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ESTIMATES FOR THE DEVIATIONS OF INTEGRAL OPERATORS IN SEMILINEAR METRIC SPACES AND THEIR APPLICATIONS

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We develop the theory of approximations in functional semilinear metric spaces that allows us to consider the classes of multi- and fuzzy-valued functions, as well as the classes of functions with values in Banach spaces, including the classes of random processes. For integral operators on the classes of functions with values in semilinear metric spaces, we obtain estimates of their deviations and discuss possible applications of these estimates to the investigation of the problems of approximation by generalized trigonometric polynomials, optimization of approximate integration formulas, and reconstruction of functions according to incomplete information.

1. Introduction

The aim of the present paper is to develop some branches of the approximation theory in semilinear metric function spaces. The research carried out in this direction can be motivated as follows: At present, the approximation theory of functions taking numerical values is a well-developed part of analysis (see, e.g., the monographs [1–7]) and has numerous applications. For several last decades, the researchers made attempts to develop, for various theoretical and practical reasons, the approximation theory of set-valued (see [8]) and fuzzy-valued (see [9]) functions. It is quite natural that they first tried to generalize the existing methods developed for numerical-valued functions to the new types of functions. As a rule, this led and still leads to significant difficulties.

Note that the development of approximation theory in semilinear metric spaces enables one, first, to consider the corresponding problems for set- and fuzzy-valued functions, as well as for the functions with values in Banach spaces (in particular, for random processes) and, second, these results can be used as a basis for the development of computational algorithms aimed at the solution of various problems posed for these functions. In addition, the possibility of generalization of the results known for numerical functions to the case of functions with values in semilinear metric spaces is of significant theoretical interest.

We now briefly describe the structure of the present paper. In the second section, we present necessary results from the theory of semilinear metric spaces (L-spaces). In Sec. 3, we study the problem of estimation of the deviations of integral operators in the spaces of functions with values in L-spaces. In the fourth section, we discuss possible applications of the obtained results to the problems of approximation by generalized trigonometric polynomials, optimization of the formulas of approximate integration, and reconstruction of functions according to incomplete information.

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2. Definitions and the Required Information about L-Spaces

2.1.

Definition 1. *A set X is called a semilinear space if the operations of addition of its elements and multiplication by a real number are defined in this set and, for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$, the following conditions are satisfied:*

(i) $x + y = y + x$;

(ii) $x + (y + z) = (x + y) + z$;

(iii) $\exists \theta \in X: \ x + \theta = x$;

(iv) $\alpha(x + y) = \alpha x + \alpha y$;

(v) $\alpha(\beta x) = (\alpha\beta)x$;

(vi) $1 \cdot x = x, \ 0 \cdot x = \theta$,

where θ denotes a neutral element of the space X specified by property (iii). It is easy to see that this element θ is unique.

In what follows, for $\alpha \in \mathbb{R}$ and $x \in X$, we sometimes write $x\alpha$ instead of αx and $-\alpha x$ instead of $(-\alpha)x$. In particular, the notation $-x$ means $(-1) \cdot x$.

Definition 2. *An element $x \in X$ is called convex if, for all $\alpha, \beta \geq 0$,*

$$(\alpha + \beta)x = \alpha x + \beta x. \tag{1}$$

By X^c we denote the subspace of all convex elements of the space X.

Remark 1. Some authors (see, e.g., [10]) include the condition $X = X^c$ to the set of axioms of semilinear space.

Definition 3. *A semilinear metric space X with metric $h = h_X$ is called an L-space if it is complete and separable and, moreover, for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$,*

$$h(\alpha x, \alpha y) = |\alpha| h(x, y),$$

$$h(x + z, y + z) \leq h(x, y). \tag{2}$$

Remark 2. In view of the triangle inequality (2), we obtain

$$h(x + z, y + w) \leq h(x, y) + h(z, w) \quad \forall x, y, z, w \in X.$$

Definition 4. *An L-space X is called isotropic if inequality (2) turns into the equality for all $x, y, z \in X$.*

Any separable Banach spaces and any complete and separable quasilinear normed spaces are L-spaces (see [11]). The space $\Omega(X)$ of nonempty compact subspaces of a separable Banach space X equipped with an ordinary Hausdorff metric, the space $\Omega_{\mathrm{conv}}(X)$ of convex elements from $\Omega(X)$, and the spaces of fuzzy sets (see, e.g., [12]) are also examples of L-spaces. All spaces mentioned above are isotropic. An example of non-isotropic L-space was constructed in [13]. More examples of L-spaces are presented in [14–16].

Definition 5. *We say that an element $x \in X$ is invertible if there exists an element $x' \in X$ such that $x + x' = \theta$. The element x' is called inverse to x. By X^{inv} we denote the set of invertible elements of the space X. An element x is called strongly invertible if $x' = -x$.*

Remark 3. In any L-space, which is not only semilinear but also linear and, in particular, in any Banach space, every element is convex and strongly invertible. In the space $\Omega(X)$, any element of the form $\{x\}$, $x \in X$, is convex and strongly invertible. In the spaces of fuzzy sets, every function $u_{x_0} = \chi_{\{x_0\}}$, where χ_A is the characteristic function of the set A, is also a convex and strongly invertible element.

The following statements are true (see [13, 14, 17]):

Lemma 1. *If $x \in X^{\mathrm{inv}}$, then the inverse element x' is unique.*

Lemma 2. *If $x \in X^{\mathrm{inv}} \cap X^{\mathrm{c}}$, then $x' \in X^{\mathrm{c}}$.*

Lemma 3. *For all $x \in X^{\mathrm{c}}$ and $\alpha, \beta \in \mathbb{R}$,*

$$h(\alpha x, \beta x) \le |\alpha - \beta| h(x, \theta). \tag{3}$$

If X is isotropic, then inequality (3) turns into the equality for $x \in X^{\mathrm{c}}$ and $\alpha \cdot \beta \ge 0$.

Lemma 4. *Let X be an isotropic L-space. Then, for any $x \in X^{\mathrm{c}} \cap X^{\mathrm{inv}}$,*

$$h(x, x') = d(x + x, \theta) = 2h(x, \theta).$$

Lemma 5. *For any $x \in X^{\mathrm{inv}} \cap X^{\mathrm{c}}$, the following equality is true: $h(x', \theta) = h(x, \theta)$.*

We also need the following statement:

Lemma 6. *If a convex element x of the isotropic space X is strongly invertible, then, for all $\alpha, \beta \in \mathbb{R}$, equality (1) is true and inequality (3) turns into the equality.*

Proof. We first prove equality (1). If $\alpha \cdot \beta = 0$, then the statement is obvious. At the same time, if $\alpha, \beta < 0$, then, by Lemma 2,

$$\alpha x + \beta x = (-\alpha)x' + (-\beta)x' = (-\alpha - \beta)x' = (\alpha + \beta)x.$$

If $\alpha \cdot \beta < 0$, then we can assume that $\beta < 0 < \alpha$ and $\alpha + \beta \ge 0$. Thus,

$$\begin{aligned} h(\alpha x + \beta x, (\alpha + \beta)x) &= h(\alpha x + (-\beta)x', (\alpha + \beta)x) \\ &= h(\alpha x, (\alpha + \beta)x + (-\beta)x) \\ &= h(\alpha x, (\alpha + \beta - \beta)x) = 0, \end{aligned}$$

and, hence, equality (1) is proved.

We now prove inequality (3). By Lemma 3, it suffices to show that inequality (3) turns into the equality for $\alpha \cdot \beta < 0$. We can assume that $\beta < 0 < \alpha$. Therefore,

$$h(\alpha x, \beta x) = h(\alpha x, -\beta(-x)) = h(\alpha x, -\beta x')$$

$$= h(\alpha x + (-\beta)x, \theta) = h((\alpha - \beta)x, \theta) = |\alpha - \beta| h(x, \theta),$$

Q.E.D.

2.2. Integration in the L-Spaces. Let $(S, \mathcal{F})$ be a measurable space (i.e., S is a set and $\mathcal{F}$ is a σ-algebra of its subsets) with full finite positive measure μ. By $L_p(S)$, $1 \leq p \leq \infty$, we denote the spaces of functions $f\colon S \to \mathbb{R}$ with the corresponding norms $\|f\|_{L_p(S)}$.

Definition 6. *Let X be an L-space. A function $f : S \to X$ is called measurable if, for any element $x \in X$, the real-valued function $t \mapsto h(f(t), x)$ is measurable.*

For an L-space (X, h), by $L_p(S, X)$, $1 \leq p \leq \infty$, we denote a space of measurable functions $f\colon S \to X$ such that $h(f(\cdot), \theta) \in L_p(S)$. If $f, g \in L_p(S, X)$, then the function $h(f(\cdot), g(\cdot)))$ is measurable (see [18], Theorem 1.4.22) and belongs to the space $L_p(S, X)$. Thus,

$$h_{L_p(S,X)}(f, g) = \|h(f(\cdot), g(\cdot))\|_{L_p(S)}$$

is a metric in the space $L_p(S, X)$.

We now present the definition and some properties of the Lebesgue integral for the functions $f \in L_1(S, X)$ (see [19] and [11], Sec. 5).

Definition 7. *A surjective operator $P : X \to X^c$ is called covexifying if*

$$h(P(x), P(y)) \leq h(x, y) \quad \textit{for all} \quad x, y \in X,$$

$$P \circ P = P,$$

$$P(\alpha x + \beta y) = \alpha P(x) + \beta P(y) \quad \textit{for all} \quad x, y \in X \quad \textit{and} \quad \alpha, \beta \in \mathbb{R}.$$

Note that, for all $x \in X^c$, we have $P(x) = x$ (see, e.g., [15], Remark 4).

The operator $\operatorname{conv}\colon \Omega(\mathbb{R}^m) \to \Omega(\mathbb{R}^m)$, which associates each $x \in \Omega(\mathbb{R}^m)$ with its convex hull $\operatorname{conv} x$, is a covexifying operator.

Let X be an L-space and let P be a covexifying operator. A mapping $f\colon S \to X$ is called simple if it has an at most countable set of values $\{f_k\}$ on mutually disjoint measurable sets S_k whose union is equal to S. It is said that a simple mapping is Lebesgue integrable if the series $\sum_k h(P(f_k), \theta)\mu(S_k)$ converges. The Lebesgue integral of a simple mapping f is defined as follows:

$$\int_S f(s)\, ds := \sum_k P(f_k)\mu(S_k),$$

where μ is the Lebesgue measure.

For simple f and g, the following properties are true:

(i) for all $\alpha, \beta \in \mathbb{R}$,

$$\int_S (\alpha f(t) + \beta g(t))\, dt = \alpha \int_S f(t)\, dt + \beta \int_S g(t)\, dt;$$

(ii) the function $t \mapsto h(f(t), g(t))$ is integrable and

$$h\left(\int_S f(t)\, dt, \int_S g(t)\, dt\right) \le \int_S h(f(t), g(t))\, dt;$$

(iii) the function $P(f(\cdot))$ is integrable and

$$\int_S f(t)\, dt = P\left(\int_S f(t)\, dt\right) = \int_S P(f(t))\, dt;$$

(iv) for continuous measurable sets S_1 and S_2 such that $S = S_1 \cup S_2$, the following relation is true:

$$\int_S f(t)\, dt = \int_{S_1} f(t)\, dt + \int_{S_2} f(t)\, dt.$$

A function $f \in L_1(S, X)$ is called integrable if there exists a sequence $\{f^k\}$ of simple functions convergent to f in the space $L_1(S, X)$. By definition,

$$\int_S f(t)\, dt = \lim_{k\to\infty} \int_S f^k(t)\, dt.$$

This definition is correct. It is known that any map $f \in L_1(S, X)$ is integrable (see [11], Theorem 9).

It is clear that the properties (i)–(iv) of the Lebesgue integral for simple functions are true for any functions from $L_1(S, X)$. Note that, in the case where X is a Banach space, the analyzed integral is a Bochner integral (see [20], Secs. 3.7 and 3.8). For $X = \Omega(\mathbb{R}^m)$, this integral coincides with the Aumann integral (see [11], Theorem 12).

We also need the following statement:

Lemma 7. *Suppose that $f \in L_1(S, \mathbb{R})$ and $a \in X$ is a convex strongly invertible element. Then*

$$\int_S f(s) \cdot a ds = \left(\int_S f(s) ds\right) \cdot a.$$

Proof. We set $S_\pm = \{s \in S\colon\ \pm f(s) > 0\}$. Thus, we get

$$\int_S f(s) \cdot a ds = \int_{S_+} f(s) \cdot a ds + \int_{S_-} f(s) \cdot a ds$$

$$= \left(\int_{S_+} f(s) ds\right) \cdot a + \int_{S_-} (-f(s)) \cdot a' ds$$

$$= \left(\int\limits_{S_+} f(s)ds \right) a + \left(\int\limits_{S_-} (-f(s))ds \right) a'$$

$$= \left(\int\limits_{S_+} f(s)ds \right) a + \left(\int\limits_{S_-} f(s)ds \right) a = \left(\int\limits_{S} f(s)ds \right) a.$$

3. Estimates for the Deviations of Integral Operators

As usual, for $p \in [1, \infty]$, we set $p' = p/(p-1)$. Let Q be a set, let $(S, \mathcal{F})$ be the measurable space with measure μ described above, let $K, N \colon Q \times S \to \mathbb{R}$, and let $\phi \colon S \to X$. Consider a problem of deviation of two integral operators

$$(\widetilde{K}\phi)(t) = \int\limits_{S} K(t,s)\phi(s)d\mu(s) \qquad \text{and} \qquad (\widetilde{N}\phi)(t) = \int\limits_{S} N(t,s)\phi(s)d\mu(s), \quad t \in Q. \tag{4}$$

Let $Q = S = [0, 2\pi]^d$, $d \in \mathbb{N}$, and let μ be the Lebesgue measure. For the spaces of functions $\phi \colon \mathbb{R}^d \to X$ 2π-periodic in each variable, we use the notation L_p^X instead of $L_p\big([0,2\pi]^d, X\big)$, $L_p = L_p^{\mathbb{R}}$. Note that the convolution operator is an important example of operators (4) in the spaces of functions 2π-periodic in each variable. Namely, if $K(t,s) = \mathcal{K}(t-s)$ and $N(t,s) = \mathcal{N}(t-s)$, where $\mathcal{K}, \mathcal{N} \in L_1$, then operators (4) turn into the convolution operators

$$(\mathcal{K}*\phi)(t) = \int\limits_{S} \mathcal{K}(t-s)\phi(s)d\mu(s) \qquad \text{and} \qquad (\mathcal{N}*\phi)(t) = \int\limits_{S} \mathcal{N}(t-s)\phi(s)d\mu(s), \quad t \in Q.$$

Theorem 1. *Suppose that* $p \in (1, \infty]$ *and that* $K, N \colon Q \times S \to \mathbb{R}$ *are functions such that* $K(t,\cdot), N(t,\cdot) \in L_{p'}(S)$ *for every* $t \in Q$. *Then, for any function* $\phi \in L_p(S, X)$ *and any* $t \in Q$, *the inequality*

$$h\Big((\widetilde{K}\phi)(t), (\widetilde{N}\phi)(t)\Big) \le \|K(t,\cdot) - N(t,\cdot)\|_{L_{p'}(S)} \|h(\phi, \theta)\|_{L_p(S)} \tag{5}$$

is true. In particular, for periodic functions and convolution operators,

$$h((\mathcal{K} * \phi)(t), (\mathcal{N} * \phi)(t)) \le \|\mathcal{K} - \mathcal{N}\|_{L_{p'}} \|h(\phi, \theta)\|_{L_p(S)}. \tag{6}$$

If the space X *is isotropic and the set* $X^{\mathrm{c}} \cap X^{\mathrm{inv}}$ *has a nonzero strongly invertible element* a, *then inequality (5) is unimprovable and turns into the equality for any function of the form*

$$\phi_t(s) = \varphi_t(s) \cdot a, \quad t \in Q,$$

where

$$\varphi_t(s) = |K(t,s) - N(t,s)|^{p'-1} \operatorname{sgn}(K(t,s) - N(t,s)), \qquad s \in S.$$

Proof. By using the properties of the integral and covexifying operator, Lemma 3, and the Hölder inequality, we get

$$h((\widetilde{K}\phi)(t),(\widetilde{N}\phi)(t)) = h\left(\int\limits_S K(t,s)P(\phi(s))d\mu(s), \int\limits_S N(t,s)P(\phi(s))d\mu(s)\right)$$

$$\leq \int\limits_S h(K(t,s)P(\phi(s)), N(t,s)P(\phi(s)))d\mu(s)$$

$$\leq \int\limits_S |K(t,s)-N(t,s)|h(P(\phi(s)),\theta)d\mu(s)$$

$$\leq \|K(t,\cdot)-N(t,\cdot)\|_{L_{p'}(S)}\|h(\phi,\theta)\|_{L_p(S)}.$$

Inequality (5) is proved.

Further, assume that the space X be isotropic. We now show that inequality (5) is sharp. By using Lemma 6, the definition of the function ϕ_t, and the fact that this function is convex-valued, we obtain

$$h((\widetilde{K}\phi_t)(s),(\widetilde{N}\phi_t)(s)) = h\left(\int\limits_S K(t,s)\phi_t(s)d\mu(s), \int\limits_S N(t,s)\phi_t(s)d\mu(s)\right)$$

$$= h\left(\left[\int\limits_S K(t,s)\varphi_t(s)d\mu(s)\right]a, \left[\int\limits_S N(t,s)\varphi_t(s)d\mu(s)\right]a\right)$$

$$= \left|\int\limits_S (K(t,s)-N(t,s))\varphi_t(s)d\mu(s)\right| h(a,\theta)$$

$$= \int\limits_S |K(t,s)-N(t,s)|^{p'}d\mu(s)h(a,\theta)$$

$$= \|K(t,\cdot)-N(t,\cdot)\|_{L_{p'}(S)}\left(\int\limits_S |K(t,s)-N(t,s)|^{(p'-1)p}d\mu(s)\right)^{\frac{1}{p}} h(a,\theta)$$

$$= \|K(t,\cdot)-N(t,\cdot)\|_{L_{p'}(S)}\|h(\phi_t(\cdot),\theta)\|_{L_p(S)}.$$

Theorem 1 is proved.

Consider the deviation in the integral metric. For $p,q \in [1,\infty]$, by $L_{q,p}(Q\times S)$ we denote a collection of functions $K(\cdot,\cdot)$ such that

$$\big\|\|K(t,s)\|_{L_q(Q)}\big\|_{L_p(S)} = \|K(\cdot,\cdot)\|_{L_{q,p}(Q\times S)} < \infty.$$

Theorem 2. *Suppose that* $p, q \in [1, \infty)$ *and* $K, N \in L_{q,p'}(Q \times S)$. *Then, for any function* $\phi \in L_p(S, X)$, *the following inequality is true:*

$$\left\| h(\widetilde{K}\phi, \widetilde{N}\phi) \right\|_{L_q(Q)} \le \|K - N\|_{L_{q,p'}(Q\times S)} \cdot \|h(\phi, \theta)\|_{L_p(S)}. \tag{7}$$

In particular, for functions periodic in each variable,

$$\|h(\mathcal{K} * \phi, \mathcal{N} * \phi)\|_{L_q} \le (2\pi)^{d/p'} \|\mathcal{K} - \mathcal{N}\|_{L_q} \|h(\phi, \theta)\|_{L_p}. \tag{8}$$

If the space X *is isotropic and the set* $X^c \cap X^{\rm inv}$ *contains a nonzero strongly invertible element* a, *then inequalities (7) and (8) are unimprovable for* $p = q = 1$.

Proof. By using the properties of the metric in X and the integral, the generalized Minkowski inequality, and the Hölder inequality, we obtain

$$\begin{aligned}
\left\| h(\widetilde{K}\phi, \widetilde{N}\phi) \right\|_{L_q(Q)} &= \left\| h\left(\int_S K(\cdot, s) P(\phi(s)) d\mu(s), \int_S N(\cdot, s) P(\phi(s)) d\mu(s) \right) \right\|_{L_q(Q)} \\
&\le \left\| \int_S |K(\cdot, s) - N(\cdot, s)| h(P(\phi(s)), \theta) d\mu(s) \right\|_{L_q(Q)} \\
&\le \int_S h(P(\phi(s)), \theta) \|K(\cdot, s) - N(\cdot, s)\|_{L_q(Q)} d\mu(s) \\
&\le \|K - N\|_{L_{q,p'}(Q\times S)} \|h(\phi, \theta)\|_{L_p(S)}.
\end{aligned}$$

Inequality (7) is proved. Inequality (8) follows from inequality (7).

We now establish the unimprovability of inequality (8) for $p = q = 1$. For $\varepsilon > 0$, by F_ε we denote the Steklov function for the function $F \in L_1$:

$$F_\varepsilon(t) = \frac{1}{(2\varepsilon)^d} \int_{[-\varepsilon,\varepsilon]^d} F(t - s) d\mu(s).$$

We set

$$\phi_{ae}(s) = \frac{1}{(2\varepsilon)^d} \chi_{[-\varepsilon,\varepsilon]^d}(s) \cdot a,$$

where $a \in X^c$ is strongly invertible and such that $h(a, \theta) = 1$. In view of the isotropy of the space X and Lemma 7, we conclude that

$$\begin{aligned}
&\left\| h\left(\int_{[0,2\pi]^d} \mathcal{K}(\cdot - s) \phi_{ae}(s) d\mu(s), \int_{[0,2\pi]^d} \mathcal{N}(\cdot - s) \phi_{ae}(s) d\mu(s) \right) \right\|_{L_1} \\
&\qquad = 1/(2\varepsilon)^d \left\| h\left(\int_{[-\varepsilon,\varepsilon]^d} \mathcal{K}(\cdot - s) a d\mu(s), \int_{[-\varepsilon,\varepsilon]^d} \mathcal{N}(\cdot - s) a d\mu(s) \right) \right\|_{L_1}
\end{aligned}$$

$$= 1/(2\varepsilon)^d \left\| h\left(\left[\int\limits_{[-\varepsilon,\varepsilon]^d} \mathcal{K}(\cdot - s)d\mu(s) \right] a, \left[\int\limits_{[-\varepsilon,\varepsilon]^d} \mathcal{N}(\cdot - s)d\mu(s) \right] a \right) \right\|_{L_1}$$

$$= 1/(2\varepsilon)^d \left\| \left| \int\limits_{[-\varepsilon,\varepsilon]^d} (\mathcal{K}(\cdot - s) - \mathcal{N}(\cdot - s))d\mu(s) \right| \cdot h(a,\theta) \right\|_{L_1} = \|\mathcal{K}_h - \mathcal{N}_h\|_{L_1}.$$

Since $\|h(\phi_{ae},\theta)\|_{L_1} = 1$ and $\|\mathcal{K}_\varepsilon - \mathcal{N}_\varepsilon\|_{L_1} \to \|\mathcal{K} - \mathcal{N}\|_{L_1}$ as $\varepsilon \to 0$, inequality (8) is unimprovable for $p = q = 1$.

Theorem 2 is proved.

4. Applications

4.1. Trigonometric Approximations. If the kernel $\mathcal{N}$ of the convolution operator is a trigonometric polynomial, then, for any function $\phi \in L_1^X$, the convolution $\mathcal{N} * \phi$ is a generalized trigonometric polynomial with coefficients from the space X. In view of inequalities (6) and (8), the estimates for the approximation of the kernel $\mathcal{K}$ by the polynomial $\mathcal{N}$ yield estimates for the approximation of the function $\mathcal{K} * \phi$ by generalized trigonometric polynomials of the form $\mathcal{N} * \phi,\ \phi \in L_1^X$. The quantities $\|\mathcal{K} - \mathcal{N}\|_{L_p}$ were investigated by numerous mathematicians. Thus, sharp, asymptotically sharp, or exact-order estimates of these quantities are known in many cases. Numerous results obtained in this direction and also the corresponding references can be found in the monographs [1–7].

In what follows, we consider the one-dimensional case in more detail. We set

$$\Phi_p^X := \left\{ \phi \in L_p^X : \|h(\phi,\theta)\|_{L_p} \le 1 \right\}.$$

We write Φ_p instead of $\Phi_p^{\mathbb{R}}$. Also let $\mathcal{N} \in L_1$ be a given real-valued kernel. Consider the problem of approximation of the classes

$$\mathcal{K} * \Phi_p^X = \left\{ f = \mathcal{K} * \phi \colon\ \phi \in \Phi_p^X \right\}.$$

Note that the functions from $\mathcal{K} * \Phi_p^X$ are convex-valued. It is known (see, e.g., [1, 21–24]) that many important classes of numerical functions are classes of the form $\mathcal{K} * \Phi_p$.

Let $f \in L_p^X$ and $H \subset L_p^X$. We set

$$E(f,H)_{L_p^X} = \inf_{\tau \in H} \|h(f,\tau)\|_{L_p} \qquad \text{and} \qquad E\left(\mathcal{K} * \Phi_p^X, H\right)_{L_p^X} = \sup_{\phi \in \Phi_p^X} E(\mathcal{K} * \phi, H)_{L_p^X}. \tag{9}$$

Quantities (9) are called the best approximations of the function f and the class $\mathcal{K} * \Phi_p^X$ by the set H in the metric of the space L_p^X, respectively.

If a mapping $A\colon\ L_p^X \to H$ is given, then we set

$$U\left(\mathcal{K} * \Phi_p^X, A\right)_{L_p} = \sup_{\phi \in \Phi_p^X} \|h(\mathcal{K} * \phi, A\phi)\|_{L_p}.$$

The quantity $U\left(\mathcal{K} * \Phi_p^X, A\right)_{L_p}$ is called the approximation error of the class $\mathcal{K} * \Phi_p^X$ by a given approximating method.

For a given collection $\mathcal{A}$ of mappings $A\colon L_p^X \to H$, the quantity

$$\mathcal{E}\big(\mathcal{K} * \Phi_p^X, \mathcal{A}\big)_{L_p} = \inf_{A\in\mathcal{A}} U(\mathcal{K} * \Phi_p^X, A)_{L_p}$$

is called the best $\mathcal{A}$-approximation of the class $\mathcal{K} * \Phi_p^X$ by the set H.

By $H_{2n-1}^{T,X}$ $\Big(n = 1, 2, \ldots,\ H_{2n-1}^{T,\mathbb{R}} = H_{2n-1}^{T}\Big)$ we denote the set of generalized trigonometric polynomials $T(t)$ of degree at most $n-1$, i.e., the set of functions of the form

$$T(t) = \frac{a_0}{2} + \sum_{k=1}^{n-1} a_k \cos kt + b_k \sin kt, \quad a_k, b_k \in X.$$

In this section, as $\mathcal{A}$, we use a collection of mappings of the form $T * \phi,\ T \in H_{2n-1}^T,\ \phi \in L_p^X$.

Definition 8 (Nikol'skii [21]). *Let $\varphi_n(t) := \operatorname{sgn}\sin nt$. We say that a kernel $\mathcal{K}$ satisfies the condition N_n^* if there exist a polynomial $T^* \in H_{2n-1}^T$ and a point $\theta \in [0, \pi/n]$ such that*

$$(\mathcal{K}(t) - T^*(t))\varphi_n(t-\theta) \geq 0$$

for almost all t.

It is known that almost all kernels important for the approximation theory satisfy the condition N_n^* (see [21–24]).

If the kernel $\mathcal{K}$ satisfies the condition N_n^*, then [21]

$$E\big(\mathcal{K}, H_{2n-1}^T\big)_{L_1} = \|\mathcal{K} - T^*\|_{L_1} = \|\mathcal{K} * \varphi_n\|_{L_\infty}.$$

This result and Theorems 1 and 2 yield the following corollary:

Corollary 1. *If $p = \infty$ or $p = 1$ and the kernel $\mathcal{K}$ satisfies the condition N_n^*, then*

$$E\big(\mathcal{K} * \Phi_p^X, H_{2n-1}^T\big)_{L_p} \leq \mathcal{E}\big(\mathcal{K} * \Phi_p^X, \mathcal{A}\big)_{L_p} = \|\mathcal{K} - T^*\|_{L_1} = \|\mathcal{K} * \varphi_n\|_{L_\infty},$$

where $T^* \in H_{2n-1}^T$ is the polynomial of the best L_1-approximation for the kernel $\mathcal{K}$.

By using inequality (6), we obtain the following generalization of Theorem 1 in [25]:

Corollary 2. *Let $p > 1$ and $\mathcal{K} \in L_{p'}$. Then*

$$E\Big(\mathcal{K} * \Phi_p^X, H_{2n-1}^{T,X}\Big)_{L_\infty} \leq \mathcal{E}\big(\mathcal{K} * \Phi_p^X, \mathcal{A}\big)_{L_\infty} = E\big(\mathcal{K}, H_{2n-1}^T\big)L_{p'}.$$

The cases of sharpness of the estimates for the best approximations, which follow from the analyzed corollaries will be considered elsewhere. Here, we only note that the estimate established in Corollary 1 is sharp in the case where $X = \Omega(\mathbb{R}^d)$ (see [26]) and also if X is a Banach space.

4.2. Errors of Approximate Integration. By applying Theorem 1 to integral operators with the kernels

$$K'(t,s) = \int\limits_Q K(u,s)d\mu(u) \qquad \text{and} \qquad N'(t,s) = \sum_{j=1}^n c_j K(t_j, s),$$

where $t_j \in Q,\ c_j \in \mathbb{R},\ j = 1,\ldots,n,$ we arrive at the following estimates for the errors of the formulas of approximate integration of functions of the form $(\widetilde{K}\phi)(t)$:

Corollary 3. *Let $p \in (1,\infty]$ and $t_j \in Q,\ c_j \in \mathbb{R},\ j = 1,\ldots,n$. Then, for any function $\phi \in L_p(S,X)$ such that $\|h(\phi,\theta)\|_{L_p(S)} \le 1$, the following inequality is true:*

$$h\left(\int\limits_Q (\widetilde{K}\phi)(t)d\nu(t), \sum_{j=1}^n c_j(\widetilde{K}\phi)(t_j)\right) \le \left\|\int\limits_Q K(t,\cdot)d\nu(t) - \sum_{j=1}^n c_j K(t_j,\cdot)\right\|_{L_{p'}(S)}.$$

If the space X is isotropic and the set $X^c \cap X^{\rm inv}$ contains a nonzero strongly invertible element a, then this inequality is unimprovable.

We now illustrate the application of this corollary to the problems of optimization of the formulas of approximate integration on the classes of periodic functions of one variable. For $\overline{t} = \{t_1,\ldots,t_n\} \in [0,2\pi)$, $\overline{c} = \{c_1,\ldots,c_n\} \in \mathbb{R}^n$, and a continuous function f, we set

$$M_{\overline{t},\overline{c}}(f) = \sum_{j=1}^n c_j f(t_j).$$

Let

$$R(f, M_{\overline{t},\overline{c}}) = \int\limits_0^{2\pi} f(t)d\mu(t) - M_{\overline{t},\overline{c}}(f), \qquad \mathcal{R}\big(\mathcal{K} * \Phi_p^X, M_{\overline{t},\overline{c}}\big) = \sup_{f \in \mathcal{K}*\Phi_p^X} |R(f, M_{\overline{t},\overline{c}})|,$$

and

$$\mathcal{R}_n\big(\mathcal{K} * \Phi_p^X\big) = \inf_{\overline{t},\overline{c}} \mathcal{R}\big(\mathcal{K} * \Phi_p^X, M_{\overline{t},\overline{c}}\big).$$

In the problem of the best quadrature formula in the class $\mathcal{K} * \Phi_p^X$, it is necessary to determine the quantity $\mathcal{R}_n\big(\mathcal{K} * \Phi_p^X\big)$ and collections $\overline{t}$ and $\overline{c}$ realizing the infimum on the right-hand side of the last equality.

For numerical functions, this problem is well studied (see, e.g., [7, 27]). In particular, the exact value of the quantity

$$\left\|\int_0^{2\pi} \mathcal{K}(t)d\mu(t) - \sum\nolimits_{j=1}^n c_j \mathcal{K}(t_j - \cdot)\right\|_{L_{p'}}$$

was found in many cases. Hence, by virtue of Corollary 3, this also gives estimates for the quantities $\mathcal{R}_n\big(\mathcal{K} * \Phi_p^X\big)$.

4.3. Errors of Reconstruction of the Functions. Applying Theorem 1 to the integral operators with kernels

$$K(t,s) \qquad \text{and} \qquad N(t,s) = \sum_{j=1}^n c_j K(t_j, s),$$

we get the following estimates for the errors of the formulas of approximate reconstruction of the value of a function of the form $(\widetilde{K}\phi)(t)$ at the point t according to its values at the points t_j:

Corollary 4. *Let* $p \in (1,\infty]$ *and* $t_j \in Q,\ c_j \in \mathbb{R},\ j = 1,\ldots,n$. *Then, for any function* $\phi \in L_p(S,X)$ *such that* $\|h(\phi,\theta)\|_{L_p(S)} \le 1$ *and any* $t \in Q$, *the following inequality is true:*

$$h\left((\widetilde{K}\phi)(t), \sum_{j=1}^{n} c_j (\widetilde{K}\phi)(t_j)\right) \le \left\| K(t,\cdot) - \sum_{j=1}^{n} c_j K(t_j,\cdot)\right\|_{L_{p'}(S)}.$$

We also consider the problem of reconstruction of the function $\widetilde{K}\phi$ according to its values at n points $t_j \in Q$ in the integral metrics. The method of reconstruction is specified as follows: We choose n functions $c_j\colon Q \to \mathbb{R}$ and set

$$\Phi(t) = \sum_{j=1}^{n} c_j(t)(\widetilde{K}\phi)(t_j).$$

By using Theorem 2, for the integral operators with kernels $K(t,s)$ and

$$N(t,s) = \sum_{j=1}^{n} c_j(t) K(t_j,s),$$

we obtain the following assertion:

Corollary 5. *Let* $p,q \in [1,\infty),\ t_j \in Q$, *and* $c_j \in L_\infty(Q),\ j = 1,\ldots,n$. *Then, for any function* $\phi \in L_p(S,X)$ *such that* $\|h(\phi,\theta)\|_{L_p(S)} \le 1$, *the following inequality is true:*

$$\left\| h\left((\widetilde{K}\phi)(\cdot), \sum_{j=1}^{n} c_j(\cdot)(\widetilde{K}\phi)(t_j)\right)\right\|_{L_q(Q)} \le \left\| K(\cdot,\cdot) - \sum_{j=1}^{n} c_j(\cdot) K(t_j,\cdot)\right\|_{L_{q,p'}(Q\times S)}.$$

The problem of reconstruction of functions and operators according to the available incomplete information is important both from the theoretical and from the practical point of view. The corresponding optimization problem is also of high importance. For the general approaches to the solution of these problems and their specific solutions, see the monographs [28, 29]. The problem of the optimal reconstruction of operators in L-spaces was considered in [13–15, 17].

The discussion of these and similar problems in semilinear metric spaces will be continued elsewhere.

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