in \mathbf{N},\ M, \mu > 0,$ and μ has the form*

$$\mu = m\frac{\pi}{n} + 2\Theta, \qquad m \in \mathbf{N}\bigcup\{0\}, \quad \Theta \in \big(0, \pi/(2n)\big).$$

Among all pairs (x, I) of polynomials $x \in T_n$ with given uniform norm M and segments I from the family

$$S := \big\{I \subset \mathbf{R}\colon\ \mu\big(\operatorname{supp}_I x'_\pm\big) \le \mu\big\},$$

the maximal total length of arcs of the graph of positive (negative) part $x_\pm$ on the segment I has the polynomial

$$x(t) = M \sin n(t + \tau^\pm)$$

on the segment $[A, B]$ such that

$$B - A = 2m\frac{\pi}{n} + 2\Theta, \tag{7.1}$$

$$\big(\sin n\left(A + \Theta + \tau^\pm\right)\big)_\pm = \big(\sin n\left(B - \Theta + \tau^\pm\right)\big)_\pm = 1.$$

Among all pairs (x, I) of shifts of the splines $x \in \tilde{S}_{n,r}$ with given uniform norm M and segments I of the family S, the maximal total length of arcs of the graph of positive (negative) part $x_\pm$ on the segment I is observed for the shift of the spline

$$x(t) = \frac{M}{\|\varphi_{n,r}\|_\infty}\varphi_{n,r}(t + \tau^\pm)$$

on the segment $[A, B]$ such that equality (7.1) is true and, in addition,

$$(\varphi_{n,r-1})_\pm\left(A + \Theta + \tau^\pm\right) = (\varphi_{n,r-1})_\pm\left(B - \Theta + \tau^\pm\right) = \|\varphi_{n,r-1}\|_\infty.$$

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