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REALIZATION OF THE EXACT THREE-POINT FINITE-DIFFERENCE SCHEMES FOR THE SYSTEM OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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We consider an exact three-point finite-difference scheme (EDS) for the Dirichlet boundary-value problem for a system of second-order ordinary differential equations with boundary conditions of the first kind. We find weaker (as compared with known) conditions under which the analyzed scheme can be represented in the divergence form. The coefficient stability of the EDS and the accuracy of the perturbed scheme are investigated. It is shown that the matrix coefficients and the right-hand side of the equation can be represented via the solutions of four initial-value problems on the intervals whose length is equal to the length of grid step. The solutions of these problems can be obtained by using an arbitrary one-step method, which leads to a truncated difference scheme of a certain rank.

1. Introduction

Finite-difference method is one of the most widespread methods used for approximate solving of the equations of mathematical physics [1]. Among difference schemes proposed for second-order ordinary differential equations, a special place is occupied by exact three-point difference schemes (EDS). The investigation of these schemes was originated in [2], where the EDS in the divergence form was proposed for the second-order ordinary differential equations with piecewise smooth coefficients and boundary conditions of the first kind. The realization of EDS in the form of truncated schemes of any order of accuracy was performed and substantiated in [3]. These results were extended to boundary conditions of the third kind and nonuniform grids in [4], to equations with generalized solutions in the monograph [5], and to systems of second-order ordinary differential equations in [6].

In this connection, we especially mention the works [7–9], which strongly stimulated the development of two new directions in the theory and realization of EDS. Thus, in [7, 8], a new structure of the coefficients and right-hand sides of these schemes was proposed by using solely the solutions of Cauchy problems on the intervals of length equal to the length of grid step. In [9], the existence of EDS for quasilinear ordinary differential equations (ODE) of the second order was proved in a constructive way and an algorithm of their realization was theoretically substantiated. In the monograph [10], the results accumulated in these directions at that time were generalized and summarized. In this monograph, the authors studied the boundary-value problems for systems of nonlinear ODE of the first order, the boundary-value problems for nonlinear ODE of the second order with monotone operator, boundary-value problems on the semiaxis, etc.

The urgency of investigation of exact schemes and their algorithmic realizations is illustrated by the publication of specialized issues devoted to the analysis of the state-of-the-art in this field of numerical analysis. Thus, in the collective monograph [11], parallel with the classical methodology [12], the authors considered the concept of

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so-called nonstandard difference schemes [13] constructed on the basis of some other mathematical and philosophical principles. A new efficient algorithmic realization of EDS based on the truncated schemes on nonuniform grids for systems of nonlinear ODE of the second order with derivative on the right-hand side was proposed in [14] (see also the references therein).

It is known that the construction of exact three-point difference schemes is essentially based on the properties of so-called template (scalar or matrix) functions obtained as solutions of two auxiliary Cauchy problems for the corresponding homogeneous (scalar or vector) equation. Since these functions can be found in the explicit form only in some special cases, the problem of construction of efficient algorithmic realizations of the EDS, namely, of the truncated schemes of any order of accuracy, proves to be quite urgent both from the theoretical and practical points of view.

Note that one of approaches (see, e.g., [3]) is based on the expansions of template functions in power series (in powers of the grid step h) whose coefficients are expressed via multiple integrals by the recurrence relations with subsequent replacement of these series by their finite sums.

As already indicated, another approach was proposed in [7, 8]. In this case, to find the coefficients and the right-hand side of the EDS at the node of the grid x_i, it is necessary to solve four auxiliary Cauchy problems: two problems on the segment $[x_{i-1}; x_i]$ (forward) and two problems on the segment $[x_i; x_{i+1}]$ (backward). Solving each of these problems by an arbitrary one-step method (e.g., by expanding in the Taylor formula, by the Runge–Kutta method, etc.) whose accuracy is consistent with the smoothness of the coefficients and the right-hand side of equation, we get a truncated scheme of the required order of accuracy. The aim of the present paper is to generalize the results obtained in [6] and extend the results presented in [7, 8] to the first boundary-value problem for the vector equation [6].

The paper consists of the introduction and four sections. In Sec. 2, we complement the available results [6] on the properties of template matrix functions. Note that sufficient conditions under which the EDS can be reduced to a homogeneous divergence form were established in [6]. These conditions are connected with the requirement of commutativity of matrix elements and, in fact, reduce the original matrix-vector problem to the scalar case. Up to now, the problem of existence of an EDS in the divergence form for systems of ODE of the second order with symmetric matrix coefficients but without requirements of their commutativity remains open. The problem of weakening of the indicated sufficient conditions is considered in Sec. 3. In Sec. 4, we analyze the coefficient stability and the accuracy of the perturbed EDS. Finally, in Sec. 5, we prove the theorem on accuracy of the truncated scheme in the case where its coefficients are expressed via the solutions of four Cauchy problems found by using one-step methods of the corresponding order of accuracy.

2. Exact Three-Point Difference Scheme

Consider a boundary-value problem

$$L^{(P,Q)}u \equiv \frac{d}{dx}\left(P(x)\frac{d\vec{u}}{dx}\right) - Q(x)\vec{u} = -\vec{f}(x), \quad x \in (0;1), \tag{1}$$

$$\vec{u}(0) = \vec{u}(1) = \vec{0}, \tag{2}$$

where $P(x)$ and $Q(x)$ are given square matrices of order $n,$ $\vec{f}(x)$ is a given vector, and $\vec{u}(x)$ is the required solution. Elements of the matrix functions $P(x)$ and $Q(x)$ and the vector function $\vec{f}(x)$ are real functions of a certain order of smoothness on $[0;1]$.

We introduce a scalar product in $\mathbb{R}^n$ as follows:

$$(\vec{u}(x), \vec{v}(x)) = \sum_{i=1}^{n} u_i(x) v_i(x).$$

Let $\bar{\omega} = \{x_i = ih,\ i = 0, 1, \ldots, N\ (h = 1/N)\}$ be a uniform grid on $[0; 1]$. In what follows, we need the grid sets

$$\omega = \{x_i = ih,\ i = 1, \ldots, N-1\} \quad \text{and} \quad \omega^+ = \{x_i = ih,\ i = 1, \ldots, N\}.$$

We define template matrix functions $V_\alpha^{(i)}(x),\ \alpha = 1, 2,\ i = 1, 2, \ldots, N-1,$ as solutions of the Cauchy problems

$$L^{(P,Q)} V_\alpha^{(i)}(x) = 0, \qquad x \in (x_{i-1}; x_{x+1}), \quad \alpha = 1, 2, \tag{3}$$

$$V_1^{(i)}(x_{i-1}) = 0, \qquad P(x_{i-1}) \frac{dV_1^{(i)}}{dx}(x_{i-1}) = E, \tag{4}$$

$$V_2^{(i)}(x_{i+1}) = 0, \qquad P(x_{i+1}) \frac{dV_2^{(i)}}{dx}(x_{i+1}) = -E, \tag{5}$$

where 0 is the null matrix and E is the identity matrix.

We define a vector function $\vec{v}_3^{\,i}(x),\ i = 1, 2, \ldots, N-1,$ as a solution of the boundary-value problem

$$\begin{gathered} L^{(P,Q)} \vec{v}_3^{\,(i)}(x) = -\vec{f}(x), \quad x \in (x_{i-1}; x_{x+1}), \\ \vec{v}_3^{\,(i)}(x_{i-1}) = \vec{0}, \qquad \vec{v}_3^{\,(i)}(x_{i+1}) = \vec{0}. \end{gathered} \tag{6}$$

For the sake of convenience, we present some main results necessary in what follows (their proofs can be found in [6]).

Lemma 1 [6]. *Suppose that the matrices $P(x)$ and $Q(x)$ satisfy the conditions*

$$\begin{gathered} (P(x)\vec{u}(x), \vec{u}(x)) > 0, \qquad (Q(x)\vec{u}(x), \vec{u}(x)) \geq 0 \\ \forall x \in [0; 1] \quad \forall \vec{u}, \vec{v} \in C\left([0; 1]; \mathbb{R}^n\right). \end{gathered} \tag{7}$$

Then the template matrix functions $V_\alpha^{(i)}(x),\ \alpha = 1, 2,\ i = 1, 2, \ldots, N-1,$ have the following properties:

(i) $V_\alpha^{(i)}(x)$ are linearly independent;

(ii) $V_\alpha^{(i)}(x)$ are nondegenerate, i.e.,

$$\det V_1^{(i)}(x) \neq 0 \quad \forall x \in (x_{i-1}; x_{x+1}], \qquad \det V_2^{(i)}(x) \neq 0 \quad \forall x \in [x_{i-1}; x_{x+1}).$$

In the next statement, the matrix functions $V_\alpha^{(i)}(x)$ are used for the construction of an EDS.

Lemma 2 [6]. *Suppose that conditions (7) are satisfied. Then, for problem (1), (2), there exists an EDS, which has the form*

$$\vec{u}_i = V_1^{(i)}(x_i)[V_1^{(i)}(x_{i+1})]^{-1}\vec{u}_{i+1} + V_2^{(i)}(x_i)[V_2^{(i)}(x_{i-1})]^{-1}\vec{u}_{i-1} + \vec{v}_3^{(i)}(x_i),$$
$$\vec{u}_0 = \vec{u}_N = \vec{0}. \tag{8}$$

We now study the matrix functions $V_\alpha^{(i)}(x)$ by complementing Lemma 3 in [6] with new properties.

Lemma 3. *Suppose that conditions (7) and the conditions*

$$P(x) = P^*(x), \qquad Q(x) = Q^*(x) \quad \forall x \in [0; 1], \tag{9}$$

are satisfied.

Then the template matrix functions $V_\alpha^{(i)}(x),\ \alpha = 1, 2,\ i = 1, 2, \dots, N-1,$ *have the following properties:*

(i) $V_1^{(i)}(x_{i+1}) = V_2^{(i)^*}(x_{i-1})$;

(ii) $V_1^{(i+1)}(x_{i+1}) = V_2^{(i)^*}(x_i)$;

(iii) $V_1^{(i)}(x_{i+1}) = V_2^{(i)^*}(x_i)\Big\{E + \int_{x_{i-1}}^{x_i} Q(\xi)V_1^{(i)}(\xi)d\xi\Big\} + \Big\{E + \int_{x_i}^{x_{i+1}} V_2^{(i)^*}(x_i)Q(\xi)d\xi\Big\}V_1^{(i)}(x_i)$;

(iv) $V_\alpha^{(i)^*}(x)P(x)\dfrac{dV_\alpha^{(i)}(x)}{dx} = \dfrac{dV_\alpha^{(i)^*}(x)}{dx}P(x)V_\alpha^{(i)}(x),\ x \in [x_{i-1}; x_{i+1}],\ \alpha = 1, 2$;

(v) $V_1^{(i)^*}(x)\Big\{E + \int_{x_{i-1}}^{x} Q(\xi)V_1^{(i)}(\xi)d\xi\Big\} = \Big\{E + \int_{x_{i-1}}^{x} V_1^{(i)^*}(\xi)Q(\xi)d\xi\Big\}V_1^{(i)}(x),\ x \in [x_{i-1}; x_{i+1}]$;

(vi) $V_2^{(i)^*}(x)\Big\{E + \int_{x}^{x_{i-1}} Q(\xi)V_2^{(i)}(\xi)d\xi\Big\} = \Big\{E + \int_{x}^{x_{i+1}} V_2^{(i)^*}(\xi)Q(\xi)d\xi\Big\}V_2^{(i)}(x),\ x \in [x_{i-1}; x_{i+1}].$

Proof. Properties (i)–(iii) are proved as in [1, p. 189] with insignificant changes. We prove Property (iv) for $\alpha = 1$ (for $\alpha = 2$, the proof is similar). By using two equations

$$V_1^{(i)^*}(\xi)\frac{d}{d\xi}\left(P(\xi)\frac{dV_1^{(i)}(\xi)}{d\xi}\right) = V_1^{(i)^*}(\xi)Q(\xi)V_1^{(i)}(\xi), \quad \xi \in [x_{i-1}; x_{i+1}],$$

$$\frac{d}{d\xi}\left(\frac{dV_1^{(i)^*}(\xi)}{d\xi}P(\xi)\right)V_1^{(i)}(\xi) = V_1^{(i)^*}(\xi)Q(\xi)V_1^{(i)}(\xi), \quad \xi \in [x_{i-1}; x_{i+1}],$$

we get the inequality

$$V_1^{(i)^*}(\xi)\frac{d}{d\xi}\left(P(\xi)\frac{dV_1^{(i)}(\xi)}{d\xi}\right) = \frac{d}{d\xi}\left(\frac{dV_1^{(i)^*}(\xi)}{d\xi}P(\xi)\right)V_1^{(i)}(\xi), \quad \xi \in [x_{i-1}; x_{i+1}].$$

Integrating this inequality from $\xi = x_{i-1}$ to $\xi = x$, we get

$$V_1^{(i)^*}(\xi)P(\xi)\frac{dV_1^{(i)}(\xi)}{d\xi}\Bigg|_{x_{i-1}}^{x} - \int\limits_{x_{i-1}}^{x} \frac{dV_1^{(i)^*}}{d\xi}P(\xi)\frac{dV_1^{(i)}(\xi)}{d\xi}d\xi$$

$$= \frac{dV_1^{(i)^*}(\xi)}{d\xi}P(\xi)V_1^{(i)}(\xi)\Bigg|_{x_{i-1}}^{x} - \int\limits_{x_{i-1}}^{x} \frac{dV_1^{(i)^*}}{d\xi}P(\xi)\frac{dV_1^{(i)}(\xi)}{d\xi}d\xi.$$

In view of the initial conditions, we obtain

$$V_1^{(i)^*}(x)P(x)\frac{dV_1^{(i)}(x)}{dx} = \frac{dV_1^{(i)^*}(x)}{dx}P(x)V_1^{(i)}(x), \quad x \in [x_{i-1}; x_{i+1}].$$

We now prove Property (v). Integrating the equalities

$$\frac{d}{d\xi}\left(P(\xi)\frac{dV_1^{(i)}(\xi)}{d\xi}\right) = Q(\xi)V_1^{(i)}(\xi), \quad \xi \in [x_{i-1}; x_{i+1}],$$

$$\frac{d}{d\xi}\left(\frac{dV_1^{(i)^*}(\xi)}{d\xi}P(\xi)\right) = V_1^{(i)^*}(\xi)Q(\xi), \quad \xi \in [x_{i-1}; x_{i+1}],$$

from $\xi = x_{i-1}$ to $\xi = x$ and using the initial conditions, we get

$$P(x)\frac{dV_1^{(i)}(x)}{dx} = E + \int\limits_{x_{i-1}}^{x} Q(\xi)V_1^{(i)}(\xi)d\xi, \quad x \in [x_{i-1}; x_{i+1}],$$

$$\frac{dV_1^{(i)^*}(x)}{dx}P(x) = E + \int\limits_{x_{i-1}}^{x} V_1^{(i)^*}(\xi)Q(\xi)d\xi, \quad x \in [x_{i-1}; x_{i+1}].$$

We multiply the first equation by $V_1^{(i)^*}(x)$ from the left and the second equation by $V_1^{(i)}(x)$ from the right. Then Property (v) follows from Property (iv).

The proof of Property (vi) is similar to the proof of Property (v). To this end, it suffices to replace $V_1^{(i)}$ by $V_2^{(i)}$, integrate from $\xi = x$ to $\xi = x_{i+1}$, and use the corresponding initial conditions.

Lemma 3 is proved.

Definition. *A matrix function $G^{(i)}(x,\xi)$ satisfying the conditions*

$$L_x^{(P,Q)}G^{(i)}(x,\xi) \equiv \frac{d}{dx}\left(P(x)\frac{dG^{(i)}(x,\xi)}{dx}\right) - Q(x)G^{(i)}(x,\xi) = 0, \qquad x \in (x_{i-1}; x_{i+1}), \quad x \neq \xi,$$

$$G^{(i)}(x_{i-1},\xi) = G^{(i)}(x_{i+1},\xi) = 0,$$

$$\left[G^{(i)}(x,\xi)\right]_{x=\xi}=0,\qquad \left[P(x)\frac{dG^{(i)}(x,\xi)}{dx}\right]_{x=\xi}=-E,$$

is called the Green function of the operator $L^{(P,Q)}$ *on the segment* $[x_{i-1};x_{i+1}]$.

Here,

$$[v(x)]_{x=\xi}\equiv v(\xi+0)-v(\xi-0)$$

denotes the jump of the function $v(x)$ *at the point* $x=\xi$.

The following lemma contains the explicit form of the Green function $G^{(i)}(x,\xi)$ obtained by using the template matrix functions $V_\alpha^{(i)}(x)$:

Lemma 4 [6]. *Suppose that conditions (7) and (9) are satisfied. Then the matrix Green function* $G^{(i)}(x,\xi)$ *has the form*

$$G^{(i)}(x,\xi)=\begin{cases}V_1^{(i)}(x)\left[V_1^{(i)}(x_{i+1})\right]^{-1}V_2^{(i)^*}(\xi), & x\le\xi,\\ V_2^{(i)}(x)\left[V_1^{(i)^*}(x_{i+1})\right]^{-1}V_1^{(i)^*}(\xi), & \xi\le x,\end{cases}\qquad x,\xi\in[x_{i-1};x_{i+1}]. \tag{10}$$

Corollary 1. *For the matrix Green function* $G^{(i)}(x,\xi)$, *the following relations are true:*

$$G^{(i)}(x,\xi)=G^{(i)^*}(\xi,x)\quad \forall x,\xi\in[x_{i-1};x_{i+1}],$$

$$\left[G^{(i)}(x_i,x_i)\right]^{-1}=\left[V_1^{(i)^*}(x_i)\right]^{-1}+\left[V_1^{(i)^*}(x_i)\right]^{-1}\int\limits_{x_{i-1}}^{x_i}V_1^{(i)^*}(\xi)Q(\xi)d\xi$$

$$+\left[V_2^{(i)^*}(x_i)\right]^{-1}+\left[V_2^{(i)^*}(x_i)\right]^{-1}\int\limits_{x_i}^{x_{i+1}}V_2^{(i)^*}(\xi)Q(\xi)d\xi. \tag{11}$$

Proof. As in [6] (Corollary 1), the first relation directly follows from representation (10). We prove the second relation. By Property (iii) in Lemma 3, we get

$$\left[G^{(i)}(x_i,x_i)\right]^{-1}=\left[V_2^{(i)^*}(x_i)\right]^{-1}V_1^{(i)}(x_{i+1})\left[V_1^{(i)}(x_i)\right]^{-1}$$

$$=\left[V_1^{(i)}(x_i)\right]^{-1}+\int\limits_{x_{i-1}}^{x_i}Q(\xi)V_1^{(i)}(\xi)d\xi\,\left[V_1^{(i)}(x_i)\right]^{-1}$$

$$+\left[V_2^{(i)^*}(x_i)\right]^{-1}+\left[V_2^{(i)^*}(x_i)\right]^{-1}\int\limits_{x_i}^{x_{i+1}}V_2^{(i)^*}(\xi)Q(\xi)d\xi. \tag{12}$$

In view of Property (v) in Lemma 3, we arrive at the equality

$$\left[V_1^{(i)}(x_i)\right]^{-1} + \int\limits_{x_{i-1}}^{x_i} Q(\xi)V_1^{(i)}(\xi)d\xi \left[V_1^{(i)}(x_i)\right]^{-1}$$

$$= \left[V_1^{(i)^*}(x_i)\right]^{-1} + \left[V_1^{(i)^*}(x_i)\right]^{-1} \int\limits_{x_{i-1}}^{x_i} V_1^{(i)^*}(\xi)Q(\xi)d\xi. \quad (13)$$

Substituting (13) in (12), we obtain relation (11).

Corollary 2 [6]. *The vector solution* $\vec{v}_3^{(i)}(x)$ *of the boundary-value problem (6) has the form*

$$\vec{v}_3^{(i)}(x) = V_2^{(i)}(x)\left[V_1^{(i)^*}(x_{i+1})\right]^{-1} \int\limits_{x_{i-1}}^{x} V_1^{(i)^*}(\xi)\vec{f}(\xi)d\xi$$

$$+ V_1^{(i)}(x)\left[V_1^{(i)}(x_{i+1})\right]^{-1} \int\limits_{x}^{x_{i+1}} V_2^{(i)^*}(\xi)\vec{f}(\xi)d\xi, \quad x \in [x_{i-1}; x_{i+1}].$$

Theorem 1 [6]. *Suppose that conditions (7) and (9) are satisfied. Then the EDS (8) can be reduced to the form*

$$\Lambda\vec{u} \equiv \frac{1}{h}\left\{\left[\frac{1}{h}V_2^{(i)^*}(x_i)\right]^{-1}\vec{u}_{x,i} - \left[\frac{1}{h}V_1^{(i)^*}(x_i)\right]^{-1}\vec{u}_{\bar{x},i}\right\} - D_i\vec{u}_i = -\vec{\varphi}_i, \quad i = 1, 2, \ldots, N-1,$$

$$\vec{u}_0 = \vec{0}, \qquad \vec{u}_N = \vec{0}, \quad (14)$$

where

$$D_i = \frac{1}{h}\left[V_1^{(i)^*}(x_i)\right]^{-1} \int\limits_{x_{i-1}}^{x_i} V_1^{(i)^*}(\xi)Q(\xi)d\xi + \frac{1}{h}\left[V_2^{(i)^*}(x_i)\right]^{-1} \int\limits_{x_i}^{x_{i+1}} V_2^{(i)^*}(\xi)Q(\xi)d\xi, \quad (15)$$

$$\vec{\varphi}_i = \frac{1}{h}\left[V_1^{(i)^*}(x_i)\right]^{-1} \int\limits_{x_{i-1}}^{x_i} V_1^{(i)^*}(\xi)\vec{f}(\xi)d\xi + \frac{1}{h}\left[V_2^{(i)^*}(x_i)\right]^{-1} \int\limits_{x_i}^{x_{i+1}} V_2^{(i)^*}(\xi)\vec{f}(\xi)d\xi. \quad (16)$$

Proof. By using Lemmas 3 and 4 and Corollary 1, we rewrite the EDS (8) in the form

$$\left[G^{(i)}(x_i, x_i)\right]^{-1}\vec{u}_i = [V_2^{(i)^*}(x_i)]^{-1}\vec{u}_{i+1}$$

$$+ [V_1^{(i)^*}(x_i)]^{-1}\vec{u}_{i-1} + \left[G^{(i)}(x_i, x_i)\right]^{-1}\vec{v}_3^{(i)}(x_i), \quad (17)$$

where

$$\Big[G^{(i)}(x_i,x_i)\Big]^{-1} = \Big[V_2^{(i)^*}(x_i)\Big]^{-1} V_1^{(i)}(x_{i+1})\Big[V_1^{(i)}(x_i)\Big]^{-1}$$

$$= \Big[V_1^{(i)^*}(x_i)\Big]^{-1} V_2^{(i)}(x_{i-1})\Big[V_2^{(i)}(x_i)\Big]^{-1}.$$

By virtue of Corollary 2, we can rewrite the term $\big[G^{(i)}(x_i,x_i)\big]^{-1}\vec{v}_3^{(i)}(x_i)$ in (17) as follows:

$$\Big[G^{(i)}(x_i,x_i)\Big]^{-1}\vec{v}_3^{(i)}(x_i)$$

$$= \Big[V_1^{(i)^*}(x_i)\Big]^{-1}\int\limits_{x_{i-1}}^{x_i} V_1^{(i)^*}(\xi)\vec{f}(\xi)d\xi + \Big[V_2^{(i)^*}(x_i)\Big]^{-1}\int\limits_{x_i}^{x_{i+1}} V_2^{(i)^*}(\xi)\vec{f}(\xi)d\xi. \tag{18}$$

Substituting (18) and (11) in Eq. (17), we arrive at the following scheme:

$$[V_2^{(i)^*}(x_i)]^{-1}\,(\vec{u}_{i+1}-\vec{u}_i) + [V_1^{(i)^*}(x_i)]^{-1}\,(\vec{u}_{i-1}-\vec{u}_i)$$

$$+\left\{\Big[V_1^{(i)^*}(x_i)\Big]^{-1}\int\limits_{x_{i-1}}^{x_i} V_1^{(i)^*}(\xi)\vec{f}(\xi)d\xi + \Big[V_2^{(i)^*}(x_i)\Big]^{-1}\int\limits_{x_i}^{x_{i+1}} V_2^{(i)^*}(\xi)\vec{f}(\xi)d\xi\right\}$$

$$-\left\{\Big[V_1^{(i)^*}(x_i)\Big]^{-1}\int\limits_{x_{i-1}}^{x_i} V_1^{(i)^*}(\xi)Q(\xi)d\xi + \Big[V_2^{(i)^*}(x_i)\Big]^{-1}\int\limits_{x_i}^{x_{i+1}} V_2^{(i)^*}(\xi)Q(\xi)d\xi\right\}\vec{u}_i \;=\; \vec{0},$$

which proves the theorem.

Corollary 3 ([6], Lemma 6)**.** *Suppose that conditions (7) and (9) are satisfied and the relations*

$$V_\alpha^{(i)}(x) = V_\alpha^{(i)^*}(x), \qquad i=1,2,\dots,N-1, \quad \alpha=1,2, \tag{19}$$

are true. Then the EDS (8) can be represented in the divergence form as follows:

$$\begin{gathered}\Lambda u \equiv (A\vec{u}_{\bar{x}})_x - D\vec{u} = -\vec{\varphi}, \quad x\in\omega,\\ \vec{u}(0)=\vec{0}, \qquad \vec{u}(1)=\vec{0},\end{gathered} \tag{20}$$

where

$$A_i = A(x_i) = \left[\frac{1}{h}V_1^{(i)}(x_i)\right]^{-1}. \tag{21}$$

In order to reduce the EDS to the divergence form, we use the factorization method. We multiply (14) by the matrix Ψ_i from the left and seek it from the condition of divergence

$$\Psi_{i+1}\left[\frac{1}{h}V_1^{(i+1)^*}(x_{i+1})\right]^{-1} = \Psi_i\left[\frac{1}{h}V_1^{(i+1)}(x_{i+1})\right]^{-1}.$$

We now determine the solution of this matrix difference equation of the first order:

$$\begin{aligned}\Psi_{i+1} &= \Psi_0\left[V_1^{(1)}(x_1)\right]^{-1}\left[V_1^{(1)^*}(x_1)\right] \\ &\qquad \times\left[V_1^{(2)}(x_2)\right]^{-1}\left[V_1^{(2)^*}(x_2)\right]\times\ldots\times\left[V_1^{(i+1)}(x_{i+1})\right]^{-1}\left[V_1^{(i+1)^*}(x_{i+1})\right] \\ &= \Psi_0\prod_{j=1}^{i+1}\left[V_1^{(j)}(x_j)\right]^{-1}\left[V_1^{(j)^*}(x_j)\right].\end{aligned}$$

Then the EDS (14) takes the following form:

$$\frac{1}{h}\left\{\Psi_i\left[\frac{1}{h}V_1^{(i)^*}(x_i)\right]^{-1}\vec{u}_{\bar{x},i}\right\}_{x,i} - \Psi_i D_i\vec{u}_i = -\Psi_i\vec{\varphi}_i, \quad i=1,2,\ldots,N-1,$$

$$\vec{u}_0 = \vec{u}_N = \vec{0}.$$

3. Conditions for the Divergence Form of the EDS

It follows from Corollary 3 that the self-adjointness of template matrix functions is a sufficient condition for the reduction of the EDS (8) [or (14)] to a homogeneous divergence form. A sufficient condition for the existence of these template matrix functions was established in [6]. More exactly, if the following commutativity conditions are satisfied parallel with conditions (7) and (9):

$$\begin{gathered}P(x)P(\xi)=P(\xi)P(x), \qquad Q(x)Q(\xi)=Q(\xi)Q(x), \qquad P(x)Q(\xi)=Q(\xi)P(x) \\ \forall x,\xi\in[x_{i-1};x_{i+1}], \quad i=1,\ldots,N-1,\end{gathered} \tag{22}$$

then the template matrix functions $V_\alpha^{(i)}(x)$ have property (19). However, as indicated in [6, p. 1201], under conditions (7), (9), and (22), the original system decomposes (reduces to the scalar case). We construct an example illustrating that the commutativity condition (22) is not necessary for the self-adjointness of template matrix functions $V_\alpha^{(i)}(x)$.

Consider the Cauchy problem (3), (4) for $n=2$:

$$P(x)=\begin{bmatrix} m & 0 \\ 0 & p\end{bmatrix}, \qquad m=\text{const}>0, \quad p=\text{const}>0, \quad m\neq p,$$

$$Q(x) = \begin{bmatrix} a(x) & 1 \\ 1 & b(x) \end{bmatrix},$$

$$a(x), b(x) \in C^2[0;1], \quad (Q(x)\vec{u}, \vec{u}) \geq 0 \quad \forall x \in [0;1] \quad \forall \vec{u} \in \mathbb{R}^2.$$

Denoting its solution by

$$V_1^{(i)}(x) = \begin{bmatrix} v_{11}(x) & v_{12}(x) \\ v_{21}(x) & v_{22}(x) \end{bmatrix},$$

we rewrite system (3) and the initial conditions (4) in the form

$$\begin{bmatrix} m & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} v''_{11}(x) & v''_{12}(x) \\ v''_{21}(x) & v''_{22}(x) \end{bmatrix} = \begin{bmatrix} a(x) & 1 \\ 1 & b(x) \end{bmatrix} \begin{bmatrix} v_{11}(x) & v_{12}(x) \\ v_{21}(x) & v_{22}(x) \end{bmatrix}, \quad x \in (x_{i-1};\ x_{i+1}), \tag{23}$$

$$V_1^{(i)}(x_{i-1}) \equiv \begin{bmatrix} v_{11}(x_{i-1}) & v_{12}(x_{i-1}) \\ v_{21}(x_{i-1}) & v_{22}(x_{i-1}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\frac{dV_1^{(i)}(x_{i-1})}{dx} \equiv \begin{bmatrix} v'_{11}(x_{i-1}) & v'_{12}(x_{i-1}) \\ v'_{21}(x_{i-1}) & v'_{22}(x_{i-1}) \end{bmatrix} = P^{-1} = \begin{bmatrix} 1/m & 0 \\ 0 & 1/p \end{bmatrix}.$$

In what follows, we need the derivatives

$$\frac{d^2 V_1^{(i)}(x_{i-1})}{dx^2} = 0,$$

$$\frac{d^3 V_1^{(i)}(x_{i-1})}{dx^3} = P^{-1} Q(x_{i-1}) P^{-1}$$

$$= \begin{bmatrix} 1/m & 0 \\ 0 & 1/p \end{bmatrix} \begin{bmatrix} a(x_{i-1}) & 1 \\ 1 & b(x_{i-1}) \end{bmatrix} \begin{bmatrix} 1/m & 0 \\ 0 & 1/p \end{bmatrix} = \begin{bmatrix} \dfrac{a(x_{i-1})}{m^2} & \dfrac{1}{mp} \\ \dfrac{1}{mp} & \dfrac{b(x_{i-1})}{p^2} \end{bmatrix}.$$

By using system (23), we get

$$m v''_{11} = a(x) v_{11} + v_{21},$$

$$p v''_{21} = v_{11} + b(x) v_{21}.$$

Excluding

$$v_{11} = p v''_{21} - b(x) v_{21}$$

from the first equation, we conclude that the function $v_{21}(x)$ is a solution of the Cauchy problem for the fourth-order ODE:

$$mp\frac{d^4v_{21}(x)}{dx^4} + \left[-mb(x) - pa(x)\right]\frac{d^2v_{21}(x)}{dx^2} - 2mb'(x)\frac{dv_{21}(x)}{dx} + \left[a(x)b(x) - mb''(x) - 1\right]v_{21} = 0, \tag{24}$$

$$v_{21}(x_{i-1}) = 0, \qquad \frac{dv_{21}(x_{i-1})}{dx} = 0, \qquad \frac{d^2v_{21}(x_{i-1})}{dx^2} = 0, \qquad \frac{d^3v_{21}(x_{i-1})}{dx^3} = \frac{1}{mp}.$$

In a similar way, we derive the following system from (23):

$$mv_{12}'' = a(x)v_{12} + v_{22},$$

$$pv_{22}'' = v_{12} + b(x)v_{22}.$$

Thus, eliminating

$$v_{22} = mv_{12}'' - a(x)v_{12}$$

from the second equation, we arrive at the Cauchy problem for the fourth-order ODE for the function $v_{12}(x)$:

$$pm\frac{d^4v_{12}(x)}{dx^4} + \left[-pa(x) - mb(x)\right]\frac{d^2v_{12}(x)}{dx^2} - 2pa'(x)\frac{dv_{12}(x)}{dx} + \left[b(x)a(x) - pa''(x) - 1\right]v_{12} = 0, \tag{25}$$

$$v_{12}(x_{i-1}) = 0, \qquad \frac{dv_{12}(x_{i-1})}{dx} = 0, \qquad \frac{d^2v_{12}(x_{i-1})}{dx^2} = 0, \qquad \frac{d^3v_{12}(x_{i-1})}{dx^3} = \frac{1}{mp}.$$

It is clear that if the functions $a(x)$ and $b(x)$ satisfy the condition

$$a'(x) = b'(x) \quad \forall x \in [x_{i-1};\ x_{i+1}],$$

then (24) and (25) turn into the same Cauchy problem. Hence, $v_{12}(x) = v_{21}(x)$ $\forall x \in [x_{i-1}; x_{i+1}]$ and the matrix function $V_1^{(i)}(x)$ is symmetric.

In particular, for $m = 1,\ p = 1/2,\ a(x) \equiv 1$, and $b(x) \equiv 1$, the positive definiteness of the matrix $P(x)$ and the nonnegativeness of the matrix $Q(x)$ for all $x \in [0; 1]$ are obvious. We directly find

$$V_1^{(i)}(x) = \begin{bmatrix} \dfrac{2(x-x_{i-1})}{3} + \dfrac{\sinh\left(\sqrt{3}(x-x_{i-1})\right)}{3\sqrt{3}} & -\dfrac{2(x-x_{i-1})}{3} + \dfrac{2\sinh\left(\sqrt{3}(x-x_{i-1})\right)}{3\sqrt{3}} \\[2ex] -\dfrac{2(x-x_{i-1})}{3} + \dfrac{2\sinh\left(\sqrt{3}(x-x_{i-1})\right)}{3\sqrt{3}} & \dfrac{2(x-x_{i-1})}{3} + \dfrac{4\sinh\left(\sqrt{3}(x-x_{i-1})\right)}{3\sqrt{3}} \end{bmatrix}.$$

Similarly, we get

$$V_2^{(i)}(x) = \begin{bmatrix} -\dfrac{2(x-x_{i+1})}{3} - \dfrac{\sinh\left(\sqrt{3}(x-x_{i+1})\right)}{3\sqrt{3}} & \dfrac{2(x-x_{i+1})}{3} - \dfrac{2\sinh\left(\sqrt{3}(x-x_{i+1})\right)}{3\sqrt{3}} \\ \dfrac{2(x-x_{i+1})}{3} - \dfrac{2\sinh\left(\sqrt{3}(x-x_{i+1})\right)}{3\sqrt{3}} & -\dfrac{2(x-x_{i+1})}{3} - \dfrac{4\sinh\left(\sqrt{3}(x-x_{i+1})\right)}{3\sqrt{3}} \end{bmatrix}.$$

It is clear that the matrices $P(x)$ and $Q(x)$ do not satisfy the commutativity condition (22) because

$$\begin{aligned} P(x)Q(\xi) - Q(\xi)P(x) &= \begin{bmatrix} m & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} a(\xi) & 1 \\ 1 & b(\xi) \end{bmatrix} - \begin{bmatrix} a(\xi) & 1 \\ 1 & b(\xi) \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & p \end{bmatrix} \\ &= \begin{bmatrix} ma(\xi) & m \\ p & pb(\xi) \end{bmatrix} - \begin{bmatrix} ma(\xi) & p \\ m & pb(\xi) \end{bmatrix} \\ &= \begin{bmatrix} p & m-p \\ p-m & 0 \end{bmatrix} \neq 0 \quad \text{for} \quad m \neq p \end{aligned}$$

$$\forall x, \xi \in [x_{i-1}; x_{i+1}].$$

4. Coefficient Stability of the Exact Three-Point Difference Scheme

We study the coefficient stability of the EDS (20). As in the case of $G^{(i)}(x,\xi)$, we denote the matrix Green function of the operator $L^{(P,Q)}$ on the segment $[0;1]$ by $G(x,\xi)$. We define the template matrix functions $V_\alpha(x)$, $\alpha = 1, 2,$ as solutions of the Cauchy problems

$$\begin{gathered} L^{(P,Q)} V_\alpha(x) = 0, \qquad 0 < x < 1, \quad \alpha = 1, 2, \\ V_1(0) = 0, \qquad P(0)\frac{dV_1(0)}{dx} = E, \\ V_2(1) = 0, \qquad P(1)\frac{dV_2(1)}{dx} = -E. \end{gathered} \tag{26}$$

By analogy with Lemma 4, we can show that the Green function $G(x,\xi)$ has the form

$$G(x,\xi) = \begin{cases} V_1(x)[V_1(1)]^{-1} V_2^*(\xi), & x \le \xi, \\ V_2(x)\left[V_1^*(1)\right]^{-1} V_1^*(\xi), & \xi \le x, \end{cases} \qquad x, \xi \in [0;1]. \tag{27}$$

Let $G^h(x,\xi)$ be the matrix Green function of the operator Λ of the difference problem (20) and let $V_\alpha^h(x)$, $\alpha = 1,2,$ be solutions of the difference Cauchy problems

$$\begin{gathered}\Lambda V_\alpha^h(x) = 0, \qquad x \in \omega, \quad \alpha = 1,2, \\ V_1^h(0) = 0, \qquad \left[\frac{1}{h} V_1^{(1)}(x_1)\right]^{-1} V_{1\,\bar{x}}^h(x_1) = E, \\ V_2^h(1) = 0, \qquad \left[\frac{1}{h} V_2^{(N-1)}(x_{N-1})\right]^{-1} V_{2\,\bar{x}}^h(1) = -E.\end{gathered} \tag{28}$$

Lemma 5. *Suppose that conditions (7), (9), and (19) are satisfied. Then the equality*

$$G^h(x,\xi) = G(x,\xi) \quad \forall x, \xi \in \bar{\omega},$$

is true.

Proof. Since $V_1^{(1)}(x) = V_1(x)\ \forall x \in [0; x_2]$, the matrix function $V_1(x)$ satisfies the initial conditions of the Cauchy problem (28):

$$V_1(0) = 0, \qquad \left[\frac{1}{h} V_1^{(1)}(x_1)\right]^{-1} \frac{V_1(x_1) - V_1(0)}{h} = E,$$

and, hence, for $\alpha = 1$, the difference scheme (28) is exact for (26) and $V_1^h(x) = V_1(x)\ \forall x \in \bar{\omega}$. Similarly, $V_2^h(x) = V_2(x)\ \forall x \in \bar{\omega}$. This yields the assertion of the lemma.

We introduce the norm of a matrix consistent with the norm of a vector

$$\|A\| = \sup_{\vec{u} \neq \vec{0}} \frac{\|A\vec{u}\|}{\|\vec{u}\|}, \qquad \|\vec{u}\| = \sqrt{(\vec{u},\vec{u})} = \left(\sum_{k=1}^{n} u_k^2\right)^{1/2}.$$

Lemma 6. *Suppose that the matrices $P(x)$ and $Q(x)$ satisfy condition (9) and the following conditions:*

$$\begin{gathered}C_1(\vec{u},\ \vec{u}) \le (P(x)\vec{u},\ \vec{u}) \le C_2(\vec{u},\ \vec{u}), \qquad 0 \le (Q(x)\vec{u},\ \vec{u}) \le C_2(\vec{u},\ \vec{u}) \\ \forall x \in [0;1] \quad \forall \vec{u} \in \mathbb{R}^n \quad (C_1 > 0).\end{gathered} \tag{29}$$

Then, for the matrix functions $V_\alpha(x)$, $\alpha = 1,2,$ of problem (26), the following inequalities are true:

$$\|V_\alpha(x)\| \le \frac{1}{C_1} \exp\left(\frac{C_2}{2C_1}\right) \quad \forall x \in [0;1], \tag{30}$$

$$\|V_{\alpha\bar{x}}(x)\| \le \frac{1}{C_1} \exp\left(\frac{C_2}{2C_1}\right) \quad \forall x \in [x_1;1], \quad \alpha = 1,2. \tag{31}$$

Proof. By using conditions (9) and (29), we arrive at the estimates

$$\|P^{-1}(x)\| \le \frac{1}{C_1}, \qquad \|Q(x)\| \le C_2 \quad \forall x \in [0;1].$$

Consider the case $\alpha = 1$ (for $\alpha = 2$, the proof is similar). Integrating the equation

$$\frac{d}{d\eta}\left(P(\eta)\frac{dV_1(\eta)}{d\eta}\right) = Q(\eta)V_1(\eta)$$

from $\eta = 0$ to $\eta = \xi$ with regard for the initial condition and multiplying both sides by the matrix $P^{-1}(\xi)$ from the left, we obtain

$$\frac{dV_1(\xi)}{d\xi} = P^{-1}(\xi) + P^{-1}(\xi)\int_0^\xi Q(\eta)V_1(\eta)d\eta.$$

Further, integrating from $\xi = 0$ to $\xi = x$ and using the initial condition, we obtain

$$V_1(x) = \int_0^x P^{-1}(\xi)d\xi + \int_0^x P^{-1}(\xi)\int_0^\xi Q(\eta)V_1(\eta)d\eta d\xi, \quad x \in [0;1]. \tag{32}$$

In view of conditions (29), this yields the inequality

$$\|V_1(x)\| \le \int_0^x \|P^{-1}(\xi)\|d\xi + \int_0^x \|P^{-1}(\xi)\| \int_0^\xi \|Q(\eta)\|\|V_1(\eta)\|d\eta\, d\xi$$

$$\le \frac{x}{C_1} + \frac{C_2 x}{C_1}\int_0^x \|V_1(\eta)\|d\eta, \quad x \in [0;1],$$

which can be rewritten in the form

$$\frac{\|V_1(x)\|}{x} \le \frac{1}{C_1} + \frac{C_2}{C_1}\int_0^x \eta\frac{\|V_1(\eta)\|}{\eta}d\eta, \quad x \in (0;1].$$

By using the Bellman–Gronwall lemma [15, pp. 134–135], we get

$$\frac{\|V_1(x)\|}{x} \le \frac{1}{C_1}\exp\left(\frac{C_2}{C_1}\int_0^x \eta d\eta\right) = \frac{1}{C_1}\exp\left(\frac{C_2 x^2}{2C_1}\right), \quad x \in (0;1],$$

i.e.,

$$\|V_1(x)\| \le \frac{x}{C_1}\exp\left(\frac{C_2 x^2}{2C_1}\right), \quad x \in [0;1], \tag{33}$$

and, hence, estimate (30) is true.

We now prove estimate (31). In view of (32), we get

$$V_1(x_i) - V_1(x_{i-1}) = \int\limits_{x_{i-1}}^{x_i} P^{-1}(\xi)d\xi + \int\limits_{x_{i-1}}^{x_i} P^{-1}(\xi)\int\limits_0^{\xi} Q(\eta)V_1(\eta)d\eta\, d\xi. \quad i=1,2,\ldots,N.$$

This yields the inequality

$$\left\|V_1(x_i) - V_1(x_{i-1})\right\| \le \frac{x_i - x_{i-1}}{C_1} + \frac{C_2(x_i - x_{i-1})}{C_1}\int\limits_0^{x_i} \|V_1(\eta)\|d\eta, \quad i=1,2,\ldots,N.$$

By using estimate (33), we get

$$\begin{aligned}\left\|\frac{V_1(x_i)-V_1(x_{i-1})}{x_i - x_{i-1}}\right\| &\le \frac{1}{C_1} + \frac{C_2}{C_1}\int\limits_0^{x_i}\frac{\eta}{C_1}\exp\left(\frac{C_2\eta^2}{2C_1}\right)d\eta \\ &= \frac{1}{C_1} + \frac{1}{C_1}\left(\exp\left(\frac{C_2 x_i^2}{2C_1}\right) - 1\right) \\ &= \frac{1}{C_1}\exp\frac{C_2x_i^2}{2C_1}, \qquad i=1,2,\ldots,N,\end{aligned}$$

which proves estimate (31).

Lemma 6 is proved.

We now establish estimates for the Green function $G^h(x,\xi)$ and its difference derivatives.

Lemma 7. *Suppose that conditions (7), (9), (19), and (29) are satisfied. Then the following estimates are true:*

$$\max_{x\in\omega,\ \xi\in\omega^+}\|G_{\bar\xi}(x,\xi)\| \le M_1, \tag{34}$$

$$\max_{x\in\omega^+}\sum_{\xi\in\omega^+} h\|G_{\bar\xi\bar x}(x,\xi)\| \le 3M_1, \tag{35}$$

where

$$M_1 = \frac{1}{C_1^2}\exp^2\left(\frac{C_2}{2C_1}\right)\left\|[V_1(1)]^{-1}\right\|.$$

Proof. Estimate (34) directly follows from the representation

$$G_{\bar\xi}(x,\xi) = \begin{cases} V_1(x)[V_1(1)]^{-1}V^*_{2\bar\xi}(\xi), & x\le\xi, \\ V_2(x)\left[V_1^*(1)\right]^{-1}V^*_{1\bar\xi}(\xi), & \xi\le x,\end{cases} \qquad x\in\omega, \quad \xi\in\omega^+,$$

and Lemma 6. To prove estimate (35), we find

$$G_{\bar{\xi}\bar{x}}(x,\xi)=\begin{cases} V_{1\bar{x}}(x)[V_1(1)]^{-1}V_{2\bar{\xi}}^{*}(\xi), & x<\xi,\\ V_{2\bar{x}}(x)\left[V_1^{*}(1)\right]^{-1}V_{1\bar{\xi}}^{*}(\xi), & \xi<x, \quad x,\xi\in\omega^{+}.\\ \dfrac{1}{h}\left\{V_{1\bar{x}}(x)[V_1(1)]^{-1}V_2^{*}(x)-V_{2\bar{x}}(x)\left[V_1^{*}(1)\right]^{-1}V_1^{*}(x-h)\right\}, & \xi=x, \end{cases}$$

By virtue of Lemma 6, we obtain

$$\sum_{\xi\in\omega^{+}} h\|G_{\bar{\xi}\bar{x}}(x,\xi)\| = \sum_{\xi\in\omega^{+},\,\xi\neq x} h\|G_{\bar{\xi}\bar{x}}(x,\xi)\| + h\|\,G_{\bar{\xi}\bar{x}}(x,\xi)\big|_{\xi=x}\|$$

$$\le \frac{3}{C_1^2}\exp^2\left(\frac{C_2}{2C_1}\right)\left\|[V_1(1)]^{-1}\right\|, \quad x\in\omega^{+}.$$

Lemma 7 is proved.

Parallel with the EDS (20), we consider a perturbed difference scheme

$$\tilde{\Lambda}\vec{y}\equiv\frac{1}{h}\left\{\left[\frac{1}{h}\tilde{V}_1^{(i+1)}(x_{i+1})\right]^{-1}\vec{y}_{x,\,i}-\left[\frac{1}{h}\tilde{V}_1^{(i)}(x_i)\right]^{-1}\vec{y}_{\bar{x},\,i}\right\}-\tilde{D}_i\vec{y}_i=-\tilde{\vec{\varphi}}_i, \quad i=1,2,\ldots,N-1,$$
$$\vec{y}_0=\vec{y}_N=\vec{0}, \tag{36}$$

or, in the index-free form,

$$\tilde{\Lambda}\vec{y}\equiv(\tilde{A}\vec{y}_{\bar{x}})_x-\tilde{D}\vec{y}=-\tilde{\vec{\varphi}}(x), \quad x\in\omega,$$
$$\vec{y}(0)=0, \qquad \vec{y}(1)=\vec{0}. \tag{37}$$

Then the error $\vec{z}(x)=\vec{u}(x)-\vec{y}(x),\ x\in\bar{\omega},$ is a solution of the discrete problem

$$\Lambda\vec{z}\equiv(A\vec{z}_{\bar{x}})_x-D\vec{z}=-\vec{\psi}(x), \quad x\in\omega,$$
$$\vec{z}(0)=0, \quad \vec{z}(1)=\vec{0}, \tag{38}$$

where

$$\vec{\psi}(x)=\Big((A(x)-\tilde{A}(x))\vec{y}_{\bar{x}}\Big)_x-(D(x)-\tilde{D}(x))\vec{y}+\vec{\varphi}(x)-\tilde{\vec{\varphi}}(x).$$

Theorem 2 (on the coefficient stability). *Suppose that conditions (7), (9), (19), and (29) are satisfied. Then the exact three-point scheme (20) is coefficient stable and the following estimates are true:*

$$\max_{x\in\omega}\|\vec{z}(x)\|\le M_1\left\{\max_{\xi\in\omega^{+}}\|\vec{y}_{\bar{\xi}}(\xi)\|\sum_{\xi\in\omega^{+}}h\|A(\xi)-\tilde{A}(\xi)\|\right.$$

$$+\max_{\eta\in\omega}\|\vec{y}(\eta)\|\sum_{\xi\in\omega^+}h\sum_{\eta=0}^{\xi-h}h\|D(\eta)-\tilde{D}(\eta)\|+\sum_{\xi\in\omega^+}h\sum_{\eta=0}^{\xi-h}h\|\vec{\varphi}(\eta)-\tilde{\vec{\varphi}}(\eta)\|\Bigg\}, \quad (39)$$

$$\max_{x\in\omega^+}\|\vec{z}_{\bar{x}}(x)\|\le 3M_1\Bigg\{\max_{\xi\in\omega^+}\|\vec{y}_{\bar{\xi}}(\xi)\|\max_{\xi\in\omega^+}\|A(\xi)-\tilde{A}(\xi)\|$$

$$+\max_{\eta\in\omega}\|\vec{y}(\eta)\|\max_{\xi\in\omega^+}\sum_{\eta=0}^{\xi-h}h\|D(\eta)-\tilde{D}(\eta)\|+\max_{\xi\in\omega^+}\sum_{\eta=0}^{\xi-h}h\|\vec{\varphi}(\eta)-\tilde{\vec{\varphi}}(\eta)\|\Bigg\}, \quad (40)$$

where M_1 *is the positive constant specified in Lemma 7.*

Proof. In view of Lemma 5, the solution of problem (38) can be represented in the form

$$\vec{z}(x)=\sum_{\xi\in\omega}hG(x,\xi)\vec{\psi}(\xi)=\sum_{\xi\in\omega}hG(x,\xi)\Big[\Big(A(\xi)-\tilde{A}(\xi)\Big)\vec{y}_{\bar{\xi}}(\xi)\Big]_{\xi}$$

$$-\sum_{\xi\in\omega}hG(x,\xi)\left[\sum_{\eta=0}^{\xi-h}h\left(D(\eta)-\tilde{D}(\eta)\right)\vec{y}(\eta)\right]_{\xi}$$

$$-\sum_{\xi\in\omega}hG(x,\xi)\left[\sum_{\eta=0}^{\xi-h}h\left(\vec{\varphi}(\eta)-\tilde{\vec{\varphi}}(\eta)\right)\right]_{\xi},\quad x\in\omega.$$

Summarizing here by parts, we get

$$\vec{z}(x)=-\sum_{\xi\in\omega^+}hG_{\bar{\xi}}(x,\xi)\left[\Big(A(\xi)-\tilde{A}(\xi)\Big)\vec{y}_{\bar{\xi}}(\xi)\right]$$

$$+\sum_{\xi\in\omega^+}hG_{\bar{\xi}}(x,\xi)\left[\sum_{\eta=0}^{\xi-h}h\left(D(\eta)-\tilde{D}(\eta)\right)\vec{y}(\eta)\right]$$

$$+\sum_{\xi\in\omega^+}hG_{\bar{\xi}}(x,\xi)\left[\sum_{\eta=0}^{\xi-h}h\left(\vec{\varphi}(\eta)-\widetilde{\vec{\varphi}}(\eta)\right)\right],\quad x\in\omega. \quad (41)$$

This yields the inequality

$$\|\vec{z}(x)\|\le\max_{\xi\in\omega^+}\|G_{\bar{\xi}}(x,\xi)\|\Bigg\{\max_{\xi\in\omega^+}\|\vec{y}_{\bar{\xi}}(\xi)\|\sum_{\xi\in\omega^+}h\|A(\xi)-\tilde{A}(\xi)\|+$$

$$+\max_{\eta\in\omega}\|\vec{y}(\eta)\|\sum_{\xi\in\omega^+}h\sum_{\eta=0}^{\xi-h}h\|D(\eta)-\tilde{D}(\eta)\|+\sum_{\xi\in\omega^+}h\sum_{\eta=0}^{\xi-h}h\|\vec{\varphi}(\eta)-\tilde{\vec{\varphi}}(\eta)\|\Bigg\},\quad x\in\omega.$$

Thus, by virtue of Lemma 7, we obtain estimate (39).

To establish estimate (40), we use relation (41). We get

$$\vec{z}_{\bar{x}}(x) = -\sum_{\xi\in\omega^+} hG_{\bar{\xi}\bar{x}}(x,\xi)\left[\left(A(\xi)-\tilde{A}(\xi)\right)\vec{y}_{\bar{\xi}}(\xi)\right]$$

$$+\sum_{\xi\in\omega^+} hG_{\bar{\xi}\bar{x}}(x,\xi)\left[\sum_{\eta=0}^{\xi-h} h\left(D(\eta)-\tilde{D}(\eta)\right)\vec{y}(\eta)\right]$$

$$+\sum_{\xi\in\omega^+} hG_{\bar{\xi}\bar{x}}(x,\xi)\left[\sum_{\eta=0}^{\xi-h} h\left(\vec{\varphi}(\eta)-\tilde{\vec{\varphi}}(\eta)\right)\right], \quad x\in\omega.$$

Thus, we can write

$$\|\vec{z}_{\bar{x}}(x)\| \le \sum_{\xi\in\omega^+} h\|G_{\bar{\xi}\bar{x}}(x,\xi)\|\left\{\max_{\xi\in\omega^+}\|\vec{y}_{\bar{\xi}}(\xi)\|\max_{\xi\in\omega^+}\|A(\xi)-\tilde{A}(\xi)\|\right.$$

$$\left.+\max_{\eta\in\omega}\|\vec{y}(\eta)\|\max_{\xi\in\omega^+}\sum_{\eta=0}^{\xi-h} h\|D(\eta)-\tilde{D}(\eta)\| + \max_{\xi\in\omega^+}\sum_{\eta=0}^{\xi-h} h\|\vec{\varphi}(\eta)-\tilde{\vec{\varphi}}(\eta)\|\right\}, \quad x\in\omega^+.$$

Hence, by using Lemma 7, we arrive at inequality (40).

Theorem 2 is proved.

As a corollary of Theorem 2, we obtain the following statement:

Theorem 3 (on exactness)**.** *Suppose that the assumptions of Theorem 2 are true and the coefficients of the EDS (20) and the perturbed scheme (37) satisfy the conditions*

$$\begin{gathered}\max_{\xi\in\omega^+}\|A(\xi)-\tilde{A}(\xi)\| \le M_2h^n, \qquad \max_{\xi\in\omega^+}\sum_{\eta=0}^{\xi-h} h\|D(\eta)-\tilde{D}(\eta)\| \le M_2h^n,\\ \max_{\xi\in\omega^+}\sum_{\eta=0}^{\xi-h} h\|\vec{\varphi}(\eta)-\tilde{\vec{\varphi}}(\eta)\| \le M_2h^n,\end{gathered} \tag{42}$$

$$\max_{\xi\in\omega^+}\|A(\xi)\| \le M_3, \qquad \max_{\xi\in\omega^+}\sum_{\eta=0}^{\xi-h} h\|D(\eta))\| \le M_3, \qquad \max_{\xi\in\omega^+}\sum_{\eta=0}^{\xi-h} h\|\vec{\varphi}(\eta)\| \le M_3, \tag{43}$$

where M_2 and M_3 are positive constants independent of h. Then the error $\vec{z}(x)=\vec{u}(x)-\vec{y}(x)$ has the estimate

$$\max_{x\in\omega}\|\vec{z}(x)\| + \max_{x\in\omega^+}\|\vec{z}_{\bar{x}}(x)\| \le Mh^n, \tag{44}$$

where $M>0$ is a constant independent of $h,\ h\in(0;h_0)$, and $h_0>0$ is a fixed number.

Proof. We first establish estimates for the quantities $\max_{x\in\omega}\|\vec{y}(x)\|$ and $\max_{x\in\omega^+}\|\vec{y}_{\bar{x}}(x)\|$ in inequalities (39) and (40). Substituting $\vec{u}(x)=\vec{z}(x)+\vec{y}(x)$ in the EDS (20), we get the following difference problem:

$$\begin{gathered}\Lambda\vec{y}\equiv(A\vec{y}_{\bar{x}})_x-D\vec{y}=-(A\vec{z}_{\bar{x}})_x+D\vec{z}-\vec{\varphi}(x),\quad x\in\omega,\\ \vec{y}(0)=0,\quad \vec{y}(1)=\vec{0}.\end{gathered}\tag{45}$$

By analogy with the proof of Theorem 1, by using conditions (43), we get the following estimates:

$$\begin{aligned}\max_{x\in\omega}\|\vec{y}(x)\|&\le M_1\left\{\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\|\sum_{\xi\in\omega^+}h\|A(\xi)\|\right.\\ &\qquad\left.+\max_{\eta\in\omega}\|\vec{z}(\eta)\|\sum_{\xi\in\omega^+}h\sum_{\eta=0}^{\xi-h}h\|D(\eta)\|+\sum_{\xi\in\omega^+}h\sum_{\eta=0}^{\xi-h}h\|\vec{\varphi}(\eta)\|\right\}\\ &\le M_1M_3\left[\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\|+\max_{\eta\in\omega}\|\vec{z}(\eta)\|+1\right],\end{aligned}\tag{46}$$

$$\begin{aligned}\max_{x\in\omega^+}\|\vec{y}_{\bar{x}}(x)\|&\le 3M_1\left\{\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\|\max_{\xi\in\omega^+}\|A(\xi)\|\right.\\ &\qquad\left.+\max_{\eta\in\omega}\|\vec{z}(\eta)\|\max_{\xi\in\omega^+}\sum_{\eta=0}^{\xi-h}h\|D(\eta)\|+\max_{\xi\in\omega^+}\sum_{\eta=0}^{\xi-h}h\|\vec{\varphi}(\eta)\|\right\}\\ &\le 3M_1M_3\left[\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\|+\max_{\eta\in\omega}\|\vec{z}(\eta)\|+1\right].\end{aligned}\tag{47}$$

In view of inequalities (46) and (47), with the help of condition (42), we transform estimates (39) and (40) as follows:

$$\begin{aligned}\max_{x\in\omega}\|\vec{z}(x)\|&\le M_1\left\{3M_1M_3\left[\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\|+\max_{\eta\in\omega}\|\vec{z}(\eta)||+1\right]M_2h^n\right.\\ &\qquad\left.+M_1M_3\left[\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\|+\max_{\eta\in\omega}\|\vec{z}(\eta)||+1\right]M_2h^n+M_2h^n\right\}\\ &=4M_1^2M_2M_3\left[\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\|+\max_{\eta\in\omega}\|\vec{z}(\eta)||\right]h^n+M_1M_2(4M_1M_3+1)h^n,\end{aligned}$$

$$\begin{aligned}\max_{x\in\omega^+}\|\vec{z}_{\bar{x}}(x)\|&\le 3M_1\left\{M_1M_3\left[\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\|+\max_{\eta\in\omega}\|\vec{z}(\eta)||+1\right]M_2h^n\right.\\ &\qquad\left.+3M_1M_3\left[\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\|+\max_{\eta\in\omega}\|\vec{z}(\eta)||+1\right]M_2h^n+M_2h^n\right\}\end{aligned}$$

$$= 12M_1^2M_2M_3\left[\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\| + \max_{\eta\in\omega}\|\vec{z}(\eta)\|\right]h^n + 3M_1M_2(4M_1M_3+1)h^n.$$

Finding the sum of these inequalities, we get

$$\max_{x\in\omega}\|\vec{z}(x)\| + \max_{x\in\omega^+}\|\vec{z}_{\bar{x}}(x)\|$$

$$\leq 16M_1^2M_2M_3\left[\max_{\xi\in\omega^+}\|\vec{z}_{\bar{\xi}}(\xi)\| + \max_{\eta\in\omega}\|\vec{z}(\eta)\|\right]h^n + 4M_1M_2(4M_1M_3+1)h^n.$$

If $h \in (0; h_0)$, where $h_0 = \left(32M_1^2M_2M_3\right)^{-1/n}$, then we find

$$\max_{x\in\omega}\|\vec{z}(x)\| + \max_{x\in\omega^+}\|\vec{z}_{\bar{x}}(x)\| \leq 8M_1M_2(4M_1M_3+1)h^n.$$

Denoting

$$M = 8M_1M_2(4M_1M_3+1),$$

we obtain estimate (44).

Theorem 3 is proved.

5. Algorithmic Realization of the EDS

According to the main idea indicated in the introduction, it is necessary to express the coefficients A_i and D_i and the right-hand side $\vec{\varphi}_i$ of the EDS (20) solely via the solutions of the Cauchy problems. Note that this can be done without the assumption of symmetry for the template matrix functions $V_\alpha^{(i)}(x)$. For A_i, the required representation is obvious:

$$A_i = A(x_i) = \left[\frac{1}{h}V_1^{(i)}(x_i)\right]^{-1}. \tag{48}$$

We now express the matrix coefficient D_i via the solutions of the Cauchy problems. Integrating the equation

$$\frac{d}{dx}\left(P(x)\frac{dV_1^{(i)}}{dx}\right) - Q(x)V_1^{(i)}(x) = 0$$

from $x = x_{i-1}$ to $x = x_i$, with regard for the initial condition, we get

$$\int\limits_{x_{i-1}}^{x_i} Q(x)V_1^{(i)}(x)dx = \left(P(x)\frac{dV_1^{(i)}}{dx}\right)\Bigg|_{x_{i-1}}^{x_i} = P(x_i)\frac{dV_1^{(i)}(x_i)}{dx} - E.$$

Thus, we can write

$$\int\limits_{x_{i-1}}^{x_i} V_1^{(i)^*}(x)Q(x)dx = \frac{dV_1^{(i)^*}(x_i)}{dx}P(x_i) - E. \tag{49}$$

Similarly, we find

$$\int_{x_i}^{x_{i+1}} V_2^{(i)^*}(x)Q(x)dx = -E - \frac{dV_2^{(i)^*}(x_i)}{dx}P(x_i). \tag{50}$$

Hence, D_i takes the form

$$D_i = \frac{1}{h}\left[V_1^{(i)^*}(x_i)\right]^{-1}\left\{\frac{dV_1^{(i)^*}(x_i)}{dx}P(x_i) - E\right\} + \frac{1}{h}\left[V_2^{(i)^*}(x_i)\right]^{-1}\left\{-\frac{dV_2^{(i)^*}(x_i)}{dx}P(x_i) - E\right\},$$

i.e.,

$$D_i = \frac{1}{h}\left[V_1^{(i)^*}(x_i)\right]^{-1}\left[M_1^{(i)}(x_i) - E\right] - \frac{1}{h}\left[V_2^{(i)^*}(x_i)\right]^{-1}\left[M_2^{(i)}(x_i) + E\right], \tag{51}$$

where

$$M_\alpha^{(i)}(x) = \frac{dV_\alpha^{(i)^*}(x)}{dx}P(x), \quad \alpha = 1, 2. \tag{52}$$

We now consider $\vec{\varphi}_i$. We introduce two new auxiliary vector functions $\vec{w}_1^{(i)}(x)$ and $\vec{w}_2^{(i)}(x)$ as solutions of the Cauchy problems

$$L^{(P,Q)}\vec{w}_\alpha^{(i)}(x) = -\vec{f}(x), \quad x_{i-1} < x < x_{i+1}, \quad \alpha = 1, 2, \quad i = 1, 2, \ldots, N-1, \tag{53}$$

$$\vec{w}_1^{(i)}(x_{i-1}) = \frac{d\vec{w}_1^{(i)}}{dx}(x_{i-1}) = 0, \tag{54}$$

$$\vec{w}_2^{(i)}(x_{i+1}) = \frac{d\vec{w}_2^{(i)}}{dx}(x_{i+1}) = 0. \tag{55}$$

After simple transformations, we obtain

$$\int_{x_{i-1}}^{x_i} V_1^{(i)^*}(x)\vec{f}(x)dx = \frac{dV_1^{(i)^*}(x_i)}{dx}P(x_i)\vec{w}_1^{(i)}(x_i) - V_1^{(i)^*}(x_i)P(x_i)\frac{d\vec{w}_1^{(i)}(x_i)}{dx},$$

$$\int_{x_i}^{x_{i+1}} V_2^{(i)^*}(x)\vec{f}(x)dx = -\frac{dV_2^{(i)^*}(x_i)}{dx}P(x_i)\vec{w}_2^{(i)}(x_i) + V_2^{(i)^*}(x_i)P(x_i)\frac{d\vec{w}_2^{(i)}(x_i)}{dx}.$$

Hence, $\vec{\varphi}_i$ takes the following form:

$$\vec{\varphi}_i = \frac{1}{h}\left[V_1^{(i)^*}(x_i)\right]^{-1}\frac{dV_1^{(i)^*}(x_i)}{dx}P(x_i)\vec{w}_1^{(i)}(x_i) - \frac{1}{h}P(x_i)\frac{d\vec{w}_1^{(i)}(x_i)}{dx}$$

$$- \frac{1}{h}\left[V_2^{(i)^*}(x_i)\right]^{-1}\frac{dV_2^{(i)^*}(x_i)}{dx}P(x_i)\vec{w}_2^{(i)}(x_i) + \frac{1}{h}P(x_i)\frac{d\vec{w}_2^{(i)}(x_i)}{dx}.$$

Introducing the notation

$$\vec{l}_\alpha^{(i)}(x) = P(x)\frac{d\vec{w}_\alpha^{(i)}(x)}{dx}, \qquad \alpha = 1, 2,$$

we get

$$\vec{\varphi}_i = \frac{1}{h}\left[V_1^{(i)*}(x_i)\right]^{-1} M_1^{(i)}(x_i)\vec{w}_1^{(i)}(x_i) - \frac{1}{h}\vec{l}_\alpha^{(i)}(x_i)$$

$$- \frac{1}{h}\left[V_2^{(i)*}(x_i)\right]^{-1} M_2^{(i)}(x_i)\vec{w}_2^{(i)}(x_i) + \frac{1}{h}\vec{l}_\alpha^{(i)}(x_i). \tag{56}$$

Note that, in the scalar case, relations (48), (51), and (56) turn into similar formulas deduced in [8].

Thus, it follows from relations (48), (51), and (56) that, in order to determine the coefficients A_i and D_i and the right-hand side $\vec{\varphi}_i$ of the EDS (20) at each node $x_i \in \omega$, it is necessary to solve four Cauchy problems, namely, problem (3), (4) and problem (53), (54) on the segment $[x_{i-1}; x_i]$ (forward) and problem (3), (5) and problem (53), (55) on the segment $[x_i; x_{i+1}]$ (backward). For the approximate solution of each of these problems, one may apply any one-step method (the method of expansion in the Taylor formula, the Runge–Kutta method, etc.).

To simplify our presentation, we apply the method of expansion in the Taylor formula (see also [8], Lemmas 4 and 5). We mark the obtained approximations of the matrix functions $V_\alpha^{(i)}(x)$ and $M_\alpha^{(i)}(x)$ and the vector functions $\vec{w}_\alpha^{(i)}(x)$ and $\vec{l}_\alpha^{(i)}(x)$ by the superscripts; see, e.g.,

$$\overset{(n)}{V_\alpha^{(i)}}(x) = \sum_{k=0}^{n} \frac{1}{k!}\frac{d^k V_\alpha^{(i)}(x_{i+(-1)^\alpha})}{dx^k}(x - x_{i+(-1)^\alpha})^k, \quad \alpha = 1, 2.$$

Lemma 8. *Suppose that the matrices $P(x)$ and $Q(x)$ satisfy conditions (7) and (9), the elements of the matrix $P(x)$ belong to the class $C^{n+1}[0;1]$, and the elements of the matrix $Q(x)$ and the components of the vector $\vec{f}(x)$ belong to the class $C^n[0;1]$. Then the following relations are true:*

$$V_\alpha^{(i)}(x_i) = \overset{(n)}{V_\alpha^{(i)}}(x_i) + \overset{(n)}{R_\alpha^{(i)}}, \quad \textit{where} \quad \left\| \overset{(n)}{R_\alpha} \right\| = O(h^{n+1}), \tag{57}$$

$$M_\alpha^{(i)}(x_i) = \overset{(n+1)}{M_\alpha}{}^{(i)}(x_i) + \overset{(n+1)}{S_\alpha}{}^{(i)}, \quad \textit{where} \quad \left\| \overset{(n+1)}{S_\alpha}{}^{(i)} \right\| = O(h^{n+2}), \tag{58}$$

$$\vec{w}_\alpha^{(i)}(x_i) = \overset{(n+1)}{\vec{w}_\alpha}{}^{(i)}(x_i) + \overset{(n+1)}{\vec{b}_\alpha}{}^{(i)}, \quad \textit{where} \quad \left\| \overset{(n+1)}{\vec{b}_\alpha}{}^{(i)} \right\| = O(h^{n+2}), \tag{59}$$

$$\vec{l}_\alpha^{(i)}(x_i) = \overset{(n)}{\vec{l}_\alpha^{(i)}}(x_i) + \overset{(n)}{\vec{g}_\alpha^{(i)}}, \quad \textit{where} \quad \left\| \overset{(n)}{\vec{g}_\alpha^{(i)}} \right\| = O(h^{n+1}), \tag{60}$$

$$\alpha = 1, 2, \quad i = 1, 2, \dots, N-1.$$

Proof. It follows from the conditions of the lemma that the elements of the matrix function $V_\alpha^{(i)}(x)$ and the components of the vector function $\vec{w}_\alpha(x)$ belong to the class $C^{n+2}[0;1]$, while the components of the matrix function $M_\alpha^{(i)}(x)$ and the vector function $\vec{l}_\alpha(x),\ \alpha = 1, 2,$ belong to the class $C^{n+1}[0;1]$. Thus, relations (57), (59), and (60) follow from the Taylor-formula expansions at the points $x_{i+(-1)^\alpha}$ with remainder represented in the integral form.

We now prove relation (58). In view of (52) and the Leibniz formula for the derivative of the product of matrices, we get

$$M_\alpha^{(i)}(x_i) = \overset{(n)}{M}{}_\alpha^{(i)}(x_i) + \frac{((-1)^{\alpha+1}h)^{n+1}}{(n+1)!}\frac{d^{n+1}M_\alpha^{(i)}(\tilde{x}_\alpha)}{dx^{n+1}}$$

$$= \overset{(n)}{M}{}_\alpha^{(i)}(x_i) + \frac{\Big((-1)^{\alpha+1}h\Big)^{n+1}}{(n+1)!}\frac{d^n}{dx^n}\Big(V_\alpha^{(i)^*}(x)Q(x)\Big)(\tilde{x}_\alpha)$$

$$= \overset{(n)}{M}{}_\alpha^{(i)}(x_i) + \frac{\Big((-1)^{\alpha+1}h\Big)^{n+1}}{(n+1)!}\left\{V_\alpha^{(i)^*}(\tilde{x}_\alpha)Q^{(n)}(\tilde{x}_\alpha) + \sum_{k=1}^{n} C_n^k \frac{d^k V_\alpha^{(i)^*}(\tilde{x}_\alpha)}{dx^k} Q^{(n-k)}(\tilde{x}_\alpha)\right\},$$

where $\tilde{x}_\alpha \in (x_{i-2+\alpha}; x_{i-1+\alpha}),\ \alpha = 1, 2$. We transform the expression in braces in the following form:

$$V_\alpha^{(i)^*}(\tilde{x}_\alpha)Q^{(n)}(\tilde{x}_\alpha) = \int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} \frac{dV_\alpha^{(i)^*}(t)}{dt}dt\, Q^{(n)}(\tilde{x}_\alpha),$$

$$\frac{d^k V_\alpha^{(i)^*}(\tilde{x}_\alpha)}{dx^k} = \frac{d^k V_\alpha^{(i)^*}(x_{i+(-1)^\alpha})}{dx^k} + \int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} \frac{d^{k+1}V_\alpha^{(i)^*}(t)}{dt^{k+1}}dt,$$

$$Q^{(n-k)}(\tilde{x}_\alpha) = Q^{(n-k)}(x_{i+(-1)^\alpha}) + \int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} Q^{(n-k+1)}(t)dt.$$

This yields the representation

$$M_\alpha^{(i)}(x_i) = \overset{(n)}{M}{}_\alpha^{(i)}(x_i) + \frac{\Big((-1)^{\alpha+1}h\Big)^{n+1}}{(n+1)!}\sum_{k=1}^{n} C_n^k \frac{d^k V_\alpha^{(i)^*}(x_{i+(-1)^\alpha})}{dx^k} Q^{(n-k)}(x_{i+(-1)^\alpha})$$

$$+ \frac{\Big((-1)^{\alpha+1}h\Big)^{n+1}}{(n+1)!}\left\{\int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} \frac{dV_\alpha^{(i)^*}(t)}{dt}dt\, Q^{(n)}(\tilde{x}_\alpha)\right.$$

$$+ \sum_{k=1}^{n} C_n^k \frac{d^k V_\alpha^{(i)^*}(x_{i+(-1)^\alpha})}{dx^k}\int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} Q^{(n-k+1)}(t)dt$$

$$+\sum_{k=1}^{n} C_n^k \int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} \frac{d^{k+1}V_\alpha^{(i)^*}(t)}{dt^{k+1}} dt Q^{(n-k)}(x_{i+(-1)^\alpha})$$

$$\left.+\sum_{k=1}^{n} C_n^k \int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} \frac{d^{k+1}V_\alpha^{(i)^*}(t)}{dt^{k+1}} dt \int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} Q^{(n-k+1)}(t)dt\right\}$$

$$= \overset{(n)}{M}{}_\alpha^{(i)}(x_i) + \frac{\left((-1)^{\alpha+1}h\right)^{n+1}}{(n+1)!} \frac{d^{n+1}M_\alpha^{(i)}(x_{i+(-1)^\alpha})}{dx^{n+1}} + \overset{(n+1)}{S_\alpha}{}^{(i)}$$

$$= \overset{(n+1)}{M_\alpha}{}^{(i)}(x_i) + \overset{(n+1)}{S_\alpha}{}^{(i)},$$

where the remainder $\overset{(n+1)}{S_\alpha}{}^{(i)}$ admits the following estimate:

$$\left\|\overset{(n+1)}{S_\alpha}{}^{(i)}\right\| \le \frac{h^{n+1}}{(n+1)!}\left\{\int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} \left\|\frac{dV_\alpha^{(i)^*}(t)}{dt}\right\| dt \left\|Q^{(n)}(\tilde{x}_\alpha)\right\|\right.$$

$$+\sum_{k=1}^{n} C_n^k \left\|\frac{d^k V_\alpha^{(i)^*}(x_{i+(-1)^\alpha})}{dx^k}\right\| \int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} \left\|Q^{(n-k+1)}(t)\right\| dt$$

$$+\sum_{k=1}^{n} C_n^k \left\|\int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} \frac{d^{k+1}V_\alpha^{(i)^*}(t)}{dt^{k+1}} dt\right\| \left\|Q^{(n-k)}(x_{i+(-1)^\alpha})\right\|$$

$$\left.+\sum_{k=1}^{n} C_n^k \int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} \left\|\frac{d^{k+1}V_\alpha^{(i)^*}(t)}{dt^{k+1}}\right\| dt \int\limits_{x_{i+(-1)^\alpha}}^{\tilde{x}_\alpha} \left\|Q^{(n-k+1)}(t)\right\| dt\right\} = O(h^{n+2}).$$

This proves relation (58) and, hence, Lemma 8.

Lemma 9. *Suppose that conditions (7), (9), (19), and (29) are satisfied and*

$$\overset{(n)}{A}(x_i) = \left[\frac{1}{h}\overset{(n)}{V}{}_\alpha^{(i)}(x_i)\right]^{-1}, \tag{61}$$

$$\overset{(n)}{D}(x_i) = \frac{1}{h}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[\overset{(n)}{V}{}_\alpha^{(i)}(x_i)\right]^{-1}\left[\overset{(n+1)}{M_\alpha}{}^{(i)}(x_i) + (-1)^\alpha E\right], \tag{62}$$

$$\overset{(n)}{\vec{\varphi}}_i(x_i) = \frac{1}{h}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left\{\left[\overset{(n)}{V}{}_\alpha^{(i)}(x_i)\right]^{-1}\overset{(n+1)}{M}{}_\alpha^{(i)}(x_i)\,\overset{(n+1)}{\vec{w}}{}_\alpha^{(i)}(x_i) - \overset{(n)}{\vec{l}}{}_\alpha^{(i)}\right\}, \tag{63}$$

$$\alpha = 1, 2, \quad i = 1, 2, \ldots, N-1.$$

Then the relations

$$\left\| \overset{(n)}{A}(x_i) - A(x_i)\right\| = O(h^n), \tag{64}$$

$$\left\| \overset{(n)}{D}(x_i) - D(x_i)\right\| = O(h^n), \tag{65}$$

$$\left\| \overset{(n)}{\vec{\varphi}}_i(x_i) - \vec{\varphi}_i(x_i)\right\| = O(h^n), \tag{66}$$

$$i = 1, 2, \ldots, N-1,$$

are true.

Proof. Equality (64) follows from relations (21), (61), and (57) and the estimate [16, p. 157]

$$\left\| \overset{(n)}{A}(x_i) - A(x_i)\right\| = \left\| \left[\frac{1}{h}\overset{(n)}{V}{}_\alpha^{(i)}(x_i)\right]^{-1} - \left[\frac{1}{h}V_\alpha^{(i)}(x_i)\right]^{-1}\right\|$$

$$\leq \frac{\left\|\frac{1}{h}\overset{(n)}{V}{}_\alpha^{(i)}(x_i) - \frac{1}{h}V_\alpha^{(i)}(x_i)\right\| \left\|\left[\frac{1}{h}V_\alpha^{(i)}(x_i)\right]^{-1}\right\|^2}{1 - \left\|\frac{1}{h}\overset{(n)}{V}{}_\alpha^{(i)}(x_i) - \frac{1}{h}V_\alpha^{(i)}(x_i)\right\| \left\|\left[\frac{1}{h}V_\alpha^{(i)}(x_i)\right]^{-1}\right\|} = \frac{O(h^n)O(1)}{1 - O(h^n)O(1)} = O(h^n).$$

By using relations (51), (62), (58), and (64) and the representation

$$\overset{(n)}{D}(x_i) - D_i = \frac{1}{h}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[\overset{(n)}{V}{}_\alpha^{(i)}(x_i)\right]^{-1}\left[\overset{(n+1)}{M}{}_\alpha^{(i)}(x_i) + (-1)^\alpha E\right]$$

$$- \frac{1}{h}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[V_\alpha^{(i)}(x_i)\right]^{-1}\left[M_\alpha^{(i)}(x_i) + (-1)^\alpha E\right]$$

$$= \frac{1}{h^2}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left\{\left[\frac{1}{h}V_\alpha^{(i)}(x_i)\right]^{-1}\left[\overset{(n+1)}{M}{}_\alpha^{(i)}(x_i) - M_\alpha^{(i)}(x_i)\right]\right.$$

$$\left. + \left(\left[\frac{1}{h}\overset{(n)}{V}{}_\alpha^{(i)}(x_i)\right]^{-1} - \left[\frac{1}{h}V_\alpha^{(i)}(x_i)\right]^{-1}\right)\left[\overset{(n+1)}{M}{}_\alpha^{(i)}(x_i) + (-1)^\alpha E\right]\right\},$$

we obtain equality (65), whence it follows that

$$\left\| \overset{(n)}{D}(x_i) - D_i \right\| \leq \frac{1}{h^2} \sum_{\alpha=1}^{2} \left\{ \left\| \left[\frac{1}{h} V_\alpha^{(i)}(x_i) \right]^{-1} \right\| \left\| \overset{(n+1)}{M_\alpha^{(i)}}(x_i) - M_\alpha^{(i)}(x_i) \right\| \right.$$

$$\left. + \left\| \left[\frac{1}{h} \overset{(n)}{V_\alpha^{(i)}}(x_i) \right]^{-1} - \left[\frac{1}{h} V_\alpha^{(i)}(x_i) \right]^{-1} \right\| \left\| \overset{(n+1)}{M_\alpha^{(i)}}(x_i) + (-1)^\alpha E \right\| \right\}$$

$$= \frac{1}{h^2} \sum_{\alpha=1}^{2} \left[O(1) O(h^{n+2}) + O(h^n) O(h^2) \right] = O(h^n).$$

Similarly, by using relations (56), (63), (58)–(60), and (64), we arrive at relation (66).

As a consequence of the established statements, we get the following result:

Theorem 4. *Suppose that the conditions of Lemma 9 are satisfied. Then, for sufficiently small h, the error*

$$\vec{z}(x) = \vec{u}(x) - \vec{y}(x), \quad x \in \bar{\omega},$$

of scheme (37) with the following coefficients and right-hand side:

$$\tilde{A}(x_i) = \overset{(n)}{A}(x_i), \qquad \tilde{D}(x_i) = \overset{(n)}{D}(x_i), \qquad \tilde{\vec{\varphi}}(x_i) = \overset{(n)}{\vec{\varphi}_i}(x_i)$$

given by (61)–(63), satisfies estimate (44).

Proof. The proof of the theorem directly follows from Lemma 9 and Theorem 3.

Remark. By using the method proposed in [17], we can show that for even n, the accuracy of the difference scheme (37) with the coefficient

$$\overset{(n)}{D}(x_i) = \frac{1}{h} \sum_{\alpha=1}^{2} (-1)^{\alpha+1} \left[\overset{(n)}{V_\alpha^{(i)}}(x_i) \right]^{-1} \left[\overset{(n)}{M_\alpha^{(i)}}(x_i) + (-1)^\alpha E \right]$$

instead of (62) and the right-hand side

$$\overset{(n)}{\vec{\varphi}_i}(x_i) = \frac{1}{h} \sum_{\alpha=1}^{2} (-1)^{\alpha+1} \left\{ \left[\overset{(n)}{V_\alpha^{(i)}}(x_i) \right]^{-1} \overset{(n)}{M_\alpha^{(i)}}(x_i) \overset{(n)}{\vec{w}_\alpha^{(i)}}(x_i) - \overset{(n)}{\vec{l}_\alpha^{(i)}} \right\}$$

instead of (63) is also a quantity of the order $O(h^n)$.

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