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BOUNDARY-VALUE PROBLEM FOR A CLASS OF NONLINEAR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

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We study a boundary-value problem for a class of nonlinear systems of partial differential equations of higher orders. For this problem, we establish the existence, uniqueness, and absence of solutions.

1. Statement of the Problem

In the Euclidean space $\mathbb{R}^{n+1}$ of variables $x=(x_1,\ldots,x_n)$ and $t,$ we consider a nonlinear system of partial differential equations of the form

$$L_f := \frac{\partial^{4k}u}{\partial t^{4k}} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(A_{ij}\frac{\partial u}{\partial x_i}\right) + f(u) = F, \tag{1.1}$$

where $f=(f_1,\ldots,f_N),$ $F=(F_1,\ldots,F_N)$ are given vector functions, $u=(u_1,\ldots,u_N),$ $N\geq 2,$ is the required vector function, A_{ij} are given square matrices of order N such that, in addition, $A_{ij}=A_{ji},$ $i,j=1,\ldots,n,$ $n\geq 2,$ and k is a natural number.

For system (1.1), we consider the following boundary-value problem: In a cylindrical domain $D_T := \Omega\times(0,T),$ where Ω is an open Lipschitz domain in $\mathbb{R}^n$, it is necessary to find the solution $u=u(x,t)$ of system (1.1) with the following boundary conditions:

$$\left.\left(\frac{\partial u}{\partial N} + Bu\right)\right|_{\Gamma} = 0, \tag{1.2}$$

$$\left.\frac{\partial^i u}{\partial t^i}\right|_{\partial\Omega_0\cup\Omega_T} = 0, \quad i=0,\ldots,2k-1, \tag{1.3}$$

where $\Gamma := \partial\Omega\times(0,T)$ is the lateral part of the boundary of a cylindrical domain $D_T,$ $\Omega_0\colon x\in\Omega,$ $t=0,$ and $\Omega_T\colon x\in\Omega,$ $t=T,$ are, respectively, the lower and upper bases of the cylinder, $B\colon \overline{\Gamma}\to\mathbb{R}^{N\times N}$ is a given continuous square matrix of order $N;$

$$\frac{\partial u}{\partial N} = \sum_{i,j=1}^{N} A_{ij}\frac{\partial u}{\partial x_i}\nu_j$$

(in the scalar case, this derivative coincides with the derivative along the conormal), $\nu=(\nu_1,\ldots,\nu_n,\nu_{n+1})$ is the unit vector of the outer normal to $\partial D_T,$ and $\nu_{n+1}|_{\Gamma}=0.$

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Note that, in [1], we considered the boundary-value problem for Eq. (1.1) with conditions (1.3) in the scalar case, i.e., for $N = 1$, in the cylindrical domain D_T but with a homogeneous Dirichlet condition $u|_\Gamma = 0$ instead of (1.2). The initial and mixed problems for semilinear partial differential equations of higher orders with structures different from (1.1) were studied in numerous works (see, e.g., [2–12] and the references therein).

By $C^{2,4k}(\bar{D}_T)$ we denote a space of vector functions $u = (u_1, \ldots, u_N)$ continuous in $\bar{D}_T$ with continuous partial derivatives $\dfrac{\partial u}{\partial x_i}$, $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$, $\dfrac{\partial^l u}{\partial t^l}$, $i, j = 1, \ldots, n,\ l = 1, \ldots, 4k$, in $\bar{D}_T$. We set

$$C_0^{2,4k}(\bar{D}_T) := \left\{ u \in C^{2,4k}(\bar{D}_T) : \left.\frac{\partial^i u}{\partial t^i}\right|_{\Omega_0 \cup \Omega_T} = 0,\ i = 0, \ldots, 2k-1 \right\}.$$

We also introduce a Hilbert space $W_0^{1,2k}(D_T)$ obtained by completion with respect to the norm

$$\|u\|^2_{W_0^{1,2k}(D_T)} = \int\limits_{D_T} \left[u^2 + \sum_{i=1}^{2k} \left(\frac{\partial^i u}{\partial t^i} \right)^2 + \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \right] dx\,dt \tag{1.4}$$

of the classical space $C_0^{2,4k}(\bar{D}_T)$, $u^2 = \sum_{i=1}^N u_i^2$.

Remark 1.1. It follows from (1.4) that $u \in W_2^1(D_T)$ and $\dfrac{\partial^i u}{\partial t^i} \in L_2(D_T)$, $i = 1, \ldots, 2k$, if $u \in W_0^{1,2k}(D_T)$. Here, $W_2^1(D_T)$ is the well-known Sobolev space formed by elements of $L_2(D_T)$ with generalized first-order derivatives that belong to $L_2(D_T)$ [13, p. 56].

Further, we impose the following restrictions on the nonlinear vector function $f = (f_1, \ldots, f_N)$ from (1.1):

$$f \in C(\mathbb{R}^N), \qquad |f(u)| \le M_1 + M_2 |u|^\alpha, \quad u \in \mathbb{R}^N, \tag{1.5}$$

where $|\cdot|$ is the norm in the space $\mathbb{R}^N$, $M_i = \text{const} \ge 0$, $i = 1, 2$, and

$$0 \le \alpha = \text{const} < \frac{n+1}{n-1}. \tag{1.6}$$

Remark 1.2. The embedding operator $i\colon W_2^1(d_t) \to L_q(D_T)$ is a linear and continuous compact operator for $1 < q < \dfrac{2(n+1)}{n-1}$, $n > 1$ [13, p. 81]. At the same time, the Nemytskii operator $K\colon L_q(D_T) \to L_2(D_T)$ acting by the formula

$$Ku = f(u),$$

where $u = (u_1, \ldots, u_N) \in L_q(D_T)$ and the vector function $f = (f_1, \ldots, f_N)$ satisfies condition (1.5), is continuous and bounded for $q \ge 2\alpha$ [14, pp. 66, 67]. Thus, if $\alpha < \dfrac{n+1}{n-1}$, then there exists a number q such that

$$1 < q < \frac{2(n+1)}{n-1} \quad \text{and} \quad q \ge 2\alpha.$$

Hence, in this case, the operator

$$K_0 = KI\colon W_2^1(D_T) \to L_2(D_T) \tag{1.7}$$

is continuous and compact. Since $u \in W_2^1(D_T)$, we conclude that $f(u) \in L_2(D_T)$ and if $u^m \to u$ in the space $W_2^1(D_T)$, then

$$f(u^m) \to f(u) \quad \text{in} \quad L_2(D_T).$$

Here and in what follows, the fact that a vector function $v = (v_1, \ldots, v_N)$ belongs to a certain space X means that each component $v_i,\ 1 \le i \le N,$ of this vector belongs to the space $X.$

Remark 1.3. Let $A_{ij} = A_{ij}(x) \in C^1(\Omega),\ i, j = 1, \ldots, n,$ and let $u \in C_0^{2,4k}(\bar{D}_T)$ be the solution of problem (1.1)–(1.3). Multiplying scalarly both sides of system (1.1) by an arbitrary vector function $\varphi \in C_0^{2,4k}(\bar{D}_T)$ and integrating the obtained equality by parts over the domain $D_T,$ we obtain

$$\int\limits_{D_T} \left[\frac{\partial^{2k} u}{\partial t^{2k}} \frac{\partial^{2k} \varphi}{\partial t^{2k}} + \sum_{i,j=1}^{n} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right] dx\, dt$$

$$+ \int\limits_{\Gamma} Bu \cdot \varphi d\Gamma + \int\limits_{D_T} f(u) \cdot \varphi \, dx\, dt$$

$$= \int\limits_{D_T} F \cdot \varphi \, dx\, dt \qquad \forall \varphi \in C_0^{2,4k}(\bar{D}_T), \tag{1.8}$$

where the symbol $\eta \cdot \xi$ denotes the scalar product of N-dimensional vectors, i.e., $\sum_{i=1}^{N} \eta_i \cdot \xi_i.$

Equality (1.8) is used as basic in the definition of a weak generalized solution of problem (1.1)–(1.3).

Definition 1.1. *Assume that a vector function f satisfies conditions (1.5) and (1.6) and that $F \in L_2(D_T).$ A vector function $u \in W_0^{1,2k}(D_T)$ is called a weak generalized solution of problem (1.1)–(1.3) if the integral equality (1.8) holds for any vector function $\varphi \in W_0^{1,2k}(D_T),$ i.e.,*

$$\int\limits_{D_T} \left[\frac{\partial^{2k} u}{\partial t^{2k}} \frac{\partial^{2k} \varphi}{\partial t^{2k}} + \sum_{i,j=1}^{n} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right] dx\, dt$$

$$+ \int\limits_{\Gamma} Bu \cdot \varphi d\Gamma + \int\limits_{D_T} f(u) \cdot \varphi \, dx\, dt$$

$$= \int\limits_{D_T} F \cdot \varphi \, dx\, dt \qquad \forall \varphi \in W_0^{1,2k}(D_T). \tag{1.9}$$

Note that, according to Remark 1.2, the integral $\int_{D_T} f(u) \cdot \varphi \, dx\, dt$ in equality (1.9) is well defined because the fact that $u \in W_0^{1,2k}(D_T)$ implies that $f(u) \in L_2(D_T)$ and, hence, $f(u) \cdot \varphi \in L_1(D_T).$

It is easy to see that if the solution u of problem (1.1)–(1.3) belongs to the class $C_0^{2,4k}(\overline{D}_T)$ in a sense of Definition 1.1, then it is also a classical solution of this problem.

2. Solvability of Problem (1.1)–(1.3)

In what follows, we assume that the operator

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(A_{ij}\frac{\partial}{\partial x_i}\right) \tag{2.1}$$

is strongly elliptic [15, p. 96], i.e.,

$$\sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j \geq c_0 \sum_{i=1}^{n} |\xi_i|^2 \quad \forall x \in \Omega \quad \text{and} \quad \xi_1, \dots, \xi_n \in \mathbb{R}^N, \tag{2.2}$$

where $c_0 = \text{const} > 0$.

Note that in the scalar case, under condition (2.2), the operator in (2.1) is an ordinary uniformly elliptic operator. In this case, the linear part of the operator L_f in (1.1), i.e., L_0, is semielliptic for any fixed $x \in \Omega$ [16, p. 142].

If, in addition to condition (2.2), the following condition is satisfied:

$$B(x)\eta \cdot \eta \geq 0 \quad \forall x \in \Gamma, \quad \eta \in \mathbb{R}^N, \tag{2.3}$$

then, in the space $C_0^{2,4k}(\bar{D}_T)$, parallel with the scalar product

$$(u, v)_0 = \int\limits_{D_T} \left[u \cdot v + \sum_{i=1}^{n} \frac{\partial^i u}{\partial t^i}\frac{\partial^i v}{\partial t^i} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} \right] dx\, dt \tag{2.4}$$

and the norm $\| \cdot \|_0 = \| \cdot \|_{W_0^{1,2k}(D_T)}$ given by the right-hand side of equality (1.4), we can introduce a scalar product

$$(u, v)_1 = \int\limits_{D_T} \left[\frac{\partial^{2k} u}{\partial t^{2k}}\frac{\partial^{2k} v}{\partial t^{2k}} + \sum_{i,j=1}^{n} A_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} \right] dx\, dt + \int\limits_{\Gamma} Bu \cdot v d\Gamma \tag{2.5}$$

with the norm

$$\|u\|_1^2 = \int\limits_{D_T} \left[\left(\frac{\partial^{2k} u}{\partial t^{2k}}\right)^2 + \sum_{i,j=1}^{n} A_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} \right] dx\, dt + \int\limits_{\Gamma} Bu \cdot u d\Gamma, \tag{2.6}$$

where $u, v \in C_0^{2,4k}(\bar{D}_T)$.

Lemma 2.1. *Under conditions (2.2) and (2.3), the inequalities*

$$c_1\|u\|_0 \leq \|u\|_1 \leq c_2\|u\|_0 \quad \forall u \in C_0^{2,4k}(\bar{D}_T) \tag{2.7}$$

are true with positive constants c_1 and c_2 independent of u.

Proof. If $u \in C_0^{2,4k}(\bar{D}_T)$, then $u(x,0)=0,\ x \in \Omega$ and, hence,

$$u(x,t) = \int_0^t \frac{\partial u(x,\tau)}{\partial t} d\tau, \quad (x,t) \in D_T.$$

Thus, by using standard arguments, we obtain [13, p. 69]

$$\int_{D_T} u^2 \, dx \, dt \le T \int_{D_T} \left(\frac{\partial u}{\partial t}\right)^2 dx \, dt. \tag{2.8}$$

We now estimate the norms $\left\| \frac{\partial^i u}{\partial t^i} \right\|_{L_2(D_T)}$, $i = 1, \dots, 2k-1$, in terms of the norm $\left\| \frac{\partial^{2k} u}{\partial t^{2k}} \right\|_{L_2(D_T)}$. Since $u \in C_0^{2,4k}(\bar{D}_T)$ satisfies equality (1.3), it is easy to see that

$$\frac{\partial^i u(\cdot\,,t)}{\partial t^i} = \frac{1}{(2k-i-1)!} \int_0^t (t-\tau)^{2k-i-1} \frac{\partial^{2k} u(\cdot\,,\tau)}{\partial t^{2k}} d\tau, \quad i = 1, \dots, 2k-1. \tag{2.9}$$

In view of the Cauchy inequality, it follows from (2.9) that

$$\left(\frac{\partial^i u(\cdot\,,t)}{\partial t^i}\right)^2 \le \frac{1}{((2k-i-1)!)^2} \int_0^t (t-\tau)^{2(2k-i-1)} d\tau \int_0^t \left(\frac{\partial^{2k} u(\cdot\,,t)}{\partial t^{2k}}\right)^2 d\tau$$

$$= \frac{t^{4k-2i-1}}{((2k-i-1)!)^2(4k-2i-1)} \int_0^t \left(\frac{\partial^{2k} u(\cdot\,,t)}{\partial t^{2k}}\right)^2 d\tau$$

$$\le T^{4k-2i-1} \int_0^T \left(\frac{\partial^{2k} u(\cdot\,,\tau)}{\partial t^{2k}}\right)^2 d\tau,$$

whence it follows that

$$\int_0^T \left(\frac{\partial^i u(\cdot\,,t)}{\partial t^i}\right)^2 d\tau \le T^{4k-2i} \int_0^T \left(\frac{\partial^{2k} u(\cdot\,,\tau)}{\partial t^{2k}}\right)^2 d\tau, \quad i = 1, \dots, 2k-1. \tag{2.10}$$

Since $A_{ij} = A_{ij}(x) \in C(\overline{\Omega}),\ i,j = 1, \dots, n,$ the elements of these matrices are bounded in $\overline{\Omega}$ and, hence,

$$\sum_{i,j=1}^n A_{ij}(x)\xi_i \cdot \xi_j \le \tilde{c_0} \sum_{i=1}^n |\xi_i|^2 \quad \forall x \in \overline{\Omega} \quad \text{and} \quad \xi_1, \dots, \xi_n \in \mathbb{R}^N \tag{2.11}$$

with a positive constant $\tilde{c_0}$ independent of $x \in \overline{\Omega}$ and $\xi_1, \dots, \xi_n \in \mathbb{R}^N$.

In view of (2.2) and (2.11), for any $u \in C_0^{2,4k}(\bar{D}_T)$, we obtain

$$c_0 \int\limits_{D_T} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 dxdt \le \int\limits_{D_T} \sum_{i,j=1}^{n} A_{ij} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} dxdt$$

$$\le \tilde{c_0} \int\limits_{D_T} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 dxdt. \tag{2.12}$$

By virtue of (2.3) and the embedding theorem, for the trace $v|_\Gamma$ of the vector function $v \in W_2^1(D_T)$, we get [13, p. 72]

$$0 \le \int\limits_{\Gamma} B(x)v \cdot v d\Gamma \le \tilde{c}_3 \int\limits_{D_T} \left[v^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i}\right)^2 \right] dx\, dt \tag{2.13}$$

with a positive constant $\tilde{c}_3$ independent of v.

Finally, by using (1.4), (2.4), (2.6), (2.8), (2.12), and (2.13), we easily obtain (2.7).

Lemma 2.1 is proved.

Remark 2.1. According to Lemma 2.1, if we complete the space $C_0^{2,2k}(\bar{D}_T)$ with respect to norm (2.5), then, in view of (2.4), we obtain the same Hilbert space $W_0^{1,2k}(D_T)$ with equivalent scalar products (2.4) and (2.5).

Consider a condition

$$\lim_{|u|\to\infty} \inf \frac{u \cdot f(u)}{u^2} \ge 0. \tag{2.14}$$

Lemma 2.2. *Suppose that $F \in L_2(D_T)$ and conditions (1.5), (1.6), (2.2), (2.3), and (2.14) are satisfied. Then, for any weak generalized solution $u \in W_0^{1,2k}(D_T)$ of problem (1.1)–(1.3), the a priori estimate*

$$\|u\|_0 = \|u\|_{W_0^{1,2k}(D_T)} \le c_3 \|F\|_{L_2(D_T)} + c_4 \tag{2.15}$$

is true with constants $c_3 > 0$ and $c_4 \ge 0$ independent of u and F.

Proof. Since $f \in C(\mathbb{R}^N)$, inequality (2.14) implies that, for any $\varepsilon > 0$, there exists a number $M_\varepsilon \ge 0$ such that

$$u \cdot f(u) \ge -M_\varepsilon - \varepsilon u^2 \quad \forall u \in \mathbb{R}^N. \tag{2.16}$$

Setting $\varphi = u \in W_0^{1,2k}(D_T)$ in equality (1.9) and taking into account (2.16) and (2.6), for any $\varepsilon > 0$, we get

$$\|u\|_1^2 = -\int\limits_{D_T} u \cdot f(u)\, dx\, dt + \int\limits_{D_T} F \cdot u\, dx\, dt$$

$$\le M_\varepsilon \operatorname{mes} D_T + \varepsilon \int\limits_{D_T} u^2\, dx\, dt + \int\limits_{D_T} \left(\frac{1}{4\varepsilon} F^2 + \varepsilon u^2\right) dx\, dt$$

$$= \frac{1}{4\varepsilon}\|F\|^2_{L_2(D_T)} + M_\varepsilon \operatorname{mes} D_T + 2\varepsilon\|u\|^2_{L_2(D_T)}$$

$$\le \frac{1}{4\varepsilon}\|F\|^2_{L_2(D_T)} + M_\varepsilon \operatorname{mes} D_T + 2\varepsilon\|u\|^2_0. \tag{2.17}$$

In view of (2.7), it follows from (2.17) that

$$c_1^2\|u\|_0^2 \le \|u\|_1^2 \le \frac{1}{4\varepsilon}\|F\|^2_{L_2(D_T)} + M_\varepsilon \operatorname{mes} D_T + 2\varepsilon\|u\|^2_0.$$

For $\varepsilon = \frac{1}{4}c_1^2$, this yields

$$\|u\|_0^2 \le 2c_1^{-4}\|F\|^2_{L_2(D_T)} + 2c_1^{-2}M_\varepsilon \operatorname{mes} D_T.$$

The last inequality gives (2.15), where $c_3^2 = 2c_1^{-4}$ and $c_4^2 = 2c_1^{-2}M_\varepsilon \operatorname{mes} D_T$ for $\varepsilon = \frac{1}{4}c_1^2$.

Lemma 2.2 is proved.

Remark 2.2. Prior to study the solvability of problem (1.1)–(1.3) in the nonlinear case, we consider the corresponding linear problem (1.1)–(1.3), i.e., the case $f = 0$. In this case, for $F \in L_2(D_T)$, we introduce, in a similar way, the definition of a weak generalized solution $u \in \overset{\circ}{W}{}^{1,2k}_0(D_T)$ of this problem for which the following integral equality us true:

$$(u,\varphi)_1 = \int\limits_{D_T}\left[\frac{\partial^{2k}u}{\partial t^{2k}}\frac{\partial^{2k}\varphi}{\partial t^{2k}} + \sum_{i,j=1}^n A_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \varphi}{\partial x_j}\right]dx\,dt + \int\limits_\Gamma Bu\cdot\varphi d\Gamma$$

$$= \int\limits_{D_T} F\cdot\varphi\,dx\,dt \quad \forall\varphi \in W^{1,2k}_0(D_T). \tag{2.18}$$

By using (1.4), (2.4), and (2.7), we obtain

$$\left|\int\limits_{D_T} F\cdot\varphi\,dx\,dt\right| \le \|F\|_{L_2(D_T)}\|\varphi\|_{L_2(D_T)}$$

$$\le \|F\|_{L_2(D_T)}\|\varphi\|_0 \le c_1^{-1}\|F\|_{L_2(D_T)}\|\varphi\|_1. \tag{2.19}$$

According to Remark 2.1, in view of relations (2.18) and (2.19), it follows from the Riesz theorem that there exists a unique vector function $u \in W^{1,2k}_0(D_T)$ satisfying equality (2.18) for any $\varphi \in W^{1,2k}_0(D_T)$, and its norm can be estimated as follows:

$$\|u\|_1 \le c_1^{-1}\|F\|_{L_2(D_T)}. \tag{2.20}$$

In view of (2.7), it follows from (2.20) that

$$\|u\|_0 = \|u\|_{W^{1,2k}_0(D_T)} \le c_1^{-2}\|F\|_{L_2(D_T)}. \tag{2.21}$$

Thus, introducing the notation $u = L_0^{-1}F$, we conclude that the linear problem corresponding to (1.1)–(1.3) i.e., with $f = 0$ has the following linear bounded operator:

$$L_0^{-1}:\ L_2(D_T) \to \overset{\circ}{W}{}^{1,2k}(D_T).$$

According to (2.21), its norm can be estimated as follows;

$$\|L_0^{-1}\|_{L_2(D_T)\to \overset{\circ}{W}{}^{1,2k}(D_T)} \le c_1^{-2}. \tag{2.22}$$

By using Definition 1.1 and Remark 2.2, we rewrite the integral identity (1.9) equivalent to problem (1.1)–(1.3) in the form of the following functional equation:

$$u = L_0^{-1}[-f(u) + F] \tag{2.23}$$

in the Hilbert space $\overset{\circ}{W}{}^{1,2k}(D_T)$.

Remark 2.3. Note that, in view of (1.4) and Remark 1.1, the space $\overset{\circ}{W}{}^{1,2k}(D_T)$ is continuously embedded in the space $W_2^1(D_T)$. Thus, by virtue of (1.7) and Remark 1.2, under conditions (1.5) and (1.6), the operator

$$K_1 = KII_1:\ \overset{\circ}{W}{}^{1,2k}(D_T) \to L_2(D_T),$$

where $I_1:\ \overset{\circ}{W}{}^{1,2k}(D_T) \to W_2^1(D_T)$ is the embedding operator, is also continuous and compact.

We rewrite Eq. (2.23) in the form

$$u = Au := L_0^{-1}(K_1 u + F). \tag{2.24}$$

By virtue of (2.23) and Remark 2.3, we conclude that the operator $A\colon\ \overset{\circ}{W}{}^{1,2k}(D_T) \to \overset{\circ}{W}{}^{1,2k}(D_T)$ in (2.24) is continuous and compact. At the same time, by using the scheme of the proof of the *a priori* estimate (2.15) with

$$c_3^2 = 2c_1^{-4} \qquad \text{and} \qquad c_4^2 = 2c_1^{-2}M_\varepsilon \operatorname{mes} D_T, \quad \varepsilon = \frac{1}{4}c_1^2,$$

we can easily show that, for any value of the parameter $\tau \in [0,1]$ and any solution $u \in \overset{\circ}{W}{}^{1,2k}(D_T)$ of the equation $u = \tau Au$, the same *a priori* estimate (2.15) holds with the same constants $c_3 > 0$ and $C_4 \ge 0$ independent of u, F, and τ. Thus, by the Leray–Schauder fixed-point theorem [16, p. 375], Eq. (2.24) and, hence, also problem (1.1)–(1.3) have at least one weak generalized solution u in the space $\overset{\circ}{W}{}^{1,2k}(D_T)$. Thus, the following theorem is true:

Theorem 2.1. *Suppose that conditions (1.5), (1.6), (2.2), (2.3), and (2.14) are satisfied. Then, for any $F \in L_2(D_T)$, problem (1.1)–(1.3) has at least one weak generalized solution u in the space $\overset{\circ}{W}{}^{1,2k}(D_T)$.*

3. Uniqueness of the Solution of Problem (1.1)–(1.3)

Consider the condition of monotonicity of the Nemytskii operator

$$K(u) = f(u)\colon \mathbb{R}^N \to \mathbb{R}^N,$$

i.e.,

$$(K(u) - K(v))(u - v) \geq 0 \quad \forall u, v \in \mathbb{R}^N. \tag{3.1}$$

Remark 3.1. It is easy to see that condition (3.1) is satisfied if $f = (f_1, \dots, f_N) \in C^1(\mathbb{R}^N)$ and the matrix $\left(\dfrac{\partial f_i}{\partial u_j}\right)_{i,j=1}^{n}$ is nonnegative-definite, i.e.,

$$\sum_{i,j=1}^{N} \frac{\partial f_i}{\partial u_j}(u)\xi_i\xi_j \geq 0 \quad \forall \xi = (\xi_1, \dots, \xi_N), \quad u = (u_1, \dots, u_N) \in \mathbb{R}^N.$$

Theorem 3.1. *Suppose that a vector function f satisfies conditions (1.5) and (1.6) and that the corresponding Nemytskii operator $K(u) = f(u)\colon \mathbb{R}^N \to \mathbb{R}^n$ is monotone. Also let conditions (2.2) and (2.3) be satisfied. Then, for any vector function $f \in L_2(D_T)$, problem (1.1)–(1.3) cannot have more than one weak generalized solution in the space $W_0^{1,2k}(D_T)$.*

Proof. Let $f \in L_2(D_T)$ and let u_1 and u_2 be two weak generalized solutions of problem (1.1)–(1.3) in the space $W_0^{1,2k}(D_T)$, i.e., according to (1.9), the following equalities are true:

$$\int\limits_{D_T} \left[\frac{\partial^{2k} u_i}{\partial t^{2k}} \frac{\partial^{2k} \varphi}{\partial t^{2k}} + \sum_{i,j=1}^{n} A_{ij} \frac{\partial u_i}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\right] dx\, dt$$

$$+ \int\limits_{\Gamma} B u_i \cdot \varphi d\Gamma + \int\limits_{D_T} f(u_i) \cdot \varphi\, dx\, dt$$

$$= \int\limits_{D_T} F \cdot \varphi\, dx\, dt \quad \forall \varphi \in W_0^{1,2k}(D_T), \quad i = 1,\, 2. \tag{3.2}$$

By using (3.2), for the difference $v = u_2 - u_1$, we obtain

$$\int\limits_{D_T} \left[\frac{\partial^{2k} v}{\partial t^{2k}} \frac{\partial^{2k} \varphi}{\partial t^{2k}} + \sum_{i,j=1}^{n} A_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \varphi}{\partial x_j}\right] dx\, dt + \int\limits_{\Gamma} B v \cdot \varphi d\Gamma$$

$$= -\int\limits_{D_T} (f(u_2) - f(u_1)) \cdot \varphi\, dx\, dt \qquad \forall \varphi \in W_0^{1,2k}(D_T). \tag{3.3}$$

Setting $\varphi = v \in W_0^{1,2k}(D_T)$ in equality (3.3), in view of (2.6), we get

$$\|v\|_1 = -\int\limits_{D_T} (f(u_2) - f(u_1))(u_2 - u_1)\, dx\, dt. \tag{3.4}$$

Since, by the condition of the theorem, the Nemytskii operator

$$K(u) = f(u)\colon \mathbb{R}^N \to \mathbb{R}^n$$

satisfies inequality (3.1), in view of (2.7) and (3.4), we obtain

$$c_1\|v\|_0 \le \|v\|_1 \le 0.$$

This yields $v = 0$ and, hence, $u_2 = u_1$.

Theorem 3.1 is proved.

Theorems 2.1 and 3.1 yield the following assertion:

Theorem 3.2. *Assume that conditions (1.5), (1.6), (2.2), (2.3), (2.14), and (3.1) are satisfied. Then, for any $F \in L_2(D_T)$, problem (1.1)–(1.3) possesses a unique weak generalized solution u in the space $\overset{\circ}{W}{}^{1,2k}_0(D_T)$.*

4. Cases of Absence of the Solutions of Problem (1.1)–(1.3)

We now consider a special case of system (1.1) in which it is split in the leading part, i.e., $A_{ij} = a_{ij} I_N$, where I_N is the identity matrix of order N and $a_{ij} = a_{ij}(x)$ are scalar functions such that the operator

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right)$$

is elliptic. We also assume that, in the boundary condition (1.2), $B = bI_N$, where b is a nonnegative number.

We consider the following condition imposed on the vector function f: there exist numbers $l_1, \ldots, l_N$, $\sum_{i=1}^{N} |l_i| \neq 0$, such that

$$\sum_{i=1}^{N} l_i f_i(u) \le -\left|\sum_{i=1}^{N} l_i u_i\right|^{\beta} \quad \forall u \in \mathbb{R}^N, \quad 1 < \beta = \text{const} < \frac{n+1}{n-1}. \tag{4.1}$$

To simplify our presentation, we consider the case where $\Omega\colon |x| < 1$.

Theorem 4.1. *Suppose that a vector function f satisfies conditions (1.5), (1.6), and (4.1),*

$$F^0 = (F_1^0, \ldots, F_N^0) \in L_2(D_T), \qquad G = \sum_{i=1}^{N} l_i F_i^0 \ge 0, \qquad \textit{and} \qquad \|G\|_{L_2(D_T)} \neq 0.$$

Then there exists a number $\mu_0 = \mu_0(G, \beta) > 0$ such that, for $\mu > \mu_0$, problem (1.1)–(1.3) cannot have weak generalized solutions in the space $W_0^{1,2k}(D_T)$ for $F = \mu F_0$.

Proof. Assume that the conditions of the theorem are satisfied and that a weak generalized solution $u \in W_0^{1,2k}(D_T)$ of problem (1.1)–(1.3) exists for any fixed $\mu > 0$. We also assume that $\varphi = (l_1\varphi_0, \ldots, l_N\varphi_0)$ in equality (1.9), where φ_0 is a scalar function satisfying the conditions

$$\varphi_0 \in C_0^{2,4k}(\bar{D}_T), \qquad \varphi_0\big|_{\Gamma} = 0, \qquad \frac{\partial \varphi_0}{\partial x_i}\bigg|_{\Gamma} = 0, \quad i = 1, \ldots, n, \qquad \varphi_0\big|_{D_T} > 0. \tag{4.2}$$

The space $C_0^{2,4k}(\bar{D}_T) \subset W_0^{1,2k}(D_T)$ was introduced in the introduction. In this case, as the function $\varphi_0,$ we can take the function

$$\varphi_0(x,t) = \big[(1-|x|^2)t(T-t)\big]^{2k}.$$

We set

$$v = \sum_{i=1}^{N} l_i u_i.$$

Thus, by using the facts that system (1.1) is split in the leading part and $B = BI_N,$ by virtue of (1.9), we conclude that

$$\int\limits_{D_T} \left[\frac{\partial^{2k} v}{\partial t^{2k}} \frac{\partial^{2k} \varphi_0}{\partial t^{2k}} + \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial \varphi_0}{\partial x_j} \right] dx\, dt + \int\limits_{\Gamma} bv\varphi_0 d\Gamma$$

$$= \int\limits_{D_T} \left(-\sum_{i=1}^{N} l_i f_i(u) \right) \varphi_0 \, dx\, dt + \mu \int\limits_{D_T} G\varphi_0 \, dx\, dt. \tag{4.3}$$

In view of (4.2) and $V \in W_0^{1,2k}(D_T),$ we can integrate (4.3) by parts and obtain

$$\int\limits_{D_T} \left(-\sum_{i=1}^{N} l_i f_i(u) \right) \varphi_0 \, dx\, dt + \mu \int\limits_{D_T} G\varphi_0 \, dx\, dt$$

$$= \int\limits_{D_T} v \left[\frac{\partial^{2k} \varphi_0}{\partial t^{2k}} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial \varphi_0}{\partial x_i} \right) \right] dx\, dt$$

$$= \int\limits_{D_T} vL_0\varphi_0 \, dx\, dt, \tag{4.4}$$

where L_0 is the scalar operator corresponding to the operator in (1.1) with $f = 0.$

It follows from (4.1) and (4.4) that

$$\int\limits_{D_T} |v|^\beta \varphi_0 \, dx\, dt \le \int\limits_{D_T} vL_0\, \varphi_0 \, dx\, dt - \mu \int\limits_{D_T} G\varphi_0 \, dx\, dt. \tag{4.5}$$

Further, we apply the method of test functions [18, pp. 10–12]. In the Young inequality with parameter $\varepsilon > 0,$

$$ab \le \frac{\varepsilon}{\beta} a^\beta + \frac{1}{\beta' \varepsilon^{\beta'-1}} b^{\beta'}, \quad a, b \ge 0, \qquad \beta' = \frac{\beta}{\beta - 1},$$

we set

$$a = |u|\varphi_0^{1/\beta} \qquad \text{and} \qquad b = |L_0\varphi_0| \,/\, \varphi_0^{1/\beta}$$

and take into account the fact that $\beta'/\beta = \beta' - 1.$ This yields

$$|vL_0\varphi_0| = |v|\varphi_0^{1/\beta}\frac{|L_0\varphi_0|}{\varphi_0^{1/\beta}} \le \frac{\varepsilon}{\beta}|v|^\beta\varphi_0 + \frac{1}{\beta'\varepsilon^{\beta'-1}}\frac{|L_0\varphi_0|^{\beta'}}{\varphi_0^{\beta'-1}}. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$\left(1 - \frac{\varepsilon}{\beta}\right)\int\limits_{D_T}|v|^\beta\varphi_0\,dx\,dt = \frac{1}{\beta'\varepsilon^{\beta'-1}}\int\limits_{D_T}\frac{|L_0\varphi_0|^{\beta'}}{\varphi_0^{\beta'-1}}\,dx\,dt - \mu\int\limits_{D_T}G\varphi_0\,dx\,dt.$$

For $\varepsilon < \beta,$ this yields

$$\int\limits_{D_T}|v|^\beta\varphi_0\,dx\,dt \le \frac{\beta}{(\beta-\varepsilon)\beta'\varepsilon^{\beta'-1}}\int\limits_{D_T}\frac{|L_0\varphi_0|^{\beta'}}{\varphi_0^{\beta'-1}}\,dx\,dt - \frac{\beta\mu}{\beta-\varepsilon}\int\limits_{D_T}G\varphi_0\,dx\,dt. \tag{4.7}$$

By using the equalities

$$\beta' = \frac{\beta}{\beta-1} \quad \text{and} \quad \beta = \frac{\beta'}{\beta'-1}$$

and also the fact that

$$\min_{0<\varepsilon<\beta}\frac{\beta}{(\beta-\varepsilon)\beta'\varepsilon^{\beta'-1}} = 1$$

(this minimum is attained for $\varepsilon = 1),$ we get the following inequality from (4.7):

$$\int\limits_{D_T}|v|^\beta\varphi_0\,dx\,dt \le \int\limits_{D_T}\frac{|L_0\varphi_0|^{\beta'}}{\varphi_0^{\beta'-1}}\,dx\,dt - \beta'\mu\int\limits_{D_T}G\varphi_0\,dx\,dt. \tag{4.8}$$

It is easy to see that there exists a test function φ_0 such that, in addition to (4.2),

$$\kappa_0 = \int\limits_{D_T}\frac{|L_0\varphi_0|^{\beta'}}{\varphi_0^{\beta'-1}}\,dx\,dt < \infty. \tag{4.9}$$

Indeed, we can easily show that the function

$$\varphi_0(x,t) = \big[(1-|x|^2)t(T-t)\big]^m,$$

satisfies condition (4.9) for a sufficiently large positive m.

Since, under the condition of the theorem, $G \in L_2(D_T),$ $\|G\|_{L_2(D_T)} \ne 0,$ $G \ge 0,$ and $\operatorname{mes} D_T < +\infty,$ in view of the fact that $\varphi_0\,|_{D_T} > 0,$ we obtain

$$0 < \kappa_1 = \int\limits_{D_T}G\varphi_0\,dx\,dt < +\infty. \tag{4.10}$$

By $g(\mu)$ we denote the right-hand side of inequality (4.8), which is a linear function of μ. Thus, according to (4.9) and (4.10), we obtain

$$g(\mu) < 0 \quad \text{for} \quad \mu > \mu_0 \qquad \text{and} \qquad g(\mu) > 0 \quad \text{for} \quad \mu < \mu_0, \tag{4.11}$$

where

$$g(\mu) = \kappa_0 - \beta'\mu\kappa_1, \qquad \mu_0 = \frac{\kappa_0}{\beta'\kappa_1} > 0.$$

According to (4.11), for $\mu > \mu_0$, the right-hand side of inequality (4.8) is negative, whereas the left-hand side of this inequality is nonnegative. The obtained contradiction proves the theorem.

Note that in the case where condition (4.1) is satisfied, condition (2.14) is violated. Indeed, in this case, it suffices to take $u = \lambda(l_1, \ldots, l_N)$ as $\lambda \to +\infty$.

Remark 4.1. In Theorem 4.1, for the sake of simplicity, it has been assumed that $\Omega\colon |x| < 1$. However, this theorem remains true in a more general case where Ω is a sufficiently smooth domain. This assumption is explained by the structure of the test function φ_0 satisfying condition (4.9) according to the formula

$$\varphi_0(x,t) = \big[(1 - |x|^2)t(T-t)\big]^m \tag{4.12}$$

for sufficiently large positive m. If the boundary of the domain Ω is given by the equation $\partial\Omega\colon \omega(x) = 0$, where

$$\nabla_x\omega|_{\partial\Omega} \neq 0, \qquad \omega|_\Omega > 0, \qquad \nabla_x = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \qquad \text{and} \qquad \omega \in C^2(\mathbb{R}^n),$$

then the test function given by Eq. (4.12) should be replaced by

$$\varphi_0(x,t) = \big[\omega(x)t(T-t)\big]^m,$$

where m is a sufficiently large positive number. In this case, Theorem 4.1 also remains true.

Remark 4.2. In the proof of Theorem 4.1, condition (4.1) can be replaced by a more general condition

$$\sum_{i=1}^N l_i f_i(u) \le -d_0 \left|\sum_{i=1}^N l_i u_i\right|^\beta \quad \forall u \in \mathbb{R}^N, \quad 1 < \beta = \text{const} < \frac{n+1}{n-1}, \tag{4.13}$$

where $d_0 = \text{const} > 0$. Indeed, case (4.13) is reduced to case (4.1) if we pass from l_i to $\tilde{l}_i$ by the formula

$$l_i = \lambda \tilde{l}_i,$$

where $\lambda = d_0^{\frac{1}{1-\beta}}$. As a result, we obtain inequality (4.1) with $\tilde{l}_i$ instead of l_i. We now present a class of vector functions f satisfying condition (4.13):

$$f_i(u_1, \ldots, u_N) = \sum_{j=1}^N a_{ij}|u_j|^{\beta_{ij}} + b_i, \quad i = 1, \ldots, N, \tag{4.14}$$

where the constant numbers a_{ij}, β_{ij}, and b_i satisfy the inequalities

$$a_{ij} > 0, \quad 1 < \beta_{ij} < \frac{n+1}{n-1}, \qquad \sum_{i=1}^{N} b_i > 0, \quad i,j = 1,\ldots,N. \tag{4.15}$$

In this case, it is necessary to set $l_1 = \ldots = l_N = -1$ in (4.13). Indeed, according to (4.15), we choose constant numbers α_0 and β such that

$$0 < a_0 \le \min_{i,j} a_{ij}, \qquad \sum_{i=1}^{N} b_i - a_0 N^2 \ge 0, \qquad 1 < \beta < \beta_{ij}, \quad i,j = 1,\ldots,N. \tag{4.16}$$

It is easy to see that $|s|^{\beta_{ij}} \ge |s|^{\beta} - 1 \ \ \forall s \in (-\infty, \infty)$. By using the well-known inequality [19, p. 302]

$$\sum_{i=1}^{N} |y_i|^{\beta} > N^{1-\beta} \left| \sum_{i=1}^{N} y_i \right|^{\beta} \quad \forall y = (y_1,\ldots,y_N) \in \mathbb{R}^N, \qquad \beta = \text{const} > 1,$$

by virtue of (4.14) and (4.15), we obtain

$$\begin{aligned} \sum_{i=1}^{N} f_i(u_1,\ldots,u_N) &\ge a_0 \sum_{i,j=1}^{N} |u_j|^{\beta_{ij}} + \sum_{i=1}^{N} b_i \\ &\ge a_0 \sum_{i,j=1}^{N} (|u_j|^{\beta} - 1) + \sum_{i=1}^{N} b_i \ge a_0 N \sum_{j=1}^{N} |u_j|^{\beta} - a_0 N^2 + \sum_{i=1}^{N} b_i \\ &\ge a_0 N^{2-\beta} \left| \sum_{j=1}^{N} u_j \right|^{\beta} + \sum_{i=1}^{N} b_i - a_0 N^2 \ge a_0 N^{2-\beta} \left| \sum_{j=1}^{N} u_j \right|^{\beta}. \end{aligned} \tag{4.17}$$

According to (4.17), we conclude that, under conditions (4.14) and (4.15), inequality (4.13) is true with

$$l_1 = \ldots = l_N = -1 \qquad \text{and} \qquad d_0 = a_0 N^{2-\beta}.$$

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