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LIE–BÄCKLUND SYMMETRY, REDUCTION, AND SOLUTIONS OF NONLINEAR EVOLUTIONARY EQUATIONS

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We study the problem of symmetry reduction of nonlinear partial differential equations used to describe diffusion processes in an inhomogeneous medium. We find ansatzes reducing partial differential equations to systems of ordinary differential equations. These ansatzes are constructed by using the operators of Lie–Bäcklund symmetry of the third-order ordinary differential equations. The proposed method enables us to find solutions that cannot be obtained by using the classical Lie method. These solutions are constructed for nonlinear diffusion equations invariant under one-, two-, and three-parameter Lie groups of point transformations.

1. Introduction

It is known that the method of classical [1] and nonclassical (conditional) symmetry [2–5] is an efficient tool for finding the exact solutions of nonlinear partial differential equations. This method is based on the construction of special ansatzes obtained in the form of general invariant or conditionally invariant solutions reducing the original partial differential equations to equations with smaller numbers of independent variables and, in particular, to ordinary differential equations. The method of conditional Lie–Bäcklund symmetry of the evolutionary equations with two independent variables was proposed in [6, 7]. Within the framework of this approach, the evolutionary partial differential equations are reduced to systems of ordinary differential equations. For the applications of this method to nonlinear diffusion equations, see, e.g., [8]. The relationship between the generalized conditional symmetry of evolutionary equations and consistency of the system of equations was studied in [11]. The procedure of reduction of nonlinear evolutionary equations to systems of ordinary equations was proposed in [9]. This procedure is based on the Lie–Bäcklund symmetry of ordinary linear homogeneous equations and offers the theoretical-group substantiation of the method of "nonlinear separation of variables."

In the present paper, we use the method proposed in [10], which can be interpreted as a generalization of the Svirshchevskii method. It is based on the Lie–Bäcklund symmetry of ordinary differential equations, which are not necessarily linear and homogeneous. In the general case, they are nonlinear. Moreover, this method can be applied not only to the evolutionary equations but also, generally speaking, to any partial differential equations and admits generalizations to the multidimensional cases [13, 14].

We demonstrate the efficiency of application of this method by using, as an example, the equation that describes the processes of nonlinear diffusion in inhomogeneous media.

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2. Application of the Method with the Use of Ordinary Differential Equations of the Third Order

Consider an ordinary differential equation of the third order

$$u_{xxx} - U(x, u, u_x, u_{xx}) = 0, \tag{1}$$

where $u(t, x)$ is a function of an independent variable x and a variable t. In the analyzed case, t can be interpreted as a parameter. We seek the function U in the form

$$U = \sum_{j_0, j_1, j_2 \in \mathbb{Z}} a_{j_0, j_1, j_2}(x)\, u^{j_0}\, u_x^{j_1}\, u_{xx}^{j_2},$$

where $a_{j_0, j_1, j_2}(x)$ are smooth functions that should be found. Assume that Eq. (1) admits the Lie–Bäcklund symmetry operator $X = F(x, u, u_x, u_{xx})\partial_u$, where F is the right-hand side of the evolutionary equation

$$u_t = F(x, u, u_x, u_{xx}). \tag{2}$$

In the present paper, we illustrate the efficiency of the proposed method by analyzing an example of the following diffusion-type equation:

$$u_t = \left(\frac{H(x)}{u}\right)_{xx} + \eta(x, u, u_x). \tag{3}$$

Consider a function H of the form

$$H(x) = \frac{1}{C_2 x^2 + C_1 x + C_0},$$

where $C_1,\ C_2,\ C_0 \in R$ are real constants. For $C_1 = C_2 = 0$, $C_0 = -1$, and $\eta = 0$, Eq. (3) turns into the well-known nonlinear differential equation with variable heat-conduction coefficient that describes nonlinear diffusion processes and is used in the problems of plasma physics and solid-state physics, as well as in many other applied problems. This equation possesses the infinite Lie–Bäcklund symmetry and, with the help of a certain transformation, is reduced to the linear heat-conduction equation.

The criterion of invariance for Eq. (1) has the form

$$X^{(3)}\big(u_{xxx} - U(x, u, u_x, u_{xx})\big)\big|_{u_{xxx}=U} = 0, \tag{4}$$

where $X^{(3)}$ is the third-order extension of the Lie–Bäcklund symmetry operator X. The procedure of finding of the function U is quite cumbersome. In the present paper, we omit this procedure and present only the final result in the following formulation:

Proposition 1. *Equation (1) admits the Lie–Bäcklund symmetry operator*

$$X = \left(\frac{H(x)}{u}\right)_{xx} \partial_u,$$

where

$$H(x) = \frac{1}{C_2 x^2 + C_1 x + C_0},$$

if it has the following form:

$$u_{xxx} = 9\frac{u_{xx}u_x}{u} - 12\frac{u_x^3}{u^2} + \frac{6(2C_2x + C_1)}{C_2x^2 + C_1x + C_0}u_{xx}$$

$$- \frac{18(2C_2x + C_1)}{C_2x^2 + C_1x + C_0}\frac{u_x^2}{u} - \frac{6(10C_2^2x^2 + 10C_1x - 2C_0C_2 + 3C_1^2)}{(C_2x^2 + C_1x + C_0)^2}u_x$$

$$- \frac{6(2C_2x + C_1)(5C_2^2x^2 + 5C_1x - 3C_0C_2 + 2C_1^2)}{(C_2x^2 + C_1x + C_0)^3}u,$$

and $C_1, C_2, C_0 \in R$ are real constants.

2.1. Determination of Solutions of the Equation $u_t = \left(\frac{H(x)}{u}\right)_{xx} + \eta(x, u, u_x)$. We use the property of reduction of the equation

$$u_t = \left(\frac{H(x)}{u}\right)_{xx}, \tag{5}$$

where

$$H(x) = \frac{1}{C_2x^2 + C_1x + C_0},$$

to a system of three ordinary differential equations of the first order with the help of an ansatz in the form of a general solution of the ordinary differential equation of the third order in Proposition 1. It is clear that, for the operator of contact symmetry of this ordinary differential equation $X = \eta(x, u, u_x)\partial_u$ rewritten in the Lie–Bäcklund form, Eq. (3) is also reduced to a system of three ordinary differential equations with the help of the same ansatz, i.e., the property of reduction of the modified equation is preserved.

Consider several special cases of Proposition 1.

Let $C_2 = C_0 = 0$ and $C_1 \neq 0$. Then we can take $H(x) = \frac{\kappa}{x}$, $\kappa = \text{const}$. Thus, the ordinary differential equation generated by the ansatz has the form

$$u_{xxx} = 9\frac{u_{xx}u_x}{u} - 12\frac{u_x^3}{u^2} + \frac{6}{x}u_{xx} - \frac{18}{x}\frac{u_x^2}{u} - \frac{18}{x^2}u_x - \frac{12}{x^3}u. \tag{6}$$

Its solution is given by the formula

$$u(x) = \pm\frac{1}{x\sqrt{\varphi_2x^2 + \varphi_1x + \varphi_0}}, \tag{7}$$

where φ_1, φ_2, and φ_3 are arbitrary functions of the variable t. At the same time, the Lie algebra of the Lie group of contact transformations, which is a symmetry group of Eq. (6), can be determined by the basis elements:

$$X_1 = u\partial_u, \quad X_2 = xu_x\partial_u, \quad X_3 = x^2u^3\partial_u, \quad X_4 = x^3u^3\partial_u, \quad X_5 = x^4u^3\partial_u, \tag{8}$$

$$X_6 = \left(\frac{u}{x} + u_x\right)\partial_u, \quad X_7 = (2xu + x^2u_x)\partial_u, \tag{9}$$

$$X_8 = \frac{x^2u_x^2 + 4xuu_x + 4u^2}{x^2u^3}\partial_u, \quad X_9 = \frac{x^2u_x^2 + 3xuu_x + 2u^2}{x^3u^3}\partial_u, \quad X_{10} = \frac{x^2u_x^2 + 2xuu_x + u^2}{x^4u^3}\partial_u. \tag{10}$$

Thus, the reduction method can be applied to an arbitrary equation from the class

$$u_t = \left(\frac{\kappa}{xu}\right)_{xx} + \sum_{i=1}^{10} a_i X_i u, \qquad a_i = \text{const}, \quad i = 1, \ldots, 10.$$

We modify the original equation by adding the terms containing derivatives whose order is smaller than two. In the present paper, we consider solely the characteristics of the operators of point symmetry. Thus, consider an equation

$$u_t = \left(\frac{\kappa}{xu}\right)_{xx} + a_1 u + a_2 x u_x + a_4 (xu)^3, \quad \kappa, a_i \in \mathbb{R}. \tag{11}$$

For $\kappa \neq 0$ and any $a_1, a_2, a_4 \in \mathbb{R}$, this equation admits a two-dimensional Lie algebra with the basis elements

$$X_1 = \partial_t,$$

$$X_2 = 2x\partial_x - 3u\partial_u.$$

Moreover, this algebra is maximal in the analyzed case, i.e., this is a Lie algebra of the complete symmetry group of Eq. (11). In what follows, if we consider a group of point symmetry transformations of the diffusion equation, then we always mean the complete group (i.e., the maximal Lie algebra) without repeating this remark.

If $2a_1 \neq 3a_2$ and $a_4 = 0$, then the equation admits the following additional operator:

$$X_3 = e^{(2a_1 - 3a_2)t}\big(-a_2 x\partial_x + \partial_t + a_1 u\partial_u\big).$$

If the conditions $2a_1 = 3a_2$ and $a_4 = 0$ are satisfied, then the equation admits an additional operator of the form

$$X_3' = -a_2 x\partial_x + t\partial_t + \left(a_1 t + \frac{1}{2}\right)u\partial_u.$$

Ansatz (7) reduces Eq. (11) to a system of three ordinary differential equations

$$\varphi_2' + 2(a_1 - 2a_2)\varphi_2 = 0,$$

$$\varphi_0' + 2(a_1 - a_2)\varphi_0 = 0,$$

$$\varphi_1' - \frac{1}{2}\kappa\varphi_1^2 + 2\kappa\varphi_0\varphi_2 + 2\left(a_1 - \frac{3}{2}a_2\right)\varphi_1 + 2a_4 = 0.$$

For $2a_1 \neq 3a_2$ and $a_4 = 0$, we get the following solutions of the system of reduced equations:

$$u(x,t) = \pm \frac{1}{x\sqrt{c_2 e^{2(2a_2 - a_1)t} x^2 + 2A(t)\dfrac{c_3 \cos\left(\dfrac{\kappa A(t)}{2a_1 - 3a_2}\right) - c_4 \sin\left(\dfrac{\kappa A(t)}{2a_1 - 3a_2}\right)}{c_3 \sin\left(\dfrac{\kappa A(t)}{2a_1 - 3a_2}\right) + c_4 \cos\left(\dfrac{\kappa A(t)}{2a_1 - 3a_2}\right)} x + c_0 e^{2(a_2 - a_1)t}}}, \tag{12}$$

where $c_0c_2 < 0,\ A(t) = \sqrt{-c_0c_2}e^{(3a_2-2a_1)t}$, and $c_3^2 + c_4^2 > 0$, or

$$u(x,t) = \pm\frac{1}{x\sqrt{c_2e^{2(2a_2-a_1)t}x^2 + 2B(t)\dfrac{c_3\cosh\left(\dfrac{\kappa B(t)}{2a_1-3a_2}\right) + c_4\sinh\left(\dfrac{\kappa B(t)}{2a_1-3a_2}\right)}{c_3\sinh\left(\dfrac{\kappa B(t)}{2a_1-3a_2}\right) + c_4\cosh\left(\dfrac{\kappa B(t)}{2a_1-3a_2}\right)}x + c_0e^{2(a_2-a_1)t}}}, \quad (13)$$

where $c_0c_2 > 0,\ B(t) = \sqrt{c_0c_2}e^{(3a_2-2a_1)t}$, and $c_3^2 + c_4^2 > 0$.

For $a_1 = \dfrac{3}{2}a_2 = \dfrac{3}{2}a$ and arbitrary $a_4 \in \mathbb{R}$, we obtain the following solutions of the system of reduced equations:

$$u(x,t) = \pm\frac{1}{x\sqrt{c_2e^{at}x^2 - \dfrac{2\sqrt{-C}}{\kappa}\dfrac{c_3\cos\left(\sqrt{-C}t\right) + c_4\sin\left(\sqrt{-C}t\right)}{c_3\sin\left(\sqrt{-C}t\right) - c_4\cos\left(\sqrt{-C}t\right)}x + c_0e^{-at}}}, \quad (14)$$

where $C = \kappa^2c_0c_2 + \kappa a_4 < 0$ and $c_3^2 + c_4^2 > 0$, or

$$u(x,t) = \pm\frac{1}{x\sqrt{c_2e^{at}x^2 - \dfrac{2\sqrt{C}}{\kappa}\dfrac{c_3\cosh\left(\sqrt{C}t\right) + c_4\sinh\left(\sqrt{C}t\right)}{c_3\sinh\left(\sqrt{C}t\right) + c_4\cosh\left(\sqrt{C}t\right)}x + c_0e^{-at}}}, \quad (15)$$

where $C = \kappa^2c_0c_2 + \kappa a_4 > 0$ and $c_3^2 + c_4^2 > 0$.

Setting $c_3 = 0$ and $c_4 \neq 0$ in (13), we obtain a particular solution that contains the function $\tanh\left(\dfrac{\kappa B(t)}{2a_1-3a_2}\right)$. Replacing $\tanh\left(\dfrac{\kappa B(t)}{2a_1-3a_2}\right)$ in the obtained formula with $\coth\left(\dfrac{\kappa B(t)}{2a_1-3a_2}\right)$, we again obtain the solution of Eq. (11). This can be shown by setting $c_3 \neq 0$ and $c_4 = 0$ in relation (13). Note that solutions (15) have a similar property.

By the condition of invariance of solution (12) under a certain one-parameter subgroup

$$\sum_{i=1}^{3}\alpha_iX_i(u - u(x,t))\big|_{u=u(x,t)} = 0,$$

we conclude that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus, the solution (12) of Eq. (11) is not invariant and, hence, it cannot be obtained with the help of the classical Lie method. In a similar way, we show that solutions (13), (14), and (15) are not invariant.

Further, we consider a special case where $C_1 = C_0 = 0$ and $C_2 \neq 0$. Thus, we can take $H(x) = \dfrac{\kappa}{x^2}$, $\kappa = \text{const}$. In this case, the ordinary differential equation generated by the ansatz takes the form:

$$u_{xxx} = 9\frac{u_{xx}u_x}{u} - 12\frac{u_x^3}{u^2} + \frac{12}{x}u_{xx} - \frac{36}{x}\frac{u_x^2}{u} - \frac{60}{x^2}u_x - \frac{60}{x^3}u. \quad (16)$$

Thus, we get the solution

$$u(x) = \pm\frac{1}{x^2\sqrt{\varphi_2x^2 + \varphi_1x + \varphi_0}} \quad (17)$$

of the analyzed equation, where $\varphi_1,\ \varphi_2$, and φ_3 are arbitrary functions of the variable t.

The Lie algebra of the Lie group of contact transformations of the symmetry group of Eq. (16) can be specified by the basis elements as follows:

$$X_1 = u\partial_u, \qquad X_2 = xu_x\partial_u, \qquad X_3 = x^4u^3\partial_u, \qquad X_4 = x^5u^3\partial_u, \qquad X_5 = x^6u^3\partial_u, \tag{18}$$

$$X_6 = \left(2\frac{u}{x} + u_x\right)\partial_u, \qquad X_7 = (3xu + x^2u_x)\partial_u, \tag{19}$$

$$X_8 = \frac{x^2u_x^2 + 6xuu_x + 9u^2}{x^4u^3}\partial_u, \quad X_9 = \frac{x^2u_x^2 + 5xuu_x + 6u^2}{x^5u^3}\partial_u, \quad X_{10} = \frac{x^2u_x^2 + 4xuu_x + 4u^2}{x^6u^3}\partial_u. \tag{20}$$

Thus, by using ansatz (17), we can apply the reduction method to an arbitrary equation from the class of equations

$$u_t = \left(\frac{\kappa}{x^2u}\right)_{xx} + \sum_{i=1}^{10} a_iX_iu, \qquad a_i = \text{const}, \quad i = 1, \dots, 10.$$

Consider the following equation from this class:

$$u_t = \left(\frac{\kappa}{x^2u}\right)_{xx} + a_4x^5u^3 + a_5x^6u^3 + a_7(3xu + x^2u_x), \qquad \kappa, a_i \in \mathbb{R}, \quad \kappa \neq 0. \tag{21}$$

For any $a_4, a_5, a_7 \in \mathbb{R}$, this equation admits a one-parameter Lie group with the infinitesimal generator

$$Y_1 = \partial_t.$$

At the same time, if $a_4 \neq 0$ and $a_7 = 0$, then the equation admits the following additional operator:

$$Y_2' = \left(-2x^2\frac{a_5}{a_4} - 2x\right)\partial_x + t\partial_t + \frac{3}{2}\left(4x\frac{a_5}{a_4} + 3\right)u\partial_u.$$

At the same time, for $a_4 = 0$, the equation admits the following two additional operators:

$$Y_2 = -x\partial_x + t\partial_t + \frac{5}{2}u\partial_u, \qquad Y_3 = x^2\partial_x - 3xu\partial_u.$$

Moreover, if $a_4 = a_5 = 0$, then we additionally get the fourth operator

$$Y_4 = -a_7tx^2\partial_x + t\partial_t + \left(3a_7tx + \frac{1}{2}\right)u\partial_u.$$

Ansatz (17) reduces Eq. (21) to the system of ordinary differential equations

$$\varphi_2' + 2\kappa\varphi_0\varphi_2 - \frac{1}{2}\kappa\varphi_1^2 + 2a_5 + a_7\varphi_1 = 0,$$

$$\varphi_1' + 2a_4 + 2a_7\varphi_0 = 0,$$

$$\varphi_0' = 0.$$

The solutions of the system of reduced equations have the form

$$\varphi_2 = \frac{(c_0a_7 + a_4)^2}{c_0}t^2 - (c_0a_7 + a_4)\left(\frac{c_1}{c_0} + \frac{a_4}{\kappa c_0^2}\right)t$$

$$+ \frac{c_1^2}{4c_0} + \frac{c_1a_4 - 2c_0a_5}{2\kappa c_0^2} + \frac{a_4(c_0a_7 + a_4)}{2\kappa^2 c_0^3} + c_2e^{-2c_0\kappa t},$$

$$\varphi_1 = -2(c_0a_7 + a_4)t + c_1,$$

$$\varphi_0 = c_0, \quad c_0 \neq 0,$$

or

$$\varphi_2 = \frac{2}{3}\kappa a_4^2t^3 + a_4(a_7 - \kappa c_1)t^2 + \left(\frac{1}{2}c_1^2\kappa - c_1a_7 - 2a_5\right)t + c_2,$$

$$\varphi_1 = -2a_4t + c_1,$$

$$\varphi_0 = 0.$$

Substituting $\varphi_0(t)$, $\varphi_1(t)$, and $\varphi_2(t)$ in (17), we arrive at the solutions invariant under the corresponding one-parameter subgroup with generator in the form of a nontrivial linear combination of the operators $\{Y_1, Y_2', Y_2, Y_3, Y_4\}$ if and only if $a_4 = a_5 = 0$ or $a_4 = \varphi_0 = 0$; otherwise, they are not invariant solutions.

Further, we consider the equation

$$u_t = \left(\frac{\kappa}{x^2u}\right)_{xx} + a_1u + a_4x^5u^3 + a_5x^6u^3, \qquad \kappa, a_i \in \mathbb{R}, \quad a_1 \neq 0, \quad \kappa \neq 0. \tag{22}$$

Equation (22) admits the operator $Z_1 = \partial_t$ if a_4 and a_5 are arbitrary real numbers. If $a_4 = 0$, then the equation admits an additional operator

$$Z_2 = x^2\partial_x - 3xu\partial_u.$$

In addition, if $a_4 = a_5 = 0$, then we observe the appearance of two more operators

$$Z_3 = x\partial_x - 2u\partial_u, \qquad Z_4 = e^{2a_1t}(\partial_t + a_1u\partial_u).$$

Ansatz (17) reduces Eq. (22) to the following system of differential equations:

$$\varphi_2' + 2\kappa\varphi_0\varphi_2 - \frac{1}{2}\kappa\varphi_1^2 + 2a_1\varphi_2 + 2a_5 = 0,$$

$$\varphi_1' + 2a_1\varphi_1 + 2a_4 = 0,$$

$$\varphi_0' + 2a_1\varphi_0 = 0.$$

The solutions of this system are given by the formulas

$$\varphi_2 = e^{-2a_1 t}\left(e^{\frac{c_0\kappa}{a_1}e^{-2a_1 t}}\left(c_2 - \frac{\kappa}{4a_1^2}\left(2c_1 a_4 - c_0\left(4a_5 - \frac{a_4^2\kappa}{a_1^2}\right)\right)\Gamma\left(0, \frac{c_0\kappa}{a_1}e^{-2a_1 t}\right)\right)\right.$$
$$\left. + \frac{c_1^2}{4c_0}\right) - \frac{a_5}{a_1} + \frac{\kappa a_4^2}{4a_1^3},$$

$$\varphi_1 = -\frac{a_4}{a_1} + c_1 e^{-2a_1 t},$$

$$\varphi_0 = c_0 e^{-2a_1 t}, \quad c_0 \neq 0,$$

or

$$\varphi_2 = \left(c_2 - \frac{c_1 a_4 \kappa}{a_1}t\right)e^{-2a_1 t} - \frac{\kappa c_1^2}{4a_1}e^{-4a_1 t} - \frac{a_5}{a_1} + \frac{\kappa a_4^2}{4a_1^3},$$

$$\varphi_1 = -\frac{a_4}{a_1} + c_1 e^{-2a_1 t},$$

$$\varphi_0 = 0,$$

where $\Gamma(0, z)$ is the upper incomplete gamma-function. The solution $u(x,t)$ [obtained with the help of (17) and the solutions $\varphi_0,\ \varphi_1,$ and φ_2 of the reduced system of equations] is not invariant for $a_4 \neq 0$ or $a_5 \neq 0.$

Finally, we consider the equation

$$u_t = \left(\frac{\kappa}{x^2 u}\right)_{xx} + a_3 x^4 u^3 + a_4 x^5 u^3 + a_5 x^6 u^3, \qquad \kappa, a_i \in \mathbb{R}, \quad a_3 \neq 0, \quad \kappa \neq 0. \tag{23}$$

For any a_4 and $a_5,$ this equation admits the following symmetry operator:

$$W_1 = \partial_t.$$

If $a_4 = a_5 = 0,$ then the operator admits the additional operator

$$W_2 = x\partial_x - 2u\partial_u.$$

Then ansatz (17) reduces Eq. (23) to the system of ordinary differential equations

$$\varphi_2' + 2\kappa\varphi_0\varphi_2 - \frac{1}{2}\kappa\varphi_1^2 + 2a_5 = 0,$$

$$\varphi_1' + 2a_4 = 0,$$

$$\varphi_0' + 2a_3 = 0,$$

which has the following solution:

$$\varphi_2 = e^{2\kappa(a_3t^2 - c_0t)}\left(\frac{-\sqrt{\pi}\left(\kappa\left(c_1 - \frac{c_0a_4}{a_3}\right)^2 + \left(\frac{a_4^2}{a_3} - 4a_5\right)\right)}{4\sqrt{2a_3\kappa}} e^{\frac{\kappa c_0^2}{2a_3}} \operatorname{erf}\left(\frac{\kappa(-2a_3t + c_0)}{\sqrt{2a_3\kappa}}\right) + c_2\right)$$

$$+ \frac{a_4}{4a_3}\left(-2a_4t + 2c_1 - \frac{c_0a_4}{a_3}\right),$$

$$\varphi_1 = -2a_4t + c_1,$$

$$\varphi_0 = -2a_3t + c_0,$$

where $\operatorname{erf}(z)$ is the error function. To find the solution $u(x,t)$ of Eq. (23), we substitute $\varphi_0,\ \varphi_1,$ and φ_2 in (17). For $a_4 \neq 0$ or $a_5 \neq 0,$ the obtained solution is not invariant under the one-parameter group with the infinitesimal generator $W_1 = \partial_t$ and also under the one-parameter group with the generator $\alpha_1W_1 + \alpha_2W_2,$ where α_1 and α_2 are arbitrary constants in the case where $a_4 = a_5 = 0.$

It is clear that we can also use the Lie–Bäcklund operators of ordinary differential equations of the first order. Thus, we consider a differential equation

$$\psi_x = u\psi^2 + v. \tag{24}$$

It turns out that Eq. (24) admits the following Lie–Bäcklund symmetry operator:

$$Q = \left(-\psi_t + \exp\left(\frac{\psi_x - v}{\psi^2}\right)\cdot\left(\frac{\psi_x - v}{\psi^2}\right)_x\psi^2 + \beta\right)\partial_\psi$$

provided that u and v satisfy the following system of determining equations:

$$u_t = (e^u u_x)_x, \qquad v_t = \left(\frac{e^u u_x v}{u}\right)_x. \tag{25}$$

In this case, it follows from the results presented above that the reduction method can be applied to the nonlinear differential equation

$$\psi_t = \exp\left(\frac{\psi_x - v}{\psi^2}\right)\cdot\left(\frac{\psi_x - v}{\psi^2}\right)_x\psi^2 + \frac{e^u u_x v}{u},$$

where $u(t,x)$ and $v(t,x)$ are the solutions of system (25).

3. Conclusions

In the present paper, we find the solutions of nonlinear evolutionary equations that describe the processes of diffusion in nonlinear inhomogeneous media by the method proposed in [10], which is a generalization of the Svirshchevskii method [9]. It is shown that this method enables us to obtain solutions that are not invariant in the classical Lie sense. To this end, we use the Lie–Bäcklund symmetry operators of ordinary differential equations of the third order. The ansatzes in the form of general solutions of ordinary differential equations reduce the nonlinear diffusion equation to a system of three ordinary differential equations. It turns out that the solutions obtained within the framework of this approach cannot be found by the classical Lie method only in the case where the invariance algebra of the diffusion equation is one-dimensional, two-dimensional, or three-dimensional. At the same time, if the algebra is four-dimensional, then the obtained solutions can be also found by the classical Lie method, as shown in Sec. 2.1. These results agree with the results obtained in [12], where the solutions are obtained by the method of conditional point symmetry.

It is clear that this method can be also used for the construction of some other classes of diffusion equations and their solutions with the help of the Lie–Bäcklund symmetry operators of other ordinary differential equations.

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