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CONVERGENCE AND ESTIMATION OF THE TRUNCATION ERROR FOR THE CORRESPONDING TWO-DIMENSIONAL CONTINUED FRACTIONS

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For the corresponding two-dimensional continued fractions with complex partial numerators that belong to certain subsets of the Cartesian product of two angular sets in the right half plane and partial denominators equal to one, we establish sufficient conditions for the uniform convergence and an estimate for the truncation error by using an analog of the method of fundamental inequalities, relations for the real and imaginary parts of the tails of figured approximants, and a multidimensional analog of the Stieltjes–Vitali theorem.

1. Introduction

As one of the most widespread methods used for the decomposition of analytic functions of many variables into branched continued fractions (discrete multidimensional generalizations of continued fractions), we can mention the construction of fractions corresponding to given formal multiple power series [9, 12, 15–19, 23, 24, 29, 30]. The problem of correspondence between a formal power series and a sequence of holomorphic functions of one variable and, in particular, a sequence of approximations to a continuous functional fraction was considered in [27], whereas the correspondence between a formal multiple power series and a sequence of approximations to some generalizations of continued fractions was studied in [19, 24, 26]. Note that the corresponding branched continued fractions are constructed ambiguously. In particular, there exist different structures of branched continued fractions corresponding to a given formal double power series [18, 29, 30]. Later, the branched continued fractions proposed in [18, 29] were called two-dimensional continued fractions.

The property of convergence of continued fractions and their multidimensional generalizations is important for their applications [2, 9–11, 25]. The problem of the pointwise convergence of the two-dimensional continued fraction (1) is reduced to the investigation of convergence of a number two-dimensional continued fraction obtained from the functional two-dimensional continued fraction for fixed values of the variables. The method used for the investigation of number two-dimensional continued fractions (1) whose elements belong to disk, angular, parabolic, and paired domains of the complex plane, and curvilinear trapezoids was described in [6–9, 19]. The problem of convergence of one class of functional two-dimensional continued fractions, namely, two-dimensional continued g-fractions in some domains of the space $\mathbb{C}^2$, was considered in [13, 14, 19, 28]. The investigation of convergence of functional two-dimensional continued fractions in other domains proves to be an urgent problem.

Consider a two-dimensional continued fraction

$$a_{0,0} + \Phi_0(z) + \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{a_{k,k}(z)}{1+\Phi_k(z)}, \quad \Phi_l(z) = \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{l+j,l}(z)}{1} + \mathop{\mathrm{D}}_{j=1}^{\infty} \frac{a_{l,l+j}(z)}{1}, \quad l = 0, 1, \ldots, \tag{1}$$

where $z = (z_1, z_2) \in \mathbb{C}^2$,

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$$a_{j,j}(z) = c_{j,j} z_1 z_2, \qquad a_{k+j,k}(z) = c_{k+j,k} z_1, \qquad a_{k,k+j}(z) = c_{k,k+j} z_2, \tag{2}$$

$$k = 0, 1, \ldots, \quad j = 1, 2, \ldots,$$

and $a_{0,0},\ c_{j,k},\ j,k = 0,1,\ldots,\ j+k \geq 1$, are complex constants.

The approximation obtained from the correspondence problem (the nth figured approximation or the nth figured convergent) has the form

$$f_1(z) = a_{0,0} + \Phi_0^{(1)}(z), \quad f_n(z) = a_{0,0} + \Phi_0^{(n)}(z) + \mathop{\mathrm{D}}_{k=1}^{[n/2]} \frac{a_{k,k}(z)}{1 + \Phi_k^{(n-2k)}(z)}, \quad n = 2, 3, \ldots, \tag{3}$$

$$\Phi_l^{(0)}(z) = 0, \quad \Phi_l^{(p)}(z) = \mathop{\mathrm{D}}_{j=1}^{p} \frac{a_{l+j,l}(z)}{1} + \mathop{\mathrm{D}}_{j=1}^{p} \frac{a_{l,l+j}(z)}{1}, \quad l = 0, 1, \ldots, \quad p = 1, 2, \ldots, \tag{4}$$

where $[\alpha]$ is the integral part of a real number α.

Definition 1. *The two-dimensional continued fraction (1) is called convergent if, starting from a certain number n_0, all its approximations are meaningful and the limit*

$$f(z) = \lim_{n\to\infty} f_n(z)$$

exists. The value of this limit is regarded as the value of the two-dimensional continued fraction. The difference $f(z) - f_n(z)$ is called the truncation error of the approximation of two-dimensional continued fraction by the nth figured convergent, $n \geq 1$.

Definition 2. *The functional two-dimensional continued fraction (1) is called uniformly convergent on a compact subset K of the domain $D \subset \mathbb{C}^2$ if, starting from a certain number $n(K)$, its approximations $f_k(z)$, $k \geq n(K)$, for all z from a certain domain that contains K, are meaningful and finite and, for any $n_\varepsilon > n(K)$, there exists a number $n_\varepsilon > n(K)$ such that the inequality $|f_n(z) - f_m(z)| < \varepsilon$ holds for all $n, m \geq n_\varepsilon$ and $z \in K$.*

One of the methods used for the investigation of convergence and capable of estimation of the truncation errors of multidimensional analogs of continued fractions is the method of fundamental inequalities [1, 3, 9]. Examples of its application were described in [1, 4, 5, 22].

Analogs of the method of fundamental inequalities were also developed for the two-dimensional continued fraction (1) [19–21]. As an important component of this method, we can mention the estimation of so-called remainders.

Expressions of the form

$$Q_j^{(0)}(z) = 1, \qquad Q_j^{(1)}(z) = 1 + \Phi_j^{(1)}(z), \qquad Q_j^{(p+2)}(z) = 1 + \Phi_j^{(p+2)}(z) + \frac{a_{j+1,j+1}(z)}{Q_{j+1}^{(p)}(z)}, \tag{5}$$

$$j = 1, 2, \ldots, \quad p = 0, 1, \ldots,$$

$$Q^{(0)}_{k+j,k}(z) = Q^{(0)}_{k,k+j}(z) = 1, \quad Q^{(p+1)}_{k+j,k}(z) = 1 + \frac{a_{k+j+1,k}(z)}{Q^{(p)}_{k+j+1,k}(z)}, \quad Q^{(p+1)}_{k,k+j} = 1 + \frac{a_{k,k+j+1}(z)}{Q^{(p)}_{k,k+j+1}(z)}, \tag{6}$$

$$j = 1, 2, \ldots, \quad k, \; p = 0, 1, \ldots,$$

are called remainders of the two-dimensional continued fraction (1).

By using relations (3)–(6), we get

$$\Phi^{(p)}_k(z) = \frac{a_{k+1,k}(z)}{Q^{(p-1)}_{k+1,k}(z)} + \frac{a_{k,k+1}(z)}{Q^{(p-1)}_{k,k+1}(z)}, \quad k = 0, 1, \ldots, \quad p = 1, 2, \ldots, \tag{7}$$

$$f_1(z) = a_{0,0} + \Phi^{(1)}_0(z), \qquad f_n(z) = a_{0,0} + \Phi^{(n)}_0(z) + \frac{a_{1,1}(z)}{Q^{(n-2)}_1(z)}, \quad n = 2, 3, \ldots. \tag{8}$$

To establish convergence and find the estimates for the rate of convergence of two-dimensional continued fractions studied in the present paper, we use an analog of the method of fundamental inequalities for figured approximations of the form (3), (4), as well as the formulas for the real and imaginary parts of the remainders of figured approximations for two-dimensional continued fractions established in [20]. This procedure was also used for the investigation of convergence of the corresponding branched continued fractions with different structures in [2, 4].

2. An Analog of the Method of Fundamental Inequalities

We now prove an auxiliary theorem (an analog of the method of fundamental inequalities) and use it to substantiate our main results. For the sake of brevity, we omit the dependence of elements of the two-dimensional continued fraction (1) on z.

Theorem 1. *Suppose that the remainders of the two-dimensional continued fraction (1) satisfy the inequalities*

$$Q^{(p)}_{i+k,i} \neq 0, \quad Q^{(p)}_{i,i+k} \neq 0, \quad Q^{(p)}_k \neq 0, \quad i = 0, 1, \ldots, \quad k, p = 1, 2, \ldots, \tag{9}$$

and there exist positive constants $M,\ M_{1,0},\ M_{0,1},\ H_1,\ H_2,\ \rho,\ \rho_1,$ *and* $\rho_2,\ \rho < 1,\ \rho_1 < 1,\ \rho_2 < 1,$ *such that*

$$\frac{|a_{1,0}|}{|Q^{(p)}_{1,0}|} \leq M_{1,0}, \qquad \frac{|a_{0,1}|}{|Q^{(p)}_{0,1}|} \leq M_{0,1}, \qquad \frac{|a_{1,1}|}{|Q^{(p)}_1|} \leq M, \qquad p = 0, 1, \ldots, \tag{10}$$

$$\frac{|a_{k+1,k}|}{|Q^{(p+1)}_k Q^{(p)}_{k+1,k}|} \leq H_1, \qquad \frac{|a_{k,k+1}|}{|Q^{(p+1)}_k Q^{(p)}_{k,k+1}|} \leq H_2, \qquad p = 0, 1, \ldots, \quad k = 1, 2, \ldots, \tag{11}$$

$$\frac{|a_{k+j+1,k}|}{|Q^{(m+1)}_{k+j,k} Q^{(m)}_{k+j+1,k}|} \leq \rho_1, \qquad \frac{|a_{k,k+j+1}|}{|Q^{(m+1)}_{k,k+j} Q^{(m)}_{k,k+j+1}|} \leq \rho_2, \qquad k, m = 0, 1, \ldots, \quad j = 1, 2, \ldots, \tag{12}$$

$$\frac{|a_{j+1,j+1}|}{|Q^{(p+2)}_j Q^{(p)}_{j+1}|} \leq \rho, \qquad j = 1, 2, \ldots, \quad p = 0, 1, \ldots. \tag{13}$$

Then the two-dimensional continued fraction (1) converges to the value f and the truncation error of its approximation by the mth figured convergent $f_m,\ m \geq 2$, can be estimated as follows:

$$|f - f_m| \leq M_{1,0}(\rho_1)^m + M_{0,1}(\rho_2)^m + M\rho^{[m/2]} + MH_1S_{1,m} + MH_2S_{2,m}, \tag{14}$$

where

$$S_{i,m} = \begin{cases} (\rho_i)^{m-2[m/2]}[m/2](\delta_i)^{[m/2-1]} & \text{for} \quad \delta_i = \tilde{\delta}_i, \\ (\rho_i)^{m-2[m/2]}\dfrac{(\delta_i)^{[m/2]} - (\tilde{\delta}_i)^{[m/2]}}{\delta_i - \tilde{\delta}_i} & \text{for} \quad \delta_i > \tilde{\delta}_i, \end{cases} \qquad i = 1, 2, \quad m = 2, 3, \ldots, \tag{15}$$

$$\delta_i = \max\bigl((\rho_i)^2, \rho\bigr), \qquad \tilde{\delta}_i = \min\bigl((\rho_i)^2, \rho\bigr), \qquad i = 1, 2. \tag{16}$$

Proof. In order to prove convergence of the sequence of convergents $\{f_n\},\ n = 1, 2, \ldots,$ and the validity of inequality (14), we use the relation for the difference between figured convergents of the form (3) of the two-dimensional continued fraction (1) [9]:

$$f_n - f_m = \sum_{k=0}^{[m/2]} \frac{(\Phi_k^{(n-2k)} - \Phi_k^{(m-2k)}) \prod_{j=1}^{k}(-a_{j,j})}{\prod_{j=1}^{k} Q_j^{(n-2j)} Q_j^{(m-2j)}} \tag{17}$$

$$- \frac{\prod_{j=1}^{[m/2]+1}(-a_{j,j})}{\prod_{j=1}^{[m/2]+1} Q_j^{(n-2j)} \prod_{j=1}^{[m/2]} Q_j^{(m-2j)}}, \qquad n > m+1, \quad m = 2, 3, \ldots. \tag{18}$$

We now show that

$$|f_n - f_m| \leq M_{1,0}(\rho_1)^m + M_{0,1}(\rho_2)^m + M\rho^{[m/2]} + MH_1S_{1,m} + MH_2S_{2,m}, \tag{19}$$

$$m \geq 2, \quad n > m+1,$$

where $S_{1,m}$ and $S_{2,m}$ are given by relations (15) and (16).

For $m = 2p$ and $p = 1, 2, \ldots,$ by using relation (18), we obtain

$$|f_n - f_{2p}| \leq \left|\Phi_0^{(n)} - \Phi_0^{(2p)}\right|$$

$$+ \sum_{k=1}^{p} \frac{|\Phi_k^{(n-2k)} - \Phi_k^{(2p-2k)}| \prod_{j=1}^{k}|a_{j,j}|}{\prod_{j=1}^{k}|Q_j^{(n-2j)}Q_j^{(2p-2j)}|}$$

$$+ \frac{\prod_{j=1}^{p+1}|a_{j,j}|}{\prod_{j=1}^{p+1}|Q_j^{(n-2j)}| \prod_{j=1}^{p}|Q_j^{(2p-2j)}|}. \tag{20}$$

It follows from relation (4) that

$$\left|\Phi_k^{(n-2k)} - \Phi_k^{(2p-2k)}\right| \le \left|\mathop{\mathrm{D}}_{j=1}^{n-2k} \frac{a_{k+j,k}}{1} - \mathop{\mathrm{D}}_{j=1}^{2p-2k} \frac{a_{k+j,k}}{1}\right| + \left|\mathop{\mathrm{D}}_{j=1}^{n-2k} \frac{a_{k,k+j}}{1} - \mathop{\mathrm{D}}_{j=1}^{2p-2k} \frac{a_{k,k+j}}{1}\right|, \qquad 0 \le k \le p.$$

By using relations (7) and conditions (9)–(12), we get

$$\left|\mathop{\mathrm{D}}_{j=1}^{n-2k} \frac{a_{k+j,k}}{1} - \mathop{\mathrm{D}}_{j=1}^{2p-2k} \frac{a_{k+j,k}}{1}\right| = \frac{\prod_{j=1}^{2p-2k+1} |a_{k+j,k}|}{\prod_{j=1}^{2p-2k+1} \left|Q_{k+j,k}^{(n-2k-j)}\right| \prod_{j=1}^{2p-2k} \left|Q_{k+j,k}^{(2p-2k-j)}\right|}$$

$$= \frac{|a_{k+1,k}|}{\left|Q_{k+1,k}^{(n-2k-1)}\right|} \prod_{j=1}^{p-k} \frac{|a_{k+2j,k}|}{\left|Q_{k+2j-1,k}^{(2p-2k-2j+1)} Q_{k+2j,k}^{(2p-2k-2j)}\right|}$$

$$\times \prod_{j=1}^{p-k} \frac{|a_{k+2j+1,k}|}{\left|Q_{k+2j,k}^{(n-2k-2j)} Q_{k+2j+1,k}^{(n-2k-2j-1)}\right|}$$

$$\le \frac{|a_{k+1,k}|}{\left|Q_{k+1,k}^{(n-2k-1)}\right|} (\rho_1)^{2p-2k}.$$

Similarly, we find

$$\left|\mathop{\mathrm{D}}_{j=1}^{n-2k} \frac{a_{k,k+j}}{1} - \mathop{\mathrm{D}}_{j=1}^{2p-2k} \frac{a_{k,k+j}}{1}\right| \le \frac{|a_{k,k+1}|}{\left|Q_{k,k+1}^{(n-2k-1)}\right|} (\rho_2)^{2p-2k}.$$

We now estimate the product $\prod_{j=1}^{k} \frac{|a_{j,j}|}{\left|Q_j^{(2p-2j)} Q_j^{(n-2j)}\right|}$ by using conditions (10) and (13). For $k = 2l$ and $l \ge 1$, we find

$$\prod_{j=1}^{2l} \frac{|a_{j,j}|}{\left|Q_j^{(2p-2j)} Q_j^{(n-2j)}\right|} = \frac{|a_{1,1}|}{\left|Q_1^{(n-2)} Q_{2l}^{(n-4l)}\right|} \prod_{j=1}^{l} \frac{|a_{2j,2j}|}{\left|Q_{2j-1}^{(2p-4j+2)} Q_{2j}^{(2p-4j)}\right|}$$

$$\times \prod_{j=1}^{l-1} \frac{|a_{2j+1,2j+1}|}{\left|Q_{2j}^{(n-4j)} Q_{2j+1}^{(n-4j-2)}\right|}$$

$$\le \frac{|a_{1,1}|}{\left|Q_1^{(n-2)} Q_{2l}^{(n-4l)}\right|} \rho^{2l-1} = \frac{|a_{1,1}|}{\left|Q_1^{(n-2)} Q_k^{(n-2k)}\right|} \rho^{k-1}.$$

Hence,

$$\left|\underset{j=1}{\overset{n-2k}{\mathrm{D}}}\frac{a_{k+j,k}}{1}-\underset{j=1}{\overset{2p-2k}{\mathrm{D}}}\frac{a_{k+j,k}}{1}\right|\prod_{j=1}^{k}\frac{|a_{j,j}|}{|Q_j^{(n-2j)}Q_j^{(2p-2j)}|}$$

$$\leq\frac{|a_{1,1}|}{|Q_1^{(n-2)}|}\frac{|a_{k+1,k}|}{|Q_k^{(n-2k)}Q_{k+1,k}^{(n-2k-1)}|}(\rho_1)^{2p-2k}\rho^{k-1}\leq MH_1(\rho_1)^{2p-2k}\rho^{k-1}.$$

For $k=2l-1$ and $l\geq 1$, we get

$$\prod_{j=1}^{2l-1}\frac{|a_{j,j}|}{|Q_j^{(2p-2j)}Q_j^{(n-2j)}|}=\frac{|a_{1,1}|}{|Q_1^{(2p-2)}Q_{2l-1}^{(n-4l+2)}|}\prod_{j=1}^{l-1}\frac{|a_{2j,2j}|}{|Q_{2j-1}^{(n-4j+2)}Q_{2j}^{(n-4j)}|}$$

$$\times\prod_{j=1}^{l-1}\frac{|a_{2j+1,2j+1}|}{|Q_{2j}^{(2p-4j)}Q_{2j+1}^{(2p-4j-2)}|}$$

$$\leq\frac{|a_{1,1}|}{|Q_1^{(2p-2)}Q_{2l-1}^{(n-4l+2)}|}\rho^{2l-2}=\frac{|a_{1,1}|}{|Q_1^{(2p-2)}Q_k^{(n-2k)}|}\rho^{k-1}$$

and

$$\left|\underset{j=1}{\overset{n-2k}{\mathrm{D}}}\frac{a_{k+j,k}}{1}-\underset{j=1}{\overset{2p-2k}{\mathrm{D}}}\frac{a_{k+j,k}}{1}\right|\prod_{j=1}^{k}\frac{|a_{j,j}|}{|Q_j^{(2p-2j)}Q_j^{(n-2j)}|}$$

$$\leq\frac{|a_{1,1}|}{|Q_1^{(2p-2)}|}\frac{|a_{k+1,k}|}{|Q_k^{(n-2k)}Q_{k+1,k}^{(n-2k-1)}|}(\rho_1)^{2p-2k}\rho^{k-1}\leq MH_1(\rho_1)^{2p-2k}\rho^{k-1}.$$

Therefore,

$$\sum_{k=1}^{p}\left|\underset{j=1}{\overset{n-2k}{\mathrm{D}}}\frac{a_{k+j,k}}{1}-\underset{j=1}{\overset{2p-2k}{\mathrm{D}}}\frac{a_{k+j,k}}{1}\right|\prod_{j=1}^{k}\frac{|a_{j,j}|}{|Q_j^{(2p-2j)}Q_j^{(n-2j)}|}$$

$$\leq MH_1\sum_{k=1}^{p}(\rho_1)^{2p-2k}\rho^{k-1}=MH_1S_{1,2p}.$$

Similarly, we obtain

$$\sum_{k=1}^{p}\left|\underset{j=1}{\overset{n-2k}{\mathrm{D}}}\frac{a_{k,k+j}}{1}-\underset{j=1}{\overset{2p-2k}{\mathrm{D}}}\frac{a_{k,k+j}}{1}\right|\prod_{j=1}^{k}\frac{|a_{j,j}|}{|Q_j^{(2p-2j)}Q_j^{(n-2j)}|}$$

$$\leq MH_2\sum_{k=1}^{p}(\rho_2)^{2p-2k}\rho^{k-1}=MH_2S_{2,2p}.$$

In addition,

$$\big|\Phi_0^{(n)} - \Phi_0^{(2p)}\big| \le M_{1,0}(\rho_1)^{2p} + M_{0,1}(\rho_2)^{2p},$$

$$\frac{\prod_{j=1}^{p+1} |a_{j,j}|}{\prod_{j=1}^{p+1} |Q_j^{(n-2j)}| \prod_{j=1}^{p} |Q_j^{(2p-2j)}|} \le \frac{|a_{1,1}|}{|Q_1^{(n-2)}|}\rho^p \le M\rho^p.$$

By using the estimates established above and inequality (20), we can write

$$|f_n - f_{2p}| \le M_{1,0}(\rho_1)^{2p} + M_{0,1}(\rho_2)^{2p} + M\rho^p + MH_1S_{1,2p} + MH_2S_{2,2p}.$$

The estimate for the expression $|f_n - f_m|$ with $m = 2p+1,\ p = 1, 2, \ldots,\ n > 2p+2,$ can be obtained by using the same scheme. We have

$$|f_n - f_{2p+1}| \le \sum_{k=0}^{p} \frac{|\Phi_k^{(n-2k)} - \Phi_k^{(2p-2k+1)}| \prod_{j=1}^{k} |a_{j,j}|}{\prod_{j=1}^{k} |Q_j^{(n-2j)} Q_j^{(2p-2j+1)}|} + \frac{\prod_{j=1}^{p+1} |a_{j,j}|}{\prod_{j=1}^{p+1} |Q_j^{(n-2j)}| \prod_{j=1}^{p} |Q_j^{(2p-2j+1)}|}. \tag{21}$$

By using relations (7) once again and conditions (9)–(12), we find

$$\left| \mathop{\mathrm{D}}_{j=1}^{n-2k} \frac{a_{k+j,k}}{1} - \mathop{\mathrm{D}}_{j=1}^{2p-2k+1} \frac{a_{k+j,k}}{1} \right| = \frac{|a_{k+1,k}|}{|Q_{k+1,k}^{(2p-2k)}|} \prod_{j=1}^{p-k+1} \frac{|a_{k+2j,k}|}{|Q_{k+2j-1,k}^{(n-2k-2j)} Q_{k+2j,k}^{(n-2k-2j-1)}|}$$

$$\times \prod_{j=1}^{p-k} \frac{|a_{k+2j+1,k}|}{|Q_{k+2j,k}^{(2p+1-2k-2j)} Q_{k+2j+1,k}^{(2p-2k-2j)}|}$$

$$\le \frac{|a_{k+1,k}|}{|Q_{k+1,k}^{(2p-2k)}|}(\rho_1)^{2p-2k+1},$$

$$\left| \mathop{\mathrm{D}}_{j=1}^{n-2k} \frac{a_{k,k+j}}{1} - \mathop{\mathrm{D}}_{j=1}^{2p-2k+1} \frac{a_{k,k+j}}{1} \right| \le \frac{|a_{k,k+1}|}{|Q_{k,k+1}^{(2p-2k)}|}(\rho_2)^{2p-2k+1},$$

and, by virtue of conditions (10) and (13),

$$\sum_{k=1}^{p} \left| \mathop{\mathrm{D}}_{j=1}^{n-2k} \frac{a_{k+j,k}}{1} - \mathop{\mathrm{D}}_{j=1}^{2p-2k+1} \frac{a_{k+j,k}}{1} \right| \prod_{j=1}^{k} \frac{|a_{j,j}|}{|Q_j^{(2p+1-2j)} Q_j^{(n-2j)}|}$$

$$\le MH_1 \sum_{k=1}^{p} (\rho_1)^{2p-2k+1} \rho^{k-1} = MH_1\rho_1 \sum_{k=1}^{p} (\rho_1)^{2p-2k} \rho^{k-1} = MH_1 S_{1,2p+1}. \tag{22}$$

Similarly, we get

$$\sum_{k=1}^{p}\left|\operatorname*{D}_{j=1}^{n-2k}\frac{a_{k,k+j}}{1}-\operatorname*{D}_{j=1}^{2p-2k+1}\frac{a_{k,k+j}}{1}\right|\prod_{j=1}^{k}\frac{|a_{j,j}|}{\left|Q_j^{(2p+1-2j)}Q_j^{(n-2j)}\right|}\le MH_2S_{2,2p+1}. \tag{23}$$

In view of (21)–(23) and the inequalities

$$\left|\Phi_0^{(n)}-\Phi_0^{(2p+1)}\right|\le M_{1,0}(\rho_1)^{2p+1}+M_{0,1}(\rho_2)^{2p+1},$$

$$\frac{\prod_{j=1}^{p+1}|a_{j,j}|}{\prod_{j=1}^{p+1}|Q_j^{(n-2j)}|\prod_{j=1}^{p}|Q_j^{(2p+1-2j)}|}\le\frac{|a_{1,1}|}{|Q_1^{(n-2)}|}\rho^p\le M\rho^p,$$

we conclude that inequality (19) is true for $m=2p+1$.

Passing to the limit as m tends to infinity in inequality (19) and using the conditions of Theorem (1), we establish the convergence of the two-dimensional continued fraction (1) and, as $n\to\infty$, the validity of estimate (14)–(16) for the approximation of the two-dimensional continued fraction (1) by its mth convergent.

3. Sufficient Conditions for the Uniform Convergence of the Corresponding Two-Dimensional Continued Fractions

Consider the problem of convergence of the functional two-dimensional continued fraction (1) whose elements are given by (2).

Theorem 2. *Suppose that the elements of the two-dimensional continued fraction (1), (2) satisfy the conditions*

$$0<c_{j,j}\le L,\quad 0<c_{k+j,k}\le L_1,\quad 0<c_{k,k+j}\le L_2,\quad k=0,1,\ldots,\quad j=1,2,\ldots, \tag{24}$$

where L, L_1, and L_2 are positive constants. Then:

(i) at any point z of the set

$$G_K=\left\{z\in\mathbb{C}^2:\ |z_j|\le K_j,\ \ 0\le\arg(z_j)\le\frac{\pi}{2},\ \ j=1,2;\ \ \arg(z_1)+\arg(z_2)\le\frac{\pi}{2}\right\}, \tag{25}$$

the two-dimensional continued fraction (1), (2) converges to the function $f(z)$ and, moreover,

$$|f(z)-f_m(z)|\le c_{1,0}K_1(\rho_1)^m+c_{0,1}K_2(\rho_2)^m+c_{1,1}K_1K_2\left(\rho^{[m/2]}+\rho_1S_{1,m}+\rho_2S_{2,m}\right), \tag{26}$$

where $f_m(z)$ is the value of the mth approximation for the two-dimensional continued fraction (1), (2) at the point $z=(z_1,z_2)$,

$$\rho_1=\frac{L_1K_1}{\sqrt{1+(L_1)^2(K_1)^2}},\quad \rho_2=\frac{L_2K_2}{\sqrt{1+(L_2)^2(K_2)^2}},\quad \rho=\frac{LK_1K_2}{\sqrt{1+L^2(K_1)^2(K_2)^2}}, \tag{27}$$

and $S_{1,m}$ and $S_{2,m}$ are given by relations (15) and (16);

(ii) the indicated convergence is uniform on every compact subset of $\operatorname{Int}G_K$.

Proof. We now show that inequalities (9)–(13) are true under conditions (24) for all $z \in G_K$. To this end, we consider remainders (5) and (6) of the two-dimensional continued fraction (1), (2). To estimate their values, we use the relations for their real and imaginary parts [20].

Let

$$\varphi_{k,l} = \arg a_{k,l}, \qquad u_{k,l}^{(p)} = \Re Q_{k,l}^{(p)}, \qquad v_{k,l}^{(p)} = \Im Q_{k,l}^{(p)},$$

$$k, l, p = 0, 1, \ldots, \quad k \neq l,$$

$$\varphi_{k,k} = \arg a_{k,k}, \qquad u_{k}^{(p)} = \Re Q_{k}^{(p)}, \qquad v_{k}^{(p)} = \Im Q_{k}^{(p)},$$

$$k = 1, 2, \ldots, \quad p = 0, 1, \ldots.$$

For remainders (6), we have

$$u_{k+j,k}^{(p)} = 1 + \sum_{m=1}^{p} \prod_{r=1}^{m} \frac{|a_{k+j+r,k}|}{\left|Q_{k+j+r,k}^{(p-r)}\right|^2} \cos\left(\sum_{l=1}^{m} (-1)^{l-1} \varphi_{k+j+l,k}\right), \quad k = 0, 1, \ldots, \quad j, p = 1, 2, \ldots, \tag{28}$$

$$v_{k+j,k}^{(p)} = \sum_{m=1}^{p} \prod_{r=1}^{m} \frac{|a_{k+j+r,k}|}{\left|Q_{k+j+r,k}^{(p-r)}\right|^2} \sin\left(\sum_{l=1}^{m} (-1)^{l-1} \varphi_{k+j+l,k}\right), \quad k = 0, 1, \ldots, \quad j, p = 1, 2, \ldots, \tag{29}$$

$$u_{k,k+j}^{(p)} = 1 + \sum_{m=1}^{p} \prod_{r=1}^{m} \frac{|a_{k,k+j+r}|}{\left|Q_{k,k+j+r}^{(p-r)}\right|^2} \cos\left(\sum_{l=1}^{m} (-1)^{l-1} \varphi_{k,k+j+l}\right), \quad k = 0, 1, \ldots, \quad j, p = 1, 2, \ldots, \tag{30}$$

$$v_{k,k+j}^{(p)} = \sum_{m=1}^{p} \prod_{r=1}^{m} \frac{|a_{k,k+j+r}|}{\left|Q_{k,k+j+r}^{(p-r)}\right|^2} \sin\left(\sum_{l=1}^{m} (-1)^{l-1} \varphi_{k,k+j+l}\right), \quad k = 0, 1, \ldots, \quad j, p = 1, 2, \ldots. \tag{31}$$

It follows from relations (28)–(31) and (7) that

$$\begin{aligned} \Re \Phi_k^{(p)} = & \sum_{m=1}^{p} \prod_{r=1}^{m} \frac{|a_{k+r,k}|}{\left|Q_{k+r,k}^{(p-r)}\right|^2} \cos\left(\sum_{j=1}^{m} (-1)^{j-1} \varphi_{k+j,k}\right) \\ & + \sum_{m=1}^{p} \prod_{r=1}^{m} \frac{|a_{k,k+r}|}{\left|Q_{k,k+r}^{(p-r)}\right|^2} \cos\left(\sum_{j=1}^{m} (-1)^{j-1} \varphi_{k,k+j}\right), \quad k, p = 1, 2, \ldots, \end{aligned} \tag{32}$$

$$\begin{aligned} \Im \Phi_k^{(p)} = & \sum_{m=1}^{p} \prod_{r=1}^{m} \frac{|a_{k+r,k}|}{\left|Q_{k+r,k}^{(p-r)}\right|^2} \sin\left(\sum_{j=1}^{m} (-1)^{j-1} \varphi_{k+j,k}\right) \\ & + \sum_{m=1}^{p} \prod_{r=1}^{m} \frac{|a_{k,k+r}|}{\left|Q_{k,k+r}^{(p-r)}\right|^2} \sin\left(\sum_{j=1}^{m} (-1)^{j-1} \varphi_{k,k+j}\right), \quad k, p = 1, 2, \ldots. \end{aligned} \tag{33}$$

The formulas for the real and imaginary parts of remainders (5) take the form

$$u_k^{(p)} = 1 + \Re\Phi_k^{(p)} + \sum_{m=1}^{[p/2]} \prod_{r=1}^{m} \frac{|a_{k+r,k+r}|}{|Q_{k+r}^{(p-2r)}|^2} \cos\left(\sum_{j=1}^{m} (-1)^{j-1} \varphi_{k+j,k+j}\right)$$

$$+ \sum_{l=1}^{[(p-1)/2]} \prod_{q=1}^{l} \frac{|a_{k+q,k+q}|}{|Q_{k+q}^{(p-2q)}|^2} \sum_{m=1}^{p-2l} \prod_{r=1}^{m} \frac{|a_{k+l+r,k+l}|}{|Q_{k+l+r,k+l}^{(p-2l-r)}|^2}$$

$$\times \cos\left(\sum_{j=1}^{l} (-1)^{j-1} \varphi_{k+j,k+j} + \sum_{j=1}^{m} (-1)^{j+l-1} \varphi_{k+j+l,k+l}\right)$$

$$+ \sum_{l=1}^{[(p-1)/2]} \prod_{q=1}^{l} \frac{|a_{k+q,k+q}|}{|Q_{k+q}^{(p-2q)}|^2} \sum_{m=1}^{p-2l} \prod_{r=1}^{m} \frac{|a_{k+l,k+l+r}|}{|Q_{k+l,k+l+r}^{(p-2l-r)}|^2}$$

$$\times \cos\left(\sum_{j=1}^{l} (-1)^{j-1} \varphi_{k+j,k+j} + \sum_{j=1}^{m} (-1)^{j+l-1} \varphi_{k+l,k+j+l}\right), \quad k, p = 1, 2, \ldots, \tag{34}$$

$$v_k^{(p)} = \Im\Phi_k^{(p)} + \sum_{m=1}^{[p/2]} \prod_{r=1}^{m} \frac{|a_{k+r,k+r}|}{|Q_{k+r}^{(p-2r)}|^2} \sin\left(\sum_{j=1}^{m} (-1)^{j-1} \varphi_{k+j,k+j}\right)$$

$$+ \sum_{l=1}^{[(p-1)/2]} \prod_{q=1}^{l} \frac{|a_{k+q,k+q}|}{|Q_{k+q}^{(p-2q)}|^2} \sum_{m=1}^{p-2l} \prod_{r=1}^{m} \frac{|a_{k+l+r,k+l}|}{|Q_{k+l+r,k+l}^{(p-2l-r)}|^2}$$

$$\times \sin\left(\sum_{j=1}^{l} (-1)^{j-1} \varphi_{k+j,k+j} + \sum_{j=1}^{m} (-1)^{j+l-1} \varphi_{k+j+l,k+l}\right)$$

$$+ \sum_{l=1}^{[(p-1)/2]} \prod_{q=1}^{l} \frac{|a_{k+q,k+q}|}{|Q_{k+q}^{(p-2q)}|^2} \sum_{m=1}^{p-2l} \prod_{r=1}^{m} \frac{|a_{k+l,k+l+r}|}{|Q_{k+l,k+l+r}^{(p-2l-r)}|^2}$$

$$\times \sin\left(\sum_{j=1}^{l} (-1)^{j-1} \varphi_{k+j,k+j} + \sum_{j=1}^{m} (-1)^{j+l-1} \varphi_{k+l,k+j+l}\right), \quad k, p = 1, 2, \ldots. \tag{35}$$

It is clear that relations (28)–(35) are deduced under the assumption that all remainders in the denominators of expressions on the right-hand sides of these relations are not equal to zero. We now show that this assumption is true under conditions (24) and $z \in G_K$. In this case,

$$\varphi_{j,j} = \arg z_1 + \arg z_2, \quad \varphi_{k+j,k} = \arg z_1, \quad \varphi_{k,k+j} = \arg z_2, \quad k = 0, 1, \ldots, \quad j = 1, 2, \ldots, \tag{36}$$

$$\sum_{j=1}^{m}(-1)^{j-1}\varphi_{k+j,k} = \begin{cases} \arg z_1 & \text{for} \quad m = 2r+1, \quad r = 0,1,\dots, \\ 0, & \text{otherwise}, \end{cases} \tag{37}$$

$$\sum_{j=1}^{m}(-1)^{j-1}\varphi_{k,k+j} = \begin{cases} \arg z_2 & \text{for} \quad m = 2r+1, \quad r = 0,1,\dots, \\ 0, & \text{otherwise}, \end{cases} \tag{38}$$

$$\sum_{j=1}^{m}(-1)^{j-1}\varphi_{k+j,k+j} = \begin{cases} \arg z_1 + \arg z_2 & \text{for} \quad m = 2r+1, \quad r = 0,1,\dots, \\ 0, & \text{otherwise}, \end{cases} \tag{39}$$

$$\sum_{j=1}^{l}(-1)^{j-1}\varphi_{k+j,k+j} + \sum_{j=1}^{m}(-1)^{j+l-1}\varphi_{k+j,k}$$

$$= \begin{cases} \arg z_1 + \arg z_2 & \text{for} \quad m = 2r, \quad l = 2q-1, \quad r,q = 1,2,\dots, \\ 0 & \text{for} \quad m = 2r, \quad l = 2q, \quad r,q = 1,2,\dots, \\ \arg z_2 & \text{for} \quad m = 2r-1, \quad l = 2q-1, \quad r,q = 1,2,\dots, \\ \arg z_1 & \text{for} \quad m = 2r-1, \quad l = 2q, \quad r,q = 1,2,\dots, \end{cases} \tag{40}$$

$$\sum_{j=1}^{l}(-1)^{j-1}\varphi_{k+j,k+j} + \sum_{j=1}^{m}(-1)^{j+l-1}\varphi_{k,k+j}$$

$$= \begin{cases} \arg z_1 + \arg z_2 & \text{for} \quad m = 2r, \quad l = 2q-1, \quad r,q = 1,2,\dots, \\ 0 & \text{for} \quad m = 2r, \quad l = 2q, \quad r,q = 1,2,\dots, \\ \arg z_1 & \text{for} \quad m = 2r-1, \quad l = 2q-1, \quad r,q = 1,2,\dots, \\ \arg z_2 & \text{for} \quad m = 2r-1, \quad l = 2q, \quad r,q = 1,2,\dots. \end{cases} \tag{41}$$

By using relations (6), for any $j = 1,2,\dots,\ k = 0,1,\dots,$ we get $Q_{k+j,k}^{(0)} = 1,$

$$u_{k+j,k}^{(1)} = 1 + \Re a_{k+j+1,k} = 1 + |a_{k+j+1,k}|\cos(\arg z_1) \geq 1.$$

Hence, $\left|Q_{k+j,k}^{(p)}\right| \geq 1,\ p = 0,1,$ and relation (28) is true for $p = 2.$ Thus,

$$u_{k+j,k}^{(2)} = 1 + \sum_{m=1}^{2}\prod_{r=1}^{m}\frac{|a_{k+j+r,k}|}{\left|Q_{k+j+r,k}^{(2-r)}\right|^2}\cos\left(\sum_{l=1}^{m}(-1)^{l-1}\varphi_{k+j+l,k}\right) \geq 1,$$

$$k = 0,1,\dots, \quad j = 1,2,\dots, \quad j = 1,2,\dots.$$

This implies that

$$\left|Q^{(2)}_{k+j,k}\right| \geq 1,$$

which means that relation (28) can be also used for $p = 3$. If we assume that

$$\left|Q^{(p)}_{k+j,k}\right| \geq 1, \quad 0 \leq p \leq s,$$

and take into account relation (28) for $p = s + 1$ and the conditions of the theorem, then we conclude that

$$\left|Q^{(s+1)}_{k+j,k}\right| \geq 1.$$

Hence, by induction, we find

$$\left|Q^{(p)}_{k+j,k}\right| \geq 1, \quad j = 1, 2, \ldots, \quad k, p = 0, 1, \ldots .$$

Similarly, we can show that the following inequalities are true:

$$|Q^{(p)}_{k,k+j}| \geq 1, \quad j = 1, 2, \ldots, \quad k, p = 0, 1, \ldots .$$

We now consider the remainders $Q^{(p)}_k$, $k = 1, 2, \ldots,$ $p = 0, 1, \ldots$. By virtue of relations (5), we get

$$Q^{(0)}_k = 1$$

and

$$u^{(1)}_k = 1 + \Re\Phi^{(1)}_k = 1 + |a_{k+1,k}|\cos(\arg z_1) + |a_{k,k+1}|\cos(\arg z_2) \geq 1, \quad \left|Q^{(1)}_k\right| \geq 1.$$

Hence, the formulas presented above for the real parts of the remainders $Q^{(2)}_k$ and $Q^{(3)}_k$ are correct. By using relations (32)–(41), we obtain

$$u^{(2)}_k = 1 + \Re\Phi^{(2)}_k + \Re a_{k+1,k+1} = 1 + \Re\Phi^{(2)}_k + |a_{k+1,k+1}|\cos(\arg z_1 + \arg z_2) \geq 1,$$

$$u^{(3)}_k = 1 + \Re\Phi^{(3)}_k + \frac{|a_{k+1,k+1}|}{\left|Q^{(1)}_{k+1}\right|^2}\cos(\arg z_1 + \arg z_2)$$

$$+ \frac{|a_{k+1,k+1}|}{\left|Q^{(1)}_{k+1}\right|^2}\left(\frac{|a_{k+2,k+1}|}{\left|Q^{(0)}_{k+2,k+1}\right|^2}\cos(\arg z_2) + \frac{|a_{k+1,k+2}|}{\left|Q^{(0)}_{k+1,k+2}\right|^2}\cos(\arg z_1)\right) \geq 1,$$

$$k = 1, 2, \ldots .$$

This yields

$$\left|Q^{(p)}_k\right| \geq 1, \quad k = 1, 2, \ldots, \quad p = 2, 3.$$

Reasoning similarly, by induction, we conclude that

$$\left|Q_k^{(p)}\right| \geq 1, \quad k = 1, 2, \ldots, \quad p = 0, 1, \ldots.$$

Under conditions (24), inequalities (9) are satisfied for all z from set (25), i.e., all remainders of the two-dimensional continued fraction (1), (2) are not equal to zero. Moreover,

$$\frac{|a_{1,0}|}{\left|Q_{1,0}^{(p)}\right|} \leq c_{1,0}K_1, \qquad \frac{|a_{0,1}|}{\left|Q_{0,1}^{(p)}\right|} \leq c_{0,1}K_2, \qquad \frac{|a_{1,1}|}{\left|Q_1^{(p)}\right|} \leq c_{1,1}K_1K_2, \quad p = 0, 1, \ldots,$$

i.e., condition (10) is satisfied. It follows from relations (28)–(41) that

$$u_{i,j}^{(p)} \geq 1, \qquad v_{i,j}^{(p)} \geq 0, \quad i, j, p = 0, 1, \ldots, \quad i \neq j,$$

$$u_k^{(p)} \geq 1, \qquad v_k^{(p)} \geq 0, \qquad \Re\Phi_k^{(p)} \geq 0, \qquad \Im\Phi_k^{(p)} \geq 0,$$

$$k = 1, 2, \ldots, \quad p = 0, 1, \ldots.$$

In view of these inequalities, we conclude that

$$u_k^{(p+2)} \geq 1 + \frac{|a_{k+1,k+1}|}{\left|Q_{k+1}^{(p)}\right|^2}\Big(u_{k+1}^{(p)} \cos\varphi_{k+1,k+1} + v_{k+1}^{(p)} \sin\varphi_{k+1,k+1}\Big), \tag{42}$$

$$v_k^{(p+2)} \geq \frac{|a_{k+1,k+1}|}{\left|Q_{k+1}^{(p)}\right|^2}\Big(u_{k+1}^{(p)} \sin\varphi_{k+1,k+1} - v_{k+1}^{(p)} \cos\varphi_{k+1,k+1}\Big). \tag{43}$$

It follows from inequalities (42) and (43) that

$$\begin{aligned}
\left|Q_k^{(p+2)}\right|^2 &= \Big(u_k^{(p+2)}\Big)^2 + \Big(v_k^{(p+2)}\Big)^2 \\
&\geq 1 + 2\frac{|a_{k+1,k+1}|}{\left|Q_{k+1}^{(p)}\right|^2}\Big(u_{k+1}^{(p)} \cos\varphi_{k+1,k+1} + v_{k+1}^{(p)} \sin\varphi_{k+1,k+1}\Big) \\
&\quad + \frac{|a_{k+1,k+1}|^2}{\left|Q_{k+1}^{(p)}\right|^4}\Big(u_{k+1}^{(p)} \cos\varphi_{k+1,k+1} + v_{k+1}^{(p)} \sin\varphi_{k+1,k+1}\Big)^2 \\
&\quad + \frac{|a_{k+1,k+1}|^2}{\left|Q_{k+1}^{(p)}\right|^4}\Big(u_{k+1}^{(p)} \sin\varphi_{k+1,k+1} - v_{k+1}^{(p)} \cos\varphi_{k+1,k+1}\Big)^2 \\
&\geq 1 + \frac{|a_{k+1,k+1}|^2}{\left|Q_{k+1}^{(p)}\right|^2}.
\end{aligned}$$

Therefore,

$$\frac{|a_{k+1,k+1}|^2}{|Q_k^{(p+2)}|^2|Q_{k+1}^{(p)}|^2} \le \frac{|a_{k+1,k+1}|^2}{|Q_{k+1}^{(p)}|^2+|a_{k+1,k+1}|^2}$$

$$\le \frac{|a_{k+1,k+1}|^2}{1+|a_{k+1,k+1}|^2} \le \frac{L^2(K_1)^2(K_2)^2}{1+L^2(K_1)^2(K_2)^2} = \rho^2,$$

i.e., conditions (13) are satisfied.

To check inequalities (11), we estimate the following expressions:

$$\frac{|a_{k+1,k}|}{|Q_k^{(p+1)}||Q_{k+1,k}^{(p)}|}, \qquad \frac{|a_{k,k+1}|}{|Q_k^{(p+1)}||Q_{k,k+1}^{(p)}|}, \quad p=0,1,\ldots, \quad k=1,2,\ldots.$$

Since

$$u_k^{(1)} = 1+\Re\Phi_k^{(1)}, \qquad v_k^{(1)} = \Im\Phi_k^{(1)}, \qquad u_k^{(p)} \ge 1+\Re\Phi_k^{(p)}, \qquad v_k^{(p)} \ge \Im\Phi_k^{(p)},$$

$$k=1,2,\ldots, \quad p=2,3,\ldots,$$

for $k=1,2,\ldots,\quad p=0,1,\ldots,$ we obtain

$$u_k^{(p+1)} \ge 1+\Re\frac{a_{k+1,k}}{Q_{k+1,k}^{(p)}}+\Re\frac{a_{k,k+1}}{Q_{k,k+1}^{(p)}}$$

$$\ge 1+\frac{|a_{k+1,k}|}{|Q_{k+1,k}^{(p)}|^2}\left(u_{k+1,k}^{(p)}\cos\varphi_{k+1,k}+v_{k+1,k}^{(p)}\sin\varphi_{k+1,k}\right) \ge 1,$$

$$v_k^{(p+1)} \ge \Im\frac{a_{k+1,k}}{Q_{k+1,k}^{(p)}}+\Im\frac{a_{k,k+1}}{Q_{k,k+1}^{(p)}}$$

$$\ge \frac{|a_{k+1,k}|}{|Q_{k+1,k}^{(p)}|^2}\left(u_{k+1,k}^{(p)}\sin\varphi_{k+1,k}-v_{k+1,k}^{(p)}\cos\varphi_{k+1,k}\right) \ge 0.$$

Hence,

$$|Q_k^{(p+1)}|^2 = \left(u_k^{(p+1)}\right)^2+\left(v_k^{(p+1)}\right)^2$$

$$\ge 1+2\frac{|a_{k+1,k}|}{|Q_{k+1,k}^{(p)}|^2}\left(u_{k+1,k}^{(p)}\cos\varphi_{k+1,k}+v_{k+1,k}^{(p)}\sin\varphi_{k+1,k}\right)$$

$$+\frac{|a_{k+1,k}|^2}{|Q_{k+1,k}^{(p)}|^4}\left(u_{k+1,k}^{(p)}\cos\varphi_{k+1,k}+v_{k+1,k}^{(p)}\sin\varphi_{k+1,k}\right)^2$$

$$+\frac{|a_{k+1,k}|^2}{|Q^{(p)}_{k+1,k}|^4}\left(u^{(p)}_{k+1,k}\sin\varphi_{k+1,k}-v^{(p)}_{k+1,k}\cos\varphi_{k+1,k}\right)^2$$

$$\geq 1+\frac{|a_{k+1,k}|^2}{|Q^{(p)}_{k+1,k}|^2}$$

and, therefore,

$$\frac{|a_{k+1,k}|^2}{|Q^{(p+1)}_k|^2|Q^{(p)}_{k+1,k}|^2}\leq\frac{|a_{k+1,k}|^2}{|Q^{(p)}_{k+1,k}|^2+|a_{k+1,k}|^2}$$

$$\leq\frac{|a_{k+1,k}|^2}{1+|a_{k+1,k}|^2}\leq\frac{(L_1)^2(K_1)^2}{1+(L_1)^2(K_1)^2}=(\rho_1)^2.$$

Similarly, we can show that

$$\frac{|a_{k,k+1}|^2}{|Q^{(p+1)}_k|^2|Q^{(p)}_{k,k+1}|^2}\leq\frac{(L_2)^2(K_2)^2}{1+(L_2)^2(K_2)^2}=(\rho_2)^2.$$

Thus, for the two-dimensional continued fraction (1), (2) whose elements satisfy the conditions of Theorem 2, inequalities (11) are true with $H_1=\rho_1$ and $H_2=\rho_2$. Finally, we estimate the expressions

$$\frac{|a_{k+j+1,k}|}{|Q^{(m+1)}_{k+j,k}Q^{(m)}_{k+j+1,k}|},\qquad\frac{|a_{k,k+j+1}|}{|Q^{(m+1)}_{k,k+j}Q^{(m)}_{k,k+j+1}|},\qquad k,m=0,1,\ldots,\quad j=1,2,\ldots.$$

In view the fact that

$$u^{(m+1)}_{k+j,k}=1+\Re\frac{a_{k+j+1,k}}{Q^{(m)}_{k+j+1,k}}$$

$$=1+\frac{|a_{k+j+1,k}|}{|Q^{m}_{k+j+1,k}|^2}\left(u^{(m)}_{k+j+1,k}\cos\varphi_{k+j+1,k}+v^{(m)}_{k+j+1,k}\sin\varphi_{k+j+1,k}\right),$$

$$v^{(m+1)}_{k+j,k}=\Im\frac{a_{k+j+1,k}}{Q^{(m)}_{k+j+1,k}}$$

$$=\frac{|a_{k+j+1,k}|}{|Q^{m}_{k+j+1,k}|^2}\left(u^{(m)}_{k+j+1,k}\sin\varphi_{k+j+1,k}-v^{(m)}_{k+j+1,k}\cos\varphi_{k+j+1,k}\right),$$

under the conditions of the theorem, for $k,m=0,1,\ldots,$ $j=1,2,\ldots,$ we obtain

$$|Q^{(m+1)}_{k+j,k}|^2=\left(u^{(m+1)}_{k+j,k}\right)^2+\left(v^{(m+1)}_{k+j,k}\right)^2\geq 1+\frac{|a_{k+j+1,k}|^2}{|Q^{(m)}_{k+j+1,k}|^2}.$$

This yields

$$\frac{|a_{k+j+1,k}|^2}{\left|Q^{(m+1)}_{k+j,k}\right|^2\left|Q^{(m)}_{k+j+1,k}\right|^2} \le \frac{(L_1)^2(K_1)^2}{1+(L_1)^2(K_1)^2} = (\rho_1)^2, \quad k,m=0,1,\ldots, \quad j=1,2,\ldots.$$

Similarly, we get

$$\frac{|a_{k,k+j+1}|^2}{\left|Q^{(m+1)}_{k,k+j}\right|^2\left|Q^{(m)}_{k,k+j+1}\right|^2} \le \frac{(L_2)^2(K_2)^2}{1+(L_2)^2(K_2)^2} = (\rho_2)^2, \quad k,m=0,1,\ldots, \quad j=1,2,\ldots.$$

Finally, by using Theorem 1, we complete the proof of Theorem 2.

Theorem 3. *Suppose that elements of the two-dimensional continued fraction (1), (2) satisfy conditions (24), where $L,\ L_1$, and L_2 are positive constants. Then:*

(i) at every point $z=(z_1,z_2)$ of the set

$$G_K=\left\{(z_1,z_2)\in\mathbb{C}^2:\ |z_j|\le K_j,\ -\frac{\pi}{2}\le\arg(z_j)\le 0,\ j=1,2;\ -\frac{\pi}{2}\le\arg(z_1)+\arg(z_2)\right\},$$

the two-dimensional continued fraction (1), (2) converges to the function $f(z)$ and estimate (26) is true, where $\rho,\ \rho_1$, and ρ_2 are given by relations (27) and $S_{1,m}$ and $S_{2,m}$ are given by relations (15) and (16);

(ii) the indicated convergence is uniform on each compact subset of $\operatorname{Int} G_K$.

Note that, in this case, by using relations (28)–(41) and the conditions of the theorem, we obtain

$$u^{(p)}_{i,j}\ge 1, \quad v^{(p)}_{i,j}\le 0, \quad i,j,p=0,1,\ldots, \quad i\ne j,$$

$$u^{(p)}_{k}\ge 1, \qquad v^{(p)}_{k}\le 0, \qquad \Re\Phi^{(p)}_{k}\ge 0, \qquad \Im\Phi^{(p)}_{k}\le 0, \quad k=1,2,\ldots, \quad p=0,1,\ldots.$$

The subsequent proof of Theorem 3 is performed according to the scheme of the proof of Theorem 2.

Theorem 4. *Suppose that the elements of the two-dimensional continued fraction (1), (2) satisfy conditions (24), where $L,\ L_1$, and L_2 are positive constants. Then the two-dimensional continued fraction (1), (2) converges to a function holomorphic in the domain*

$$G=\left\{(z_1,z_2)\in\mathbb{C}^2:\ |\arg(z_j)|<\frac{\pi}{2},\ j=1,2;\ |\arg(z_1)+\arg(z_2)|<\frac{\pi}{2}\right\}.$$

Moreover, this convergence is uniform on every compact subset of the domain G.

Proof. All estimates for the real parts of remainders obtained in the proof of Theorem 2 remain true under the conditions of Theorem 4. Hence,

$$u^{(p)}_1\ge 1, \qquad u^{(p)}_{1,0}\ge 1, \qquad u^{(p)}_{0,1}\ge 1, \quad p=0,1,\ldots. \tag{44}$$

Thus, all figured convergents of the two-dimensional continued fraction (1), (2) are holomorphic in the domain G of the function.

Let K be an arbitrary compact set in the domain G. Then there exists an open ball $B \subset \mathbb{C}^2$ centered at the point $(0;0)$ with radius r such that $K \subset B$.

By using relations (5), (7), and (8) and estimates (44), we get

$$|f_1(z)| \le |a_{0,0}| + \frac{|a_{1,0}|}{|Q_{1,0}^{(0)}|} + \frac{|a_{0,1}|}{|Q_{0,1}^{(0)}|} \le |a_{0,0}| + c_{1,0}|z_1| + c_{0,1}|z_2| < |a_{0,0}| + (c_{1,0} + c_{0,1})r,$$

$$|f_n(z)| \le |a_{0,0}| + \frac{|a_{1,0}|}{|Q_{1,0}^{(n-1)}|} + \frac{|a_{0,1}|}{|Q_{0,1}^{(n-1)}|} + \frac{|a_{1,1}|}{|Q_{1}^{(n-2)}|} < |a_{0,0}| + (c_{1,0} + c_{0,1})r + c_{1,1}r^2,$$

$$n = 2, 3, \ldots,$$

i.e., $\{f_n(z)\}$ is a sequence of functions uniformly bounded on K. Thus, this sequence is uniformly bounded on every compact subset of the domain G.

By virtue of Theorem 2, the sequence $\{f_m(z)\}$ converges at any point of the set

$$\Delta = \{0 < \Re z_j < \delta, \ \Im z_j = 0, \ j = 1, 2\} \subset G,$$

which is a two-dimensional real neighborhood of the point $z^{(0)} = (\delta/2, \delta/2)$. Thus, the sequence $\{f_n(z)\}$ satisfies the conditions of a multidimensional analog of the Stieltjes–Vitali theorem [9] (Theorem 2.17) and, hence, uniformly converges on any compact subset of the domain G to a holomorphic function in G.

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