DOI 10.1007/s11253-022-02102-5
Ukrainian Mathematical Journal, Vol. 74, No. 5, October, 2022 (Ukrainian Original Vol. 74, No. 5, May, 2022)

DIRECT AND INVERSE APPROXIMATION THEOREMS IN THE BESICOVITCH–MUSIELAK–ORLICZ SPACES OF ALMOST PERIODIC FUNCTIONS

S. O. Chaichenko,[1] **A. L. Shidlich,**[2,3] **and T. V. Shulyk**[4]

UDC 517.5

In terms of the best approximations of functions and generalized moduli of smoothness, direct and inverse approximation theorems are proved for the Besicovitch almost periodic functions whose Fourier exponent sequences have a single limit point at infinity and their Orlicz norms are finite. Special attention is given to the study of cases where the constants in these theorems are unimprovable.

1. Introduction

The investigations of relationships between the difference and differential properties of the approximated function and the value of the error of its approximation by certain methods were originated in the well-known works by Jackson (1911) and Bernstein (1912) in which the first direct and inverse approximation theorems were obtained. Later, similar studies were carried out by numerous authors for various functional classes and for various approximating aggregates. Their results constitute the classics of modern approximation theory. Moreover, the exact results (in particular, in a sense of unimprovable constants) deserve special attention. A fairly complete description of the results on obtaining direct and inverse approximation theorems can be found in the monographs [14, 28, 30, 31].

In the spaces of almost periodic functions, direct approximation theorems were established in [8, 12, 23, 24, 26]. Thus, in particular, Prytula [23] obtained the direct approximation theorem for Besicovitch almost periodic functions of order 2 (B^2-a.p. functions) in terms of the best approximations of functions and their moduli of continuity. In [24] and [8], theorems of this kind were obtained with moduli of smoothness of B^2-a.p. functions of any positive integer order and with generalized moduli of smoothness, respectively. In [26], the direct and inverse approximation theorems were obtained in the Besicovitch–Stepanets spaces $B\mathcal{S}^p$. The main aim of the present paper is to obtain the corresponding theorems in the Besicovitch–Musielak–Orlicz spaces $B\mathcal{S}_{\mathbf{M}}$. These spaces are obtained as natural generalizations of all above-mentioned spaces, and the accumulated results can be regarded as extensions of the corresponding results to the spaces $B\mathcal{S}_{\mathbf{M}}$.

2. Preliminaries

2.1. Definition of the spaces $B\mathcal{S}_{\mathbf{M}}$. Let B^s, $1 \le s < \infty$, be the space of all functions Lebesgue summable with the sth power in each finite interval of the real axis, where the distance is defined by the equality

$$D_{B^s}(f,g) = \left(\limsup_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} |f(x)-g(x)|^s dx \right)^{1/s}.$$

[1] Donbas State Pedagogical University, Sloviansk, Donetsk Oblast, Ukraine; e-mail: s.chaichenko@gmail.com.
[2] Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine and National University of Life and Environmental Sciences of Ukraine, Kyiv, Ukraine; e-mail: shidlich@gmail.com.
[3] Corresponding author.
[4] Donbas State Pedagogical University, Sloviansk, Donetsk Oblast, Ukraine; e-mail: schulik111@gmal.com.

Published in Ukrains'kyi Matematychnyi Zhurnal, Vol. 74, No. 5, pp. 701–716, May, 2022. Ukrainian DOI: 10.37863/umzh.v74i5.7045.
Original article submitted December 14, 2021.

Further, let $\mathcal{T}$ be the set of all trigonometric sums of the form

$$\tau_N(x) = \sum_{k=1}^{N} a_k e^{\mathrm{i}\lambda_k x}, \quad N \in \mathbb{N},$$

where λ_k and a_k are arbitrary real and complex numbers ($\lambda_k \in \mathbb{R}$, $a_k \in \mathbb{C}$).

An arbitrary function f is called a Besicovitch almost periodic function of order s (or a B^s-a.p. function) and denoted by $f \in B^s$-a.p. {see[20] (Ch. 5, §10) and [10] (Ch. 2, §7)} if there exists a sequence of trigonometric sums $\tau_1, \tau_2, \ldots$ from the set $\mathcal{T}$ such that

$$\lim_{N\to\infty} D_{B^s}(f, \tau_N) = 0.$$

If $s_1 \geq s_2 \geq 1$, then (see, e.g., [12, 13]) B^{s_1}-a.p. $\subset B^{s_2}$-a.p. $\subset B$-a.p., where B-a.p. $:= B^1$-a.p. For any B-a.p. function f, there exists the average value

$$A\{f\} := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x)dx.$$

The value of the function $A\{f(\cdot)e^{-\mathrm{i}\lambda\cdot}\}$, $\lambda \in \mathbb{R}$, can be nonzero on at most a countable set. As a result of numbering the values of this set in an arbitrary order, we obtain a set $\mathcal{S}(f) = \{\lambda_k\}_{k\in\mathbb{N}}$ of Fourier exponents, which is called the spectrum of the function f. The numbers

$$A_{\lambda_k} = A_{\lambda_k}(f) = A\{f(\cdot)e^{-\mathrm{i}\lambda_k\cdot}\}$$

are called the Fourier coefficients of the function f. Each function $f \in B$-a.p. with spectrum $\mathcal{S}(f)$ is associated with a Fourier series of the form $\sum_k A_{\lambda_k} e^{\mathrm{i}\lambda_k x}$. If, in addition, $f \in B^2$-a.p., then the Parseval equality holds (see, e.g., [10], Ch. 2, § 9)

$$A\{|f|^2\} = \sum_{k\in\mathbb{N}} |A_{\lambda_k}|^2.$$

Further, we consider only the B-a.p. functions from the spaces BS^p for which the sequences of Fourier exponents have a single limit point at infinity. For these functions f, the Fourier series can be written in the symmetric form as follows:

$$S[f](x) = \sum_{k\in\mathbb{Z}} A_k e^{\mathrm{i}\lambda_k x}, \qquad \text{where} \quad A_k = A_k(f) = A\Big\{f(\cdot)e^{-\mathrm{i}\lambda_k\cdot}\Big\}, \tag{2.1}$$

$\lambda_0 := 0$, $\lambda_{-k} = -\lambda_k$, $|A_k| + |A_{-k}| > 0$, and $\lambda_{k+1} > \lambda_k > 0$ for $k > 0$.

Let $\mathbf{M} = \{M_k(t)\}_{k\in\mathbb{Z}}$, $t \geq 0$, be a sequence of Orlicz functions. In other words, for every $k \in \mathbb{Z}$, the function $M_k(t)$ is a nondecreasing convex function for which $M_k(0) = 0$ and $M_k(t) \to \infty$ as $t \to \infty$. Let $\mathbf{M}^* = \big\{M_k^*(v)\big\}_{k\in\mathbb{Z}}$ be a sequence of functions defined by the relations

$$M_k^*(v) := \sup\{uv - M_k(u)\colon\ u \geq 0\}, \quad k \in \mathbb{Z}.$$

Consider the set $\Gamma = \Gamma(\mathbf{M}^*)$ of sequences of positive numbers $\gamma = \{\gamma_k\}_{k\in\mathbb{Z}}$ such that

$$\sum_{k\in\mathbb{Z}} M_k^*(\gamma_k) \le 1.$$

The modular space (or the Musielak–Orlicz space) $B\mathcal{S}_{\mathbf{M}}$ is the space of all functions f ($f \in B$-a.p.) such that the following quantity (which is also called the Orlicz norm of f) is finite:

$$\|f\|_{\mathbf{M}} := \|\{A_k\}_{k\in\mathbb{Z}}\|_{l_{\mathbf{M}}(\mathbb{Z})} := \sup\left\{\sum_{k\in\mathbb{Z}} \gamma_k |A_k(f)| :\ \gamma \in \Gamma(\mathbf{M}^*)\right\}. \tag{2.2}$$

By definition, the B-a.p. functions are regarded as identical in $B\mathcal{S}_{\mathbf{M}}$ if they have the same Fourier series.

The spaces $B\mathcal{S}_{\mathbf{M}}$ defined in this way are Banach spaces. Functional spaces of this kind were studied by mathematicians since the 1940s (see, e.g., the monographs [21, 22, 25]). In particular, the subspaces $\mathcal{S}_{\mathbf{M}}$ of all 2π-periodic functions from $B\mathcal{S}_{\mathbf{M}}$ were considered in [3, 5]. If all functions M_k are identical (namely, $M_k(t) \equiv M(t),\ k \in \mathbb{Z}$), then the spaces $\mathcal{S}_{\mathbf{M}}$ coincide with the ordinary Orlicz-type spaces $\mathcal{S}_M$ [15]. If $M_k(t) = \mu_k t^{p_k}$, $p_k \ge 1,\ \mu_k \ge 0$, then the spaces $\mathcal{S}_{\mathbf{M}}$ coincide with the weighted spaces $\mathcal{S}_{\mathbf{p},\,\mu}$ with variable exponents [2].

If all functions

$$M_k(u) = u^p\left(p^{-1/p}q^{-1/p'}\right)^p, \quad p > 1, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

then $B\mathcal{S}_{\mathbf{M}}$ are the Besicovitch–Stepanets spaces $B\mathcal{S}^p$ [26] of functions $f \in B$-a.p. with the norm

$$\|f\|_{\mathbf{M}} = \|f\|_{B\mathcal{S}^p} = \|\{A_{\lambda_k}(f)\}_{k\in\mathbb{N}}\|_{l_p(\mathbb{N})} = \left(\sum_{k\in\mathbb{N}} |A_{\lambda_k}(f)|^p\right)^{1/p}. \tag{2.3}$$

The subspaces of all 2π-periodic Lebesgue summable functions from $B\mathcal{S}^p$ coincide with the well-known spaces $\mathcal{S}^p$ (see, e.g., [28], Ch. XI). For $p = 2$, the sets $B\mathcal{S}^p = B\mathcal{S}^2$ coincide with the sets of B^2-a.p. functions, and the spaces $\mathcal{S}^p$ coincide with the ordinary Lebesgue spaces of 2π-periodic square-summable functions, i.e., $\mathcal{S}^2 = L_2$.

By G_{λ_n} we denote the set of all B-a.p. functions whose Fourier exponents belong to the interval $(-\lambda_n, \lambda_n)$ and define the value of the best approximation of $f \in B\mathcal{S}_{\mathbf{M}}$ by the equality

$$E_{\lambda_n}(f)_{\mathbf{M}} = E_{\lambda_n}(f)_{B\mathcal{S}_{\mathbf{M}}} = \inf_{g\in G_{\lambda_n}} \|f - g\|_{\mathbf{M}}. \tag{2.4}$$

2.2. Generalized Moduli of Smoothness. Let Φ be the set of all continuous bounded nonnegative pair functions $\varphi(t)$ such that $\varphi(0) = 0$ and the Lebesgue measure of the set $\{t \in \mathbb{R} :\ \varphi(t) = 0\}$ is equal to zero. For any fixed $\varphi \in \Phi$, we consider the generalized modulus of smoothness of a function $f \in B\mathcal{S}_{\mathbf{M}}$

$$\omega_\varphi(f, \delta)_{\mathbf{M}} := \sup_{|h|\le\delta} \sup\left\{\sum_{k\in\mathbb{Z}} \gamma_k \varphi(\lambda_k h)|A_k(f)| :\ \gamma \in \Gamma\right\}, \quad \delta \ge 0. \tag{2.5}$$

Consider the relationship between modulus (2.5) and some well-known moduli of smoothness. Let $\Theta = \{\theta_j\}_{j=0}^m$ be a nonzero collection of complex numbers such that

$$\sum_{j=0}^m \theta_j = 0.$$

We associate the collection Θ with a difference operator

$$\Delta_h^{\Theta}(f) = \Delta_h^{\Theta}(f,t) = \sum\nolimits_{j=0}^{m} \theta_j f(t-jh)$$

and the modulus of smoothness

$$\omega_{\Theta}(f,\delta)_{\mathbf{M}} := \sup_{|h|\le\delta} \big\|\Delta_h^{\Theta}(f)\big\|_{\mathbf{M}}.$$

Note that the collection $\Theta(m) = \Big\{\theta_j = (-1)^j\binom{m}{j},\ j=0,1,\ldots,m\Big\},\ m\in\mathbb{N},$ corresponds to the classical modulus of smoothness of order $m,$ i.e.,

$$\omega_{\Theta(m)}(f,\delta)_{\mathbf{M}} = \omega_m(f,\delta)_{\mathbf{M}}.$$

For any $k\in\mathbb{Z},$ the Fourier coefficients of the function $\Delta_h^{\Theta}(f)$ satisfy the equality

$$\big|A_k(\Delta_h^{\Theta}(f))\big| = |A_k(f)|\left|\sum_{j=0}^{m}\theta_j e^{-\mathrm{i}\lambda_k jh}\right|.$$

Therefore, in view of (2.2), we see that

$$\omega_{\varphi_\Theta}(f,\delta)_{\mathbf{M}} = \omega_{\Theta}(f,\delta)_{\mathbf{M}}$$

for

$$\varphi_\Theta(t) = \left|\sum_{j=0}^{m}\theta_j e^{-\mathrm{i}jt}\right|.$$

In particular, for

$$\varphi_m(t) = 2^m|\sin(t/2)|^m = 2^{\frac{m}{2}}(1-\cos t)^{\frac{m}{2}},\quad m\in\mathbb{N},$$

we get

$$\omega_{\varphi_m}(f,\delta)_{\mathbf{M}} = \omega_m(f,\delta)_{\mathbf{M}}.$$

Further, let

$$F_h(f,t) = f_h(x) := \frac{1}{2h}\int_{t-h}^{t+h} f(u)du$$

be the Steklov function of a function $f\in B\mathcal{S}_{\mathbf{M}}.$ We define the differences as follows:

$$\widetilde{\Delta}_h^1(f) := \widetilde{\Delta}_h^1(f,t) = F_h(f,t) - f(t) = (F_h - \mathbb{I})(f,t),$$

$$\widetilde{\Delta}_h^m(f) := \widetilde{\Delta}_h^m(f,t) = \widetilde{\Delta}_h^1(\Delta_h^{m-1}(f),t) = (F_h-\mathbb{I})^m(f,t) = \sum_{k=0}^{m} k^{m-k}\binom{m}{k}F_{h,k}(f,t),$$

where $m = 2, 3, \ldots,$ $F_{h,0}(f) := f,$ $F_{h,k}(f) := F_h(F_{h,k}(f)),$ and $\mathbb{I}$ is the identity operator in $B\mathcal{S}_{\mathbf{M}}$. Consider the following smoothness characteristics:

$$\widetilde{\omega}_m(f,\delta) := \sup_{0\le h\le\delta} \left\|\widetilde{\Delta}_h^m(f)\right\|_{\mathbf{M}}, \quad \delta > 0.$$

It can be shown [6] that

$$\omega_{\tilde{\varphi}_m}(f,\delta)_{\mathbf{M}} = \widetilde{\omega}_m(f,\delta)_{\mathbf{M}} \quad \text{for} \quad \tilde{\varphi}_m(t) = (1 - \operatorname{sinc} t)^m, \quad m \in \mathbb{N},$$

where $\operatorname{sinc} t = \{\sin t/t$ for $t \neq 0$ and 1 for $t = 0\}$.

In the general case, moduli similar to (2.5) were studied in [3–5, 8, 11, 19, 26, 32, 34].

3. Main Results

3.1. Jackson-Type Inequalities. In this section, we establish direct theorems for the functions $f \in B\mathcal{S}_{\mathbf{M}}$ in terms of the best approximations and generalized moduli of smoothness. Thus, in particular, for the functions $f \in B\mathcal{S}_{\mathbf{M}}$ with Fourier series of the form (2.1), we prove Jackson-type inequalities of the following kind:

$$E_{\lambda_n}(f)_{\mathbf{M}} \le K(\tau)\omega_\varphi\left(f, \frac{\tau}{\lambda_n}\right)_{\mathbf{M}}, \quad \tau > 0, \quad n \in \mathbb{N}.$$

Let $V(\tau),$ $\tau > 0,$ be a set of bounded nondecreasing functions v that differ from a constant on $[0, \tau]$.

Theorem 3.1. *Assume that the function $f \in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form (2.1). Then, for any $\tau > 0,$ $n \in \mathbb{N},$ and $\varphi \in \Phi,$ the following inequality holds:*

$$E_{\lambda_n}(f)_{\mathbf{M}} \le K_{n,\varphi}(\tau)\omega_\varphi\left(f, \frac{\tau}{\lambda_n}\right)_{\mathbf{M}}, \tag{3.1}$$

where

$$K_{n,\varphi}(\tau) := \inf_{v\in V(\tau)} \frac{v(\tau) - v(0)}{I_{n,\varphi}(\tau, v)} \tag{3.2}$$

and

$$I_{n,\varphi}(\tau, v) := \inf_{k\in\mathbb{N},\, k\ge n} \int_0^\tau \varphi\left(\frac{\lambda_k t}{\lambda_n}\right) dv(t). \tag{3.3}$$

Furthermore, there exists a function $v_* \in V(\tau)$ that realizes the greatest lower bound in (3.2).

In the spaces L_2 of 2π-periodic square-summable functions, results of this kind were obtained by Babenko [7] and Abilov and Abilova [6] for the moduli of continuity $\omega_m(f;\delta)$ and $\tilde{\omega}_m(f;\delta)$, respectively. In the spaces $\mathcal{S}^p$ of functions of one and several variables, the corresponding results for the classical moduli of smoothness were obtained in [27] and [1], respectively. In the Musielak–Orlicz spaces $\mathcal{S}_{\mathbf{M}},$ a similar result was obtained for the generalized moduli of smoothness in [3].

In the Besicovitch–Stepanets spaces $B\mathcal{S}^p$, a similar theorem was proved in [26]. It has been already indicated that, in the case where all functions

$$M_k(u) = u^p\Big(p^{-1/p}q^{-1/p'}\Big)^p, \quad p>1, \quad \frac{1}{p}+\frac{1}{p'}=1,$$

we have

$$B\mathcal{S}_{\mathbf{M}} = B\mathcal{S}^p \quad \text{and} \quad \|f\|_{\mathbf{M}} = \|f\|_{B\mathcal{S}^p}.$$

In the case $p=1$, similar equalities

$$B\mathcal{S}_{\mathbf{M}} = B\mathcal{S}^1 \quad \text{and} \quad \|f\|_{\mathbf{M}} = \|f\|_{B\mathcal{S}^1}$$

can obviously be obtained if all $M_k(u) = u$, $k\in\mathbb{Z}$, and Γ is the set of all sequences of positive numbers $\gamma=\{\gamma_k\}_{k\in\mathbb{Z}}$ such that

$$\|\gamma\|_{l_\infty(\mathbb{Z})} = \sup_{k\in\mathbb{Z}}\gamma_k \le 1.$$

Comparing estimate (3.1) with the corresponding result of Theorem 1 from [26], we see that, in the case where $B\mathcal{S}_{\mathbf{M}} = B\mathcal{S}^1$, inequality (3.1) is unimprovable in the set of all functions $f\in B\mathcal{S}^1$ such that $\|f-A_0(f)\|_{\mathbf{M}}\ne 0$. Furthermore, Theorem 1 [26] implies the existence of a function $v_*\in V(\tau)$ that realizes the greatest lower bound in (3.2).

Proof. In the proof of Theorem 3.1, we mainly use the ideas outlined in [7, 16, 17, 26, 27] and take into account specific features of the spaces $B\mathcal{S}_{\mathbf{M}}$. From relations (2.2) and (2.4), it follows that, for any $f\in B\mathcal{S}_{\mathbf{M}}$ with Fourier series of the form (2.1), we have

$$E_{\lambda_n}(f)_{\mathbf{M}} = \|f-S_n(f)\|_{\mathbf{M}} = \sup\Bigg\{\sum_{|k|\ge n}\gamma_k|A_k(f)|:\ \gamma\in\Gamma\Bigg\}, \tag{3.4}$$

where

$$S_n(f) := \sum_{|k|<n} A_k(f)e^{\mathrm{i}\lambda_k x}.$$

By the definition of supremum, for any $\varepsilon>0$, there exists a sequence $\tilde\gamma\in\Gamma$, $\tilde\gamma=\tilde\gamma(\varepsilon)$, such that the following relation holds:

$$\sum_{|k|\ge n}\tilde\gamma_k|A_k(f)| + \varepsilon \ge \sup\Bigg\{\sum_{|k|\ge n}\gamma_k|A_k(f)|:\ \gamma\in\Gamma\Bigg\}.$$

For any $\varphi\in\Phi$ and $h\in\mathbb{R}$, we consider a sequence of numbers $\{\varphi(\lambda_k h)A_k(f)\}_{k\in\mathbb{Z}}$. If there exists a function $\Delta_h^\varphi(f)\in B$-a.p. such that, for all $k\in\mathbb{Z}$,

$$A_k(\Delta_h^\varphi(f)) = \varphi(\lambda_k h)A_k(f), \tag{3.5}$$

then (here and in what follows) by $\|\Delta_h^\varphi(f)\|_{\mathbf{M}}$ we denote the Orlicz norm (2.2) of the function $\Delta_h^\varphi(f)$. If this B-a.p. function $\Delta_h^\varphi(f)$ does not exist, then, for the sake of simplicity, we also use the notation $\|\Delta_h^\varphi(f)\|_{\mathbf{M}}$ for the $l_{\mathbf{M}}$-norm of the sequence $\{\varphi(\lambda_k h)A_k(f)\}_{k\in\mathbb{Z}}$. In view of (2.2) and (3.5), we obtain

$$\|\Delta_h^{\varphi} f\|_{\mathbf{M}} \geq \sup\left\{\sum_{|k|\geq n} \gamma_k \varphi(\lambda_k h)|A_k(f)| \colon \gamma\in\Gamma\right\} \geq \sum_{|k|\geq n} \tilde{\gamma}_k \varphi(\lambda_k h)|A_k(f)|$$

$$= \frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)} \sum_{|k|\geq n} \tilde{\gamma}_k |A_k(f)| + \sum_{|k|\geq n} \tilde{\gamma}_k |A_k(f)| \left(\varphi(\lambda_k h) - \frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)}\right).$$

For any $u\in[0,\tau],$ we get

$$\|\Delta^{\varphi}_{\frac{u}{\lambda_n}} f\|_{\mathbf{M}} \geq \frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)} \sum_{|k|\geq n} \tilde{\gamma}_k |A_k(f)| + \sum_{|k|\geq n} \tilde{\gamma}_k |A_k(f)| \left(\varphi\left(\frac{\lambda_k u}{\lambda_n}\right) - \frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)}\right). \tag{3.6}$$

Both sides of inequality (3.6) are nonnegative and, in view of boundedness of the function $\varphi,$ the series on the right-hand side can be majorized on the entire real axis by an absolutely convergent series $\mathcal{K}(\varphi)\sum_{|k|\geq n}\tilde{\gamma}_k|A_k(f)|,$ where

$$\mathcal{K}(\varphi) := \max_{u\in\mathbb{R}} \varphi(u).$$

Thus, integrating this inequality with respect to $dv(u)$ from 0 to $\tau,$ we get

$$\int_0^{\tau} \left\|\Delta^{\varphi}_{\frac{u}{\lambda_n}} f\right\|_{\mathbf{M}} dv(u) \geq I_{n,\varphi}(\tau,v) \sum_{|k|\geq n} \tilde{\gamma}_k |A_k(f)| + \sum_{|k|\geq n} \tilde{\gamma}_k |A_k(f)| \left(\int_0^{\tau} \varphi\left(\frac{\lambda_k u}{\lambda_n}\right) dv(u) - I_{n,\varphi}(\tau,v)\right).$$

By virtue of the definition of $I_{n,\varphi}(\tau,v),$ we see that the second term on the right-hand side of the last relation is nonnegative. Therefore, for any function $v\in V(\tau),$ we have

$$\int_0^{\tau} \left\|\Delta^{\varphi}_{\frac{u}{\lambda_n}} f\right\|_{\mathbf{M}} dv(u) \geq I_{n,\varphi}(\tau,v) \sum_{|k|\geq n} \tilde{\gamma}_k |A_k(f)| \geq I_{n,\varphi}(\tau,v) \left(\sup\left\{\sum_{|k|\geq n} \gamma_k |A_k(f)| \colon \gamma\in\Gamma\right\} - \varepsilon\right),$$

whence, in view of the arbitrariness of choice of the number $\varepsilon,$ we conclude that

$$\int_0^{\tau} \left\|\Delta^{\varphi}_{\frac{u}{\lambda_n}} f\right\|_{\mathbf{M}} dv(u) \geq I_{n,\varphi}(\tau,v) E_{\lambda_n}(f)_{\mathbf{M}}.$$

Thus,

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \frac{1}{I_{n,\varphi}(\tau,v)} \int\limits_0^{\tau} \left\|\Delta^{\varphi}_{\frac{u}{\lambda_n}} f\right\|_{\mathbf{M}} dv(u) \le \frac{1}{I_{n,\varphi}(\tau,v)} \int\limits_0^{\tau} \omega_{\varphi}\Big(f, \frac{u}{\lambda_n}\Big)_{\mathbf{M}} dv(u). \tag{3.7}$$

Hence, in view of the fact that the function ω_{φ} is nondecreasing, we immediately arrive at relation (3.1).

Theorem 3.1 is proved.

We now consider some realizations of Theorem 3.1. Setting

$$\varphi_{\alpha}(t) = 2^{\frac{\alpha}{2}}(1-\cos t)^{\frac{\alpha}{2}}, \quad \alpha > 0, \qquad \omega_{\varphi_{\alpha}}(f,\delta)_{\mathbf{M}} =: \omega_{\alpha}(f,\delta)_{\mathbf{M}}, \quad \tau = \pi,$$

and

$$v(u) = 1 - \cos u, \quad u \in [0,\pi],$$

we get the following assertion:

Corollary 3.1. *For arbitrary numbers $n \in \mathbb{N}$ and $\alpha > 0$, and any function $f \in B\mathcal{S}_{\mathbf{M}}$ with Fourier series of the form (2.1), the following inequalities hold:*

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \frac{1}{2^{\frac{\alpha}{2}} I_n(\frac{\alpha}{2})} \int\limits_0^{\pi} \omega_{\alpha}\Big(f, \frac{u}{\lambda_n}\Big)_{\mathbf{M}} \sin u\, du, \tag{3.8}$$

where

$$I_n\Big(\frac{\alpha}{2}\Big) = \inf_{k\in\mathbb{N}, k\ge n} \int\limits_0^{\pi} \Big(1 - \cos\frac{\lambda_k u}{\lambda_n}\Big)^{\frac{\alpha}{2}} \sin u\, du. \tag{3.9}$$

If, in addition, $\dfrac{\alpha}{2} \in \mathbb{N}$, then

$$I_n\Big(\frac{\alpha}{2}\Big) = \frac{2^{\frac{\alpha}{2}+1}}{\dfrac{\alpha}{2}+1}, \tag{3.10}$$

and inequality (3.8) cannot be improved for any $n \in \mathbb{N}$.

Proof. Estimate (3.8) follows from (3.7). In [27] (relation (52)), it was shown that, for any $\theta \ge 1$ and $s \in \mathbb{N}$, the following inequality holds:

$$\int\limits_0^{\pi} (1-\cos\theta t)^s \sin t dt \ge \frac{2^{s+1}}{s+1},$$

which turns into the equality for $\theta = 1$. Therefore, setting

$$s = \frac{\alpha}{2} \quad \text{and} \quad \theta = \frac{\lambda_{\nu}}{\lambda_n}, \quad \nu = n, n+1, \ldots,$$

in view of the monotonicity of the sequence of Fourier exponents $\{\lambda_k\}_{k\in\mathbb{Z}}$, we conclude that, for $\frac{\alpha}{2} \in \mathbb{N}$, equality (3.10) is indeed true.

We now prove that, in this case, the constant $\dfrac{\frac{\alpha}{2}+1}{2^{\alpha+1}}$ in inequality (3.8) is unimprovable for $\frac{\alpha}{2} \in \mathbb{N}$. It suffices to verify that the function

$$f^*(x) = \gamma + \beta e^{-\lambda_n x} + \delta e^{\lambda_n x}, \tag{3.11}$$

where $\gamma,\ \beta,$ and δ are arbitrary complex numbers, satisfies the equality

$$E_{\lambda_n}(f^*)_{\mathbf{M}} = \frac{\frac{\alpha}{2}+1}{2^{\alpha+1}} \int\limits_0^\pi \omega_\alpha\left(f^*, \frac{t}{\lambda_n}\right)_{\mathbf{M}} \sin t\, dt, \quad \alpha > 0. \tag{3.12}$$

We have $E_{\lambda_n}(f^*)_{\mathbf{M}} = |\beta| + |\delta|$ and the function

$$\left\|\Delta^{\varphi_\alpha}_{u/\lambda_n} f^*\right\|_{\mathbf{M}} = 2^{\frac{\alpha}{2}}(|\beta|+|\delta|)(1-\cos u)^{\frac{\alpha}{2}}$$

does not decrease with respect to u on $[0,\pi]$. Therefore,

$$\omega_\alpha\left(f^*, \frac{u}{\lambda_n}\right)_{\mathbf{M}} = \left\|\Delta^{\varphi_\alpha}_{u/\lambda_n} f^*\right\|_{\mathbf{M}}$$

and

$$\frac{2^{\alpha+1}}{\frac{\alpha}{2}+1} E_{\lambda_n}(f^*)_{\mathbf{M}} - \int\limits_0^\pi \omega_\alpha\left(f^*, \frac{t}{\lambda_n}\right)_{\mathbf{M}} \sin t\, dt$$

$$= (|\beta|+|\delta|)\left(\frac{2^{\alpha+1}}{\frac{\alpha}{2}+1} - 2^{\frac{\alpha}{2}} \int\limits_0^\pi (1-\cos t)^{\frac{\alpha}{2}} \sin t\, dt\right) = 0.$$

Corollary 3.1 is proved.

In [27], it was shown that $I_n(s) \geq 2$ for $s \geq 1$ and $I_n(s) \geq 1 + 2^{s-1}$ for $s \in (0,1)$. Combining these two estimates and (3.8), we arrive at the following statement, which establishes a Jackson-type inequality with a constant uniformly bounded in the parameter $n \in \mathbb{N}$:

Corollary 3.2. *Assume that a function $f \in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form (2.1) and*

$$\|f - A_0(f)\|_{\mathbf{M}} \neq 0.$$

Then, for any $n \in \mathbb{N}$ and $\alpha > 0$,

$$E_{\lambda_n}(f)_{\mathbf{M}} < c_\alpha \omega_\alpha\left(f, \frac{\pi}{\lambda_n}\right)_{\mathbf{M}}, \tag{3.13}$$

where $c_\alpha = 2^{-\alpha/2}$ *for* $\alpha \ge 2$ *and* $c_\alpha = 4 \cdot 2^{-\alpha/2}/3$ *for* $0 < \alpha < 2$. *Furthermore, in the case where* $\alpha = m \in \mathbb{N}$, *the following more accurate estimate holds:*

$$E_{\lambda_n}(f)_{\mathbf{M}} < \frac{4-2\sqrt{2}}{2^{m/2}}\omega_m\Big(f, \frac{\pi}{\lambda_n}\Big)_{\mathbf{M}}. \tag{3.14}$$

Proof. Relation (3.14) follows from the estimate

$$I_n\Big(\frac{\alpha}{2}\Big) \ge 1 + \frac{1}{\sqrt{2}},$$

which is a consequence of the estimates presented above for the value of $I_n(s)$ in the case $\alpha = m \in \mathbb{N}$ [27].

If the weight function $v_2(t) = t$, then we obtain the following assertion:

Corollary 3.3. *Assume that the function* $f \in B\mathcal{S}_{\mathbf{M}}$ *has the Fourier series of the form (2.1) and* $\alpha \ge 1$. *Then, for any* $0 < \tau \le \dfrac{3\pi}{4}$ *and* $n \in \mathbb{N}$,

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \frac{1}{2^\alpha \displaystyle\int_0^\tau \sin^\alpha \frac{t}{2}\,dt} \int_0^\tau \omega_\alpha\Big(f, \frac{t}{\lambda_n}\Big)_{\mathbf{M}} dt. \tag{3.15}$$

Relation (3.15) turns into the equality for a function f^* *of the form (3.11).*

Inequalities (3.8) and (3.15) can be considered as an extension of the corresponding results obtained by Serdyuk and Shidlich [26] to the Besicovitch–Musielak spaces $B\mathcal{S}_{\mathbf{M}}$, and they coincide with the indicated results in the case $B\mathcal{S}_{\mathbf{M}} = B\mathcal{S}^1$. In the spaces $\mathcal{S}^p$ of functions of one and several variables, analogs of Theorem 3.1 and Corollaries 3.1 and 3.3 were proved in [27] and [1], respectively. The inequalities of this type were also investigated in [8, 17, 27, 32, 34].

Proof. It follows from inequality (3.7) that

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \frac{1}{2^{\frac{\alpha}{2}} I_n^*\Big(\dfrac{\alpha}{2}\Big)} \int_0^\tau \omega_\alpha\Big(f, \frac{t}{\lambda_n}\Big) dt,$$

where

$$I_n^*\Big(\frac{\alpha}{2}\Big) := \inf_{k\in\mathbb{N},\, k\ge n} \int_0^\tau \Big(1 - \cos\frac{\lambda_k t}{\lambda_n}\Big)^{\frac{\alpha}{2}} dt, \quad \alpha > 0, \quad n \in \mathbb{N}.$$

In [35], it was shown that for a function

$$F_\alpha(x) := \frac{1}{x}\int_0^x |\sin t|^\alpha\, dt,$$

any $h \in \Big(0, \dfrac{3\pi}{4}\Big)$, and $\alpha \ge 1$, the following relation is true:

$$\inf_{x \ge h/2} F_\alpha(x) = F_\alpha(h/2). \tag{3.16}$$

Since, for $h = \dfrac{\lambda_k}{\lambda_n} \geq 1 \ (k \geq n)$, we have

$$\int\limits_0^\tau \left(1 - \cos \frac{\lambda_k t}{\lambda_n}\right)^{\frac{\alpha}{2}} dt = 2^{\frac{\alpha}{2}} \int\limits_0^\tau \left|\sin \frac{\lambda_k t}{2\lambda_n}\right|^\alpha dt = 2^{\frac{\alpha}{2}} \tau F_\alpha\left(\frac{\lambda_k \tau}{2\lambda_n}\right),$$

it follows from (3.16) $\left(\text{with } \tau \in \left(0, \dfrac{3\pi}{4}\right] \text{ and } \alpha \geq 1\right)$ that

$$I_n^*\left(\frac{\alpha}{2}\right) = \inf_{k\in\mathbb{N}:\ k\geq n} \int\limits_0^\tau \left(1 - \cos \frac{\lambda_k t}{\lambda_n}\right)^{\frac{\alpha}{2}} dt = \inf_{k\in\mathbb{N}:\ k\geq n} 2^{\frac{\alpha}{2}} \int\limits_0^\tau \left|\sin \frac{\lambda_k t}{2\lambda_n}\right|^\alpha dt = 2^{\frac{\alpha}{2}} \int\limits_0^\tau \sin^\alpha \frac{t}{2}\, dt.$$

For functions f^* of the form (3.11), the equality

$$E_{\lambda_n}(f^*)_{\mathbf{M}} = \frac{1}{2^\alpha \displaystyle\int_0^\tau \sin^\alpha \frac{t}{2}\, dt} \int\limits_0^\tau \omega_\alpha\left(f^*, \frac{t}{\lambda_n}\right)_{\mathbf{M}} dt$$

is verified as the proof of equality (3.12).

Corollary 3.2 is proved.

In the case $\varphi(t) = \tilde{\varphi}_m(t) = (1 - \operatorname{sinc} t)^m,\ m \in \mathbb{N}$, where, by definition, $\operatorname{sinc} t = \{\sin t/t \text{ for } t \neq 0 \text{ and } 1 \text{ for } t = 0\}$, for $\tau = \pi$ and $v(u) = 1 - \cos u,\ u \in [0; \pi]$, it follows from relation (3.7) that

$$E_{\lambda_n}(f)_{\mathbf{M}} \leq \frac{1}{\tilde{I}_n(m)} \int\limits_0^\pi \tilde{\omega}_m\left(f, \frac{u}{\lambda_n}\right)_{\mathbf{M}} \sin u\, du,$$

where

$$\tilde{I}_n(m) = \inf_{k\in\mathbb{N}, k\geq n} \int\limits_0^\pi \left(1 - \operatorname{sinc} \frac{\lambda_k u}{\lambda_n}\right)^m \sin u\, du.$$

In view of the estimate [33]

$$1 - \operatorname{sinc}\left(\frac{\lambda_k u}{\lambda_n}\right) \geq 1 - \frac{\sin u}{u} \geq \left(\frac{u}{\pi}\right)^2, \quad k \geq n, \quad u \in [0; \pi],$$

we obtain

$$\tilde{I}_n(m) \geq \int\limits_0^\pi (1 - \operatorname{sinc} u)^m \sin u\, du$$

$$\geq \frac{1}{\pi^{2m}} \int\limits_0^\pi u^{2m} \sin u\, du = \frac{2m!}{\pi^{2m}} \left(\sum_{j=0}^m (-1)^j \frac{\pi^{2m-2j}}{(2m-2j)!} + \frac{\pi^{2m}}{2m!}(-1)^m\right) := \frac{2m!}{\pi^{2m}} \mathcal{K}(m).$$

Hence, Theorem 3.1 yields the following corollary:

Corollary 3.4. *For arbitrary numbers $n \in \mathbb{N}$ and $m > 0$ and any function $f \in B\mathcal{S}_{\mathbf{M}}$ with Fourier series of the form (2.1), the following inequalities are true:*

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \frac{\pi^{2m}}{2m! \cdot \mathcal{K}(m)} \int\limits_0^\pi \tilde{\omega}_m\Big(f, \frac{u}{\lambda_n}\Big)_{\mathbf{M}} \sin u \, du,$$

where

$$\mathcal{K}(m) = \sum_{j=0}^m (-1)^j \frac{\pi^{2m-2j}}{(2m-2j)!} + \frac{\pi^{2m}}{2m!}(-1)^m.$$

For $m = 1$, we have $2\mathcal{K}(1) = \pi^2 - 4$ and

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \frac{\pi^2}{\pi^2 - 4} \int\limits_0^\pi \tilde{\omega}_1\Big(f, \frac{u}{\lambda_n}\Big)_{\mathbf{M}} \sin u \, du$$

$$\le \frac{\pi^2 \lambda_n}{\pi^2 - 4} \int\limits_0^{\frac{\pi}{\lambda_n}} \tilde{\omega}_1(f, u)_{\mathbf{M}} \sin \lambda_n u \, du.$$

If the weight function $v_2(t) = u^{m+1}$, then we get the following assertion:

Corollary 3.5. *Assume that the function $f \in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form (2.1) and $m \ge 1$. Then, for any $0 < \tau \le \pi$ and $n \in \mathbb{N}$,*

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \pi^{m-1} \Big(\frac{2\lambda_n}{\pi^2 - 4}\Big)^m \lambda_n \int\limits_0^{\tau/\lambda_n} \tilde{\omega}_m(f, t)_{\mathbf{M}} t^m \, dt. \tag{3.17}$$

Indeed, applying Holder's inequality, we find

$$\int\limits_0^\pi \Big(1 - \operatorname{sinc} \frac{\lambda_k u}{\lambda_n}\Big)^m du^{m+1} \ge (m+1) \int\limits_0^\pi \Big(1 - \frac{\sin u}{u}\Big)^m u^m \, du = (m+1) \int\limits_0^\pi (u - \sin u)^m \, du$$

$$\ge \frac{m+1}{\pi^{m-1}} \Bigg(\int\limits_0^\pi (u - \sin u) \, du\Bigg)^m = \frac{m+1}{\pi^{m-1}} \Big(\frac{\pi^2 - 4}{2}\Big)^m.$$

In the spaces L_2 of 2π-periodic square-summable functions, for the moduli of smoothness $\tilde{\omega}_m(f;\delta)$, the results of this kind were obtained by Abilov and Abilova [6], and Vakarchuk [32]. Note that, in the case where $f \in B\mathcal{S}_{\mathbf{M}} = L_2$, inequality (3.17) follows from the result in [6] (see Theorem 1). For $m = 1$ and $f \in L_2$, the statements of Corollary 3.5 and Theorem 1 in [6] are identical and, moreover, the constant on the right-hand side of (3.17) cannot be reduced for any fixed n.

4. Inverse Approximation Theorem

Theorem 4.1. *Assume that* $f \in B\mathcal{S}_{\mathbf{M}}$ *has Fourier series of the form (2.1), the function* $\varphi \in \Phi$ *is nondecreasing on the interval* $[0,\tau]$, $\tau > 0$, *and* $\varphi(\tau) = \max\{\varphi(t)\colon t \in \mathbb{R}\}$. *Then, for any* $n \in \mathbb{N}$, *the following inequality holds:*

$$\omega_\varphi\left(f, \frac{\tau}{\lambda_n}\right)_{\mathbf{M}} \le \sum_{\nu=1}^{n}\left(\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right) - \varphi\left(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\right)\right)E_{\lambda_\nu}(f)_{\mathbf{M}}. \tag{4.1}$$

Proof. We use the same scheme of the proof as in [27] and [3] but modify it to take into account the specific features of the spaces $B\mathcal{S}_{\mathbf{M}}$ and the definition of the modulus ω_φ.

Let $f \in B\mathcal{S}_{\mathbf{M}}$. For any $\varepsilon > 0$ there exists a number $N_0 = N_0(\varepsilon) \in \mathbb{N},\ N_0 > n$, such that, for any $N > N_0$, we have

$$E_{\lambda_N}(f)_{\mathbf{M}} = \|f - S_{N-1}(f)\|_{\mathbf{M}} < \varepsilon/\varphi(\tau).$$

We set $f_0 := S_{N_0}(f)$. Thus, in view of (3.5), we conclude that

$$\begin{aligned}\|\Delta_h^\varphi(f)\|_{\mathbf{M}} &\le \|\Delta_h^\varphi(f_0)\|_{\mathbf{M}} + \|\Delta_h^\varphi(f-f_0)\|_{\mathbf{M}}\\ &\le \|\Delta_h^\varphi(f_0)\|_{\mathbf{M}} + \varphi(\tau)E_{\lambda_{N_0+1}}(f)_{\mathbf{M}}\\ &< \|\Delta_h^\varphi(f_0)\|_{\mathbf{M}} + \varepsilon.\end{aligned} \tag{4.2}$$

Further, let $S_{n-1} := S_{n-1}(f_0)$ be the Fourier sum of f_0. Then, by virtue of (3.5), for $|h| \le \tau/\lambda_n$, we get

$$\begin{aligned}\|\Delta_h^\varphi(f_0)\|_{\mathbf{M}} &= \left\|\Delta_h^\varphi(f_0 - S_{n-1}) + \Delta_h^\varphi S_{n-1}\right\|_{\mathbf{M}}\\ &\le \left\|\varphi(\tau)(f_0 - S_{n-1}) + \sum_{|k|\le n-1}\varphi(\lambda_k h)|A_k(f)|\right\|_{\mathbf{M}}\\ &\le \left\|\varphi(\tau)\sum_{\nu=n}^{N_0}H_\nu + \sum_{\nu=1}^{n-1}\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right)H_\nu\right\|_{\mathbf{M}},\end{aligned} \tag{4.3}$$

where

$$H_\nu(x) := |A_\nu(f)| + |A_{-\nu}(f)|, \qquad \nu = 1, 2, \ldots.$$

We now use the following assertion from [27]:

Lemma 4.1 [27]**.** *Let* $\{c_\nu\}_{\nu=1}^\infty$ *and* $\{a_\nu\}_{\nu=1}^\infty$ *be arbitrary numerical sequences. Then the following equality holds for all natural* $N_1,\ N_2$ *and* $N,\ N_1 \le N_2 < N$*:*

$$\sum_{\nu=N_1}^{N_2} a_\nu c_\nu = a_{N_1}\sum_{\nu=N_1}^{N} c_\nu + \sum_{\nu=N_1+1}^{N_2}(a_\nu - a_{\nu-1})\sum_{i=\nu}^{N} c_i - a_{N_2}\sum_{\nu=N_2+1}^{N} c_\nu. \tag{4.4}$$

Setting $a_\nu = \varphi\Big(\dfrac{\tau\lambda_\nu}{\lambda_n}\Big),\ c_\nu = H_\nu(x),\ N_1 = 1,\ N_2 = n-1,$ and $N = N_0$ in (4.4), we get

$$\sum_{\nu=1}^{n-1}\varphi\Big(\frac{\tau\lambda_\nu}{\lambda_n}\Big)H_\nu(x) = \varphi\Big(\frac{\tau\lambda_1}{\lambda_n}\Big)\sum_{\nu=1}^{N_0}H_\nu(x) + \sum_{\nu=2}^{n-1}\Big(\varphi\Big(\frac{\tau\lambda_\nu}{\lambda_n}\Big) - \varphi\Big(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\Big)\Big)\sum_{i=\nu}^{N_0}H_i(x)$$
$$- \varphi\Big(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\Big)\sum_{\nu=n}^{N_0}H_\nu(x).$$

Therefore,

$$\Big\|\varphi(\tau)\sum_{\nu=n}^{N_0}H_\nu + \sum_{\nu=1}^{n-1}\varphi\Big(\frac{\tau\lambda_\nu}{\lambda_n}\Big)H_\nu\Big\|_{\mathbf{M}} \le \Big\|\varphi(\tau)\sum_{\nu=n}^{N_0}H_\nu + \sum_{\nu=1}^{n-1}\Big(\varphi\Big(\frac{\tau\lambda_\nu}{\lambda_n}\Big) - \varphi\Big(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\Big)\Big)\sum_{i=\nu}^{N_0}H_i$$
$$-\varphi\Big(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\Big)\sum_{\nu=n}^{N_0}H_\nu\Big\|_{\mathbf{M}}$$
$$\le \Big\|\sum_{\nu=1}^{n}\Big(\varphi\Big(\frac{\tau\lambda_\nu}{\lambda_n}\Big) - \varphi\Big(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\Big)\Big)\sum_{i=\nu}^{N_0}H_i\Big\|_{\mathbf{M}}$$
$$\le \sum_{\nu=1}^{n}\Big(\varphi\Big(\frac{\tau\lambda_\nu}{\lambda_n}\Big) - \varphi\Big(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\Big)\Big)E_{\lambda_\nu}(f_0)_{\mathbf{M}}. \tag{4.5}$$

Combining relations (4.2), (4.3), and (4.5) and taking into account the definition of the function f_0, we conclude that, for $|h| \le \tau/\lambda_n$, the following inequality holds:

$$\|\Delta_h^\varphi(f)\|_{\mathbf{M}} \le \sum_{\nu=1}^{n}\Big(\varphi\Big(\frac{\tau\lambda_\nu}{\lambda_n}\Big) - \varphi\Big(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\Big)\Big)E_{\lambda_\nu}(f)_{\mathbf{M}} + \varepsilon.$$

In view of arbitrariness of ε, this gives (4.1).

Theorem 4.1 is proved.

Consider an important special case where

$$\varphi(t) = \varphi_\alpha(t) = 2^{\frac{\alpha}{2}}(1-\cos t)^{\frac{\alpha}{2}} = 2^\alpha|\sin(t/2)|^\alpha, \quad \alpha > 0.$$

In this case, the function φ satisfies the conditions of Theorem 4.1 with $\tau = \pi$. Thus, for $\alpha \ge 1$, by using the inequality

$$x^\alpha - y^\alpha \le \alpha x^{\alpha-1}(x-y), \quad x > 0, \quad y > 0$$

(see, e.g., [18], Ch. 1), and the ordinary trigonometric formulas, for $\nu = 1, 2, \ldots, n$, we obtain

$$\varphi\Big(\frac{\tau\lambda_\nu}{\lambda_n}\Big) - \varphi\Big(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\Big) = 2^\alpha\Big(\Big|\sin\frac{\pi\lambda_\nu}{\lambda_n}\Big|^\alpha - \Big|\sin\frac{\pi\lambda_{\nu-1}}{\lambda_n}\Big|^\alpha\Big)$$

$$\le 2^\alpha \alpha \left|\sin \frac{\pi\lambda_\nu}{\lambda_n}\right|^{\alpha-1} \left|\sin \frac{\pi\lambda_\nu}{\lambda_n} - \sin \frac{\pi\lambda_{\nu-1}}{\lambda_n}\right|$$

$$\le \alpha \left(\frac{2\pi}{\lambda_n}\right)^\alpha \lambda_\nu^{\alpha-1}(\lambda_\nu - \lambda_{\nu-1}).$$

If $0 < \alpha < 1$, then a similar estimate can be obtained using the inequality $x^\alpha - y^\alpha \le \alpha y^{\alpha-1}(x-y)$, which is true for any $x > 0,\ y > 0$ [18] (Ch. 1). Hence, for any $f \in B\mathcal{S}_{\mathbf{M}}$, we get the following estimate:

$$\omega_\alpha\left(f, \frac{\pi}{\lambda_n}\right)_{\mathbf{M}} \le \alpha \left(\frac{2\pi}{\lambda_n}\right)^\alpha \sum_{\nu=1}^{n} \lambda_\nu^{\alpha-1}(\lambda_\nu - \lambda_{\nu-1}) E_{\lambda_\nu}(f)_{\mathbf{M}}, \quad \alpha > 0. \tag{4.6}$$

It should be noted that the constant in this estimate can be improved as follows:

Theorem 4.2. *Assume that $f \in B\mathcal{S}_{\mathbf{M}}$ has Fourier series of the form (2.1). Then, for any $n \in \mathbb{N}$ and $\alpha > 0$,*

$$\omega_\alpha\left(f, \frac{\tau}{\lambda_n}\right)_{\mathbf{M}} \le \left(\frac{\pi}{\lambda_n}\right)^\alpha \sum_{\nu=1}^{n} (\lambda_\nu^\alpha - \lambda_{\nu-1}^\alpha) E_{\lambda_\nu}(f)_{\mathbf{M}}. \tag{4.7}$$

Proof. We prove this theorem by analogy with the proof of Theorem 4.1. For any $\varepsilon > 0$, by $N_0 = N_0(\varepsilon) \in \mathbb{N}$, $N_0 > n$, we denote a number such that, for any $N > N_0$,

$$E_{\lambda_N}(f)_{\mathbf{M}} = \|f - S_{N-1}(f)\|_{\mathbf{M}} < \varepsilon.$$

We set $f_0 := S_{N_0}(f),\ S_{n-1} := S_{n-1}(f_0)$, and

$$\|\Delta_h^\alpha(f)\|_{\mathbf{M}} := \|\Delta_h^{\varphi_\alpha}(f)\|_{\mathbf{M}}$$

and apply relations (4.2) and (4.3). This gives

$$\|\Delta_h^\alpha(f)\|_{\mathbf{M}} < \|\Delta_h^\alpha(f_0)\|_{\mathbf{M}} + \varepsilon \tag{4.8}$$

and

$$\|\Delta_h^\alpha(f_0)\|_{\mathbf{M}} \le \left\| 2^\alpha \sum_{\nu=n}^{N_0} H_\nu + 2^\alpha \sum_{\nu=1}^{n-1} \left|\sin \frac{\pi\lambda_\nu}{2\lambda_n}\right|^\alpha H_\nu \right\|_{\mathbf{M}}$$

$$\le \left(\frac{\pi}{\lambda_n}\right)^\alpha \left\| \lambda_n^\alpha \sum_{\nu=n}^{N_0} H_\nu + \sum_{\nu=1}^{n-1} \lambda_\nu^\alpha H_\nu \right\|_{\mathbf{M}}, \tag{4.9}$$

where $|h| \le \pi/\lambda_n$ and $H_\nu(x) = |A_\nu(f)| + |A_{-\nu}(f)|,\ \nu = 1, 2, \ldots$.

By virtue of (4.4), for $a_\nu = \lambda_\nu^\alpha,\ c_\nu = H_\nu(x),\ N_1 = 1,\ N_2 = n-1$, and $N = N_0$, we can write

$$\sum_{\nu=1}^{n-1} \lambda_\nu^\alpha H_\nu(x) = \lambda_1^\alpha \sum_{\nu=1}^{N_0} H_\nu(x) + \sum_{\nu=2}^{n-1} (\lambda_\nu^\alpha - \lambda_{\nu-1}^\alpha) \sum_{i=\nu}^{N_0} H_i(x) - \lambda_{\nu-1}^\alpha \sum_{\nu=n}^{N_0} H_\nu(x).$$

Therefore,

$$\left\|\lambda_n^{\alpha}\sum_{\nu=n}^{N_0} H_\nu + \sum_{\nu=1}^{n-1}\lambda_\nu^{\alpha} H_\nu\right\|_{\mathbf{M}} = \left\|\sum_{\nu=1}^{n}\big(\lambda_\nu^{\alpha}-\lambda_{\nu-1}^{\alpha}\big)\sum_{i=\nu}^{N_0} H_i\right\|_{\mathbf{M}}$$
$$\le \sum_{\nu=1}^{n}\big(\lambda_\nu^{\alpha}-\lambda_{\nu-1}^{\alpha}\big)E_{\lambda_\nu}(f_0)_{\mathbf{M}}. \tag{4.10}$$

Combining relations (4.8), (4.9), and (4.10) and taking into account the definition of the function f_0, we conclude that, for $|h|\le \tau/\lambda_n$, the following inequality holds:

$$\|\Delta_h^{\alpha}(f)\|_{\mathbf{M}} \le \left(\frac{\pi}{\lambda_n}\right)^{\alpha}\sum_{\nu=1}^{n}\big(\lambda_\nu^{\alpha}-\lambda_{\nu-1}^{\alpha}\big)E_{\lambda_\nu}(f)_{\mathbf{M}}+\varepsilon.$$

In view of the arbitrariness of ε, this gives (4.7).

Theorem 4.2 is proved.

In (4.1), the constant π^{α} is exact in a sense that, for any $\varepsilon>0$, there exists a function $f^*\in B\mathcal{S}_{\mathbf{M}}$ such that, for all n greater than a certain number n_0, we have

$$\omega_\alpha\left(f^*,\frac{\pi}{\lambda_n}\right)_{\mathbf{M}} > \frac{\pi^{\alpha}-\varepsilon}{\lambda_n^{\alpha}}\sum_{\nu=1}^{n}\big(\lambda_\nu^{\alpha}-\lambda_{\nu-1}^{\alpha}\big)E_{\lambda_\nu}(f^*)_{\mathbf{M}}. \tag{4.11}$$

Consider a function $f^*(x)=e^{\mathrm{i}\lambda_{k_0}x}$, where k_0 is an arbitrary positive integer. Then $E_{\lambda_\nu}(f^*)_{\mathbf{M}}=1$ for $\nu=1,2,\ldots,k_0$, $E_{\lambda_\nu}(f^*)_{\mathbf{M}}=0$ for $\nu>k_0$, and

$$\omega_\alpha\left(f^*,\frac{\pi}{\lambda_n}\right)_{\mathbf{M}} \ge \|\Delta^{\alpha}_{\frac{\pi}{\lambda_n}} f^*\|_{\mathbf{M}} \ge 2^{\alpha}\left|\sin\frac{\lambda_{k_0}\pi}{2\lambda_n}\right|^{\alpha}.$$

Since $\sin t/t$ tends to 1 as $t\to 0$, the inequality

$$2^{\alpha}|\sin\lambda_{k_0}\pi/(2\lambda_n)|^{\alpha} > (\pi^{\alpha}-\varepsilon)\lambda_{k_0}^{\alpha}/{\lambda_n}^{\alpha}$$

holds for all n greater than a certain number n_0. This yields (4.11).

Corollary 4.1. *Suppose that $f\in B\mathcal{S}_{\mathbf{M}}$ has Fourier series of the form (2.1). Then, for any $n\in\mathbb{N}$ and $\alpha>0$,*

$$\omega_\alpha\left(f,\frac{\pi}{\lambda_n}\right)_{\mathbf{M}} \le \alpha\left(\frac{\pi}{\lambda_n}\right)^{\alpha}\sum_{\nu=1}^{n}\lambda_\nu^{\alpha-1}(\lambda_\nu-\lambda_{\nu-1})E_{\lambda_\nu}(f)_{\mathbf{M}}. \tag{4.12}$$

If, in addition, the Fourier exponents λ_ν, $\nu\in\mathbb{N}$, satisfy the condition

$$\lambda_{\nu+1}-\lambda_\nu\le C,\quad \nu=1,2,\ldots, \tag{4.13}$$

with an absolute constant $C > 0,$ *then*

$$\omega_\alpha\left(f, \frac{\pi}{\lambda_n}\right)_{\mathbf{M}} \le C\alpha\left(\frac{\pi}{\lambda_n}\right)^\alpha \sum_{\nu=1}^{n} \lambda_\nu^{\alpha-1} E_{\lambda_\nu}(f)_{\mathbf{M}}. \tag{4.14}$$

5. Constructive Characteristics of the Classes of Functions Defined by the Generalized Moduli of Smoothness

Let ω be a function (majorant) given on $[0,1]$. For fixed $\alpha > 0,$ we set

$$B\mathcal{S}_{\mathbf{M}}H_\alpha^\omega = \big\{f \in B\mathcal{S}_{\mathbf{M}}:\ \omega_\alpha(f,\delta)_{\mathbf{M}} = \mathcal{O}(\omega(\delta)),\ \delta \to 0+\big\}. \tag{5.1}$$

Further, we consider majorants $\omega(\delta),\ \delta \in [0,1]$ satisfying the following conditions:

(1) $\omega(\delta)$ is continuous on $[0,1]$;

(2) $\omega(\delta) \uparrow$;

(3) $\omega(\delta) \ne 0$ for $\delta \in (0,1]$;

(4) $\omega(\delta) \to 0$ for $\delta \to 0$;

as well as the condition

$$\sum_{v=1}^{n} \lambda_v^{s-1}\omega\left(\frac{1}{\lambda_v}\right) = \mathcal{O}\left[\lambda_n^s\omega\left(\frac{1}{\lambda_n}\right)\right], \tag{5.2}$$

where $s > 0$ and $\lambda_\nu,\ \nu \in \mathbb{N},$ is an increasing sequence of positive numbers. In the case where $\lambda_\nu = \nu,$ condition (5.2) is the well-known Bari condition $(\mathcal{B}_s)$ (see, e.g., [9]).

Theorem 5.1. *Assume that the function* $f \in B\mathcal{S}_{\mathbf{M}}$ *has Fourier series of the form (2.1),* $\alpha > 0,$ *and the majorant* ω *satisfies the conditions (i)–(iv).*

1. *If* $f \in B\mathcal{S}_{\mathbf{M}}H_\alpha^\omega,$ *then the following relation is true:*

$$E_{\lambda_n}(f)_{\mathbf{M}} = \mathcal{O}\left[\omega\left(\frac{1}{\lambda_n}\right)\right]. \tag{5.3}$$

2. *If the numbers* $\lambda_\nu,\ \nu \in \mathbb{N}$ *satisfy condition (4.13) and the function* ω *satisfies condition (5.2) with* $s = \alpha,$ *then relation (5.3) yields the inclusion* $f \in B\mathcal{S}_{\mathbf{M}}H_\alpha^\omega.$

Proof. Let $f \in B\mathcal{S}_{\mathbf{M}}H_\alpha^\omega.$ Then relation (5.3) follows from (5.1) and (3.13).

On the other hand, if $f \in B\mathcal{S}_{\mathbf{M}},$ the numbers $\lambda_\nu,\ \nu \in \mathbb{N}$ satisfy condition (4.13), the function ω satisfies condition (5.2) with $s = \alpha,$ and relation (5.3) is true, then, by virtue of (4.14), we get

$$\omega_\alpha\left(f, \frac{1}{\lambda_n}\right)_{\mathbf{M}} \le \frac{C_1}{\lambda_n^\alpha}\sum_{\nu=1}^{n}\lambda_\nu^{\alpha-1}E_{\lambda_\nu}(f) \le \frac{C_1}{\lambda_n^\alpha}\sum_{\nu=1}^{n}\lambda_\nu^{\alpha-1}\omega\left(\frac{1}{\lambda_\nu}\right) = \mathcal{O}\left[\omega\left(\frac{1}{\lambda_n}\right)\right],$$

where $C_1 = \alpha(2\pi)^\alpha \cdot C.$ Hence, the function f belongs to the set $B\mathcal{S}_{\mathbf{M}}H_\alpha^\omega.$

Theorem 5.1 is proved.

The function t^r, $0 < r \leq \alpha$, satisfies condition (5.2) with $s = \alpha$. Hence, denoting the class $B\mathcal{S}_{\mathbf{M}}H^{\omega}_{\alpha}$ for $\omega(t) = t^r$ by $B\mathcal{S}_{\mathbf{M}}H^{r}_{\alpha}$, we arrive at the following statement:

Corollary 5.1. *Assume that $f \in B\mathcal{S}_{\mathbf{M}}$ has Fourier series of the form (2.1), $\alpha > 0$, $0 < r \leq \alpha$, and condition (4.13) is satisfied. The function f belongs to the set $B\mathcal{S}_{\mathbf{M}}H^{r}_{\alpha}$, iff the following relation is true:*

$$E_{\lambda_n}(f)_{\mathbf{M}} = \mathcal{O}(\lambda_n^{-r}).$$

In the spaces $\mathcal{S}^p$, Theorems 4.1 and 5.1 were proved for the classical moduli of smoothness ω_m in [27] and [1]. In the spaces $\mathcal{S}^p$, inequalities of the form (4.14) were also obtained in [29]. In the spaces L_p of 2π-periodic functions Lebesgue summable with the pth power, inequalities similar to (4.14) were obtained by M. Timan (see, e.g., [30], Ch. 6, [31], Ch. 2). In the Musielak–Orlicz type spaces, inequalities of the same kind as (4.1) were proved in [3].

The investigations carried out in Section 3 were financially supported by the Project “Innovative Methods in the Theory of Differential Equations, Computational Mathematics and Mathematical Modeling” (Project No. 0122U000670).

REFERENCES

1. F. G. Abdullayev, P. Özkartepe, V. V. Savchuk, and A. L. Shidlich, “Exact constants in direct and inverse approximation theorems for functions of several variables in the spaces $\mathcal{S}^p$,” Filomat, **33**, No. 5, 1471–1484 (2019).
2. F. Abdullayev, S. Chaichenko, M. Imash kyzy, and A. Shidlich, “Direct and inverse approximation theorems in the weighted Orlicz-type spaces with a variable exponent,” *Turkish J. Math.*, **44**, 284–299 (2020).
3. F. Abdullayev, S. Chaichenko, and A. Shidlich, “Direct and inverse approximation theorems of functions in the Musielak–Orlicz type spaces,” *Math. Inequal. Appl.*, **24**, No. 2, 323–336 (2021).
4. F. Abdullayev, A. Serdyuk, and A. Shidlich, “Widths of functional classes defined by majorants of generalized moduli of smoothness in the spaces $\mathcal{S}^p$,” *Ukr. Math. Zh.*, **73**, No. 6, 723–737 (2021); ***English translation:*** *Ukr. Math. J.*, **73**, No. 6, 841–858 (2021).
5. F. Abdullayev, S. Chaichenko, M. Imashkyzy, and A. Shidlich, “Jackson-type inequalities and widths of functional classes in the Musielak–Orlicz type spaces,” *Rocky Mountain J. Math.*, **51**, No. 4, 1143–1155 (2021).
6. V. A. Abilov and F. V. Abilova, “Problems in the approximation of 2π-periodic functions by Fourier sums in the space $L_2(2\pi)$,” *Math. Notes*, **76**, No. 6, 749–757 (2004).
7. A. G. Babenko, “On exact constant in the Jackson inequality in L_2,” *Math. Notes*, **39**, No. 5, 355–363 (1986).
8. V. F. Babenko and S. V. Savela, “Jackson–Stechkin-type inequalities for almost periodic functions,” *Visn. Dnipropetrovsk Univ.*, **20**, No. 6/1, 60–66 (2012).
9. N. K. Bari and S. B. Stechkin, “Best approximations and differential properties of two conjugate functions,” *Trudy Moskov. Mat. Obšč.*, **5**, 483–522 (1956).
10. A. S. Besicovitch, *Almost Periodic Functions*, Dover Publications, Inc., New York (1955).
11. J. Boman, “Equivalence of generalized moduli of continuity,” *Ark. Mat.*, **18**, 73–100 (1980).
12. E. A. Bredikhina, “Absolute convergence of Fourier series of almost periodic functions,” *Dokl. Akad. Nauk SSSR*, **179**, No. 5, 1023–1026 (1968).
13. E. A. Bredikhina, “Almost periodic functions,” *Encyclopedia Math.*, **4**, 543–545 (1984).
14. P. Butzer and R. Nessel, “Fourier analysis and approximation,” *One-Dimensional Theory*, Birkhäuser, Basel (1971).
15. S. Chaichenko, A. Shidlich, and F. Abdullayev, “Direct and inverse approximation theorems of functions in the Orlicz type spaces $\mathcal{S}_M$,” *Math. Slovaca*, **69**, No. 6, 1367–1380 (2019).
16. N. I. Chernykh, “On the Jackson inequality in L_2,” *Trudy Mat. Inst. Steklov*, **88**, 75–78 (1967).
17. N. I. Chernykh, “On the best approximation of periodic functions by trigonometric polynomials in L_2,” *Mat. Zametki*, **20**, No. 3, 513–522 (1967).
18. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press, Cambridge (1934).
19. A. I. Kozko and A. V. Rozhdestvenskii, “On Jackson’s inequality for a generalized modulus of continuity in L_2,” *Sb. Math.*, **195**, No. 8, 1073–1115 (2004).
20. B. M. Levitan, *Almost Periodic Functions* [in Russian], Gos. Izd. Tekn.-Teor. Lit., Moscow (1953).
21. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces*, Springer-Verlag, Berlin–New-York (1977).
22. J. Musielak, *Orlicz Spaces and Modular Spaces*, Springer-Verlag, Berlin (1983).

23. Ya. G. Pritula, "Jackson's inequality for B^2-almost periodic functions," *Izv. Vysš. Učebn. Zaved., Mat.*, **8**, 90–93 (1972).
24. Ya. G. Pritula and M. M. Yatsymirskyi, "Estimates of approximations for B^2-almost periodic functions," *Vīsnik Lviv. Derzh. Univ., Ser. Mekh.-Mat.*, **21**, 3–7 (1983).
25. M. M. Rao and Z. D. Ren, *Applications of Orlicz Spaces*, Marcel Dekker, New York (2002).
26. A. S. Serdyuk and A. L. Shidlich, "Direct and inverse theorems on the approximation of almost periodic functions in Besicovitch–Stepanets spaces," *Carpathian Math. Publ.*, **13**, No. 3, 687–700 (2021) (see also arXiv preprint, arXiv: 2105.06796).
27. A. I. Stepanets and A. S. Serdyuk, "Direct and inverse theorems in the theory of the approximation of functions in the space $\mathcal{S}^p$," *Ukr. Math. Zh.*, **54**, No. 1, 106–124 (2002); ***English translation:*** *Ukr. Math. J.*, **54**, No. 1, 126–148 (2002).
28. A. I. Stepanets, *Methods of Approximation Theory*, VSP, Leiden (2005).
29. M. D. Sterlin, "Exact constants in inverse theorems of approximation theory," *Sov. Math. Dokl.*, **13**, 160–163 (1972).
30. A. F. Timan, *Theory of Approximation of Functions of Real Variable* [in Russian], Fizmatgiz, Moscow (1960).
31. M. F. Timan, *Approximation and Properties of Periodic Functions* [in Russian], Naukova Dumka, Kiev (2009).
32. S. B. Vakarchuk, "Jackson-type inequalities with generalized modulus of continuity and exact values of the n-widths for the classes of (ψ, β)-differentiable functions in L_2. I," *Ukr. Math. Zh.*, **68**, No. 6, 723–745 (2016); ***English translation:*** *Ukr. Math. J.*, **68**, No. 6, 823–848 (2016).
33. S. B. Vakarchuk and V. I. Zabutnaya, "Inequalities between best polynomial approximations and some smoothness characteristics in the space L_2 and widths of classes of functions," *Math. Notes*, **99**, No. 1-2, 222–242 (2016).
34. S. N. Vasil'ev, "Jackson–Stechkin inequality in $L_2[-\pi, \pi]$," *Proc. Steklov Inst. Math.*, Suppl., **1**, S243–S253 (2001).
35. V. R. Voitsekhivs'kyj, "Jackson-type inequalities in the approximation of functions from the space $\mathcal{S}^p$," *Approx. Theory Related. Top., Proc. Inst. Math. Nat. Acad. Sci. Ukraine*, **35**, 33–46 (2002).