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SHARP REMEZ-TYPE INEQUALITIES ESTIMATING THE L_q-NORM OF A FUNCTION VIA ITS L_p-NORM

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For any $q \geq p > 0,\ \alpha = (r + 1/q)/(r + 1/p),\ f_p \in [0, \infty]$, and $\beta \in [0, 2\pi),$ we prove a sharp Remez-type inequality

$$\|x\|_q \leq \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|^{\alpha}_{L_p([0,2\pi]\setminus B_{y(\beta)})}} \|x\|^{\alpha}_{L_p([0,2\pi]\setminus B)} \|x^{(r)}\|^{1-\alpha}_{\infty}$$

for 2π-periodic functions $x \in L^r_\infty$, which have zeros and satisfy the condition

$$\|x_+\|_p \, \|x_-\|^{-1}_p = f_p, \tag{1}$$

where φ_r is Euler's perfect spline of order r, the number c is such that the function $x = \varphi_r + c$ satisfies condition (1), B is an arbitrary Lebesgue-measurable set such that

$$\mu B \leq \beta \left(\|\varphi_r + c\|_p \left\| x^{(r)} \right\|_\infty \|x\|^{-1}_p \right)^{-1/(r+1/p)},$$

the set $B_{y(\beta)}$ is defined by $B_{y(\beta)} := \{t \in [0, 2\pi] : |\varphi_r(t) + c| > y(\beta)\}$, and moreover, $\mu B_{y(\beta)} = \beta$. We also establish sharp Remez-type inequalities of various metrics for trigonometric polynomials and polynomial splines satisfying relation (1).

1. Introduction

Let G be a Lebesgue-measurable subset of the numerical axis and let $L_p(G)$ be a Lebesgue-measurable space of functions $x \colon G \to \mathbf{R}$ with finite norm (quasinorm)

$$\|x\|_{L_p(G)} := \begin{cases} \left(\displaystyle\int_G |x(t)|^p dt \right)^{1/p} & \text{for} \quad 0 < p < \infty, \\ \operatorname*{vrai\,sup}_{t \in G} |x(t)| & \text{for} \quad p = \infty. \end{cases}$$

By I_d we denote a circle realized in the form of a segment $[0, d]$ whose ends are identified. For the sake of brevity, we write $\|x\|_p$ instead of $\|x\|_{L_p(I_{2\pi})}$.

For $r \in \mathbf{N},\ G = \mathbf{R}$ or $G = I_d$, by $L^r_\infty(G)$ we denote the set of all functions $x \in L_\infty(G)$ with locally absolutely continuous derivatives up to the $(r-1)$th order satisfying the condition $x^{(r)} \in L_\infty(G)$.

By $\varphi_r(t),\ r \in \mathbf{N}$, we denote the shift of the rth 2π-periodic integral of the function $\varphi_0(t) = \operatorname{sgn} \sin t$ with mean value over the period equal to zero satisfying the condition $\varphi_r(0) = 0$. For $\lambda > 0$, we set

$$\varphi_{\lambda,r}(t) := \lambda^{-r} \varphi_r(\lambda).$$

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The following theorem was proved in [1]:

Theorem A. *Suppose that $r \in \mathbf{N}$ and $q > p > 0$. Then, for any function $x \in L^r_\infty(I_{2\pi})$ which has zeros, the following sharp inequality is true in the class $L^r_\infty(I_{2\pi})$:*

$$\|x\|_q \le \sup_{c\in[0,K_r]} \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_p^\alpha} \|x\|_p^\alpha \|x^{(r)}\|_\infty^{1-\alpha}, \tag{1.1}$$

where $\alpha = \dfrac{r+1/q}{r+1/p}$ and $K_r := \|\varphi_r\|_\infty$ is the Favard constant.

In the proof of inequality (1.1) in [1], it was established that if, for a given function $x \in L^r_\infty(I_{2\pi})$ which has zeros, the number $c \in [-K_r, K_r]$ is chosen to guarantee that the condition

$$\frac{\|x_+\|_p}{\|x_-\|_p} = \frac{\|(\varphi_r + c)_+\|_p}{\|(\varphi_r + c)_-\|_p}$$

is satisfied, then the inequality

$$\|x_\pm\|_q \le \frac{\|(\varphi_r + c)_\pm\|_q}{\|(\varphi_r + c)_\pm\|_p^\alpha} \|x_\pm\|_p^\alpha \|x^{(r)}\|_\infty^{1-\alpha} \tag{1.2}$$

is true.

An analog of inequality (1.1) in which the L_q-norm of a periodic function is estimated via its local L_p-norm was established in [2]. Sufficient conditions under which the least upper bound in inequality (1.1) is attained for $c = 0$ were established in [3].

In the present paper, we generalize inequalities (1.1) and (1.2) to the classes of functions with given comparison function. Moreover, these generalizations contain the “Remez effect.” We now present necessary definitions.

A function $f \in L^1_\infty(\mathbf{R})$ is called the comparison function for a function $x \in L^1_\infty(\mathbf{R})$ if there exists $c \in \mathbf{R}$ such that

$$\min_{t\in\mathbf{R}} f(t) + c \le x(t) \le \max_{t\in\mathbf{R}} f(t) + c, \quad t \in \mathbf{R},$$

and the equality $x(\xi) = f(\eta) + c$, where $\xi, \eta \in \mathbf{R}$, yields the inequality $|x'(\xi)| \le |f'(\eta)|$ provided that the indicated derivatives exist.

An odd 2ω-periodic function $\varphi \in L^1_\infty(I_{2\omega})$ is called an S-function if it has the following properties: φ is even with respect to $\omega/2$ and $|\varphi|$ is convex upward on $[0, \omega]$ and strictly monotone on $[0, \omega/2]$.

For a 2ω-periodic S-function φ, by $S_\varphi(\omega)$ we denote the class of functions $x \in L^1_\infty(I_d)$ for which φ is a comparison function. Note that the classes $S_\varphi(\omega)$ were considered in [4, 5]. As examples of the classes $S_\varphi(\omega)$, we can mention the Sobolev classes $L^r_\infty(I_d)$ with comparison function $\varphi_{\lambda,r}$, the bounded subsets of the spaces T_n (trigonometric polynomials of degree at most n) with comparison function $\sin nt$, and $S_{n,r}$ (2π-periodic splines of order r with defect 1 and nodes at the points $k\pi/n,\ k \in \mathbf{Z}$) with comparison function $\varphi_{n,r}$.

An important role in the approximation theory is played by the Remez-type inequalities

$$\|T\|_{L_\infty(I_{2\pi})} \le C(n, \beta)\|T\|_{L_\infty(I_{2\pi}\setminus B)} \tag{1.3}$$

on the class T_n, where B is an arbitrary Lebesgue-measurable set $B \subset I_{2\pi},\ \mu B \le \beta$.

The foundations of this direction were laid by Remez [6] who determined the sharp constant $C(n,\beta)$ in an inequality of the form (1.3) for algebraic polynomials. In inequality (1.3) for trigonometric polynomials, two-sided estimates for the sharp constants $C(n,\beta)$ were established in a series of works. Moreover, the asymptotic behaviors of the constants $C(n,\beta)$ as $\beta\to 2\pi$ [7] and as $\beta\to 0$ [8] are known. For the bibliography in this field, see [7–10]. In [8], the inequality

$$\|T\|_{L_\infty(I_{2\pi})}\le\left(1+2\tan^2\frac{n\beta}{4m}\right)\|T\|_{L_\infty(I_{2\pi}\setminus B)} \tag{1.4}$$

was proved for any polynomial $T\in T_n$ with the minimal period $2\pi/m$ and any Lebesgue-measurable set $B\subset I_{2\pi}$, $\mu B\le\beta$, where $\beta\in(0,2\pi m/n)$. The equality in (1.4) is attained for the polynomial

$$T(t)=\cos nx+\frac{1}{2}(1-\cos\beta/2).$$

Recently, a sharp constant for the Remez-type inequality (1.3) for trigonometric polynomials has been found in [11].

In [12], the result obtained in [8] was generalized to the classes $S_\varphi(\omega)$. As a consequence, an analog of inequality (1.4) for polynomial splines and functions from the classes $L^r_\infty(I_{2\pi})$ was obtained. In [13–17], some sharp Remez-type inequalities of different metrics and Kolmogorov–Remez-type inequalities were proved for the classes $S_\varphi(\omega)$ and, in particular, for the differentiable periodic functions, trigonometric polynomials, and splines. In addition, the relationship between the sharp constants for the Kolmogorov-type and Kolmogorov–Remez-type inequalities was investigated in [17]. Furthermore, the relationship between the sharp constants in the Kolmogorov-type inequalities for periodic functions and functions on the real axis was studied in [18].

In the present paper, we obtain sharp Remez-type inequalities of different metrics for the functions $x\in S_\varphi(\omega)$ with given ratio of the L_p-norms of their positive and negative parts (Theorem 1). As a consequence, we prove these inequalities for functions from the classes $L^r_\infty(I_{2\pi})$, trigonometric polynomials, and polynomial splines with given ratio of the L_p-norms of their positive and negative parts (Theorems 2–4). Note that the corollary of Theorem 2 contains inequality (1.1) with “Remez effect.”

2. Remez-Type Inequalities of Different Metrics on the Classes $S_\varphi(\omega)$

Theorem 1. *Suppose that $q,p>0$, $q\ge p$, φ is an S-function with period 2ω, and $\beta\in[0,2\omega)$. If, for a d-periodic function $x\in S_\varphi(\omega)$ with zeros, there exists $c\in[-\|\varphi\|_\infty,\|\varphi\|_\infty]$ satisfying the condition*

$$\|x_\pm\|_{L_p(I_d)}=\|(\varphi+c)_\pm\|_{L_p(I_{2\omega})}, \tag{2.1}$$

then, for any Lebesgue-measurable set $B\subset I_d$, $\mu B\le\beta$, the following inequality is true:

$$\|x\|_{L_q(I_d)}\le\frac{\|\varphi+c\|_{L_q(I_{2\omega})}}{\|\varphi+c\|_{L_p(I_{2\omega}\setminus B_{y(\beta)})}}\|x\|_{L_p(I_d\setminus B)}, \tag{2.2}$$

where

$$B_y:=\{t\in[0,2\omega]\colon\,|\varphi(t)+c|>y\}$$

and, moreover, $y=y(\beta)$ is chosen such that $\mu B_{y(\beta)}=\beta$.

For any fixed $c \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$, *inequality (2.2) is sharp in the class of functions* $x \in S_\varphi(\omega)$ *with zeros satisfying condition (2.1). Equality in (2.2) is attained for the function* $x(t) = \varphi(t) + c$ *and the set* $B = B_{y(\beta)}$.

We prove Theorem 1 in the form of a series of lemmas, which are also used in the proofs of the other theorems. We set

$$E_0(x)_\infty := \inf_{a \in \mathbf{R}} \|x - a\|_\infty.$$

Lemma 1. *Under the conditions of Theorem* 1*,*

$$\|x_\pm\|_\infty \le \|(\varphi + c)_\pm\|_\infty \tag{2.3}$$

and, in addition,

$$d \ge 2\omega. \tag{2.4}$$

Proof. We fix a function $x \in S_\varphi(\omega)$ and a number $c \in [-\|\varphi\|_\infty, \|\varphi\|_\infty]$ satisfying the conditions of Theorem 1. Assume that inequality (2.3) is not true for the function x. Since φ is the comparison function for the function x, we have $E_0(x)_\infty \le E_0(\varphi)_\infty$. Hence, the assumption made above means that exactly one inequality (2.3) is not true. Thus, let

$$\|x_+\|_\infty \le \|(\varphi + c)_+\|_\infty \qquad \text{and} \qquad \|x_-\|_\infty > \|(\varphi + c)_-\|_\infty.$$

Then there exists $a > 0$ such that

$$\|(x + a)_+\|_\infty \le \|(\varphi + c)_+\|_\infty, \qquad \|(x + a)_-\|_\infty = \|(\varphi + c)_-\|_\infty. \tag{2.5}$$

It is clear that $x + a \in S_\varphi(\omega)$. By m we denote the point of minimum of the function $\varphi + c$ and assume that $t_1 (t_2)$ is the left (right) zero of this function nearest to m. In view of the second relation in (2.5), there exists a shift $x(\cdot + \tau)$ of the function x such that

$$x(m + \tau) + a = \varphi(m) + c.$$

In addition, since $\varphi + c$ is the comparison function for the function x, we get

$$x(t + \tau) + a \le \varphi(t) + c < 0, \quad t \in (t_1, t_2).$$

In view of $a > 0$, this yields the estimate

$$\|x_-\|_{L_p(I_d)} > \|(x + a)_-\|_{L_p(I_d)} \ge \|(\varphi + c)_-\|_{L_p(I_{2\omega})},$$

which contradicts condition (2.1). Thus, inequality (2.3) is proved. Relation (2.4) directly follows from (2.1) and (2.3) in view of the inclusion $x \in S_\varphi(\omega)$.

Lemma 1 is proved.

For $f \in L_1[a, b]$, by $r(f, t)$, $t \in [0, b - a]$, we denote the permutation of the function $|f|$ (see, e.g., [19] Sec. 1.3) and set $r(f, t) = 0$ for $t > b - a$.

Lemma 2. *Under the conditions of Theorem 1,*

$$\int_0^\xi r^p(\bar{x}_\pm, t)dt \le \int_0^\xi r^p(\bar{\varphi}_\pm, t)dt, \quad \xi > 0, \tag{2.6}$$

where $\bar{x}$ is the restriction of x to I_d and $\bar{\varphi}$ is the restriction of $\varphi + c$ to $I_{2\omega}$. In particular,

$$\|x_\pm\|_{L_q(I_d)} \le \|(\varphi + c)_\pm\|_{L_q(I_{2\omega})}. \tag{2.7}$$

Proof. To prove (2.6), we note that, in view of (2.3), for any $y_\pm \in [0, \|\bar{x}_\pm\|_\infty)$, there exist points

$$t_i^\pm \in I_d, \quad i = 1, 2, \ldots, m, \quad m \ge 2, \quad y_j^\pm \in I_{2\omega}, \quad j = 1, 2,$$

such that

$$y_\pm = \bar{x}_\pm\big(t_i^\pm\big) = \bar{\varphi}_\pm\Big(y_j^\pm\Big).$$

Since $\varphi + c$ is the comparison function for x, we find

$$\big|\bar{x}'_\pm(t_i^\pm)\big| \le \Big|\bar{\varphi}'_\pm(y_j^\pm)\Big|.$$

We now show that if the points $\theta_1^\pm \in [0, d]$ and $\theta_2^\pm \in [0, 2\omega]$ satisfy the condition

$$y_\pm = r\big(\bar{x}_\pm, \theta_1^\pm\big) = r\big(\bar{\varphi}_\pm, \theta_2^\pm\big),$$

then

$$\big|r'\big(\bar{x}_\pm, \theta_1^\pm\big)\big| \le \big|r'\big(\bar{\varphi}_\pm, \theta_2^\pm\big)\big|.$$

Indeed, this directly follows from the theorem on the derivative of permutation (see, e.g., [19], Proposition 1.3.2). According to this theorem, we get

$$\big|r'\big(\bar{x}_\pm, \theta_1^\pm\big)\big| = \left[\sum_{i=1}^m \big|\bar{x}'_\pm(t_i)\big|^{-1}\right]^{-1} \le \left[\sum_{j=1}^2 \Big|\bar{\varphi}'_\pm\Big(y_j^\pm\Big)\Big|^{-1}\right]^{-1} = \big|r'\big(\bar{\varphi}_\pm, \theta_2^\pm\big)\big|.$$

By using the relation

$$r(\bar{x}_\pm, 0) = \|\bar{x}_\pm\|_\infty \le \|\bar{\varphi}_\pm\|_\infty = r(\bar{\varphi}_\pm, 0),$$

which follows from (2.3), and the fact that the L_∞-norm is preserved by permutations, we conclude that the difference

$$\Delta^\pm(t) := r(\bar{x}_\pm, t) - r(\bar{\varphi}_\pm, t)$$

changes sign on $[0, \infty)$ at most once (from minus to plus). The same is also true for the difference

$$\Delta_p^\pm(t) := r^p(\bar{x}_\pm, t) - r^p(\bar{\varphi}_\pm, t).$$

We set

$$I_{\pm}(\xi) := \int_0^{\xi} \Delta_p^{\pm}(t)dt.$$

Hence, $I_{\pm}(0) = 0.$ Since permutations preserve the L_p-norm, in view of (2.1) and (2.4), we get

$$I(d) = \|\bar{x}_{\pm}\|_{L_p(I_d)} - \|\bar{\varphi}_{\pm}\|_{L_p(I_{2\omega})} = 0.$$

Moreover, $I'_{\pm}(\xi) = \Delta_p^{\pm}(\xi)$ changes sign (from minus to plus) at most once.

Thus, $I(\xi) \le 0,\ \xi > 0,$ which is equivalent to (2.6). By virtue of the Hardy–Littlewood–Pólya theorem (see, e.g., [19], Theorem 1.3.1), inequality (2.6) yields inequality (2.7).

Lemma 2 is proved.

Lemma 3. *Under the conditions of Theorem 1,*

$$\|x\|_{L_p(I_d \setminus B)} \ge \|\varphi + c\|_{L_p(I_{2\omega} \setminus B_{y(\beta)})}. \tag{2.8}$$

Proof. As above, let $\bar{x}$ be the restriction of x to I_d and let $\bar{\varphi}$ be the restriction of $\varphi + c$ to $I_{2\omega}$. For any measurable set $B \subset I_d,$ we have $\mu B \le \beta,$ in view of the well-known property

$$\int_B |x(t)|^p\, dt \le \int_0^{\beta} r^p(\bar{x}, t)dt. \tag{2.9}$$

Further, since permutations preserve the L_p-norm, we find

$$\|x\|^p_{L_p(I_d \setminus B)} = \int_{I_d} |x(t)|^p dt - \int_B |x(t)|^p dt \ge \int_0^d r^p(\bar{x}, t)dt - \int_0^{\beta} r^p(\bar{x}, t)dt.$$

By using (2.1) and the inequality

$$\int_0^{\xi} r^p(\bar{x}, t)dt \le \int_0^{\xi} r^p(\bar{\varphi}, t)dt, \quad \xi > 0,$$

which follows from (2.6) according to Proposition 1.3.6 in [19], we obtain

$$\|x\|^p_{L_p(I_d \setminus B)} \ge \int_0^{2\omega} r^p(\bar{\varphi}, t)dt - \int_0^{\beta} r^p(\bar{\varphi}, t)dt = \int_{\beta}^{2\omega} r^p(\bar{\varphi}, t)dt = \int_{I_{2\omega} \setminus B_{y(\beta)}} |\varphi(t)|^p\, dt.$$

This yields (2.8).

Lemma 3 is proved.

Proof of Theorem 1. We fix a d-periodic function $x \in S_{\varphi}(\omega)$, which has zeros and satisfies condition (2.1) with some $c \in [-\|\varphi\|_{\infty}, \|\varphi\|_{\infty}]$. By Lemmas 2 and 3, this function admits estimates (2.7) and (2.8), which directly imply inequality (2.2). It is clear that this inequality is sharp.

Theorem 1 is proved.

3. Remez-Type Inequalities of Different Metrics for the Functions $x \in L^r_{\infty}(I_{2\pi})$

Recall that the symbol $\varphi_r(t),\ r \in \mathbf{N}$, denotes a shift of the rth 2π-periodic integral with zero mean value over the period of the function $\varphi_0(t) = \operatorname{sgn} \sin t$ satisfying the condition $\varphi_r(0) = 0$. It is clear that the spline

$$\varphi_{\lambda,r}(t) := \lambda^{-r}\varphi_r(\lambda t), \quad \lambda > 0,$$

is an S-function with period $2\pi/\lambda$.

For $r \in \mathbf{N},\ p > 0,$ and $f_p \in [0, \infty]$, we consider a class

$$f_p\, L^r_{\infty}(I_{2\pi}) := \left\{ x \in L^r_{\infty}(I_{2\pi}) \colon \frac{\|x_+\|_p}{\|x_-\|_p} = f_p \right\}.$$

It is clear that, for given p and f_p, there exists a unique number $c \in [-K_r, K_r]$ for which

$$\varphi_r + c \in f_p\, L^r_{\infty}(I_{2\pi}). \tag{3.1}$$

Theorem 2. *Suppose that $r \in \mathbf{N},\ p,\ q > 0,\ q \ge p,\ f_p \in [0, \infty]$, and $\beta \in [0, 2\pi)$. For any function $x \in f_p\, L^r_{\infty}(I_{2\pi})$ with zeros and any measurable set $B \subset I_{2\pi}$ such that $\mu B \le \beta/\lambda$, where λ is chosen to guarantee that*

$$\|x\|_p = \|\varphi_{\lambda,r} + \lambda^{-r}c\|_{L_p(I_{2\pi/\lambda})} \left\| x^{(r)} \right\|_{\infty} \tag{3.2}$$

and the number c satisfies condition (3.1), the following inequality is true:

$$\|x\|_q \le \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|^{\alpha}_{L_p(I_{2\pi}\setminus B_{y(\beta)})}} \|x\|^{\alpha}_{L_p(I_{2\pi}\setminus B)} \left\| x^{(r)} \right\|^{1-\alpha}_{\infty}, \tag{3.3}$$

where

$$\alpha = \frac{r + 1/q}{r + 1/p}, \qquad B_y := \{t \in I_{2\pi} \colon |\varphi_r(t) + c| > y\},$$

and, in addition, $y = y(\beta)$ is chosen such that $\mu B_{y(\beta)} = \beta$.

Inequality (3.3) is sharp in the class of all pairs (x, B) formed by a function $x \in f_p\, L^r_{\infty}(I_{2\pi})$, which has zeros, and a measurable set $B \subset I_{2\pi}$ for which $\mu B \le \beta/\lambda$, where λ satisfies condition (3.2). The equality in (3.3) is attained for the pair $(x, B_{y(\beta)})$, where $x(t) = \varphi_r(t) + c$.

Proof. We fix a function $x \in f_p\, L^r_{\infty}(I_{2\pi})$ satisfying the conditions of the theorem. Since inequality (3.3) is homogeneous, we can assume that

$$\left\| x^{(r)} \right\|_{\infty} = 1. \tag{3.4}$$

Thus, in view of (3.1), (3.2) and the definition of the class $f_p\, L^r_\infty(I_{2\pi})$, we get

$$\|x_\pm\|_p = \left\|\left(\varphi_{\lambda,r} + \lambda^{-r}c\right)_\pm\right\|_{L_p(I_{2\pi/\lambda})}. \tag{3.5}$$

For functions $x \in f_p\, L^r_\infty(I_{2\pi})$ satisfying this condition, inequality (1.2) holds

$$\|x_\pm\|_q \le \frac{\|(\varphi_r + c)_\pm\|_q}{\|(\varphi_r + c)_\pm\|_p^\alpha} \|x_\pm\|_p^\alpha \left\|x^{(r)}\right\|_\infty^{1-\alpha}.$$

By using this inequality, relations (3.4) and (3.5), and the following obvious equality:

$$\left\|\left(\varphi_{\lambda,r} + \lambda^{-r}c\right)_\pm\right\|_{L_p(I_{2\pi/\lambda})} = \lambda^{-(r+1/p)}\|(\varphi_r + c)_\pm\|_p, \quad p > 0, \tag{3.6}$$

we arrive at the estimate

$$\|x_\pm\|_q \le \left\|\left(\varphi_{\lambda,r} + \lambda^{-r}c\right)_\pm\right\|_{L_q(I_{2\pi/\lambda})}. \tag{3.7}$$

In particular, in view of (3.4) and (3.7) (for $q = \infty$), the function x satisfies the conditions of the Kolmogorov comparison theorem [20]. According to this theorem, the spline $\varphi(t) = \varphi_{\lambda,r}(t)$ is the comparison function for the function x, i.e., $x \in S_\varphi\left(\frac{\pi}{\lambda}\right)$. Hence, in view of (3.5), the function x satisfies all conditions of Theorem 1. By virtue of this theorem, for $q \ge p$ and an arbitrary measurable set $B \subset I_{2\pi},\ \mu B \le \beta/\lambda$, the inequality

$$\|x\|_q \le \frac{\|\varphi_{\lambda,r} + \lambda^{-r}c\|_{L_q(I_{2\pi/\lambda})}}{\|\varphi_{\lambda,r} + \lambda^{-r}c\|_{L_p\left(I_{2\pi/\lambda}\setminus\frac{B_{y(\beta)}}{\lambda}\right)}} \|x\|_{L_p(I_{2\pi}\setminus B)}$$

is true. It follows from the last inequality (for $q = p$) and conditions (3.2) and (3.4) that

$$\|x\|_{L_p(I_{2\pi}\setminus B)} \ge \left\|\varphi_{\lambda,r} + \lambda^{-r}c\right\|_{L_p\left(I_{2\pi/\lambda}\setminus\frac{B_{y(\beta)}}{\lambda}\right)}.$$

Combining the obtained lower estimate with inequality (3.7), in view of the obvious relation

$$\left\|\varphi_{\lambda,r} + \lambda^{-r}c\right\|_{L_p\left(I_{2\pi/\lambda}\setminus\frac{B_{y(\beta)}}{\lambda}\right)} = \lambda^{-(r+1/p)}\|\varphi_r + c\|_{L_p(I_{2\pi}\setminus B_{y(\beta)})}$$

and the definition $\alpha = \dfrac{r+1/q}{r+1/p}$, we obtain

$$\frac{\|x\|_q}{\|x\|^\alpha_{L_p(I_{2\pi}\setminus B)}} \le \frac{\|\varphi_{\lambda,r} + \lambda^{-r}c\|_{L_q(I_{2\pi/\lambda})}}{\|\varphi_{\lambda,r} + \lambda^{-r}c\|^\alpha_{L_p\left(I_{2\pi/\lambda}\setminus\frac{B_{y(\beta)}}{\lambda}\right)}} = \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|^\alpha_{L_p(I_{2\pi}\setminus B_{y(\beta)})}}.$$

By virtue of (3.4), this estimate yields (3.3). Thus, it is clear that inequality (3.3) is sharp.

Theorem 2 is proved.

Corollary 1. *Suppose that* $r \in \mathbf{N},\ p, q > 0,\ q \geq p,\ \alpha = \dfrac{r + 1/q}{r + 1/p},\ \beta \in [0, 2\pi),$ *and the number* $\bar{c} \in [0, K_r]$ *realizes the upper bound*

$$\sup_{c \in [0, K_r]} \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|^{\alpha}_{L_p(I_{2\pi} \setminus B^c_{y(\beta)})}},$$

where

$$B^c_y := \{t \in I_{2\pi} : \ |\varphi_r(t) + c| > y\}$$

and, moreover, $y = y(\beta)$ *is chosen such that* $\mu B^c_{y(\beta)} = \beta.$

Then, for any function $x \in L^r_\infty(I_{2\pi})$ *with zeros and an arbitrary measurable set* $B \subset I_{2\pi},\ \mu B \leq \beta/\lambda,$ *where* λ *is chosen to guarantee that*

$$\|x\|_p = \left\|\varphi_{\lambda, r} + \lambda^{-r} c\right\|_{L_p(I_{2\pi/\lambda})} \left\|x^{(r)}\right\|_\infty \tag{3.8}$$

and c *satisfies the condition*

$$\|x_+\|_p \, \|x_-\|^{-1}_p = \|(\varphi_r + c)_+\|_p \, \|(\varphi_r + c)_-\|^{-1}_p,$$

the following inequality is true:

$$\|x\|_q \leq \frac{\|\varphi_r + \bar{c}\|_q}{\|\varphi_r + \bar{c}\|^{\alpha}_{L_p(I_{2\pi} \setminus B^{\bar{c}}_{y(\beta)})}} \|x\|^{\alpha}_{L_p(I_{2\pi} \setminus B)} \left\|x^{(r)}\right\|^{1-\alpha}_\infty. \tag{3.9}$$

Inequality (3.9) is sharp in the class of all pairs (x, B) *formed by a function* $x \in L^r_\infty(I_{2\pi})$ *with zeros and a measurable set* $B \subset I_{2\pi}$ *such that* $\mu B \leq \beta/\lambda,$ *where* λ *satisfies condition (3.8). Equality in (3.9) is attained for the pair* $\left(x, B^{\bar{c}}_{y(\beta)}\right),$ *where* $x(t) = \varphi_r(t) + \bar{c}.$

Remark 1.

1. For $\beta = 0,$ Theorem 2 and Corollary 1 were proved in [1].

2. For functions $x \in L^r_\infty(I_{2\pi})$ satisfying the condition $\|x_+\|_p = \|x_-\|_p,$ the constant in inequality (3.3) is equal to zero.

3. For functions of constant sign $x \in L^r_\infty(I_{2\pi})$ with zeros, inequality (3.3) turns into the inequality for the best one-sided approximations by the constant

$$E^{\pm}_0(x)_{L_sG)} := \inf_{c \in \mathbf{R}} \big\{\|x - c\|_{L_s(G)} : \ \forall t \in G \ \pm (x(t) - c)_\pm \geq 0\big\}, \tag{3.10}$$

i.e., the norms $\|x\|_q$ and $\|x\|_{L_p(I_{2\pi} \setminus B)}$ in inequality (3.3) for these functions are replaced by $E^{\pm}_0(x)_q$ and $E^{\pm}_0(x)_{L_p(I_{2\pi} \setminus B)},$ respectively. Moreover, the constant c in this inequality is replaced by the Favard constant $K_r.$

4. Remez-Type Inequalities of Different Metrics for Trigonometric Polynomials

Recall that T_n is a space of trigonometric polynomials of degree at most n. For $p > 0,\ f_p \in [0, \infty]$, we set

$$f_p T_n := \left\{ T \in T_n : \frac{\|T_+\|_p}{\|T_-\|_p} = f_p \right\}.$$

Theorem 3. *Suppose that $n, m \in \mathbf{N},\ p, q > 0,\ q \ge p$, and $f_p \in [0, \infty]$. If the trigonometric polynomial $T \in f_p T_n$ with the minimal period $2\pi/m$ has zeros, then, for any measurable set $B \subset I_{2\pi},\ \mu B \le \frac{m}{n}\beta$, $\beta \in [0, 2\pi)$, the following inequality is true:*

$$\|T\|_q \le \left(\frac{n}{m}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{\|\sin(\cdot) + c\|_q}{\|\sin(\cdot) + c\|_{L_p(I_{2\pi}\setminus B_{y(\beta)})}} \|T\|_{L_p(I_{2\pi}\setminus B)}, \tag{4.1}$$

where the number $c \in [-1, 1]$ satisfies the condition

$$\sin(\cdot) + c \in f_p T_n, \tag{4.2}$$

and $B_y := \{t \in I_{2\pi} : |\sin t + c| > y\}$; moreover, $y = y(\beta)$ is chosen to guarantee that $\mu B_{y(\beta)} = \beta$.

Inequality (4.1) is sharp in the following sense:

$$\sup_{(n,m)\in N_{n,m}} \sup_{(T,B)\in P_n^m} \frac{\|T\|_q}{(n/m)^{1/p-1/q}\|T\|_{L_p(I_{2\pi}\setminus B)}} = \frac{\|\sin(\cdot) + c\|_q}{\|\sin(\cdot) + c\|_{L_p(I_{2\pi}\setminus B_{y(\beta)})}}, \tag{4.3}$$

where $N_{n,m}$ is the set of pairs (n, m) of natural numbers such that $m \le n$ and P_n^m is the set of pairs (T, B) formed by the polynomial $T \in f_p T_n$ with zeros and the minimal period $2\pi/m$ and a measurable set $B \subset I_{2\pi}$, $\mu B \le \frac{m}{n}\beta$.

Proof. We fix a polynomial $T \in f_p T_n$ satisfying the conditions of Theorem 3. For the sake of brevity, we set $\varphi(t) := \sin nt$ and $\psi(t) := \varphi(t) + c,\ t \in \mathbf{R}$. In view of the homogeneity of inequality (4.1), we can assume that

$$\|T\|_{L_p(I_{2\pi/m})} = \|\psi\|_{L_p(I_{2\pi/n})}. \tag{4.4}$$

In view of condition (4.2) and the definition of the class $f_p T_n$, this yields the equality

$$\|T_\pm\|_{L_p(I_{2\pi/m})} = \|\psi_\pm\|_{L_p(I_{2\pi/n})}. \tag{4.5}$$

We now show that

$$\|T_\pm\|_\infty \le \|\psi_\pm\|_\infty. \tag{4.6}$$

Indeed, assume the contrary, i.e., that there exists $\gamma \in (0, 1)$ such that

$$\|\gamma T_\pm\|_\infty \le \|\psi_\pm\|_\infty.$$

Moreover, one of these inequalities turns into the equality. Thus, let

$$\|\gamma T_+\|_\infty \le \|\psi_+\|_\infty, \qquad \|\gamma T_-\|_\infty = \|\psi_-\|_\infty.$$

Then the polynomial ψ is a comparison function for the polynomial γT (see the proof of Theorem 8.1.1 in [21]). Let m be a point of minimum of the function ψ and let t_1 (t_2) be the nearest (to m) left (right) zero of this function. Passing, if necessary, to the shift of the polynomial $\gamma T,$ we can assume that

$$\|\gamma T_-\|_\infty = -\gamma T(m).$$

Since ψ is a comparison function for the polynomial γT, we find

$$\gamma T(t) \le \psi(t) < 0, \quad t \in (t_1, t_2).$$

This yields the estimate

$$\|T_-\|_{L_p(2\pi/m)} > \|\gamma T_-\|_{L_p(2\pi/m)} \ge \|\psi_-\|_{L_p(2\pi/n)},$$

which contradicts (4.5). Thus, inequality (4.6) is proved.

This inequality and the proof of Theorem 8.1.1 in [21] imply that $\varphi(t) = \sin nt$ is a comparison function for the polynomial $T(t),$ i.e., $T \in S_\varphi\left(\dfrac{\pi}{n}\right)$. Hence, in view of (4.4), the polynomial T satisfies all conditions of Theorem 1 and, therefore, also the conditions of Lemmas 1–3.

Further, we establish the inequality

$$\|T\|_q \le \left(\frac{m}{n}\right)^{1/q} \|\sin(\cdot) + c\|_q. \tag{4.7}$$

Indeed, by virtue of inequality (2.7), we obtain

$$\|T\|_{L_q(I_{2\pi/m})} \le \|\varphi + c\|_{L_q(I_{2\pi/n})}.$$

This immediately yields (4.7) because the polynomial T is $2\pi/m$-periodic and the function φ is $2\pi/n$-periodic.

We now prove the inequality

$$\|T\|_{L_p(I_{2\pi\setminus B})} \ge \left(\frac{m}{n}\right)^{1/p} \|\sin(\cdot) + c\|_{L_p(I_{2\pi\setminus B_{y(\beta)}})} \tag{4.8}$$

for any measurable set $B \subset I_{2\pi},\ \mu B \le \dfrac{m}{n}\,\beta.$

Let $\bar T$ be the restriction of the polynomial T to $I_{2\pi/m}$ and let $\bar\varphi$ be the restriction of $\varphi + c$ to $I_{2\pi/n}$. By using inequality (2.9), in view of the fact that permutation preserves the L_p-norm, we get

$$\|T\|^p_{L_p(I_{2\pi\setminus B})} = \int\limits_0^{2\pi} |T(t)|^p\,dt - \int\limits_B |T(t)|^p\,dt$$

$$\ge \int\limits_0^{2\pi} r^p(T,t)\,dt - \int\limits_0^{\frac{m}{n}\beta} r^p(T,t)\,dt$$

$$= m\left[\int\limits_0^{2\pi/m} r^p(\bar{T},t)\,dt - \int\limits_0^{\beta/n} r^p\big(\bar{T},t\big)dt\right].$$

Thus, by virtue of (4.4) and the inequality

$$\int\limits_0^{\xi} r^p\big(\bar{T},t\big)\,dt \le \int\limits_0^{\xi} r^p(\bar{\varphi},t)\,dt, \quad \xi > 0,$$

which follows from (2.6), according to Proposition 1.3.6 in [19], we arrive at the following lower estimate:

$$\|T\|^p_{L_p(I_{2\pi}\setminus B)} \ge m\left[\int\limits_0^{2\pi/n} r^p(\bar{\varphi},t)\,dt - \int\limits_0^{\beta/n} r^p(\bar{\varphi},t)\,dt\right]$$

$$= m\int\limits_{\beta/n}^{2\pi/n} r^p(\bar{\varphi},t)\,dt = \frac{m}{n}\int\limits_{\beta}^{2\pi} r^p(\varphi + c,t)\,dt$$

$$= \frac{m}{n}\int\limits_{I_{2\pi}\setminus B_y(n)} |\varphi(t)+c|^p\,dt = \frac{m}{n}\|\sin(\cdot)+c\|^p_{L_p\left(I_{2\pi}\setminus B_{y(\beta)}\right)},$$

where

$$B_y(n) := \{t \in I_{2\pi}\colon\ |\sin nt + c| > y\}$$

and, moreover, $y = y(\beta)$ is chosen such that $\mu B_y(n) = \beta$. The obtained estimate yields inequality (4.8). Combining (4.7) and (4.8), we arrive at inequality (4.1). It is clear that (4.1) is sharp in a sense of (4.3).

Theorem 3 is proved.

Corollary 2. *Suppose that $n, m \in \mathbf{N},\ q, p > 0,\ q \ge p,\ \beta \in [0, 2\pi)$, and the number $\bar{c} \in [0,1]$ realizes the upper bound*

$$\sup_{c\in[0,1]} \frac{\|\sin(\cdot)+c\|_q}{\|\sin(\cdot)+c\|_{L_p(I_{2\pi}\setminus B^c_{y(\beta)})}},$$

where $B^c_y := \{t \in I_{2\pi}\colon\ |\sin t + c| > y\}$ and, moreover, $y = y(\beta)$ is chosen to guarantee that $\mu B^c_{y(\beta)} = \beta$.

Then, for any trigonometric polynomial $T \in T_n$ with zeros and the minimal period $2\pi/m$ and any measurable set $B \subset I_{2\pi},\ \mu B \le \dfrac{m}{n}\,\beta$, the following inequality is true:

$$\|T\|_q \le \left(\frac{n}{m}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{\|\sin(\cdot)+\bar{c}\|_q}{\|\sin(\cdot)+\bar{c}\|_{L_p(I_{2\pi}\setminus B^{\bar{c}}_{y(\beta)})}}\|T\|_{L_p(I_{2\pi}\setminus B)}. \tag{4.9}$$

Inequality (4.8) is sharp in the following sense:

$$\sup_{(n,m)\in N_{n,m}}\ \sup_{(T,B)\in Q^m_n} \frac{\|T\|_q}{(n/m)^{1/p-1/q}\|T\|_{L_p(I_{2\pi}\setminus B)}} = \frac{\|\sin(\cdot)+\bar{c}\|_q}{\|\sin(\cdot)+\bar{c}\|_{L_p(I_{2\pi}\setminus B^{\bar{c}}_{y(\beta)})}},$$

where $N_{n,m}$ is the set of pairs (n,m) of natural numbers such that $m \le n$ and Q_n^m is the set of pairs (T,B) formed by a polynomial $T \in T_n$ with zeros and the minimal period $2\pi/m$ and a measurable set $B \subset I_{2\pi}$, $\mu B \le \dfrac{m}{n}\beta$.

Remark 2.

1. For $\beta = 0$ and $m = 1$, Theorem 3 and Corollary 2 are proved in [1].
2. For the polynomials $T \in T_n$ satisfying the condition $\|T_+\|_p = \|T_-\|_p$, the constant c in inequality (4.1) is equal to zero.
3. For sign-preserving polynomials $T \in T_n$ which have zeros, inequality (4.1) turns into the inequality for the best one-sided approximations by a constant [see (3.10)], i.e., the norms $\|T\|_q$ and $\|T\|_{L_p(I_{2\pi}\setminus B)}$ in inequality (4.1) for these polynomials should be replaced by $E_0^{\pm}(T)_q$ and $E_0^{\pm}(T)_{L_p(I_{2\pi}\setminus B)}$, respectively. Moreover, the constant c in this inequality is equal to 1.

5. Remez-Type Inequalities of Different Metrics for Splines

Recall that $S_{n,r}$ is a space of 2π-periodic splines of order r with defect 1 and nodes at the points $k\pi/n$, $k \in \mathbf{Z}$. For $p > 0$ and $f_p \in [0,\infty]$, we set

$$f_p\, S_{n,r} := \left\{ s \in S_{n,r} : \frac{\|s_+\|_p}{\|s_-\|_p} = f_p \right\}.$$

Theorem 4. *Suppose that $n, m \in \mathbf{N}$, $p, q > 0$, $q \ge p$, and $f_p \in [0,\infty]$. If a spline $s \in f_p\, S_{n,r}$ with the minimal period $2\pi/m$ has zeros, then, for any measurable set $B \subset I_{2\pi}$, $\mu B \le \dfrac{m}{n}\beta$, the following inequality is true:*

$$\|s\|_q \le \left(\frac{n}{m}\right)^{\frac{1}{p}-\frac{1}{q}} \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_{L_p(I_{2\pi}\setminus B_{y(\beta)})}} \|s\|_{L_p(I_{2\pi}\setminus B)}, \tag{5.1}$$

where $c \in [-K_r, K_r]$ satisfies the condition

$$\varphi_{n,r} + n^{-r} c \in f_p\, S_{n,r}, \tag{5.2}$$

and $B_y := \{t \in I_{2\pi} : |\varphi_r(t) + c| > y\}$; moreover, $y = y(\beta)$ is chosen to guarantee that $\mu B_{y(\beta)} = \beta$.

Inequality (5.1) is sharp in the following sense:

$$\sup_{(n,m)\in N_{n,m}} \sup_{(s,B)\in S_n^m} \frac{\|s\|_q}{(n/m)^{1/p-1/q}\|s\|_{L_p(I_{2\pi}\setminus B)}} = \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_{L_p(I_{2\pi}\setminus B_{y(\beta)})}}, \tag{5.3}$$

where $N_{n,m}$ is a set of pairs (n,m) of natural numbers such that $m \le n$ and S_n^m is the set of pairs (s,B) formed by a spline $s \in f_p\, S_{n,r}$ with zeros and the minimal period $2\pi/m$ and a measurable set $B \subset I_{2\pi}$, $\mu B \le \dfrac{m}{n}\beta$.

Proof. We fix a spline $s \in f_p\, S_{n,r}$ satisfying the conditions of Theorem 4. For the sake of brevity, we set $\varphi(t) := \varphi_{n,r}(t)$ and $\psi(t) := \varphi_{n,r}(t) + n^{-r}c$, $t \in \mathbf{R}$. In view of the homogeneity of inequality (5.1), we can assume that

$$\|s\|_{L_p(I_{2\pi/m})} = \|\psi\|_{L_p(I_{2\pi/n})}. \tag{5.4}$$

Thus, in view of (5.2) and the definition of the class $f_p\, S_{n,r},$ we arrive at the equality

$$\|s_\pm\|_{L_p(I_{2\pi/m})} = \|\psi_\pm\|_{L_p(I_{2\pi/n})}. \tag{5.5}$$

We now show that

$$\|s_\pm\|_\infty \le \|\psi_\pm\|_\infty. \tag{5.6}$$

Indeed, assume the contrary, i.e., that there exists $\gamma \in (0,1)$ such that $\|\gamma s_\pm\|_\infty \le \|\psi_\pm\|_\infty$ and, in addition, that one of these inequalities turns into the equality; e.g., that

$$\|\gamma s_+\|_\infty \le \|\psi_+\|_\infty \qquad \text{and} \qquad \|\gamma s_-\|_\infty = \|\psi_-\|_\infty.$$

Then

$$E_0(\gamma s)_\infty \le E_0(\psi)_\infty = \|\varphi_{n,r}\|_\infty$$

and, by virtue of the Tikhomirov inequality [22]

$$\left\|s^{(r)}\right\|_\infty \le \frac{E_0(s)_\infty}{\|\varphi_{n,r}\|_\infty},$$

where $E_0(x)_\infty$ is the best uniform approximation of the function x by constants, we arrive at the inequality

$$\left\|\gamma s^{(r)}\right\|_\infty \le 1.$$

Thus, the spline γs satisfies the conditions of the Kolmogorov comparison theorem [20]. By this theorem, the spline φ is the comparison function for the spline $\gamma s.$ Let m be the point of minimum of the function ψ and let t_1 (t_2) be the left (right) nearest (to m) zero of this function. Passing, if necessary, to a shift of the spline $\gamma s,$ we can assume that

$$\|\gamma s_-\|_\infty = -\gamma s(m).$$

Since the spline ψ is the comparison function for the spline $\gamma s,$ we get

$$\gamma s(t) \le \psi(t) < 0, \quad t \in (t_1, t_2).$$

This yields the estimate

$$\|s_-\|_{L_p(2\pi/m)} > \|\gamma s_-\|_{L_p(2\pi/m)} \ge \|\psi_-\|_{L_p(2\pi/n)},$$

which contradicts (5.5). Thus, inequality (5.6) is proved.

By using inequality (5.6), we find

$$E_0(s)_\infty \le E_0(\psi)_\infty = \|\varphi_{n,r}\|_\infty.$$

Applying the Tikhomirov inequality, we obtain

$$\left\|s^{(r)}\right\|_\infty \le \frac{E_0(s)_\infty}{\|\varphi_{n,r}\|_\infty} \le 1.$$

Therefore, the spline s satisfies the conditions of the Kolmogorov comparison theorem [20]. According to this theorem, the spline φ is the comparison function for the spline s. Hence, $s \in S_\varphi\left(\frac{\pi}{n}\right)$ and, in view of (5.5), the spline s satisfies the conditions of Theorem 1 and, thus, also the conditions of Lemmas 1–3.

We prove the inequality

$$\|s\|_q \le n^{-r}\left(\frac{m}{n}\right)^{1/q}\|\varphi + c\|_q. \tag{5.7}$$

Indeed, by virtue of inequality (2.7), we get

$$\|s\|_{L_q(I_{2\pi/m})} \le \left\|\varphi_{n,r} + n^{-r}c\right\|_{L_q(I_{2\pi/n})}.$$

This directly yields (5.7) because the spline s is $2\pi/m$-periodic and the spline $\varphi_{n,r}$ is $2\pi/n$-periodic.

We now prove the inequality

$$\|s\|_{L_q(I_{2\pi}\setminus B)} \ge n^{-r}\left(\frac{m}{n}\right)^{1/p}\|\varphi_r + c\|_{L_q(I_{2\pi}\setminus B_{y(\beta)})} \tag{5.8}$$

for any measurable set $B \subset I_{2\pi}$, $\mu B \le \frac{m}{n}\beta$, $\beta \in [0, 2\pi)$. Let $\bar{s}$ be the restriction of the spline s to $I_{2\pi/m}$ and let $\bar{\psi}$ be the restriction of the spline ψ to $I_{2\pi/n}$. As in the proof of Theorem 3, by using inequality (2.9) and taking into account the fact that permutations preserve the L_p-norm, we obtain

$$\|s\|^p_{L_p(I_{2\pi}\setminus B)} \ge m\left[\int\limits_0^{2\pi/m} r^p(\bar{s}, t)\, dt - \int\limits_0^{\beta/n} r^p(\bar{s}, t)\, dt\right].$$

Further, by using (5.4) and the inequality

$$\int\limits_0^\xi r^p(\bar{s}, t) dt \le \int\limits_0^\xi r^p\big(\bar{\psi}, t\big) dt, \quad \xi > 0,$$

which follows from (2.6) according to Proposition 1.3.6 in [19], as in the proof of Theorem 3, we obtain the following lower bound:

$$\|s\|^p_{L_p(I_{2\pi}\setminus B)} \ge m\left[\int\limits_0^{2\pi/n} r^p\big(\bar{\psi}, t\big)\, dt - \int\limits_0^{\beta/n} r^p\big(\bar{\psi}, t\big) dt\right] = m\int\limits_{\beta/n}^{2\pi/n} r^p\big(\bar{\psi}, t\big)\, dt$$

$$= \frac{m}{n} \int_{\beta}^{2\pi} r^p(\psi, t)\, dt = \frac{m}{n} n^{-rp} \int\limits_{I_{2\pi} \setminus B_{y(\beta)}(n)} |\varphi_r(nt) + c|^p\, dt$$

$$= n^{-rp} \frac{m}{n} \|(\varphi_r + c)\|^p_{L_p(I_{2\pi} \setminus B_{y(\beta)})},$$

where

$$B_{y(\beta)}(n) := \{t \in I_{2\pi} \colon |\varphi_r(nt) + c| > y\}$$

and $y = y(\beta)$ is chosen to guarantee that $\mu B_{y(\beta)}(n) = \beta$.

The obtained lower bound is equivalent to (5.8). Inequality (5.1) directly follows from (5.7) and (5.8). It is clear that inequality (5.1) is sharp in the sense of (5.3).

Theorem 4 is proved.

Corollary 3. *Suppose that $n, m \in \mathbf{N},\ q, p > 0,\ q \geq p,\ \beta \in [0, 2\pi)$, and the number $\bar{c} \in [0, K_r]$ realizes the upper bound*

$$\sup_{c \in [0, K_r]} \frac{\|\varphi_r + c\|_q}{\|\varphi_r + c\|_{L_p(I_{2\pi} \setminus B^c_{y(\beta)})}},$$

where $B^c_y := \{t \in I_{2\pi} \colon |\varphi_r(t) + c| > y\}$ and, in addition, $y = y(\beta)$ is such that $\mu B^c_{y(\beta)} = \beta$.

Then, for any spline $s \in S_{n,r}$ with zeros and the minimal period $2\pi/m$ and an arbitrary measurable set $B \subset I_{2\pi},\ \mu B \leq \frac{m}{n} \beta$, the following inequality is true:

$$\|s\|_q \leq \left(\frac{n}{m}\right)^{\frac{1}{p} - \frac{1}{q}} \frac{\|\varphi_r + \bar{c}\|_q}{\|\varphi_r + \bar{c}\|_{L_p\left(I_{2\pi} \setminus B^{\bar{c}}_{y(\beta)}\right)}} \|s\|_{L_p(I_{2\pi} \setminus B)}. \tag{5.9}$$

Inequality (5.9) is sharp in the following sense:

$$\sup_{(n,m) \in N_{n,m}} \sup_{(s,B) \in \Sigma^m_n} \frac{\|s\|_q}{(n/m)^{1/p - 1/q} \|s\|_{L_p(I_{2\pi} \setminus B)}} = \frac{\|\varphi_r + \bar{c}\|_q}{\|\varphi_r + \bar{c}\|_{L_p(I_{2\pi} \setminus B_{y(\beta)})}},$$

where $N_{n,m}$ is the set of pairs (n, m) of natural numbers such that $m \leq n$ and Σ^m_n is a set of pairs (s, B) formed by a spline $s \in S_{n,r}$ with zeros and the minimal period $2\pi/m$ and a measurable set $B \subset I_{2\pi},\ \mu B \leq \frac{m}{n} \beta$.

Remark 3.

1. For $\beta = 0$ and $m = 1$, Theorem 4 and Corollary 3 were obtained in [1].

2. For the splines $s \in S_{n,r}$ satisfying the condition $\|s_+\|_p = \|s_-\|_p$, the constant c in inequality (5.1) is equal to zero.

3. For splines of constant sign $s \in S_{n,r}$ with zeros, inequality (5.1) turns into the inequality for the best one-sided approximations by a constant [see (3.10)], i.e., the norms $\|s\|_q$ and $\|s\|_{L_p(I_{2\pi} \setminus B)}$ in inequality (5.1) for these splines should be replaced by $E^{\pm}_0(s)_q$ and $E^{\pm}_0(s)_{L_p(I_{2\pi} \setminus B)}$, respectively. Moreover, the constant c in this inequality is equal to the Favard constant K_r.

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