DOI 10.1007/s11253-023-02156-z
Ukrainian Mathematical Journal, Vol. 74, No. 10, March, 2023 (Ukrainian Original Vol. 74, No. 10, October, 2022)

GENERALIZATIONS OF STARLIKE HARMONIC FUNCTIONS DEFINED BY SĂLĂGEAN AND RUSCHEWEYH DERIVATIVES

Á. O. Páll-Szabo UDC 517.5

We investigate some generalizations of the classes of harmonic functions defined by the Sălăgean and Ruscheweyh derivatives. By using the extreme-points theory, we obtain the coefficient-estimates distortion theorems and mean integral inequalities for these classes of functions.

1. Preliminaries

Let $\mathcal{A}$ denote a class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in an open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $\mathcal{G}$ if both u and v are real and harmonic in $\mathcal{G}$. In any simply connected domain $D \subset \mathcal{G}$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We say that h is the analytic part and g is the coanalytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]).

Let $\mathcal{H}$ denote the family of continuous complex-valued functions that are harmonic in U. By $S_{\mathcal{H}}$ we denote the family of functions $f \in \mathcal{H}$ of the form

$$f = h + \overline{g}, \quad h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=2}^{\infty} b_k z^k, \tag{2}$$

which are univalent and orientation preserving in the open unit disc U. Thus, $f(z)$ is given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=2}^{\infty} b_k z^k}. \tag{3}$$

A function f of the form (3) is said to be in $\mathcal{S}^*_{\mathcal{H}}(\alpha)$ if and only if (see [2, 4, 5])

$$\frac{\partial}{\partial \theta}\left(\arg f\left(re^{i\theta}\right)\right) > \alpha, \quad 0 \le \theta < 2\pi, \quad |z| = r < 1, \quad 0 \le \alpha < 1. \tag{4}$$

Babeş-Bolyai University, Cluj-Napoca, Romania; e-mail: pallszaboagnes@math.ubbcluj.ro, agnes.pallszabo@econ.ubbcluj.ro.

Published in Ukrains'kyi Matematychnyi Zhurnal, Vol. 74, No. 10, pp. 1388–1400, October, 2022. Ukrainian DOI: 10.37863/umzh.v74i10.6157. Original article submitted June 7, 2020.

 0041-5995/23/7410–1584

Similarly, a function f of the form (3) is said to be in $\mathcal{S}_{\mathcal{H}}^{c}(\alpha)$ if and only if

$$\frac{\partial}{\partial\theta}\left(\arg\frac{\partial}{\partial\theta}\left(f\left(re^{i\theta}\right)\right)\right)>\alpha,\quad 0\leq\theta<2\pi,\quad |z|=r<1. \tag{5}$$

We note that (see [7]) a harmonic function $f\in\mathcal{S}_{\mathcal{H}}^{*}(\alpha)$ if and only if

$$\Re\frac{J_{\mathcal{H}}f(z)}{f(z)}>\alpha,\quad |z|=r<1,\quad \text{where}\quad J_{\mathcal{H}}f(z)=zh'(z)-\overline{zg'(z)}.$$

Definition 1 [1]. *For $f\in\mathcal{A}$, $\lambda\geq 0$, and $n\in\mathbb{N}$, the operator $\mathscr{D}_{\lambda}^{n}$, $\mathscr{D}_{\lambda}^{n}:\mathcal{A}\to\mathcal{A}$, is defined as follows:*

$$\mathscr{D}_{\lambda}^{0}f(z)=f(z),$$

$$\mathscr{D}_{\lambda}^{n+1}f(z)=(1-\lambda)\mathscr{D}_{\lambda}^{n}f(z)+\lambda z\left(\mathscr{D}_{\lambda}^{n}f(z)\right)'=\mathscr{D}_{\lambda}\left(\mathscr{D}_{\lambda}^{n}f(z)\right),\quad z\in U.$$

Remark 1. If $f\in\mathcal{A}$, then

$$\mathscr{D}_{\lambda}^{n}f(z)=z+\sum_{k=2}^{\infty}\left[1+(k-1)\lambda\right]^{n}a_{k}z^{k},\quad z\in U.$$

Remark 2. For $\lambda=1$ in Definition 1, we get the Sălăgean differential operator [13].

Definition 2 [12]. *For $f\in\mathcal{A}$, $n\in\mathbb{N}$, the operator $\mathscr{R}^{n}$, $\mathscr{R}^{n}:\mathcal{A}\to\mathcal{A}$, is defined as follows:*

$$\mathscr{R}^{0}f(z)=f(z),$$

$$(n+1)\mathscr{R}^{n+1}f(z)=z\left(\mathscr{R}^{n}f(z)\right)'+n\mathscr{R}^{n}f(z),\quad z\in U.$$

Remark 3. If $f\in\mathcal{A}$, then

$$\mathscr{R}^{n}f(z)=z+\sum_{k=2}^{\infty}\frac{(n+k-1)!}{n!(k-1)!}a_{k}z^{k},\quad z\in U,$$

which is the Ruscheweyh differential operator [12].

Definition 3. *Let $\gamma,\lambda\geq 0$ and $n\in\mathbb{N}$. By $\mathscr{L}^{n}$ we denote the operator given by $\mathscr{L}^{n}:\mathcal{A}\to\mathcal{A}$,*

$$\mathscr{L}^{n}f(z)=(1-\gamma)\mathscr{R}^{n}f(z)+\gamma\mathscr{D}_{\lambda}^{n}f(z),\quad z\in U.$$

Remark 4. If $f\in\mathcal{A}$, then

$$\mathscr{L}^{n}f(z)=z+\sum_{k=2}^{\infty}\left\{\gamma\left[1+(k-1)\lambda\right]^{n}+(1-\gamma)\frac{(n+k-1)!}{n!(k-1)!}\right\}a_{k}z^{k},\quad z\in U.$$

We consider a linear operator $\mathscr{L}_{\mathcal{H}}^n\colon \mathcal{H} \to \mathcal{H}$ defined for a function $f = h + \overline{g} \in \mathcal{H}$ by

$$\mathscr{L}_{\mathcal{H}}^n f := \mathscr{L}^n h + (-1)^n \overline{\mathscr{L}^n g}.$$

For a function $f \in \mathcal{H}$ of the form (3), we have

$$\mathscr{L}_{\mathcal{H}}^n f(z) = z + \sum_{k=2}^{\infty} [\gamma\eta(k,n,\lambda) + (1-\gamma)\mu(k,n)]\, a_k z^k$$

$$+ (-1)^n \sum_{k=2}^{\infty} [\gamma\eta(k,n,\lambda) + (1-\gamma)\mu(k,n)]\, \overline{b_k}\overline{z}^k, \quad z \in U,$$

where

$$\eta(k,n,\lambda) = [1 + (k-1)\lambda]^n \qquad \text{and} \qquad \mu(k,n) = \frac{(n+k-1)!}{n!(k-1)!}.$$

Definition 4. *For $-B \le A < B \le 1$ and $n \in \mathbb{N}$, by $\widetilde{\mathcal{S}}_{\mathcal{H}}^n(A,B)$ we denote the class of functions $f \in \mathcal{H}$ of the form (3) such that*

$$\left| \frac{\mathscr{L}_{\mathcal{H}}^{n+1} f(z) - \mathscr{L}_{\mathcal{H}}^n f(z)}{B\mathscr{L}_{\mathcal{H}}^{n+1} f(z) - A\mathscr{L}_{\mathcal{H}}^n f(z)} \right| < 1, \quad z \in U. \tag{6}$$

Remark 5. Dziok, et al. studied the case $\gamma = 0$ in [3], while the case where $\gamma = 1$ and $\lambda = 1$ was studied in [4].

Note that the classes $\widetilde{\mathcal{S}}_{\mathcal{H}}^0(A,B)$ for the analytic case, i.e., $g \equiv 0$, were introduced by Janowski [8]. Jahangiri [6, 7] and Silverman [14] studied the classes $\mathcal{S}_{\mathcal{H}}^*(\alpha) = \widetilde{\mathcal{S}}_{\mathcal{H}}^0(2\alpha - 1, 1)$ and $\mathcal{S}_{\mathcal{H}}^c(\alpha) = \widetilde{\mathcal{S}}_{\mathcal{H}}^1(2\alpha - 1, 1)$ for the harmonic case.

2. Coefficient Estimates

Theorem 1. *A function $f \in \mathcal{H}$ of the form (3) belongs to the class $\widetilde{\mathcal{S}}_{\mathcal{H}}^n(A,B)$ if it satisfies the condition*

$$\sum_{k=2}^{\infty} (\alpha_k |a_k| + \beta_k |b_k|) \le B - A, \tag{7}$$

where

$$\alpha_k = \sigma(A,B,n,\gamma,\lambda,k) + \sigma(1,1,n,\gamma,\lambda,k),$$

$$\beta_k = \delta(A,B,n,\gamma,\lambda,k) + \delta(1,1,n,\gamma,\lambda,k),$$

$$\sigma(A,B,n,\gamma,\lambda,k) = \gamma\eta(k,n,\lambda)[(k-1)\lambda B + B - A]$$

$$+ (1-\gamma)\mu(k,n)\frac{(B-A)n + Bk - A}{n+1},$$

$$\delta(A,B,n,\gamma,\lambda,k)=\gamma\eta(k,n,\lambda)[(k-1)\lambda B+B+A] + (1-\gamma)\mu(k,n)\frac{(B+A)n+Bk+A}{n+1}.$$

Proof. It follows from Definition 4 that $f\in\widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$ if and only if

$$\left|\frac{\mathscr{L}^{n+1}_{\mathcal{H}}f(z)-\mathscr{L}^{n}_{\mathcal{H}}f(z)}{B\mathscr{L}^{n+1}_{\mathcal{H}}f(z)-A\mathscr{L}^{n}_{\mathcal{H}}f(z)}\right|<1,\quad z\in U.$$

It is sufficient to prove that

$$\left|\mathscr{L}^{n+1}_{\mathcal{H}}f(z)-\mathscr{L}^{n}_{\mathcal{H}}f(z)\right|-\left|B\mathscr{L}^{n+1}_{\mathcal{H}}f(z)-A\mathscr{L}^{n}_{\mathcal{H}}f(z)\right|<0,\quad z\in U\setminus\{0\}.$$

Letting $|z|=r,\ 0<r<1,$ we get

$$\begin{aligned}
&\left|\mathscr{L}^{n+1}_{\mathcal{H}}f(z)-\mathscr{L}^{n}_{\mathcal{H}}f(z)\right|-\left|B\mathscr{L}^{n+1}_{\mathcal{H}}f(z)-A\mathscr{L}^{n}_{\mathcal{H}}f(z)\right|\\
&\le\sum_{k=2}^{\infty}\left[\gamma\eta(k,n,\lambda)(k-1)\lambda+(1-\gamma)\mu(k,n)\,\frac{k-1}{n+1}\right]|a_k|r^k\\
&\quad+\sum_{k=2}^{\infty}\left[\gamma\eta(k,n,\lambda)\,[2+(k-1)\lambda]+(1-\gamma)\mu(k,n)\frac{2n+k+1}{n+1}\right]|b_k|r^k-(B-A)r\\
&\quad+\sum_{k=2}^{\infty}\left[\gamma\eta\,(k,n,\lambda)\,[(k-1)\lambda B+B-A]+(1-\gamma)\mu(k,n)\left(B\frac{n+k}{n+1}-A\right)\right]|a_k|r^k\\
&\quad+\sum_{k=2}^{\infty}\left[\gamma\eta\,(k,n,\lambda)\,[(k-1)\,\lambda B+B+A]+(1-\gamma)\,\mu\,(k,n)\left(B\frac{n+k}{n+1}+A\right)\right]|b_k|\,r^k\\
&\le r\left\{\sum_{k=2}^{\infty}(\alpha_k|a_k|+\beta_k|b_k|)r^{k-1}-(B-A)\right\}<0,
\end{aligned}$$

whence $f\in\widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$.

Theorem 1 is proved.

Lemma 1. *If* $\lambda\ge1,\ \gamma\in[0,1],\ n\ge0,\ -B\le A<B\le1,\ k\in\mathbb{N},\ k\ge2,$ *then*

$$\alpha_k\ge k(B-A),\quad \beta_k\ge k(B-A),$$

where α_k *and* β_k *are defined in (7).*

Proof. It is known that

$$\eta(k,n,\lambda)=[1+(k-1)\lambda]^n\ge k^n. \tag{8}$$

First, we prove that

$$\mu(k,n)=\frac{(n+k-1)!}{n!(k-1)!}\geq n+1. \tag{9}$$

For the proof, we proceed by induction.

1. Let $k\geq 2$ be fixed and $n=0$. Then

$$\mu(k,0)=\frac{(k-1)!}{0!(k-1)!}=1.$$

Let $k\geq 2$ be fixed and $n=1$. Then we get

$$\mu(k,1)=\frac{k!}{1!(k-1)!}\geq 2 \quad\Leftrightarrow\quad k!\geq 2(k-1)! \quad\Leftrightarrow\quad k\geq 2.$$

2. Assume that the following formula holds for $n=l$:

$$\mu(k,l)=\frac{(l+k-1)!}{l!(k-1)!}\geq l+1 \quad\Leftrightarrow\quad (l+k-1)!\geq l!(k-1)!(l+1)=(l+1)!(k-1)!\,.$$

3. Let $n=l+1$. Thus, it is necessary to prove that

$$\mu(k,l+1)=\frac{(l+k)!}{(l+1)!(k-1)!}\geq l+2\Leftrightarrow (l+k)!\geq (l+1)!(k-1)!(l+2).$$

This is true, in view of the previous item:

$$(l+k)!=(l+k)(l+k-1)!\geq (l+k)(l+1)!(k-1)!\geq (l+2)(l+1)!(k-1)!.$$

By using (8) and (9), we now prove that $\alpha_k\geq k(B-A)$:

$$\begin{aligned}\alpha_k&=\sigma\left(A,B,n,\gamma,\lambda,k\right)+\sigma\left(1,1,n,\gamma,\lambda,k\right)\\&\geq \gamma k^n[(k-1)\lambda B+B-A]\\&\quad+(1-\gamma)[(B-A)n+Bk-A]+\gamma k^n(k-1)\lambda+(1-\gamma)(k-1).\end{aligned}$$

However,

$$\begin{aligned}&k^n[(k-1)\lambda B+B-A]+k^n(k-1)\lambda\\&\qquad=k^n[(B-A)+\underbrace{(k-1)\lambda(B+1)}_{>0}]>k^n(B-A)>k(B-A)\end{aligned}$$

and

$$\begin{aligned}&(B-A)n+Bk-A+(k-1)\\&\qquad\geq B(k-1)+B-A+k-1=(k-1)(B+1)+B-A\\&\qquad\geq (k-1)(B-A)+B-A=k(B-A).\end{aligned}$$

Hence,

$$\alpha_k \geq \gamma(B-A)k+(1-\gamma)(B-A)k=k(B-A).$$

We now prove that $\beta_k \geq k(B-A)$:

$$\begin{aligned}\beta_k &= \delta\left(A,B,n,\gamma,\lambda,k\right)+\delta\left(1,1,n,\gamma,\lambda,k\right)\\&\geq \gamma k^n[(k-1)\lambda B+B+A]+(1-\gamma)[(B+A)n+Bk+A]\\&\qquad+\gamma k^n[(k-1)\lambda+2]+(1-\gamma)[2n+k+1]\\&> \gamma k^n[(k-1)(B+1)+B+A+2]\\&\qquad+(1-\gamma)[(B+A)n+2n+Bk+k+A+1].\end{aligned}$$

But

$$(k-1)(B+1)+B+A+2=kB+k+1+A\geq k(B-A),\quad B\geq -1,\quad A\geq -1,$$

$$k+1+A\geq -kA \;\;\Leftrightarrow\;\; k(A+1)+A+1\geq 0 \;\;\Leftrightarrow\;\; (k+1)(A+1)\geq 0$$

and

$$(B+A)n+2n+Bk+k+A+1\geq Bk+k+A+1\geq Bk-Ak,$$

because

$$k+A+1\geq -Ak \;\;\Leftrightarrow\;\; k(A+1)+A+1\geq 0 \;\;\Leftrightarrow\;\; (k+1)(A+1)\geq 0.$$

Therefore,

$$\beta_k \geq \gamma(B-A)k+(1-\gamma)(B-A)k=k(B-A).$$

Lemma 1 is proved.

Lemma 2. *If* $\lambda\geq 1,\ \gamma>1,\ n\geq 0,\ -B\leq A<B\leq 1,\ k\in\mathbb{N},\ k\geq 2,$ *then*

$$\alpha_k\geq k(B-A),\quad \beta_k\geq k(B-A),$$

where α_k *and* β_k *is defined in (7).*

Proof. First, we note that

$$\mu(k,n) = \frac{(n+k-1)!}{n!(k-1)!} \le k^n, \quad k,n \in \mathbb{N}, \quad k \ge 2. \tag{10}$$

Let k be fixed. If $n = 0$ then (10) holds.

Suppose that (10) is true for n. Then, for $n+1$, we obtain

$$(n+k)! = (n+k)(n+k-1)! \le (n+k)k^n n!(k-1)!$$
$$\le (n+1)kk^n n!(k-1)! = k^n(n+1)!(k-1)!.$$

Thus,

$$\alpha_k \ge \gamma k^n[(k-1)(B+1) + B - A] - (\gamma-1)k^n \frac{(B-A)n + Bk - A}{n+1}$$

by virtue of (8) and (10).

However,

$$\frac{(B-A)n + Bk - A + k - 1}{n+1} < (B-A) + (k-1)(B+1)$$

and, hence,

$$\alpha_k \ge [\gamma - (\gamma-1)][B - A + (k-1)(B+1)]k^n \ge k(B-A),$$

$$\beta_k \ge \gamma k^n[(k-1)(B+1) + B + A + 2]$$
$$+ (1-\gamma)k^n \frac{(B+A)n + 2n + Bk + k + A + 1}{n+1}$$
$$\ge k^n[(k-1)(B+1) + B + A + 2] \ge k(B-A)$$

because

$$(B+A)n + 2n + Bk + k + A + 1 < (n+1)[(k-1)(B+1) + B + A + 2].$$

Lemma 2 is proved.

Theorem 2. *If $f \in \mathcal{H}$ has the form (3) and f satisfies condition (7), then $f \in \mathcal{S}_{\mathcal{H}}$.*

Proof. The theorem is true for the function $f(z) \equiv z$. Let $f \in \mathcal{H}$ be a function of the form (3). Assume that there exists $k \in \{2, 3, \ldots\}$ such that $a_k \ne 0$ or $b_k \ne 0$. Since the inequalities $\dfrac{\alpha_k}{B-A} \ge k$ and $\dfrac{\beta_k}{B-A} \ge k$, $k = 2, 3, \ldots,$ have been proved in Lemmas 1 and 2, in view of (7), we get

$$\sum_{k=2}^{\infty} (k|a_k| + k|b_k|) \le 1 \tag{11}$$

and

$$\begin{aligned}\left|h'(z)\right| - \left|g'(z)\right| &\geq 1 - \sum_{k=2}^{\infty} k\,|a_k|\,|z|^k - \sum_{k=2}^{\infty} k|b_k||z|^k \\ &\geq 1 - |z| \sum_{k=2}^{\infty} (k|a_k| + k|b_k|) \\ &\geq 1 - \frac{|z|}{B-A} \sum_{k=2}^{\infty} (\alpha_k|a_k| + \beta_k|b_k|) \geq 1 - |z| > 0, \quad z \in U.\end{aligned}$$

In this case, the function f is locally univalent and sense-preserving in U. Moreover, if $z_1, z_2 \in U,\ z_1 \neq z_2$, then

$$\left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| = \left|\sum_{l=1}^{k} z_1^{l-1} z_2^{k-l}\right| \leq \sum_{l=1}^{k} |z_1|^{l-1} |z_2|^{k-1} < k, \quad k = 2, 3, \ldots.$$

Therefore, by virtue of (11), we have

$$\begin{aligned}|f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\ &\geq \left|z_1 - z_2 - \sum_{k=2}^{\infty} a_k \left(z_1^k - z_2^k\right)\right| - \left|\sum_{k=2}^{\infty} \overline{b_k \left(z_1^k - z_2^k\right)}\right| \\ &\geq |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} |a_k| \left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right| - \sum_{k=2}^{\infty} |b_k| \left|\frac{z_1^k - z_2^k}{z_1 - z_2}\right|\right) \\ &> |z_1 - z_2| \left(1 - \sum_{k=2}^{\infty} k\,|a_k| - \sum_{k=2}^{\infty} k\,|b_k|\right) \geq 0.\end{aligned}$$

This leads to the univalence of f and, hence, $f \in \mathcal{S}_{\mathcal{H}}$.

Theorem 2 is proved.

Let $\mathcal{N}$ denote a class of functions $f = h + \overline{g} \in \mathcal{H}$ of the form (see [14])

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^n \sum_{k=2}^{\infty} |b_k| \overline{z}^k, \tag{12}$$

and let $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A, B)$ denote the class $\mathcal{N} \cap \widetilde{\mathcal{S}}^n_{\mathcal{H}}(A, B)$.

Theorem 3. *Let $f = h + \overline{g}$ be defined by (12). Then $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A, B)$ if and only if condition (7) is satisfied.*

Proof. For the "if" part, see Theorem 1. For the "only if" part, we assume that $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A, B)$. Then, by (6), we get

$$\left|\frac{\sum_{k=2}^{\infty} \left[\sigma\left(1, 1, n, \gamma, \lambda, k\right) |a_k| z^{k-1} + \delta(1, 1, n, \gamma, \lambda, k) |b_k| \overline{z}^{k-1}\right]}{(B - A) - \sum_{k=2}^{\infty} \left[\sigma\left(A, B, n, \gamma, \lambda, k\right) |a_k|\, z^{k-1} + \delta\left(A, B, n, \gamma, \lambda, k\right) |b_k|\, \overline{z}^{k-1}\right]}\right| < 1, \quad z \in U.$$

For $z = r < 1,$ we obtain

$$\frac{\sum_{k=2}^{\infty}\left[\sigma\left(1,1,n,\gamma,\lambda,k\right)\left|a_k\right|+\delta\left(1,1,n,\gamma,\lambda,k\right)\left|b_k\right|\right]r^{k-1}}{(B-A)-\sum_{k=2}^{\infty}\left[\sigma\left(A,B,n,\gamma,\lambda,k\right)\left|a_k\right|+\delta\left(A,B,n,\gamma,\lambda,k\right)\left|b_k\right|\right]r^{k-1}}<1.$$

The denominator of the left-hand side cannot vanish for $r \in [0, 1)$ and, moreover, it is positive. Thus,

$$\sum_{k=2}^{\infty}(\alpha_k|a_k|+\beta_k|b_k|)r^{k-1}\le B-A.$$

Hence, letting $r \to 1^-,$ we arrive at assertion (7).

Theorem 3 is proved.

3. Extreme Points

Definition 5. *We say that a class $\mathcal{F}$ is convex if $\eta f + (1-\eta)g \in \mathcal{F}$ for all f and g in $\mathcal{F}$ and $0 \le \eta \le 1.$ The closed convex hull of $\mathcal{F}$ denoted by $\overline{\text{co}}\,\mathcal{F}$ is the intersection of all closed convex subsets of $\mathcal{H}$ (with respect to the topology of locally uniform convergence) that contain $\mathcal{F}$.*

Definition 6. *Let $\mathcal{F}$ be a convex set. A function $f \in \mathcal{F} \subset \mathcal{H}$ is called an extreme point of $\mathcal{F}$ if $f = \eta f_1 + (1-\eta) f_2$ implies that $f_1 = f_2 = f$ for all f_1 and f_2 in $\mathcal{F}$ and $0 < \eta < 1.$ We use the notation $E\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F} \subset \mathcal{F}$.*

For the extreme points, we use the Krein–Milman theorem (see [3, 4, 9]) which implies the following lemma:

Lemma 3 [3, 4]. *Let $\mathcal{F}$ be a nonempty compact convex subclass of the class $\mathcal{H}$ and let $\mathcal{J}\colon \mathcal{H} \to \mathbb{R}$ be a real-valued, continuous, and convex functional on $\mathcal{F}$. Then*

$$\max\left\{\mathcal{J}(f)\colon\ f\in\mathcal{F}\right\}=\max\left\{\mathcal{J}(f)\colon\ f\in E\mathcal{F}\right\}.$$

Since $\mathcal{H}$ is a complete metric space, we can use Montel's theorem [10].

Lemma 4 [3, 4]. *A class $\mathcal{F} \subset \mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.*

Theorem 4. *The class $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A, B)$ is a convex and compact subset of $\mathcal{H}$.*

Proof. For $0 \le \eta \le 1,$ let $f_1, f_2 \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A, B)$ be defined by (2). Then

$$\eta f_1(z)+(1-\eta)f_2(z)=z-\sum_{k=2}^{\infty}(\eta|a_{1,k}|+(1-\eta)|a_{2,k}|)z^k$$
$$+(-1)^n\sum_{k=2}^{\infty}(\eta|b_{1,k}|+(1-\eta)|b_{2,k}|\overline{z}^k)$$

and

$$\sum_{k=2}^{\infty}\left\{\alpha_k\left|\eta\left|a_{1,k}\right|+(1-\eta)|a_{2,k}|\right|+\beta_k\left|\eta\left|b_{1,k}\right|+(1-\eta)\left|b_{2,k}\right|z^k\right|\right\}$$

$$= \eta \sum_{k=2}^{\infty} \{\alpha_k |a_{1,k}| + \beta_k |b_{1,k}|\} + (1-\eta) \sum_{k=2}^{\infty} \alpha_k |a_{2,k}| + \beta_k |b_{2,k}|$$

$$\leq \eta(B-A) + (1-\eta)(B-A).$$

Therefore, the function $\phi = \eta f_1 + (1-\eta) f_2$ belongs to the class $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ and, hence, $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ is convex.

On the other hand, for $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$, $|z| \leq r$ and $0 < r < 1$, we obtain

$$|f(z)| \leq r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^n \leq r + \sum_{k=2}^{\infty} (\alpha_k |a_k| + \beta_k |b_k|) \leq r + (B-A).$$

This implies that $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ is locally uniformly bounded. Let

$$f_e(z) = z + \sum_{k=2}^{\infty} a_{e,k} z^k + \overline{\sum_{k=1}^{\infty} b_{e,k} z^k}, \quad z \in U, \quad k \in \mathbb{N},$$

and let $f \in \mathcal{H}$. By using Theorem 3, we get

$$\sum_{k=2}^{\infty} (\alpha_k |a_{e,k}| + \beta_k |b_{e,k}|) \leq B - A, \quad k \in \mathbb{N}.$$

If $f_e \to f$, then $|a_{e,k}| \to |a_k|$ and $|b_{e,k}| \to |b_k|$ as $k \to \infty$, $k \in \mathbb{N}$. This yields condition (7). Therefore, $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ and the class $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ is closed. By Lemma 3, we can now say that the class $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ is a compact subset of $\mathcal{H}$.

Theorem 4 is proved.

Theorem 5. *The set of extreme points of the class* $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ *is*

$$E\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B) = \{h_k : k \in \mathbb{N}\} \cup \{g_k : k \in \{2,3,\ldots\}\},$$

where

$$h_1 = z, \quad h_k(z) = z - \frac{B-A}{\alpha_k} z^k,$$

$$g_k(z) = z + (-1)^n \frac{B-A}{\beta_k} \overline{z}^k, \quad z \in U, \quad k \in \{2,3,\ldots\}. \tag{13}$$

Proof. If we use (7), then we can see that the functions of the indicated form are the extreme points of the class $\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$. Suppose that $f \in E\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ and f is not of the form indicated above. Thus, there exists $m \in \{2,3,\ldots\}$ such that

$$0 < |a_m| < \frac{B-A}{\alpha_m} \qquad \text{or} \qquad 0 < |b_m| < \frac{B-A}{\beta_m}.$$

If $0<|a_m|<\dfrac{B-A}{\alpha_m}$, then, setting

$$\gamma=\frac{|a_m|\,\alpha_m}{B-A} \qquad \text{and} \qquad \varphi=\frac{1}{1-\eta}\left(f-\eta h_m\right),$$

we obtain

$$0<\eta<1, \quad h_m,\varphi\in\widetilde{\mathcal{S}}^{*}_{\mathcal{HN}}(A,B), \quad h_m\neq\varphi, \quad \text{and} \quad f=\eta h_m+(1-\eta)\,\varphi.$$

Thus, $f\notin E\widetilde{\mathcal{S}}^{n}_{\mathcal{HN}}(A,B)$.

For $0<|b_m|<\dfrac{B-A}{\beta_m}$, we get the same result.

Theorem 5 is proved.

If the class $\mathcal{F}=\{f_k\in\mathcal{H}\colon\ k\in\mathbb{N}\}$ is locally uniformly bounded, then its closed convex hull is

$$\overline{\text{co}}\mathcal{F}=\left\{\sum_{k=1}^{\infty}\eta_k f_k\colon\ \sum_{k=1}^{\infty}\eta_k=1,\ \eta_k\geq 0,\ k\in\mathbb{N}\right\}.$$

Corollary 1. *Let h_k and g_k be defined by (13). Then*

$$\widetilde{\mathcal{S}}^{n}_{\mathcal{HN}}(A,B)=\left\{\sum_{k=1}^{\infty}\left(\eta_k h_k+\delta_k g_k\right)\ :\ \sum_{k=1}^{\infty}\left(\eta_k+\delta_k\right)=1,\ \delta_1=0,\ \eta_k,\delta_k\geq 0,\ k\in\mathbb{N}\right\}.$$

For each fixed value of $k\in\mathbb{N}$ and $z\in U$, the following real-valued functionals are continuous and convex on $\mathcal{H}$:

$$\mathcal{J}(f)=|a_k|, \quad \mathcal{J}(f)=|b_k|, \quad \mathcal{J}\,(f)=|f(z)|, \quad \mathcal{J}(f)=\left|\mathscr{L}^{k}_{\mathcal{H}}f(z)\right|, \quad f\in\mathcal{H}.$$

The real-valued functional

$$\mathcal{J}(f)=\left(\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\gamma}d\theta\right)^{1/\gamma}, \quad f\in\mathcal{H}, \quad \gamma\geq 1, \quad 0<r<1,$$

is continuous on $\mathcal{H}$. For $\gamma\geq 1$, it is also convex on $\mathcal{H}$ (Minkowski's inequality).

Corollary 2. *Let $f\in\widetilde{\mathcal{S}}^{n}_{\mathcal{HN}}(A,B)$ be a function of the form (12). Then*

$$|a_k|\leq\frac{B-A}{\alpha_k}, \quad |b_k|\leq\frac{B-A}{\beta_k}, \quad k=2,3,\ldots,$$

where α_k and β_k are defined by (7). The result is sharp. The extremal functions are h_k and g_k of the form (13).

Theorem 6. *Let* $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ *and* $|z| = r < 1$. *Then*

$$r - \frac{B-A}{\alpha_2} r^2 \le |f(z)| \le r + \frac{B-A}{\alpha_2} r^2,$$

$$r - \frac{(B-A)\left[\gamma(1+\lambda)^n + (1-\gamma)(n+1)\right]}{\alpha_2} r^2 \le |\mathscr{L}^n_{\mathcal{H}} f(z)| \le r + \frac{(B-A)\left[\gamma(1+\lambda)^n + (1-\gamma)(n+1)\right]}{\alpha_2} r^2.$$

The result is sharp. The extremal functions h_2 *have the form (13).*

Proof. We only prove the right inequality. The proof of the left inequality is similar and, hence, omitted. We have

$$\begin{aligned} |f(z)| &\le r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k \\ &\le r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^2 \\ &\le r + \left(\frac{1}{\alpha_2} \sum_{k=2}^{\infty} \alpha_2 |a_k| + \frac{1}{\beta_2} \sum_{k=2}^{\infty} \beta_2 |b_k| \right) r^2 \\ &\le r + \frac{1}{\alpha_2} \sum_{k=2}^{\infty} (\alpha_k |a_k| + \beta_k |b_k|) r^2 \le r + \frac{B-A}{\alpha_2} r^2, \end{aligned}$$

$$\alpha_2 \le \alpha_k, \quad \alpha_2 \le \beta_2, \quad \beta_2 \le \beta_k \quad \text{for all} \quad k \ge 2.$$

Another proof can be obtained by using Lemma 3 with extreme points.

Theorem 6 is proved.

Corollary 3. *If* $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$, *then* $U(r) \subset f(U(r))$, *where*

$$r = 1 - \frac{B-A}{\alpha_2} \qquad \text{and} \qquad U(r) := \{z \in \mathbb{C} \colon |z| < r \le 1\}.$$

Corollary 4. *Let* $0 < r < 1$ *and* $\xi \ge 1$. *If* $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$, *then*

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^{\xi} d\theta \le \frac{1}{2\pi} \int_0^{2\pi} \left| h_2\left(re^{i\theta}\right) \right|^{\xi} d\theta,$$

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \mathscr{L}^k_{\mathcal{H}} f\left(re^{i\theta}\right) \right|^{\xi} d\theta \le \frac{1}{2\pi} \int_0^{2\pi} \left| \mathscr{L}^k_{\mathcal{H}} h_2\left(re^{i\theta}\right) \right|^{\xi} d\theta, \quad \xi = 1, 2, \ldots.$$

4. Radii of Starlikeness and Convexity

We note that a harmonic function $f \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$ if and only if

$$\Re \frac{\mathscr{L}_{\mathcal{H}} f(z)}{f(z)} > \alpha, \quad |z| = r < 1,$$

where

$$\mathscr{L}_{\mathcal{H}} f(z) = zh'(z) - \overline{zg'(z)}.$$

For $0 \leq \alpha < 1,\ f \in \mathcal{S}^c_{\mathcal{H}}(\alpha)$ is equivalent to $\mathscr{L}_{\mathcal{H}} f(z) \in \mathcal{S}^*_{\mathcal{H}}(\alpha)$.

Let $\mathcal{B} \subseteq \mathcal{H}$. We now define the radius of starlikeness and the radius of convexity of the class $\mathcal{B}$:

$$R^*_\alpha(\mathcal{B}) := \inf_{f\in\mathcal{B}} \left(\sup\left\{r \in (0,1] : f \text{ is starlike of order } \alpha \in U(r)\right\}\right),$$

$$R^c_\alpha(\mathcal{B}) := \inf_{f\in\mathcal{B}} \left(\sup\left\{r \in (0,1] : f \text{ is convex of order } \alpha \in U(r)\right\}\right).$$

Theorem 7. *Let $0 \leq \alpha < 1$ and let α_k and β_k be defined by (7). Then*

$$R^*_\alpha\left(\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)\right) = \inf_{k\geq 2}\left(\frac{1-\alpha}{B-A}\min\left\{\frac{\alpha_k}{k-\alpha}, \frac{\beta_k}{k+\alpha}\right\}\right)^{\frac{1}{k-1}}.$$

Proof. Let $f \in \widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)$ be of the form (12).

We note that f is starlike of order α in $U(r)$ if and only if (see [7])

$$\sum_{k=2}^{\infty}\left(\frac{k-\alpha}{1-\alpha}|a_k| + \frac{k+\alpha}{1-\alpha}|b_k|\right) r^{k-1} \leq 1. \tag{14}$$

In addition, it follows from Theorem 3 that

$$\sum_{k=2}^{\infty}\left(\frac{\alpha_k}{B-A}|a_k| + \frac{\beta_k}{B-A}|b_k|\right) \leq 1.$$

Since $\alpha_k < \beta_k,\ k = 2, 3, \ldots,$ condition (14) is satisfied if

$$\frac{k-\alpha}{1-\alpha} r^{k-1} \leq \frac{\alpha_k}{B-A} \qquad \text{and} \qquad \frac{k+\alpha}{1-\alpha} r^{k-1} \leq \frac{\beta_k}{B-A}, \quad k = 2, 3, \ldots,$$

or

$$r \leq \left(\frac{1-\alpha}{B-A}\min\left\{\frac{\alpha_k}{k-\alpha}, \frac{\beta_k}{k+\alpha}\right\}\right)^{\frac{1}{k-1}}, \quad k = 2, 3, \ldots.$$

Hence, the function f is starlike of order α in the disk $U(r^*)$, where

$$r^* := \inf_{k\geq 2}\left(\frac{1-\alpha}{B-A}\min\left\{\frac{\alpha_k}{k-\alpha}, \frac{\beta_k}{k+\alpha}\right\}\right)^{\frac{1}{k-1}}.$$

It follows from the function

$$f_k = h_k(z) + \overline{g_k(z)} = z - \frac{B-A}{\alpha_k}\, z^k + (-1)^n \frac{B-A}{\beta_k}\, \overline{z}^k$$

that the radius r^* cannot be made larger.

Theorem 7 is proved.

Similarly, we get the following theorem:

Theorem 8. *Let* $0 \le \alpha < 1$ *and let* α_k *and* β_k *be defined by (7). Then*

$$R_\alpha^c \left(\widetilde{\mathcal{S}}^n_{\mathcal{HN}}(A,B)\right) = \inf_{k\ge 2} \left(\frac{1-\alpha}{B-A} \, \min\left\{ \frac{\alpha_k}{k\,(k-\alpha)}, \frac{\beta_k}{k(k+\alpha)} \right\} \right)^{\frac{1}{k-1}}.$$

We now examine the closure properties of the class $\widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$ under the generalized Bernardi–Libera–Livingston integral operator $\mathcal{L}_c(f),\ c > -1,$ which is defined by

$$\mathcal{L}_c(f) = \mathcal{L}_c(h) + \overline{\mathcal{L}_c(g)},$$

where

$$\mathcal{L}_c(h)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t)dt \qquad \text{and} \qquad \mathcal{L}_c(g)(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} g(t)dt.$$

Theorem 9. *Let* $f \in \widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$. *Then* $\mathcal{L}_c(f) \in \widetilde{\mathcal{S}}^n_{\mathcal{H}}(A,B)$.

Proof. It follows from the representation of $\mathcal{L}_c(f(z))$ that

$$\begin{aligned} \mathcal{L}_c(f)(z) &= \frac{c+1}{z^c} \int_0^z t^{c-1} \left[h(t) + \overline{g(t)} \right] dt \\ &= \frac{c+1}{z^c} \left[\int_0^z t^{c-1} \left(t - \sum_{k=2}^{\infty} a_k t^k \right) dt + \overline{\int_0^z t^{c-1} \left(t + (-1)^n \sum_{k=2}^{\infty} b_k t^k \right) dt} \right] \\ &= z - \sum_{k=2}^{\infty} A_k z^k + (-1)^n \sum_{k=2}^{\infty} B_k z^k, \end{aligned}$$

where

$$A_k = \frac{c+1}{c+k}\, a_k, \qquad B_k = \frac{c+1}{c+k}\, b_k.$$

Therefore,

$$\sum_{k=2}^{\infty} (\alpha_k |A_k| + \beta_k |B_k|) \le \sum_{k=2}^{\infty} \left(\alpha_k \frac{c+1}{c+k}\, |a_k| + \beta_k \frac{c+1}{c+k}\, |b_k| \right)$$

$$\leq \sum_{k=2}^{\infty}(\alpha_k|a_k| + \beta_k|b_k|) \leq B - A.$$

Since $f \in \widetilde{\mathcal{S}}^n_{\mathcal{H}}(A, B)$, by Theorem 1, we conclude that $\mathcal{L}_c(f) \in \widetilde{\mathcal{S}}^n_{\mathcal{H}}(A, B)$.

Theorem 9 is proved.

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