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ON THE RATE OF CONVERGENCE IN THE INVARIANCE PRINCIPLE FOR WEAKLY DEPENDENT RANDOM VARIABLES

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We consider nonstationary sequences of φ-mixing random variables. By using the Levy–Prokhorov distance, we estimate the rate of convergence in the invariance principle for nonstationary φ-mixing random variables. The obtained results extend and generalize several known facts established for nonstationary φ-mixing random variables.

1. Introduction

Let $\{\xi_{kn},\ k = 1,2,\ldots,k(n),\ n = 1,2,\ldots\}$ be a sequence of random variables (r.v.) in a probability space $\{\Omega,\Im,P\}$ and let

$$M_a^b(n) = \sigma\{\xi_{kn}, a \le k \le b\}, \quad 1 \le a \le b \le k(n).$$

For any $m \ge 1,$ we define (see [11])

$$\alpha(m) = \sup_{k,n} \sup_{A\in M_1^k(n),\, B\in M_{k+m}^{k(n)}(n)} |P(A\cap B) - P(A)P(B)|,$$

$$\beta(m) = E\left\{\sup_{k,n} \sup_{A\in M_{k+m}^{k(n)}(n)} \left|P(A/M_1^k(n)) - P(A)\right|\right\},$$

$$\varphi(m) = \sup_{k,n} \sup_{A\in M_1^k(n),\, B\in M_{k+m}^{k(n)}(n)} |P(B/A) - P(B)|, \quad P(A) > 0.$$

A sequence is said to be strongly mixing (s.m.), absolutely regular (a.r.), or *uniformly strong mixing* (u.s.m.) if $\alpha(m) \to 0,\ \beta(m) \to 0,$ or $\varphi(m) \to 0$ as $m \to \infty$, respectively.

Let

$$S_{kn} = \sum_{j\le k} \xi_{jn}, \qquad S_n = S_{k(n)n}, \qquad B_{kn}^2 = ES_{kn}^2, \qquad B_n^2 = B_{k(n)n}^2, \qquad S_{0n} = B_{0n}^2 = 0,$$

$$L_{ns} = B_n^{-s} \sum_{j\le k(n)} E\,|\xi_{jn}|^s\,, \qquad E\xi_{kn} = 0, \quad \varphi(0) = 1.$$

By $C(\cdot)$ with or without indices, we denote positive constants (generally speaking, different in different formulas) that depend only on the quantities in parentheses; by C we denote an absolute positive constant.

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We now consider the following points:

$$t_{kn} = \frac{\max_{1\le i\le k} B_{in}^2}{\max_{1\le i\le k(n)} B_{in}^2}$$

from the interval $[0;1]$. Further, we order these points and construct a continuous random polygon $W_n(t)$ on the interval $[0;1]$ with vertices $\left(t_{kn}; \dfrac{S_{kn}}{B_n}\right)$. If some t_{kn} are identical, i.e.,

$$B_{k_1 n}^2 = B_{k_2 n}^2 = \ldots = B_{k_r n}^2, \quad k_i \ne k_j,$$

then we choose any of these points $\left(t_{k_r n}; \dfrac{S_{k_i n}}{B_n}\right)$.

Consider the space $C[0;1]$ of continuous functions on $[0;1]$ equipped with the norm

$$\|x(t)\| = \sup_{0\le t\le 1} |x(t)|,$$

which generates a σ-algebra $\Im_C$. If W_n is the distribution of the process $\{W_n(t),\ t\in[0;1]\}$ and W is the distribution of the standard Wiener process $\{W(t),\ t\in[0;1]\}$, then the weak convergence W_n to W means that

$$\lim_{n\to\infty} P\left(W_n(t)\in A\right) = P(W(A))$$

for any Borel set A such that $W(\partial A)=0$. This fact is usually called the invariance principle (IP). Donsker [8] proved the IP for i.i.d. (independent identically distributed) random variables. At the same time, Yu. Prokhorov [16] proved the IP for triangular arrays $\left\{\xi_{kn},\ k=1,2,\ldots,k(n),\ n=1,2,\ldots\right\}$ of independent (in each series) r.v. under Lundeberg's condition:

$$\Lambda_n(\varepsilon) = \frac{1}{B_n^2}\sum_{k=1}^{n} E\left\{X_{kn}^2;\ |X_{kn}|>\varepsilon B_n\right\} \to 0 \quad \text{as} \quad n\to\infty \quad \text{for all} \quad \varepsilon>0.$$

Under Lundeberg's condition, T. Zuparov and A. Muhamedov [26] and M. Peligrad and S. Utev [15] proved the IP for nonstationary φ-mixing and α-mixing r.v., respectively.

By $L(P;Q)$ we denote the Levy–Prokhorov distance between the distributions P and Q in $C[0;1]$ (see [3, p. 327]):

$$L(P;Q) = \inf\left\{\varepsilon>0:\ P(A)\le Q\left(A^{\varepsilon}\right)+\varepsilon \ \text{ and } \ Q(A)\le P\left(A^{\varepsilon}\right)+\varepsilon \ \text{ for all } \ A\in\Im_C\right\},$$

where A^{ε} is a ε-neighborhood of A. Thus, the IP can be rewritten as $L\left(W_n;W\right)\to 0$ as $n\to\infty$.

It is known that

$$L\left(W_n;W\right) = \max\left\{\varepsilon:\ P\left(\|W_n(\cdot)-W(\cdot)\|>\varepsilon\right)\right\}. \tag{1}$$

In order to estimate (1), it is sufficient to establish the estimate for $P\left(\|W_n(\cdot)-W(\cdot)\|>\varepsilon\right)$. The rate of convergence in the IP was studied in detail in the case where the sequences of r.v. are independent. The first estimation in this case was proposed by Prokhorov [16]. He proved that

$$L(W_n;W) = o\left(L_{n3}^{1/4}\ln^2 L_{n3}\right), \quad n\to\infty.$$

This estimate was improved in the i.i.d. case by Heyde [10], Dudley [7], and other researchers. A. Borovkov [4] proved that

$$L(W_n;W) = C(s)L_{ns}^{1/(s+1)}, \quad 2 < s \le 3. \tag{2}$$

It should be emphasized that the one probability space method was used in all estimates presented above. R. Dudley [7] and A. Borovkov [4] showed that neither the Prokhorov method, nor the Skorokhod method can be used to get (2) in the case where $s > 5$. J. Komlos, P. Major, and G. Tusnady (KMT) [13] proposed a method, which allowed them to prove (1) in the i.i.d. case for all $s > 2$. Modifying the KMT method, A. Sakhanenko [17–21] extended (2) to the general case.

The fact that (2) is the best possible estimate was proved by several authors: Borovkov [4], Sakhanenko [17–21], T. Arak [1], Komlos, Major and Tusnady [14]. I. Berkes and W. Philipp [2] and Borovkov, Sakhanenko [5], Zuparov, and Muhamedov [26, 27] proposed the methods that can be used to obtain estimates for the Levy–Prokhorov distances for different classes of weakly dependent sequences.

Yoshihara [25] obtained the first result:

$$L(W_n;W) = O\left(n^{-1/8}\ln^{1/2} n\right)$$

for a.r. strictly stationary sequence $\{\xi_k,\ k\in N\}$ satisfying the inequality

$$\sum_{k=1}^{\infty} k\cdot(\beta(k))^{\delta/(4+\delta)} < \infty,$$

under the condition of existence of the absolute moment of order $4+\delta,\ \delta > 0$. Kanagawa [12] obtained the rate of convergence for the u.s.m. and s.m. strictly stationary sequences of r.v..

By the Prokhorov method, the best estimate in the IP was obtained in [9] for the stationary case with s.m. conditions, namely,

(i) if the coefficients $\alpha(k)$ of s.m. exponentially decrease to zero and

$$0 < \sigma = E\xi_1^2 + 2\sum_{i=2}^{\infty} E\xi_1\xi_i < \infty, \tag{3}$$

then

$$L(W_n;W) = O\left(n^{-\frac{s-2}{2(s-1)}}\ln^{\frac{2s+1}{6}} n\right);$$

(ii) if the coefficients $\alpha(k)$ of s.m. decrease to zero as follows:

$$\alpha(k) \le Cn^{-\theta s(s-1)/(s-2)^2}, \quad C > 0, \quad \theta > 1,$$

and condition (3) is satisfied, then

$$L(W_n;W) = O\left(n^{-\frac{(s-2)(\theta-1)}{6(\theta+1)+2(\theta-1)(s-2)}}\sqrt{\ln n}\right).$$

For the u.s.m. case and weak stationary sequences $\{\xi_k,\ k \in N\}$, S. Utev [23] showed that

$$L\left(W_n; W\right) = C(s; g; \sigma)\left(n^{-s/2}\sum_{i=1}^{n} E|\xi_i|^s\right)^{1/(s+1)}, \quad 2 < s < 5,$$

under conditions (3) for

$$\phi(k) \le A \cdot k^{-g(s)}, \quad g(s) > j(u)(j(u) - 1),$$

$$u = (2 + 5s)/2(5 - s), \quad \text{and} \quad j(u) = 2\min\{k \in N :\ 2k \ge u\}.$$

Zuparov and Muhamedov [27] announced the following estimate for nonstationary u.s.m. sequences:

$$L\left(W_n; W\right) \le C(s; \theta; K) L_{ns}^{\frac{1}{s+1}}$$

for $2 < s < 6$ and $\phi(k) \le Ak^{-\theta(s)}$; here, $\theta(s) > 2s$.

In the present paper, by using the Levy–Prokhorov distance, Bernstein's method, the Berkes–Philipp approximation theorems [2], Utev's moment inequalities [24], and the results obtained by Sakhanenko [19], we find the best possible rate of convergence for the IP and extend and generalize several known results for nonstationary φ-mixing random variables.

The paper is organized as follows. Our main results are presented in Section 2. In Section 3, we give some auxiliary lemmas. In Section 4, we present the proofs of our results.

2. Main Results

Theorem 2.1. *Suppose that, for any numbers θ and s such that*

$$\theta > \max(4, s, s(s - 2)/4), \quad s > 2,$$

the following conditions are satisfied:

$$\varphi(\tau) \le K\tau^{-\theta}, \quad K > 0,$$

$$E\ |\xi_{kn}|^s < \infty, \quad k = 1, 2, \ldots, k(n), \quad n = 1, 2, \ldots.$$

Then there exist a Wiener process $\{W(t),\ t \in [0; 1]\}$ and a constant $C(s; \theta; K)$ such that inequality

$$P\left(\|W_n(t) - W(t)\| > x\right) \le C(s; \theta; K)\frac{L_{ns}}{x^s}$$

holds for all $x > 0$.

Corollary. *Under the conditions of Theorem 2.1, the following inequality takes place:*

$$L\left(W_n; W\right) \le C(s; \theta; K) L_{ns}^{\frac{1}{s+1}}.$$

Theorem 2.2. *Under the conditions of Theorem 2.1, for* $\theta > \max(4, s, 3s(s-2)/4)$, *there exist a Wiener process* $\{W(t),\ t \in [0;1]\}$ *and a constant* $C(s;\theta;K)$ *such that inequality*

$$E\left\|W_n(t) - W(t)\right\|^s \leq C(s;\theta;K)L_{ns}$$

holds.

Remark. In [24], Utev proved that $E\left\|W_n(t) - W(t)\right\|^s$ converges to zero. The inequality in Theorem 2.2 for a nonstationary sequence of φ-mixing random variables is obtained for the first time.

Concerning the existence of sequences satisfying the conditions of Theorems 2.1 and 2.2, we can make the following remarks:

In Theorem 3.3 from [6], Bradley proved that if $X := (X_k,\ k \in Z)$ is a Markov chain (not necessarily stationary) and $\varphi(n) < 1/2$ for some $n \geq 1$, then $\varphi(n) \to 0$ at least exponentially rapidly as $n \to \infty$.

By using a strictly stationary sequence of Markov chains

$$X := (X_k,\ k \in Z),$$

we construct a nonstationary sequence $\xi := (\xi_{kn}, 1 \leq k \leq n)$ as follows: $\xi_{2k-1n} = -X_{2k-1},\ 1 \leq 2k-1 \leq n$, and $\xi_{2kn} = X_{2k},\ 1 \leq 2k \leq n$, for every series. As $X := (X_k,\ k \in Z)$, strictly stationary sequences satisfy the φ-mixing condition exponentially rapidly as $n \to \infty$. Thus, the sequence $\xi := (\xi_{kn},\ 1 \leq k \leq n)$ also satisfies the φ-mixing condition exponentially rapidly as $n \to \infty$. In addition, if $E\,|X_k|^s,\ s > 2$, then the nonstationary sequence $\xi := (\xi_{kn},\ 1 \leq k \leq n)$ satisfies the conditions of the main theorems.

3. Auxiliary Lemmas

Lemma 3.1 (see [11]). *Let the r.v.* ξ *and* η *be measurable with respect to the* σ*-algebras* M_1^k *and* $M_{k+\tau}^{k(n)}$, *respectively, where* $k \geq 1$ *and* $k + \tau \leq k(n)$. *If* $E|\xi|^p < \infty$ *and* $E|\eta|^q < \infty$ *for* $p > 1$ *and* $q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$|E\xi \cdot \eta - E\xi \cdot E\eta| \leq 2\varphi^{\frac{1}{p}}(\tau)E^{\frac{1}{p}}|\xi|^p E^{\frac{1}{q}}|\eta|^q.$$

Lemma 3.2 (see [2]). *Let* $\{(S_k, \sigma_k),\ k \geq 1\}$ *be a sequence of complete separable metric spaces. Also let* $\{X_k,\ k \geq 1\}$ *be a sequence of random variables with values in* S_k *and let* $\{B_k,\ k \geq 1\}$ *be a sequence of* σ*-fields such that* X_k *is* B_k*-measurable. Suppose that, for some* $\varphi_k \geq 0$,

$$|P(AB) - P(A)P(B)| \leq \varphi_k P(A)$$

for all $B \in B_k,\ A \in \bigcup_{j<k} B_j$. *Then, without changing the distribution, we can redefine the sequence* $\{X_k,\ k \geq 1\}$ *on a richer probability space together with a sequence* $\{Y_k,\ k \geq 1\}$ *of independent random variables such that* Y_k *has the same distribution as* X_k *and*

$$P\left(\sigma_k(X_k, Y_k) \geq 6\varphi_k\right) \leq 6\varphi_k, \quad k = 1, 2, \ldots .$$

Lemma 3.3 (see [24]). *Let* $\{X_k,\ k \geq 1\}$ *be a sequence of random variables satisfying the u.s.m. condition and let* $\varphi(p) < \frac{1}{4}$. *Then there exists a constant* $C(\varphi(p))$ *depending only on* $\varphi(p)$ *and such that, for all* $t \geq 1$ *and* *all* $1 \le q \le t$, *the following inequality takes place:*

$$E \max_{1\le k\le n} \left| \sum_{j=1}^{k} X_j \right|^t \le (C(\varphi(p)t))^t \left\{ p^t E \max_{1\le k\le n} |X_k|^t + \max_{1\le k\le n} \left(E \left| \sum_{j=1}^{k} X_j \right|^q \right)^{\frac{t}{q}} \right\}.$$

Lemma 3.4 (see [19]). *Let* $\{X_k,\ k \ge 1\}$ *be a sequence of independent random variables such that*

$$EX_k = 0 \qquad \text{and} \qquad \sum_{k=1}^{n} EX_k^2 = 1.$$

Suppose that

$$t_0 = 0, \qquad t_k = \sum_{i=1}^{k} EX_i^2, \quad k = 1, 2, \ldots, n,$$

and

$$L_{ns} = \sum_{i=1}^{n} E|X|^s.$$

Let $S(t)$ *be a continuous random polygon with vertices* $\left(t_k,\ S(t_k) = \sum_{j=1}^{k} X_j\right)$. *Then, for any numbers* $s \ge 2$ *and* $b \ge 1$, *there exists a Wiener process* $\{W(t),\ t \in [0,1]\}$ *such that the inequality*

$$P\Big(\|S(t) - W(t)\| \ge C_1 sbx\Big) \le \left(\frac{L_{ns}}{bx}\right)^b + P\left(\max_{1\le i\le n} |X_i| > x\right)$$

is true for all $x > 0$.

We introduce the following notation:

$$\xi_{jn}(x) = \xi_{jn} I\left\{|\xi_{jn}| \le CxB_n\right\} - E\xi_{jn} I\left\{|\xi_{jn}| \le CxB_n\right\}, \quad \bar{\xi}_{jn}(x) = \xi_{jn} - \xi_{jn}(x),$$

where $x > 0$ is an arbitrary real number,

$$S_{kn}(b) = \sum_{j=b+1}^{b+k} \xi_{jn}, \quad S_{kn}(b,x) = \sum_{j=b+1}^{b+k} \xi_{jn}(x), \quad \bar{S}_{kn}(b,x) = \sum_{j=b+1}^{b+k} \bar{\xi}_{jn}(x), \quad S_n(x) = S_{k(n)n}(0,x),$$

$$B_{kn}^2(b) = ES_{kn}^2(b), \quad B_{kn}^2(b,x) = ES_{kn}^2(b,x), \quad B_n^2(x) = ES_n^2(x), \quad \varphi_t = \sum_{i=0}^{k(n)+1} \varphi^{1/t}(i),$$

$$L_{ns} = B_n^{-s} \sum_{j\le k(n)} E|\xi_{jn}|^s, \qquad L_{nsx}(a,b) = B_n^{-s} \sum_{j=a+1}^{b} E|\xi_{jn}(x)|^s, \quad s > 2,$$

$$\bar{\varphi}_t = \sum_{i=0}^{k(n)+1} (i+1)\varphi^{1/t}(i).$$

We now define the positive integers m_i by using the following algorithm:

$$m_0 = 0,$$

$$m_{i+1} = \min\left\{m\colon\, m_i < m < n\colon\, E\left(\sum_{k=m_i+1}^{m+1}\xi_{kn}(x)\right)^2 > h(n)\right\} \quad \text{for} \quad i = 1, 2, \ldots, M-1,$$

where $M-1$ is the last number for which we can define m_{M-1}, i.e.,

$$E\left(\sum_{j=m_{M-1}+1}^{k(n)}\xi_{jn}(x)\right)^2 < h(n),$$

where $h(n)$ is a sequence of positive numbers.

By η_j and $\eta_j(x)$, respectively, we denote

$$\eta_j = \sum_{i=m_{j-1}+1}^{m_j}\xi_{in} \qquad \text{and} \qquad \eta_j(x) = \sum_{i=m_{j-1}+1}^{m_j}\xi_{in}(x).$$

The positive integers l_i are described by using the outlined algorithm:

$$l_0 = 0,$$

$$l_{i+1} = \min\left\{l\colon\, l_i < l < M\colon\, E\left(\sum_{k=l_i+1}^{l+1}\eta_k(x)\right)^2 > T(n)\right\} \quad \text{for} \quad i = 1, 2, \ldots, N-1,$$

where $M-1$ is the last number for which we can define l_{N-1}, i.e.,

$$E\left(\sum_{j=l_{N-1}+1}^{M}\eta_j(x)\right)^2 < T(n),$$

where $T(n)$ is a sequence of positive numbers. The sequences $T(n)$ and $h(n)$ are selected in what follows.

By ψ_j and $\psi_j(x)$, respectively, we denote

$$\psi_j = \sum_{i=l_{j-1}+1}^{l_j-1}\eta_i \qquad \text{and} \qquad \psi_j(x) = \sum_{i=l_{j-1}+1}^{l_j-1}\eta_i(x).$$

Lemma 3.5. *The following inequalities are true:*

$$\left|B_{kn}^2(b) - B_{kn}^2(b,x)\right| \le C(\varphi_s)B_n^2 x^{2-s}L_{ns}(b), \tag{4}$$

$$\max_{1\le k\le N}\left|\sum_{j=1}^{k}(D\psi_j - D\psi_j(x))\right| \le C(\varphi_s)B_n^2 x^{2-s}L_{ns}, \tag{5}$$

$$\max_{1\le k\le N}\left|B_{m_k}^2-\sum_{j=1}^{k}D\psi_j(x)\right|\le C(\varphi_2)N\cdot h(n), \tag{6}$$

$$E\psi_j^2(x)\le T(n)+\theta\cdot h(n),\quad |\theta|\le C(\varphi_2), \tag{7}$$

$$M\le C\left(\bar\varphi_2\right)\frac{B_n^2(x)}{h(n)},\qquad N\le C\left(\bar\varphi_2\right)\frac{B_n^2(x)}{T(n)}. \tag{8}$$

Proof. It is obvious that

$$\left|B_{kn}^2(b)-B_{kn}^2(b,x)\right|=\left|E\left(\sum_{j=b+1}^{b+k}\left(\xi_{jn}(x)+\bar\xi_{jn}(x)\right)\right)^2-E\left(\sum_{j=b+1}^{b+k}\xi_{jn}(x)\right)^2\right|$$

$$\le\left|\sum_{b+1\le i\ne j\le b+k}E\xi_{in}(x)\bar\xi_{jn}(x)\right|+\left|\sum_{b+1\le i\ne j\le b+k}E\bar\xi_{in}(x)\xi_{jn}(x)\right|$$

$$+\left|\sum_{b+1\le i\ne j\le b+k}E\bar\xi_{in}(x)\bar\xi_{jn}(x)\right|.$$

We now estimate the first term on the right-hand side of the inequality. The other terms can be estimated similarly. By virtue of Lemma 3.1 and the Hölder inequality, we find

$$\left|\sum_{b+1\le i\ne j\le b+k}E\xi_{in}(x)\bar\xi_{jn}(x)\right|\le\sum_{b+1\le i\ne j\le b+k}\varphi^{1/s}(|j-i|)E^{1/s}|\xi_{in}(x)|^sE^{(s-1)/s}\left|\bar\xi_{jn}(x)\right|^{s(s-1)}$$

$$\le C\left(\sum_{i=0}^{k(n)}\varphi^{1/s}(i)\right)B_n^2x^{2-s}L_{ks}(b)$$

$$\le C(\varphi_s)B_n^2x^{2-s}L_{ks}(b).$$

Inequality (4) is proved. Inequality (5) can be obtained in a similar way.

We now prove inequality (6). To do this, we estimate the difference

$$\left|B_n^2(x)-\sum_{j=1}^{N}D\psi_j(x)\right|\quad\text{for}\quad k=N.$$

The other cases are proved similarly. It is clear that

$$B_n^2(x)=E\left(\sum_{j=1}^{N}(\psi_j(x)+\eta_{l_j}(x))\right)^2.$$

By Lemma 3.1, we get

$$\left|B_n^2(x) - \sum_{j=1}^{N} E\psi_j^2(x)\right| = \left|E\left(\sum_{j=1}^{N}(\psi_j(x)+\eta_{l_j}(x))\right)^2 - \sum_{j=1}^{N} E\psi_j^2(x)\right|$$

$$\leq \left|2\sum_{1\leq j\leq l\leq N} E(\psi_j(x)+\eta_{l_j}(x))(\psi_k(x)+\eta_{l_k}(x))\right|$$

$$\leq 2\left|\sum_{j=1}^{N} E(\psi_j(x)+\eta_{l_j}(x))\left(\sum_{l=j+1}^{N} E(\psi_k(x)+\eta_{l_k}(x))\right)\right|$$

$$\leq 2\left|\sum_{j=1}^{N} E\left(\sum_{i=1}^{l_j}\eta_i(x)\right)\left(\sum_{i=l_j+1}^{l_M}\eta_i(x)\right)\right|$$

$$\leq 2\sum_{i=1}^{k(n)}(i+1)\varphi^{1/2}(i)N\cdot h(n)$$

$$\leq C(\bar{\varphi}_2)N\cdot h(n).$$

Proof of inequality (7). By the definitions of random variables $\psi_j(x)$ and $\eta_{ij}(x)$, we obtain

$$E\eta^2_{m_j+1n}(x) \leq h(n)$$

and

$$E\psi_j^2(x) \leq T(n) < E\left(\psi_j(x)+\eta_{l_j}(x)\right)^2$$

$$\leq E\psi_j^2(x) + 2E\psi_j(x)\eta_{l_j}(x) + E\eta^2_{l_j}(x)$$

$$\leq T(n) + 2E\left(\sum_{i=l_{j-1}+1}^{l_j}\eta_i(x)\right)\eta_{l_j}(x) + E\eta^2_{l_j}(x)$$

$$\leq T(n) + 2\sum_{i=1}^{N}\varphi^{1/2}(i)E^{1/2}\eta^2_{l_i}(x)E^{1/2}\eta^2_{l_j+1}(x) + E\eta^2_{l_j+1}(x)$$

$$\leq T(n) + C(\varphi_2)h(n).$$

Relations (4) and (5) imply that

$$B_n^2(x) \geq \sum_{i=1}^{N} D\psi_j(x) - C(\varphi_2)N\cdot h(n)$$

$$\geq \sum_{i=1}^{N-1} D\psi_j(x) - C(\varphi_2)N \cdot h(n)$$

$$\geq (N-1) \cdot T(n) - C(\varphi_2)N \cdot h(n).$$

Hence, we get the second inequality in (8). Since $h(n) = o(T(n))$, the first inequality in (8) is estimated similarly. Therefore, Lemma 3.5 is proved.

4. Proofs of Theorems

Proof of Theorem 2.1. We denote by $W_{nx}(t)$ a random polygon with vertices $\left(t_{kn}; \dfrac{S_k(x)}{B_n}\right)$. A polygon with vertices $\left(t_{m_k n}; \dfrac{S_{m_k}(x)}{B_n}\right)$ is denoted by $\overline{W}_{nx}(t)$. Let $\overline{\overline{W}}$ and $\widehat{W}_{nx}(t)$ be random polygons with vertices

$$\left(t_{m_k n}; \frac{\sum_{j=1}^{k} \psi_j(x)}{B_n}\right) \quad \text{and} \quad \left(t_{m_k n}; \frac{\sum_{j=1}^{k} \widehat{\psi}_j(x)}{B_n}\right),$$

respectively, where $\widehat{\psi}_j(x),\ j = 1, 2, \ldots, N,$ are independent r.v. whose marginal distributions coincide with the distributions of r.v. $\psi_j(x)$. A polygon with vertices

$$\left(\frac{\sqrt{\sum_{j=1}^{k} D\psi_j(x)}}{\sqrt{\sum_{j=1}^{N} D\psi_j(x)}}; \frac{\sum_{j=1}^{k} \widehat{\psi}_j(x)}{\sqrt{\sum_{j=1}^{N} D\psi_j(x)}}\right)$$

is denoted by $\widetilde{W}_{nx}(t)$.

It is obvious that

$$\begin{aligned}
&P\left(\|W_n(t) - W(t)\| > x\right) \\
&\quad \leq P\left(\|W_n(t) - W_{nx}(t)\| > \frac{x}{6}\right) + P\left(\left\|\overline{W}_{nx}(t) - \overline{\overline{W}}_{nx}(t)\right\| > \frac{x}{6}\right) \\
&\quad + P\left(\left\|W_{nx}(t) - \overline{W}_{nx}(t)\right\| > \frac{x}{6}\right) + P\left(\left\|\overline{\overline{W}}_{nx}(t) - \widehat{W}_{nx}(t)\right\| > \frac{x}{6}\right) \\
&\quad + P\left(\left\|\widehat{W}_{nx}(t) - \widetilde{W}_{nx}(t)\right\| > \frac{x}{6}\right) + P\left(\left\|\widetilde{W}_{nx}(t) - W(t)\right\| > \frac{x}{6}\right) = \sum_{i=1}^{6} P_i. \qquad (9)
\end{aligned}$$

To prove Theorem 2.1, we now estimate each term on the right-hand side of (9). Without loss of generality, we can assume that $L_{ns} < 1$. Let

$$T(n) = C(s, \theta, K) B_n^2 x^{\frac{2(t-s)}{t-2}} L_{ns}^{\frac{2}{t-2}}, \quad t > s.$$

Then

$$N \leq C(s,\theta,K)\frac{B_n^2(x)}{T(n)} \ll C(s,\theta,K)x^{-\frac{2(t-s)}{t-2}}L_{ns}^{-\frac{2}{t-2}}.$$

Estimate P_1. It is clear that

$$P_1 = P\left(\|W_n(t) - W_{nx}(t)\| > \frac{x}{6}\right) \leq P\left(\max_{k\leq k(n)}|\xi_{kn}| > C_1B_nx\right) \leq C\frac{L_{ns}}{x^s}.$$

Estimate P_2. By virtue of the Chebyshev inequality and Lemmas 3.3 and 3.5, for $q=2$ and $t>s,$ we get

$$\begin{aligned}P_2 = P\left(\left\|W_{nx}(t) - \overline{W}_{nx}(t)\right\| > \frac{x}{6}\right) &\leq \sum_{j\leq N}P\left(\max_{m_{j-1}\leq k\leq m_j}|S_{kn}(x) - S_{m_{j-1}n}(x)| > C\frac{xB_n}{12}\right)\\ &\leq C\frac{1}{x^tB_n^t}\sum_{j\leq N}E\max_{m_{j-1}\leq k\leq m_j}|S_{kn}(x) - S_{m_{j-1}n}(x)|^t\\ &\leq C(t,\theta,K)\left[\frac{L_{nt}(x)}{x^t} + \frac{1}{x^t}\left(\frac{T(n)}{B_n^2}\right)^{\frac{t-2}{2}}\right]\\ &\leq C(s,\theta,K)\frac{L_{ns}}{x^s}.\end{aligned}$$

Estimate P_3. It is obvious that

$$P_3 = P\left(\left\|\overline{W}_{nx}(t) - \overline{\overline{W}}_{nx}(t)\right\| > \frac{x}{6}\right) \leq P\left(\max_{k\leq N}\left|\sum_{j\leq k}\frac{\eta_{m_j}(x)}{B_n}\right| > \frac{x}{6}\right).$$

We now estimate P_3. By analogy with $P_2,$ we obtain

$$P_3 = P\left(\left\|\overline{W}_{nx}(t) - \overline{\overline{W}}_{nx}(t)\right\| > \frac{x}{6}\right) \leq C(s,\theta,K)\frac{L_{ns}}{x^s}.$$

Estimate P_4. It is clear that

$$P\left(\left\|\overline{\overline{W}}_{nx}(t) - \widehat{W}_{nx}(t)\right\| > \frac{x}{6}\right) \leq P\left(\max_{k\leq N}\left|\sum_{j\leq k}\left(\frac{\psi_j(x)}{B_n} - \frac{\widehat{\psi}_j(x)}{B_n}\right)\right| > \frac{x}{6}\right).$$

By using the Berkes–Philipp approximation theorem (see Lemma 3.2) and Lemmas 3.3 and 3.4, we get

$$P_4 \leq \sum_{j\leq N}P\left(\left|\frac{\psi_j(x)}{B_n} - \frac{\widehat{\psi}_j(x)}{B_n}\right| > \frac{x}{6N}\right)$$

$$\leq \sum_{j\leq N} P\left(\left|\frac{\psi_j(x)}{B_n} - \frac{\widehat{\psi}_j(x)}{B_n}\right| > 6\varphi(p)\right)$$

$$\leq 6N\varphi(p) \quad \text{for} \quad \frac{x}{6N\varphi(p)} > 6 \quad \text{or} \quad 36N\varphi(p) \leq x,$$

where

$$p = \min_{j\leq N}(m_j - m_{j-1}).$$

To obtain the estimate

$$P_4 \leq C(s,\theta,K)\frac{L_{ns}(x)}{x^s},$$

we find p from the condition

$$N\varphi(p) \leq Cx, \qquad N\varphi(p) \leq C\frac{L_{ns}}{x^s}.$$

In view of Lemma 3.5, this yields

$$N\varphi(p) \leq nKp^{-\theta} \leq C(s,\theta,K)x^{-\frac{2(t-s)}{t-2}}L_{ns}^{-\frac{2}{t-2}}p^{-\theta} \leq C(s,\theta,K)\min\left(x, \frac{L_{ns}}{x^s}\right).$$

Hence,

$$p \geq C(s,\theta,K)\left(\max\left(x^{-\frac{3t-2(s+1)}{t-2}}L_{ns}^{-\frac{2}{t-2}}; x^{\frac{t(s-2)}{t-2}}L_{ns}^{-\frac{t}{t-2}}\right)\right)^{\frac{1}{\theta}}.$$

Estimate P_5. It is clear that

$$P\left(\left\|\widehat{W}_{nx}(t) - \widetilde{W}_{nx}(t)\right\| > \frac{x}{5}\right)$$

$$\leq P\left(\max_{k\leq N}\left|\left(1 - \frac{B_n}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right)\sum_{j\leq k}\left(\frac{\widehat{\psi}_j}{B_n}\right)\right| > \frac{x}{5}\right)$$

$$\leq P\left(\max_{k\leq N}\left|\sum_{j\leq k}\left(\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right)\right| > \frac{xB_n\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}{5\left(B_n - \sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}\right)}\right)$$

$$\leq C\left|\frac{B_n - \sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}{xB_n}\right|^t E\left(\max_{k\leq N}\left|\sum_{j\leq k}\left(\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right)\right|^t\right).$$

Thus, by Lemma 3.3, we obtain

$$E\left(\max_{k\le N}\left|\sum_{j\le k}\left(\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\le N} D\widehat{\psi}_j(x)}}\right)\right|^t\right)\le C(t,\theta,K). \tag{10}$$

Since

$$\frac{B_n-\sqrt{\sum_{j\le N} D\widehat{\psi}_j(x)}}{B_n}=\frac{B_n^2-\sum_{j\le N} D\widehat{\psi}_j(x)}{B_n\left(B_n+\sqrt{\sum_{j\le N} D\widehat{\psi}_j(x)}\right)}$$

and

$$D\widehat{\psi}_j(x)=D\psi_j(x),$$

it follows from Lemma 3.5 that

$$\sum_{j\le N} D\psi_j(x)=B_n^2(1+o(1)).$$

As a result, it suffices to estimate $B_n^2-\sum_{j\le N} D\hat{\psi}_j(x)$. Let

$$h(n)=T(n)x^{\frac{t-s}{t}}L_{ns}^{\frac{1}{t}}.$$

Thus, Lemma 3.5 implies that

$$\left|\frac{B_n^2-\sum_{j\le N} D\psi_j(x)}{xB_n^2}\right|\le C(\varphi_2)\left(\frac{Nh(n)+B_n^2x^{2-s}L_{ns}}{xB_n^2}\right)=C(\varphi_2)\left(\frac{h(n)}{xT(n)}+x^{1-s}L_{ns}\right)$$

$$\le C(t,\varphi_2)\left(x^{-\frac{s}{t}}L_{ns}^{\frac{1}{t}}+x^{1-s}L_{ns}\right). \tag{11}$$

This yields

$$P_5=P\left(\left\|\widehat{W}_{nx}(t)-\widetilde{W}_{nx}(t)\right\|>\frac{x}{6}\right)$$

$$\le C(t,\varphi_2)\left(x^{-\frac{s}{t}}L_{ns}^{\frac{1}{t}}+x^{1-s}L_{ns}\right)^t$$

$$\le C(t,\varphi_2)\left(\frac{L_{ns}}{x^s}+\left(x\frac{L_{ns}}{x^s}\right)^t\right). \tag{12}$$

It is clear that if $0<x\le 1,$ then

$$P_5\le C(t,\varphi_2)\frac{L_{ns}}{x^s}.$$

Now let $x \geq 1$. Thus, to estimate $P_5 \leq C(t, \varphi_2)\dfrac{L_{ns}}{x^s}$, the second term of inequality (12) should satisfy the condition

$$\left(x\frac{L_{ns}}{x^s}\right)^t \leq \frac{L_{ns}}{x^s}.$$

This inequality holds for $x \geq L_{ns}^{\frac{t-1}{s(t-1)-t}}$. Hence, inequality $P_4 \leq C(t, \varphi_2)\dfrac{L_{ns}}{x^s}$ is true for all $x > 0$.

Estimate P_6. By using Lemma 3.4, we obtain

$$P_6 = P\left(\left\|\widetilde{W}_{nx}(t) - W(t)\right\| > \frac{x}{6}\right) \leq C\left(\frac{1}{x}\right)^t \left(\sum_{j\leq N} E\left|\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right|^t\right).$$

We now estimate

$$\sum_{j\leq N} E\left|\widehat{\psi}_j(x)\right|^t.$$

Since $\widehat{\psi}_j(x)$ are independent r.v. whose marginal distributions coincide with the distributions of r.v. $\psi_j(x)$, by Lemmas 3.3 and 3.5, we find

$$\sum_{j\leq N} E\,|\psi_j(x)|^t \leq \sum_{j\leq N}\left(\sum_{i=l_{j-1}}^{l_j} E|\eta_i(x)|^t + (D\psi_j(x))^{t/2}\right)$$

$$\leq C(t)\left(\sum_{i=1}^{k(n)} E|\xi_{in}(x)|^t + N(T(n))^{t/2}\right). \tag{13}$$

Hence, it follows from Lemma 3.5 and the definition of $T(n)$ that

$$P_6 = P\left(\left\|\widetilde{W}_{nx}(t) - W(t)\right\| > \frac{x}{5}\right)$$

$$\leq C(t,\varphi)\left(\frac{1}{x^t}L_{nt} + \frac{1}{x^t}\left(\frac{T(n)}{B_n^2}\right)^{\frac{t-2}{2}}\right) \leq C(t,\varphi)\frac{L_{ns}}{x^s}. \tag{14}$$

We now demonstrate the possibility of splitting of the above-mentioned isolated groups, namely, as $n \to \infty$, the conditions

$$B_n^2, \quad T(n), \quad h(n) \to \infty, \quad T(n) = o(B_n^2), \quad h(n) = o(T(n)), \quad \text{and} \quad L_{ns} \to 0$$

should be satisfied, and explain the necessity of curtailing in order to prove Theorem 2.1. These conditions are clear in the stationary case. In this case, the following asymptotic relations are true, i.e.,

$$L_{ns} \approx n^{-\frac{s-2}{2}} \quad \text{for} \quad s > 2, \qquad T(n) \approx n^{\frac{t-s}{t-2}} \quad \text{for some} \quad t, \quad t > s,$$

and

$$h(n) \approx n^{\frac{2t^2-(3s-2)t+2s-4}{2t(t-2)}} \quad \text{for some} \quad t, \quad t > t_0 = \frac{3s-2+\sqrt{9s^2-28s+36}}{4} > s,$$

$$p \gg n^{\frac{t(s-2)}{2\theta(t-2)}}, \qquad N \ll n^{\frac{t(s-2)}{s(t-2)}}, \qquad \text{and} \qquad \theta > \max\left(4, s, \frac{s(s-2)}{4}\right).$$

To obtain the required estimate for P_2 and P_4, it is necessary to have a moment of t, which is larger than s. This is why, curtailing is necessary.

Theorem 2.1 is proved.

As indicated above, the Levy–Prokhorov distance between the distributions W_n and W is determined in (1). Selecting $\varepsilon = x = L_{ns}^{\frac{1}{s+1}}$ in relation (1) and Theorem 2.1, respectively, we obtain the proof of the corollary.

Proof of Theorem 2.2. The method used to prove Theorem 2.2 remains the same as in Theorem 2.1. Here, we only list the places where it is necessary to make certain changes.

As in the proof of Theorem 2.1, the following inequality is true:

$$E\|W_n(t) - W(t)\|^s \le E\|W_n(t) - W_{nx}(t)\|^s + E\left\|W_{nx}(t) - \overline{W}_{nx}(t)\right\|^s$$
$$+ E\left\|\overline{W}_{nx}(t) - \overline{\overline{W}}_{nx}(t)\right\|^s + E\left\|\overline{\overline{W}}_{nx}(t) - \widehat{W}_{nx}(t)\right\|^s$$
$$+ E\left\|\widehat{W}_{nx}(t) - \widetilde{W}_{nx}(t)\right\|^s + E\left\|\widetilde{W}_{nx}(t) - W(t)\right\|^s = \sum_{i=1}^{6} E_i. \tag{15}$$

Thus, in order to prove Theorem 2.2, we estimate each term on the right-hand side of (15) and take $x = L_{ns}^{1/s}$. Hence, we get

$$T(n) = B_n^2 L_{ns}^{\frac{2t}{s(t-2)}}, \qquad h(n) = T(n)L_{ns}^{1/s} = B_n^2 L_{ns}^{\frac{3t-2}{s(t-2)}},$$

$$N = \frac{B_n^2}{T(n)} = L_{ns}^{-\frac{2t}{s(t-2)}}, \qquad \frac{h(n)}{T(n)} = L_{ns}^{\frac{1}{s}}.$$

Estimate E_1. It is clear that

$$E_1 = E\|W_n(t) - W_{nx}(t)\|^s \le E\left(\max_{k\le k(n)} |\xi_{kn}|^s / B_n^s\right) \le L_{ns}.$$

Estimate E_2. Based on the moment inequality, Lemmas 3.3 (for $q = 2$ and $t > s$) and 3.5, and the definition of $T(n)$, we get the following inequality:

$$E_2 = E\left\|W_{nx}(t) - \overline{W}_{nx}(t)\right\|^s \le E^{s/t}\left\|W_{nx}(t) - \overline{W}_{nx}(t)\right\|^t$$
$$\le \left(\sum_{j\le N} E\left(\max_{m_{j-1}\le k\le m_j} \left|S_{kn}(x) - S_{m_{j-1}n}(x)\right|^t / B_n^t\right)\right)^{s/t}$$

$$\leq C\left(\sum_{j\leq N} E\left(\frac{1}{B_n^t}\max_{m_{j-1}\leq k\leq m_j}\left|S_{kn}(x)-S_{m_{j-1}n}(x)\right|^t\right)\right)^{s/t}$$

$$\leq C(t,\theta,K)\left(L_{nt}(x)+\left(\frac{T(n)}{B_n^2}\right)^{\frac{t-2}{2}}\right)^{s/t}$$

$$\leq C(s,\theta,K)L_{ns}. \tag{16}$$

Estimate E_3. It is obvious that

$$E_3 = E\left\|\overline{W}_{nx}(t)-\overline{\overline{W}}_{nx}(t)\right\|^s \leq E\max_{k\leq N}\left|\sum_{j\leq k}\frac{\eta_{m_j}(x)}{B_n}\right|^s.$$

We now estimate E_3. By analogy with E_2, we obtain

$$E_3 = E\left\|\overline{W}_{nx}(t)-\overline{\overline{W}}_{nx}(t)\right\|^s \leq C(s,\theta,K)L_{ns}.$$

Estimate E_4. By Lemmas 3.2, 3.3, and 3.5, following the paper [24], we can estimate E_4 as follows:

$$E_4 \leq E\left(\max_{k\leq N}\left|\sum_{j\leq k}\left(\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right)\right|^s\right)$$

$$\leq N^s\max_{j\leq N}E\left|\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right|^s$$

$$\leq N^s\left((6\varphi(p))^s+\max_{j\leq N}\left(E\left|\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right|^s,\ 6\varphi(p)<\left|\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right|\leq 1\right)\right)$$

$$+N^s\max_{j\leq N}E\left|\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right|^t$$

$$\leq CN^s\left(\varphi^s(p)+\max_{j\leq N}P\left(\left|\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right|\geq 6\varphi(p)\right)+\left(\frac{T(n)}{B_n^2}\right)^{t/2}\right)\leq L_{ns}.$$

In this case, the mixing coefficients decrease as $N^s\varphi(p)\leq L_{ns}$. In turn,

$$N^s\varphi(p)\leq L_{ns}^{-\frac{2t}{t-2}}p^{-\theta}\leq L_{ns}\Rightarrow p\geq L_{ns}^{-\frac{3t-2}{\theta(t-2)}}$$

for $\theta > \max\left(4, s, \dfrac{3s(s-2)}{4}\right)$.

Estimate $\boldsymbol{E_5}$**.** It is obvious that

$$E\left\|\widehat{W}_{nx}(t)-\widetilde{W}_{nx}(t)\right\|^s$$

$$\leq C\left(\frac{B_n-\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}{B_n}\right)^s E\left(\max_{k\leq N}\left|\sum_{j\leq k}\left(\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right)\right|^s\right).$$

By Lemma 3.5 and inequalities (10), (11), we get

$$E\left\|\widehat{W}_{nx}(t)-\widetilde{W}_{nx}(t)\right\|^s\leq C(s,\varphi_2)\left|\frac{B_n-\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}{B_n}\right|^s\leq\left(\frac{h(n)}{T(n)}+x^{2-s}L_{ns}\right)^s\leq L_{ns}.$$

Estimate $\boldsymbol{E_6}$**.** Due to moment inequality and similar estimates for (13), (14), and (16), by Lemmas 3.3 and 3.4, we obtain

$$E\left\|\widetilde{W}_{nx}(t)-W(t)\right\|^s\leq E^{s/t}\left\|\widetilde{W}_{nx}(t)-W(t)\right\|^t$$

$$\leq\left(\sum_{j\leq N} E\left|\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right|^t\right)^{s/t}$$

$$\leq C(t,K,\theta)\left(L_{nt}(x)+\left(\frac{T(n)}{B_n^2}\right)^{\frac{t-2}{2}}\right)^{s/t}\leq C(t,K,\theta)L_{ns}.$$

Theorem 2.2 is proved.

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