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VALIRON-TYPE AND VALIRON–TITCHMARSH-TYPE THEOREMS FOR SUBHARMONIC FUNCTIONS OF SLOW GROWTH

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Let u be a subharmonic function of order zero in $\mathbb{R}^m,\ m \geq 2,$ with Riesz measure μ on the negative semiaxis $Ox_1,\ n(r,u) = \mu\big(\{x \in \mathbb{R}^m : |x| \leq r\}\big),\ d_m = m-2$ for $m \geq 3,\ d_2 = 1,$ and $N(r,u) = d_m \displaystyle\int_1^r \frac{n(t,u)}{t^{m-1}}\, dt.$ Under the condition of slow growth of $N(r,u),$ we determine the asymptotics of $u(x)$ as $|x| = r \to +\infty.$ We also study the inverse relationship between the regular growth of u and the behavior of $N(r,u)$ as $r \to +\infty.$

1. Introduction

Let f be an entire transcendental (in what follows, entire) function, let $n(r) = n(r,0,f)$ be the number of zeros of f in the disk $\{z : |z| \leq r\},$ and let ρ be the order of $f.$

If the zeros of f are negative, ρ is a noninteger number, $0 < \Delta < +\infty,$ and

$$n(r) \sim \Delta r^{\rho}, \quad r \to +\infty, \tag{1}$$

then, by the Valiron results [1],

$$\ln\big|f\big(re^{i\theta}\big)\big| \sim \frac{\pi\Delta}{\sin \pi\rho}\, r^{\rho} \cos\rho\theta, \quad |\theta| < \pi, \quad r \to +\infty.$$

Conversely, if f has only negative zeros, ρ is a noninteger number, $0 < \Delta < +\infty,$ and

$$\ln|f(r)| \sim \frac{\pi\Delta}{\sin \pi\rho}\, r^{\rho}, \quad r \to +\infty,$$

then relation (1) is true.

The simplest proof of the last assertion was proposed by Titchmarsh [2]. For this reason, theorems specifying the relationships between the regular behaviors of $n(r)$ and $\ln|f(z)|$ are called Valiron-type and Valiron–Titchmarsh-type theorems.

Similar problems for entire functions of order zero were studied in [3], where, in particular, it was proved that the asymptotic equalities

$$n(r) = r^{\lambda(r)} + o\big(r^{\lambda(r)}\big) \quad \text{and} \quad n(r) = r^{\lambda(r)} + o\big(\varepsilon(r) r^{\lambda(r)}\big), \quad r \to +\infty,$$

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are, respectively, necessary and sufficient conditions for the relation

$$\ln|f(r)| = \int_1^r t^{\lambda(t)-1}dt + o\left(r^{\lambda(r)}\right), \quad r \to +\infty,$$

to be true. Here, f is an entire function of order zero with negative roots, $\lambda(r)$ is the zero proximate order (see, e.g., [4, p. 69]), $r^{\lambda(r)} \nearrow +\infty$, and $\varepsilon(r) = \lambda(r) + r\lambda'(r)\ln r \to 0$ as $r \to +\infty$.

Later, the Valiron-type theorem and the Valiron–Titchmarsh-type theorem were proved in [5] and [6], respectively, for functions of noninteger order subharmonic in $\mathbb{R}^n$, $n \geq 3$.

In the present paper, we analyze the relationship between the regular behaviors of a function u of order zero subharmonic in $\mathbb{R}^m$, $m \geq 2$, and a Nevanlinna counting function $N(r,u)$ of its Riesz measure in the case where $N(r,u)$ is a slowly increasing function.

2. Definitions and Statement of the Results

Assume that u is a function subharmonic in $\mathbb{R}^m$, $m \geq 2$, u-harmonic in a unit neighborhood of the point O, $u(0) = 0$, μ is its Riesz measure,

$$n(t,u) = \mu(\{x\colon |x| \leq t\}), \qquad d_m = m-2 \quad \text{for} \quad m \geq 3, \qquad d_2 = 1,$$

$$N(t,u) = \int_1^r n(\tau)/\tau^{m-1}d\tau, \qquad u^+(x) = \max\{u(x);0\},$$

$c_m = 2\pi^{m/2}/\Gamma(m/2)$ is the surface area of the unit sphere $\{x \in \mathbb{R}^m\colon |x| = 1\}$, $d\sigma(x)$ is an element of the surface area of the sphere $S(0,r) = \{x\colon |x| = r\}$, and

$$T(r,u) = \frac{1}{c_m r^{m-1}} \int_{S(0,r)} u^+(x)\, d\sigma(x)$$

is the Nevanlinna characteristic of the function u. We say that u is a function of zero kind (zero order) if

$$T(r,u) = o(r) \quad (\ln T(r,u) = o(\ln r)) \quad \text{as} \quad r \to +\infty.$$

By $SH_m(0)$ we denote a class of functions of order zero subharmonic in $\mathbb{R}^m$. Further, by $SH_m^-(0)$ we denote a subclass of functions u from $SH_m(0)$ such that u are harmonic functions beyond the negative semiaxis Ox_1.

A nonnegative nondecreasing unbounded function on $[0;+\infty)$ is called a comparison function. By L we denote the set of continuously differentiable comparison functions v such that $tv'(t)/v(t) \to 0$ as $t \to +\infty$. It is easy to see that $r^{\lambda(r)} \in L$ if $\lambda(r)$ is the same zero proximate order as above. Without loss of generality, we can assume that $v(t) = 0$ on $[0,\delta]$, $0 < \delta < 1$. It is also easy to see that, with an accuracy to within equivalent functions, L coincides with the set of slowly increasing functions l, i.e., positive nondecreasing functions on $[0,+\infty)$ such that $l(2t) \sim l(t)$, $t \to +\infty$ (see, e.g., [7, p. 15]).

For $v \in L$, we set

$$v_1(r) = \int_1^r \frac{v(t)}{t}\, dt. \tag{2}$$

It is clear that $v_1 \in L$ and $v(r) = o(v_1(r))$ as $r \to +\infty$. We set

$$P(t, r, \theta; \alpha) = \left(t^2 + 2tr\cos\theta + r^2\right)^{-\alpha}, \quad \alpha > 0, \tag{3}$$

$$J(r, \theta; \alpha, \psi(t)) = \int_0^{+\infty} \psi(t) P(t, r, \theta; \alpha) dt, \tag{4}$$

$$J(r, \theta; \alpha, \beta) = J\left(r, \theta; \alpha, t^\beta\right), \quad 0 < \beta < 2\alpha - 1, \tag{5}$$

where ψ is a function locally integrable on $[0; +\infty)$ such that the integral in equality (4) converges.

For $m \geq 3$, we set

$$A(m) = \sum_{k=1}^{m-2} C_{m-1}^k I_{m-1}(m - 2 - k),$$

where

$$I_n(k) = \int_1^{+\infty} \frac{t^k dt}{(t+1)^n}, \quad n \in \mathbb{N}, \quad n \geq 2, \quad k = 0, 1, \ldots, n-2.$$

Clearly, the following recurrence relation is true:

$$I_n(k) = \frac{1}{2^{n-1}(n - 1 - k)} + kI_n(k-1), \qquad I_n(0) = \frac{1}{(n-1)2^{n-1}}.$$

Theorem 1. *Suppose that* $m \geq 3, \ u \in SH_m^-(0), \ v \in L,$

$$r = |x|, \quad x_1 = r\cos\theta, \quad \textit{and} \quad x = (r\cos\theta, x_2, \ldots, x_m).$$

If

$$N(t, u) = (1 + o(1))v(t), \quad t \to +\infty, \tag{6}$$

then

$$u(x) = \left(mJ\left(1, \theta; \frac{m+2}{2}, m-1\right)\sin^2\theta + (m-1)J\left(1, \theta; \frac{m}{2}, m-2\right)\cos\theta\right)v(r) + o(v(r)), \qquad r \to +\infty, \tag{7}$$

for $|\theta| < \pi$. *Moreover, (7) holds uniformly in* θ *on the set* $\{\theta\colon |\theta| < \pi - \delta\}$, $0 < \delta < 1$.

Remark 1. The integrals in (7) are convergent because the integrands

$$t^{m-1}P(t, 1, \theta; (m+2)/2) \sim \frac{1}{t^3} \qquad \text{and} \qquad t^{m-2}P(t, 1, \theta; m/2) \sim \frac{1}{t^2}$$

as $t \to +\infty$.

Theorem 2. *Suppose that* $u \in SH_m^-(0),\ m \geq 3,\ v \in L$, *and* $u(r) = u(r, 0, \ldots, 0)$.

(A) If

$$N(t, u) = v_1(t) + o(v(t)), \quad t \to +\infty, \tag{8}$$

then

$$u(r) = v_1(r) + A(m)v(r) + o(v(r)), \quad r \to +\infty. \tag{9}$$

(B) Conversely, if (9) is true, then

$$N(t, u) = (1 + o(1))v_1(t), \quad t \to +\infty. \tag{10}$$

Remark 2. Since

$$(m-1)J\left(1, 0; \frac{m}{2}, m-2\right) = (m-1)\int\limits_0^{+\infty} \frac{t^{m-2}}{(t+1)^m}\,dt = (m-1)B(m-1, 1) = 1,$$

for $\theta = 0$, under the condition that $N(t, u) = (1 + o(1))v_1(t)$ as $t \to +\infty$, it follows from relation (7) that

$$u(r) = u(r, 0, \ldots, 0) = (1 + o(1))v_1(r), \quad r \to +\infty,$$

which is weaker than (9).

Remark 3. In fact, we prove that (9) yields the following asymptotic equality:

$$n(t, u) = \frac{1 + o(1)}{m-2} t^{m-2} v(t), \quad t \to +\infty, \tag{11}$$

which leads to (10). The inverse implication is not true, i.e., (10) does not imply (11).

Theorem 3. *Suppose that* $u \in SH_2^-(0),\ v \in L,\ x = (r\cos\theta, r\sin\theta)$, *and*

$$n(r, u) = (1 + o(1))v(t), \quad t \to +\infty.$$

Then

$$u(x) = N(r, u) + o(v(r)), \quad r \to +\infty \tag{12}$$

for $|\theta| < \pi$. *In addition, the last asymptotic equality holds uniformly in* θ *on the set* $\{\theta\colon |\theta| < \pi - \delta\},\ 0 < \delta < 1$.

Theorem 4. *Suppose that* $u \in SH_2^-(0),\ v \in L$, *and* $u(r) = u(r, 0)$.

(A) If

$$n(t, u) = v(t) + o(tv'(t)), \quad t \to +\infty,$$

then

$$u(r) = v_1(r) + o(v(r)), \quad r \to +\infty. \tag{13}$$

(B) Conversely, if (13) is true, then

$$n(t, u) = (1 + o(1))v(t), \quad t \to +\infty.$$

The following theorem generalizes assertion (A) of Theorem 4:

Theorem 5. *Suppose that* $u \in SH_2^-(0),\ v \in L,$

$$x = (r\cos\theta, r\sin\theta), \quad \textit{and} \quad n(t,u) = v_1(t) + o(tv'(t)), \quad t \to +\infty.$$

Then, for $|\theta| < \pi,$

$$u(x) = N(r,u) + \frac{1}{2}\left(\frac{\pi^2}{3} - \theta^2\right)v(r) + o(v(r)), \quad r \to +\infty. \tag{14}$$

3. Auxiliary Results

In the proofs of the theorems, we use the following statements:

Lemma 1. *Suppose that* $v \in L,\ g$ *is a differentiable function,* g' *is a nonincreasing function on* $[1, +\infty),$ *and* $K \in \mathbb{R}.$ *If*

$$g(t) = \int_1^t \frac{v(\tau)}{\tau}\, d\tau + Kv(t) + o(v(t)), \quad t \to +\infty,$$

then

$$g'(t) = \frac{v(t)}{t} + o\left(\frac{v(t)}{t}\right), \quad t \to +\infty.$$

The proof of this lemma is similar to the proof of Lemma 4 in [3].

Lemma 2 [7, pp. 63–65]**.** *Suppose that* l *is a slowly varying function,* ϕ *is a locally integrable function on* $[0, +\infty),\ a > 0,$ *and for some* $\eta > 0,$ *the integral*

$$\int_a^{+\infty} t^{\eta}\phi(t)\, dt \qquad \left(\int_0^a t^{-\eta}\phi(t)dt\right)$$

is convergent. Then

$$\int_a^{+\infty} \phi(t)l(xt)\, dt \sim l(x)\int_a^{+\infty}\phi(t)dt \qquad \left(\int_0^a \phi(t)l(xt)dt \sim l(x)\int_0^a \phi(t)dt\right), \quad x \to +\infty.$$

Lemma 3. *Let $p \geq 3$ and $v \in L$. Then*

$$I(r) = \int\limits_0^{+\infty} \frac{d\big(t^{p-2}v(t)\big)}{(t+r)^{p-1}} = (1+o(1))\,\frac{v(r)}{r}, \quad r \to +\infty.$$

Proof. By using the assertion of Lemma 2 with $\eta = \dfrac{1}{2}$ and integrating by parts, we get

$$I(r) = (p-1)\int\limits_0^{+\infty} \frac{t^{p-2}v(t)dt}{(t+r)^p} = \frac{p-1}{r}\int\limits_0^{+\infty} v(\tau r)\frac{\tau^{p-2}d\tau}{(1+\tau)^p}$$

$$= (1+o(1))\,\frac{(p-1)v(r)}{r}B(p-1,1)$$

$$= (1+o(1))\frac{(p-1)v(r)}{r}\,\frac{\Gamma(p-1)\Gamma(1)}{\Gamma(p)}$$

$$= (1+o(1))\frac{v(r)}{r}, \quad r \to +\infty.$$

Lemma 4 [8]. *Suppose that $0 < b < a+1,\ k = [b],\ \alpha$ and β are functions positive and nondecreasing on $[0,+\infty),\ \alpha$ is a differentiable function, $\alpha(x) \to +\infty$ as $x \to +\infty,\ a\alpha(x) < x\alpha'(x) < \beta\alpha(r)$ for $x \geq x_0$, and*

$$F(x) = \int\limits_1^{+\infty} \frac{d\alpha(t)}{(x+t)^{k+1}}, \qquad G(x) = \int\limits_1^{+\infty} \frac{d\beta(t)}{(x+t)^{k+1}}.$$

If $F(x) \sim G(x)$, then $\alpha(x) \sim \beta(x)$ as $x \to +\infty$.

Lemma 5. *Suppose that $v \in L$ and $\varepsilon(r)$ is a function locally integrable on $[1,+\infty)$ and such that $\varepsilon(r) \to 0$ as $r \to +\infty$. Then*

(i) for $\alpha < 1$, $$\int\limits_1^{r} \frac{\varepsilon(t)v(t)dt}{t^\alpha} = o\left(\frac{v(r)}{r^{\alpha-1}}\right), \quad r \to +\infty;$$

(ii) for $\alpha > 1$, $$\int\limits_r^{+\infty} \frac{\varepsilon(t)v(t)dt}{t^\alpha} = o\left(\frac{v(r)}{r^{\alpha-1}}\right), \quad r \to +\infty.$$

By using the L'Hospital rule, we can easily prove the assertions of this lemma.

Lemma 6. *Suppose that $m \geq 3$ and u is a function of zero kind subharmonic in $\mathbb{R}^m$ and harmonic outside the negative semiaxis $Ox_1,\ r = |x|,\ x = (r\cos\theta, x_2, \ldots, x_m),\ |\theta| < \pi$. Then*

$$u(x) = mr^2 J\left(r,\theta;\frac{m+2}{2},t^{m-1}N(t)\right)\sin^2\theta + (m-1)rJ\left(r,\theta;\frac{m}{2},t^{m-2}N(t)\right)\cos\theta. \tag{15}$$

Proof. Under the conditions of the lemma, we have (see, e.g., [9, p. 174])

$$u(x) = \int\limits_{|\xi|<+\infty} \left(|\xi|^{2-m} - |x-\xi|^{2-m}\right) d\mu_\xi, \quad \xi \in \mathbb{R}^m,$$

where μ is the Riesz measure of u. If u is harmonic in $\mathbb{R}^m$ everywhere except the negative semiaxis $Ox_1,\ t = |\xi|$, and $j = (-1, 0, \dots, 0)$ is an m-dimensional vector, then [see (3)]

$$\begin{aligned} u(x) &= \int\limits_0^{+\infty} \left(t^{2-m} - |x+jt|^{2-m}\right) dn(t) \\ &= \int\limits_0^{+\infty} \left(t^{2-m} - P\left(t, r, \theta; \frac{m-2}{2}\right)\right) dn(t) \\ &= (m-2) \int\limits_0^{+\infty} \left(t^{1-m} - (t + r\cos\theta) P\left(t, r, \theta; \frac{m}{2}\right)\right) n(t) dt \\ &= \int\limits_0^{+\infty} \left(1 - t^{m-1}(t + r\cos\theta) P\left(t, r, \theta; \frac{m}{2}\right)\right) dN(t) \end{aligned} \tag{16}$$

because

$$n(0) = 0, \quad n(t) = \frac{t^{m-1}}{m-2} \frac{d}{dt} N(t),$$

$$\frac{P\left(t, r, \theta; \dfrac{2-m}{2}\right) - t^{m-2}}{t^{m-2} P\left(t, r, \theta; \dfrac{2-m}{2}\right)} = (1 + o(1)) \frac{(m-2) r \cos\theta}{t^{m-1}},$$

$$n(t) = o\left(t^{m-1}\right) \quad \text{as} \quad t \to +\infty.$$

Since

$$N(0) = 0, \qquad 1 - t^{m-1}(t + r\cos\theta) P\left(t, r, \theta; \frac{m}{2}\right) = (1 + o(1)) \frac{(m-1) r \cos\theta}{t},$$

$$N(t) = o(t) \quad \text{as} \quad t \to +\infty,$$

as a result of integration by parts, it follows from (16) that

$$u(x) = m r^2 \sin^2\theta \int\limits_0^{+\infty} t^{m-1} N(t) P\left(t, r, \theta; \frac{m+2}{2}\right) dt$$

$$+ (m-1) r \cos\theta \int_0^{+\infty} t^{m-2} N(t) P\left(t, r, \theta; \frac{m}{2}\right) dt,$$

i.e., we get (15) [see (4) and (5)].

4. Proofs of the Results

Proof of Theorem 1. By the change of variables $\tau = rt,$ we get the following relation from (15):

$$u(x) = m \sin^2\theta \int_0^{+\infty} \tau^{m-1} N(r\tau) P\left(\tau, 1, \theta; \frac{m+2}{2}\right) d\tau$$

$$+ (m-1)\cos\theta \int_0^{+\infty} \tau^{m-2} N(r\tau) P\left(\tau, 1, \theta; \frac{m}{2}\right) d\tau. \tag{17}$$

If $N(t) = v(t) + \varepsilon(t)v(t),$ where $\varepsilon(t) \to 0$ as $t \to +\infty,$ then, by using Lemma 2 with $\eta = 1/2$ and relations (15) and (17), as $r \to +\infty,$ we obtain [see (4) and (5)]:

$$u(x) = \left(mJ\left(1, \theta; \frac{m+2}{2}, m-1\right)\sin^2\theta + (m-1)J\left(1, \theta; \frac{m}{2}, m-2\right)\cos\theta\right) v(r)$$

$$+ mr^2 J\left(r, \theta; \frac{m+2}{2}; t^{m-1}\varepsilon(t)v(t)\right)\sin^2\theta + (m-1)rJ\left(r, \theta; \frac{m}{2}; t^{m-2}\varepsilon(t)v(t)\right)\cos\theta$$

$$+ o(v(r)). \tag{18}$$

Since [4, p. 92]

$$\left|t^2 + 2tr\cos\theta + r^2\right| = \left|t + re^{i\theta}\right|^2 \geq (t+r)^2 \sin^2\delta$$

for $|\theta| \leq \pi - \delta,\ 0 < \delta < 1,$ by virtue of Lemma 5 and the estimate

$$r^2 \sin^{m+2}\delta \left| J\left(r, \theta; \frac{m+2}{2}, t^{m-1}\varepsilon(t)v(t)\right)\right| \leq \frac{1}{r^m} \int_1^r \frac{\varepsilon(t)v(t)}{t^{1-m}}\, dt + r^2 \int_r^{+\infty} \frac{\varepsilon(t)v(t)}{t^3}\, dt,$$

we get

$$mr^2 J\left(r, \theta; \frac{m+2}{2}, t^{m-1}\varepsilon(t)v(t)\right)\sin^2\theta = o(v(r)), \quad r \to +\infty.$$

Similarly, we can show that

$$(m-1)rJ\left(r, \theta; \frac{m}{2}, t^{m-2}\varepsilon(t)v(t)\right)\cos\theta = o(v(r)), \quad r \to +\infty.$$

By using this result and relation (18), we obtain (7).

Theorem 1 is proved.

Proof of Theorem 2. If $N(t) = v_1(t) + \varepsilon(t)v(t)$, where $\varepsilon(t) \to 0$ as $t \to +\infty$, then, as above, for $\theta = 0$, we obtain

$$u(r) = (m-1)rJ\left(r, 0; \frac{m}{2}, t^{m-2}v_1(t)\right) + o(v(r))$$

$$= (m-1)r\left\{\int\limits_0^r \frac{t^{m-2}v_1(t)}{(t+r)^m}\,dt + \int\limits_r^{+\infty} \frac{t^{m-2}v_1(t)}{(t+r)^m}dt\right\} + o(v(r)), \quad r \to +\infty. \tag{19}$$

Since, for $t < r$ and $t > r$, we get

$$\frac{1}{(t+r)^m} = \frac{1}{(m-1)!}\sum_{n=0}^{+\infty} \frac{(-1)^n(m+n-1)!}{n!r^{m+n}}\,t^n,$$

$$\frac{1}{(t+r)^m} = \frac{1}{(m-1)!}\sum_{n=0}^{+\infty} \frac{(-1)^n(m+n-1)!}{n!r^{-n}}\,t^{-m-n},$$

respectively, it follows from relation (19) that

$$u(r) = \frac{1}{(m-2)!}\left(\sum_{n=0}^{+\infty} \frac{(-1)^n(m+n-1)!}{n!r^{m+n-1}}\,a_n(r;v_1)\right.$$
$$\left. + \sum_{n=0}^{+\infty} \frac{(-1)^n(m+n-1)!}{n!r^{-n-1}}\,b_n(r;v_1)\right) + o(v(r)), \quad r \to +\infty,$$

where

$$a_n(r;v_1) = \int\limits_0^r t^{m+n-2}v_1(t)dt,$$

$$b_n(r;v_1) = \int\limits_r^{+\infty} t^{-n-2}v_1(t)dt.$$

The possibility of term-by-term integration of the series is substantiated as in [3]. Further, $(rv_1'(r) = v(r))$,

$$a_n(r;v_1) = \frac{r^{m+n-1}}{m+n-1}v_1(r) - \frac{1}{m+n-1}\int\limits_0^r t^{m+n-2}tv_1'(t)dt$$

$$= \frac{r^{m+n-1}}{m+n-1}\,v_1(r) - \frac{1}{m+n-1}\,a_n(r;v),$$

and, similarly,

$$b_n(r;v_1) = \frac{r^{-n-1}}{n+1}\,v_1(r) + \frac{1}{n+1}\,b_n(r;v).$$

Hence,

$$u(r) = \frac{1}{(m-2)!}\left(\sum_{n=0}^{+\infty}\left(\frac{(-1)^n(m+n-2)!}{n!} + \frac{(-1)^n(m+n-1)!}{(n+1)!}\right)v_1(r)\right.$$

$$\left. - \sum_{n=0}^{+\infty}\frac{(-1)^n(m+n-2)!}{n!r^{m+n-1}}a_n(r;v) + \sum_{n=0}^{+\infty}\frac{(-1)^n(m+n-1)!}{(n+1)!r^{-n-1}}\,b_n(r;v)\right) + o(v(r))$$

$$= v_1(r) - \frac{1}{(m-2)!}\int_0^r \sum_{n=0}^{+\infty}\frac{(-1)^n(m+n-2)!}{n!r^{m+n-1}}\,t^{m+n-2}v(t)dt$$

$$+ \frac{1}{(m-2)!}\int_r^{+\infty}\sum_{n=0}^{+\infty}\frac{(-1)^n(m+n-1)!}{(n+1)!r^{-n-1}}\,t^{-n-2}v(t)dt + o(v(r))$$

$$= v_1(r) - \int_0^r \frac{t^{m-2}v(t)}{(t+r)^{m-1}}\,dt$$

$$+ \frac{1}{(m-2)!}\int_r^{+\infty} t^{m-2}\sum_{n=1}^{+\infty}\frac{(-1)^n(m+n-2)!}{n!r^{-n}}\,t^{-m-n+1}v(t)dt + o(v(r))$$

$$= v_1(r) - \int_0^r \frac{t^{m-2}v(t)}{(t+r)^{m-1}}\,dt + \int_r^{+\infty}\left(\frac{1}{t} - \frac{t^{m-2}}{(t+r)^{m-1}}\right)v(t)dt + o(v(r)), \quad r\to+\infty.$$

By the change of variables $t = r\tau,$ in view of Lemma 2 with $\eta = 1/2,$ from the last equality, we obtain

$$u(r) = v_1(r) - \int_0^1 \frac{\tau^{m-2}v(r\tau)}{(\tau+1)^{m-1}}\,d\tau + \int_1^{+\infty}\frac{(\tau+1)^{m-1} - \tau^{m-1}}{\tau(\tau+1)^{m-1}}\,v(r\tau)d\tau + o(v(r))$$

$$= v_1(r) - v(r)\int_0^1 \frac{\tau^{m-2}d\tau}{(\tau+1)^{m-1}} + \sum_{k=1}^{m-1}C_{m-1}^k\int_1^{+\infty}\frac{\tau^{m-1-k}v(r\tau)}{\tau(\tau+1)^{m-1}}\,d\tau + o(v(r))$$

$$= v_1(r) + A(m)v(r) + o(v(r)), \quad r\to+\infty,$$

where

$$A(m) = \sum_{k=1}^{m-2}C_{m-1}^k\int_1^{+\infty}\frac{t^{m-2-k}}{(t+1)^{m-1}}\,dt,$$

which proves assertion (A) in Theorem 2.

We now prove assertion (B). In view of (16), the function

$$u'(r) = (m-2) \int_0^{+\infty} \frac{dn(t)}{(t+r)^{m-1}}$$

does not increase on $[0, +\infty)$. Hence, by virtue of Lemma 1, it follows from relation (9) that

$$u'(r) = \frac{v(r)}{r} + o\left(\frac{v(r)}{r}\right), \quad r \to +\infty.$$

In view of Lemma 3, the last two relations imply that

$$(m-2) \int_0^{+\infty} \frac{dn(t)}{(t+r)^{m-1}} = (1+o(1)) \int_0^{+\infty} \frac{d(t^{m-2}v(t))}{(t+r)^{m-1}}, \quad r \to +\infty.$$

Since

$$t\left(t^{m-2}v(t)\right)' = (1+o(1))(m-2)t^{m-2}v(t) \quad \text{as} \quad t \to +\infty,$$

we find

$$\left((m-2) - \frac{1}{4}\right) t^{m-2}v(t) < t\left(t^{m-2}v(t)\right)' < \left((m-2) + \frac{1}{4}\right) t^{m-2}v(t), \quad t \geq 2.$$

Further, by Lemma 5 with

$$k = \left[(m-2) + \frac{1}{4}\right] = m-2,$$

we get

$$n(t,u) = \frac{1+o(1)}{m-2}\, t^{m-2}v(t), \quad t \to +\infty.$$

By using this fact and the definition of $N(t,u)$, we obtain relation (10), which completes the proof of Theorem 2.

Proof of Theorems 3–5. For $m = 2$, instead of $x = (x_1, x_2) = (|x|\cos\theta, |x|\sin\theta)$, we write $z = re^{i\theta}$, $-\pi \leq \theta < \pi$. Further, if $u \in SH_2^-(0)$, then (see, e.g., [9, p. 174]) we get $(\xi = (-t, 0), |\xi| = t)$

$$u(z) = \int_0^{+\infty} \ln\left|\frac{\xi - z}{\xi}\right| d\mu_\xi = \int_0^{+\infty} \ln\left|1 + \frac{z}{t}\right| dn(t) = Re \int_0^{+\infty} \ln\left(1 + \frac{z}{t}\right) dn(t). \tag{20}$$

If $n(t,u) = (1+o(1))v(t)$ or $n(t,u) = v_1(t) + o(v(t))$ as $t \to +\infty$, then, as in the proof of Theorem 1 in [3], for $-\pi < \theta < \pi$, we obtain (12) and, hence,

$$\int_0^{+\infty} \ln\left(1 + \frac{z}{t}\right) dn(t) = v_1(r) + i\theta v(r) + o(v(r)), \quad r \to +\infty.$$

By using the last asymptotic equality and relation (20) for $\theta = 0$, we obtain (13). Together with (12), this proves Theorem 3 and Assertion (A) in Theorem 4.

Assume that (13) is true. It follows from relation (20) with $\theta = 0$ that the function

$$u'(r) = \int_0^{+\infty} \frac{1}{t+r}\, dn(t)$$

does not increase on $[0, +\infty)$ and, in view of (13) and Lemma 1, we find

$$u'(r) = \frac{v(r)}{r} + o\left(\frac{v(r)}{r}\right), \quad r \to +\infty.$$

Since [3] (Lemma 4)

$$\int_0^{+\infty} \frac{dv(t)}{t+r} = (1+o(1))\frac{v(r)}{r}, \quad r \to +\infty,$$

the last relations imply that

$$\int_0^{+\infty} \frac{dn(t)}{t+r} = (1+o(1)) \int_0^{+\infty} \frac{dv(t)}{t+r}, \quad r \to +\infty.$$

By virtue of Lemma 4 with $k = 0$, we get

$$n(t,u) = (1+o(1))v(t), \quad t \to +\infty,$$

because

$$tv'(t) = o(v(t)) \quad \text{as} \quad t \to +\infty$$

and, hence,

$$tv'(t) < \frac{1}{2}\, v(t).$$

Assertion (B) of Theorem 4 is thus proved.

Assume that the conditions of Theorem 5 are satisfied. Then, as in the proof of Theorem 1 in [10], we get

$$\int_0^{+\infty} \ln\left(1 + \frac{z}{t}\right) dn(t) = N(r,u) + i\theta v_1(r) + \frac{1}{2}\left(\frac{\pi^2}{3} - \theta^2\right) v(r) + o(v(r)), \quad r \to +\infty.$$

This result, together with (20), yields relation (14), which completes the proof of Theorem 5.

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