DOI 10.1007/s11253-022-02038-w
Ukrainian Mathematical Journal, Vol. 73, No. 12, May, 2022 (Ukrainian Original Vol. 73, No. 12, December, 2021)

TOPOLOGICAL AND GEOMETRIC PROPERTIES OF THE SET OF 1-NONCONVEXITY POINTS OF A WEAKLY 1-CONVEX SET IN THE PLANE

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UDC 514.172

We consider a class of generalized convex sets in the real plane known as weakly 1-convex sets. For a set in the real Euclidean space $\mathbb{R}^n$, $n \geq 2$, we say that a point of the complement of this set to the entire space $\mathbb{R}^n$ is an *m-nonconvexity point of the set*, $m = \overline{1, n-1}$, if any m-dimensional plane passing through this point crosses the indicated set. An open set in the space $\mathbb{R}^n$, $n \geq 2$, is called *weakly m-convex*, $m = \overline{1, n-1}$, if its boundary does not contain any m-nonconvexity points of the set. Moreover, in the class of open weakly 1-convex sets in the plane, we select a subclass of sets with finitely many connected components and a nonempty set of 1-nonconvexity points. We mainly analyze the properties of the set of 1-nonconvexity points for the sets from the indicated subclass. In particular, for any set in this subclass, it is proved that the set of its 1-nonconvexity points is open, that any connected component of the set of its 1-nonconvexity points is the interior of a convex polygon, and that, for any convex polygon, there exists a set from the indicated subclass such that its set of 1-nonconvexity points coincides with the interior of a polygon.

1. Introduction

It is known that a set from the multidimensional real Euclidean space $\mathbb{R}^n$ is called ***convex*** if, together with any two points of this set, it also contains the entire segment connecting these points [3]. Moreover, the intersection of arbitrarily many convex points is also a convex set. This property of convex sets enables us to determine the minimum convex set that contains an arbitrary given set as follows:

Definition 1 [3]. *The convex intersection of all convex sets that contains a given set $X \subset \mathbb{R}^n$ is called the **convex hull** of the set X and denoted by*

$$\operatorname{conv} X = \bigcap_{K \supset X} K, \quad \textit{where } K \textit{ are convex sets.}$$

We consider some other classes of sets with the property that the intersection of arbitrarily many convex points also belongs to this class.

Any m-dimensional affine subspace of the space $\mathbb{R}^n$, $1 \leq m < n$, is called an ***m-plane*** [3].

Definition 2 [2]. *A set $E \subset \mathbb{R}^n$ is called m-convex, $1 \leq m < n$, if, for any point $x \in \mathbb{R}^n \setminus E$, there exists an m-plane L such that $x \in L$ and $L \cap E = \varnothing$.*

The properties of m-convex compact sets in the space $\mathbb{R}^n$ connected with the analysis of their cohomology groups were investigated in [2]. The properties of $(n-1)$-convex sets in the space $\mathbb{R}^n$ were studied in [12] and, under certain additional conditions, in [5, 6]. In particular, a topological classification of $(n-1)$-convex sets of the space $\mathbb{R}^n$, $n \geq 2$, with smooth boundary was obtained in [12], namely: *an arbitrary $(n-1)$-convex set*

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Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 73, No. 12, pp. 1657–1672, December, 2021. Ukrainian DOI: 10.37863/umzh.v73i12.6890. Original article submitted August 25, 2021.

 0041-5995/22/7312–1918

in the space $\mathbb{R}^n$ *with smooth boundary is either convex or consists of at most two unbounded connected components, or is given by the Cartesian product* $E^1 \times \mathbb{R}^{n-1}$, *where* $E^1 \subset \mathbb{R}$.

It is easy to see that the intersection of an arbitrary number of m-convex sets is also an m-convex set [14]. Thus, by analogy with the notion of convex hull, we can introduce the definition of the minimum m-convex set containing an arbitrary given set of the space $\mathbb{R}^n$"

Definition 3 [14]**.** *For fixed* m, *the intersection of all* m*-convex sets containing a given set* $X \subset \mathbb{R}^n$ *is called the* ***m-convex hull*** *of the set* X *and denoted by*

$$\operatorname{conv}_m X = \bigcap_{K \supset X} K, \quad \textit{where } K \textit{ are } m\textit{-convex sets.}$$

This notion naturally appeared in the solution of the problem of shadow posed in 1982 by Khudaiberganov [16, 17]:

To find the minimum number of open (closed) pairwise disjoint balls in the space $\mathbb{R}^n$ *whose centers lie on the sphere* S^{n-1} *(see [4]) that do not contain the center of the sphere and are such that any straight line passing through the center of the sphere crosses at least one of these balls.*

In [9], Zelins'kyi reformulated this problem in terms of 1-convex hull as follows: *What is the minimum number of open (closed) pairwise disjoint balls in the space* $\mathbb{R}^n$ *whose centers are located on a sphere* S^{n-1} *and the radii are smaller than the radius of the sphere guaranteeing that the center of the sphere belongs to the* 1*-convex hull of the family of these balls?*

This problem was completely solved in [9].

Definition 4 [8, 15]**.** *An open set* $E \subset \mathbb{R}^n$ *is called weakly* m*-convex,* $1 \leq m < n$, *if, for every point* $x \in \partial E$, *there exists an* m*-plane* L *such that* $x \in L$ *and* $L \cap E = \varnothing$.

We use the standard notation. For a set $G \subset \mathbb{R}^n$, let $\overline{G}$ be its closure, let $\operatorname{Int} G$ be its interior, and let $\partial G = \overline{G} \setminus \operatorname{Int} G$ be its measure.

Definition 5 [1]**.** *We say that a set* A *is approximated from the outside by a family of open sets* A^k, $k = 1, 2, \ldots,$ *if* $\overline{A^{k+1}}$ *is contained in* A^k *and* $A = \cap_k A^k$.

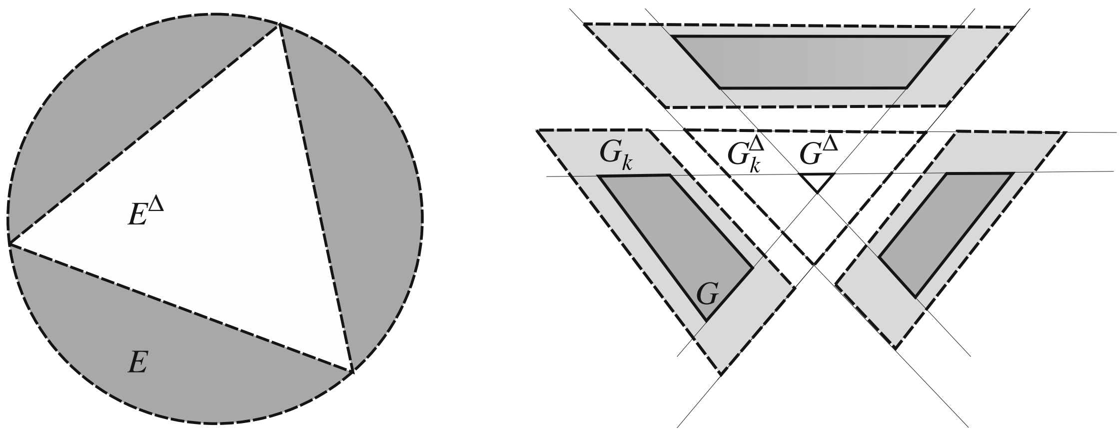
Definition 6 [8, 15]**.** *A closed set from the space* $\mathbb{R}^n$ *is called weakly* m*-convex if it is approximated from the outside by a family of open weakly* m*-convex sets.*

The geometric and topological properties of weakly m-convex sets were investigated in [7]. In particular, in [7], it was established that *if* E_1 *and* E_2 *are weakly* k*-convex and weakly* m*-convex sets, respectively,* $k \leq m$, *then the set* $E_1 \cap E_2$ *is weakly* k*-convex.* The properties of a class of generalized convex sets on Grassmann manifolds closely related to the properties of so-called conjugate sets (see Definition 2 in [8]) were investigated in [8]. This class contains m-convex and weakly m-convex sets from the space $\mathbb{R}^n$.

The maximal connected subset A_i, $i = 1, 2, \ldots,$ of a nonempty set $A \subset \mathbb{R}^n$ is called a *connected component (component)* of the set A. Moreover, $A = \cup_i A_i$.

Lemma 1 [15]**.** *Every component of a weakly* $(n-1)$*-convex open set* $E \subset \mathbb{R}^n$ *is convex.*

Let $\mathbf{C^n_m}$ and $\mathbf{WC^n_m}$ be, respectively, the classes of m-convex and weakly m-convex sets in the space $\mathbb{R}^n$, $n \geq 2$. It is clear that any open set from the class $\mathbf{C^n_m}$ is also a set from the class $\mathbf{WC^n_m}$. The converse statement is not true. It turns out that the class $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$, $n \geq 2$, of open weakly m-convex and not m-convex sets is nonempty for each $m = 1, 2, \ldots, n-1$ [13]. Moreover, open sets from the class $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ are disconnected.



(a) **(b)**

Fig. 1

The theorem presented below establishes the lower bound for the number of connected components of the sets from the class $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$.

Theorem 1 [15]. *An open set from the class* $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ *consists of at least three connected components.*

An open set E from the class $\mathbf{WC^2_1} \setminus \mathbf{C^2_1}$ with three connected components presented in [15] is shown in Fig. 1a.

A different estimate is obtained for the number of connected components of the open sets from the class $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$, $1 \le m < n-1$, $n \ge 3$.

Theorem 2 [13]. *In the space* $\mathbb{R}^n$, $n \ge 3$, *there exist domains from the class* $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$, $1 \le m < n-1$.

In [13], one can find examples of open and closed sets from the class $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ with three or more connected components. It was also proved that, as in the case of open sets, compact sets from the class $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$ consist of at least three connected components. A closed set G from the class $\mathbf{WC^2_1} \setminus \mathbf{C^2_1}$ with three connected components proposed in [13] is depicted in Fig. 1b.

Definition 7 [13]. *A point* $x \in \mathbb{R}^n \setminus E$ *is called an* m*-nonconvexity point of the set* $E \subset \mathbb{R}^n$, $1 \le m < n$, *if all* m*-planes containing* x *intersect with the set* E.

It is clear that Definition 4 is equivalent to the following definition:

Definition 8. *An open set from the space* $\mathbb{R}^n$, $n \ge 2$, *is called weakly* m*-convex,* $1 \le m < n$, *if the boundary of the set does not contain* m*-nonconvexity points of this set.*

By $E^{\triangle}_m$, $1 \le m < n$, we denote the ***set of*** m***-nonconvexity points*** of the set $E \subset \mathbb{R}^n$. Let

$$E^{\triangle} := E^{\triangle}_1, \quad E \subset \mathbb{R}^2.$$

In Fig. 1, the sets $E^{\triangle}$ and $G^{\triangle}$ for the open and closed sets $E, G \in \mathbf{WC^2_1} \setminus \mathbf{C^2_1}$ are the interior and closure of the corresponding triangles, respectively. Moreover, in Sec. 3.1, we prove that the m-convex hull, $m = 1, 2, \ldots, n-1$, of an arbitrary set $E \subset \mathbb{R}^n$ is the union of the set E itself and its set of m-nonconvexity points $E^{\triangle}_m$.

In the present paper, we continue the investigations originated by Zelins'kyi and his colleagues and study (mainly) the properties of open sets $E \subset \mathbb{R}^2$ from the class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ and their 1-convex hulls $\mathrm{conv}_1 E$ on the basis of the results of investigation of the topological and geometric properties of the sets $E^{\triangle}$.

In Sec. 2, we present some well-known definitions and statements from the theory of convex sets and prove some auxiliary statements required to establish our main results.

In Sec. 3.1, we establish some simple properties of the sets $\mathrm{conv}_m E$ and $E_m^{\triangle},\ E \subset \mathbb{R}^n,\ 1 \le m < n.$ In particular, we show that, for a bounded but not m-convex set $E \subset \mathbb{R}^n$, the set $E_m^{\triangle}$ is bounded and the m-convex hull of an arbitrary open set from the class $\mathbf{WC_m^n} \setminus \mathbf{C_m^n},\ 1 \le m < n,$ is disconnected.

In Sec. 3.2, we investigate the properties of the sets $E^{\triangle}$ for open sets $E \in \mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ with finitely many connected components. In particular, it is established that the set $E^{\triangle}$ is open; an arbitrary connected component of the set $E^{\triangle}$ is the interior of a convex polygon; for any convex polygon P, there exist an open set $E \in \mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ such that $E^{\triangle} = \mathrm{Int}\, P$ and also an open set $E_* \in \mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ such that $E_*^{\triangle} \in \mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ and $(E_*^{\triangle})^{\triangle} = \mathrm{Int}\, P.$

2. Additional Assertions

Here and in what follows, unless otherwise stated, points of the space $\mathbb{R}^n$ are denoted by small Latin letters with indices or without them; xy denotes an open segment between the points $x, y \in \mathbb{R}^n$, and the distance between these points is denoted by $|x - y|$. Moreover, an ε-neighborhood of the point x is understood as an open ball centered at x with radius ε and denoted by $U(x, \varepsilon)$; by $\gamma(x)$ we denote the straight line passing through a point x, and, finally, a ray leaving a point of the space is denoted by a small Greek letter with subscript indicating this point, e.g., $\gamma_x,\ \eta_x,\ x \in \mathbb{R}^n$.

We now present additional definitions and statements from the theory of convex sets required to prove the theorems in Sec. 3.

Definition 9. *The set obtained as the union of points of all straight lines passing trough a point $x \in \mathbb{R}^n$ and crossing the set $E \subset \mathbb{R}^n$ is called a* ***(bihollow) supporting cone of the set E for the point x*** *and denoted by $C_x E$. Moreover, it is assumed that $x \notin C_x E$ if E is open and $x \in \mathbb{R}^n \setminus E$, and $x \in C_x E$, otherwise.*

Definition 10. *The set obtained as the union of points of all rays leaving a point $x \in \mathbb{R}^n$ and crossing the set $E \subset \mathbb{R}^n$ is called a* ***(one-hollow) supporting cone of the set E for the point*** *x and denoted by $S_x E$. Moreover, it is assumed that $x \notin S_x E$ if E is open and $x \in \mathbb{R}^n \setminus E$, and $x \in S_x E$, otherwise.*

Lemma 2. *Suppose that $E \subset \mathbb{R}^2$ is an open convex set and a point $x \in \mathbb{R}^2 \setminus E$. Then $S_x E$ is an open angle that does not exceed π.*

Proof. Since the set E is open and connected, it is clear that the one-hollow supporting cone $S_x E$ is also an open connected set and, hence, $S_x E$ is an open angle. Assume that its value is greater than π. Then there exists a straight line $\gamma(x) \subset S_x E$. Let η_x^1 and η_x^2 be complementing rays with origin at a point x on the straight line $\gamma(x)$. By the definition of supporting cone $S_x E$, the points $x^1 \in E \cap \eta_x^1$ and $x^2 \in E \cap \eta_x^2$ exist. Hence, the set E is not convex because $x \in \overline{x^1 x^2}$ and $x \notin E$. We arrive at a contradiction. Thus, our assumption is not true and the angle $S_x E$ cannot be larger than π.

Corollary 1. *Let $E \subset \mathbb{R}^2$ be an open convex set and let $x \in \mathbb{R}^2 \setminus E$. Then the supporting cone $C_x E$ is the union of two open vertical angles that are not larger than π.*

Definition 11 [3]**.** *A point $y \in \partial E$ is called a* ***vertex*** *of the open set $E \subset \mathbb{R}^2$ if its supporting cone $S_y E$ is an angle whose value is smaller than π. A point $y \in \partial E$ of the open set $E \subset \mathbb{R}^2$ is called* ***smooth*** *if its supporting cone $S_y E$ is a straight angle equal to π.*

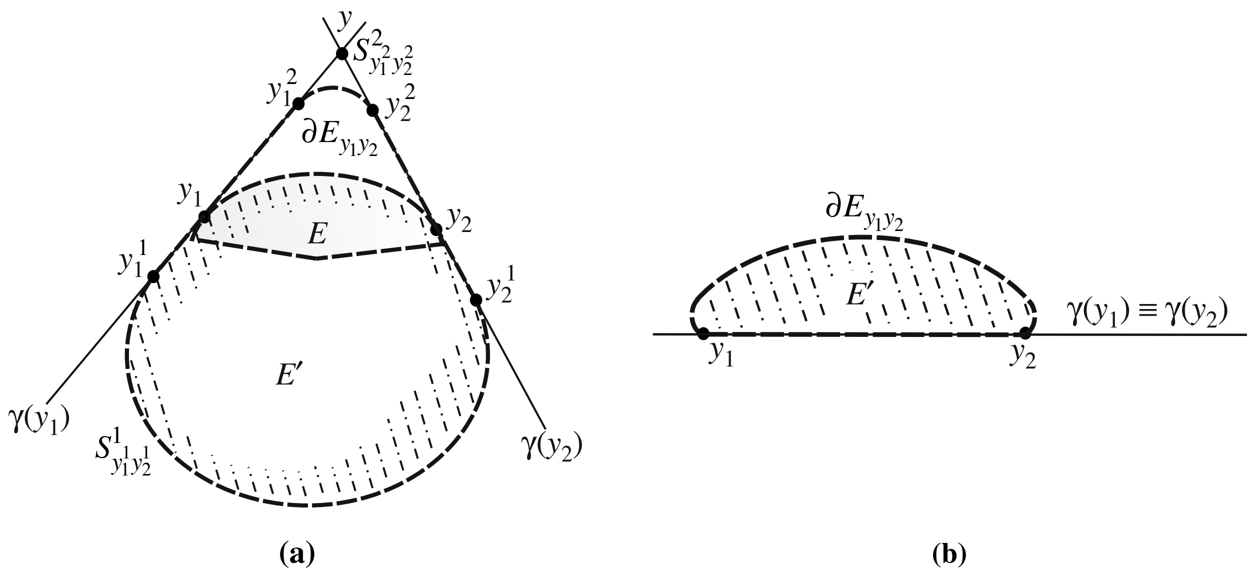


Fig. 2

Any point $y \in \partial E$ of a convex set $E \subset \mathbb{R}^2$ is either a smooth point or a vertex of the set E.

Lemma 3 [3]. *Any convex set $E \subset \mathbb{R}^n$ has an at most countable set of vertices.*

Definition 12 [3]. *A straight line γ is called supporting for the set $E \subset \mathbb{R}^2$ if E is completely contained in the closed half plane $\overline{L}$ bounded by the straight line γ and is not contained in any other closed half plane located in $\overline{L}$.*

Lemma 4. *A straight line γ is supporting for the domain $D \subset \mathbb{R}^2$ if $\gamma \cap \partial D \neq \varnothing$ and $\gamma \cap D = \varnothing$.*

We also say that γ is ***supporting for the domain E at the point*** $y \in \gamma \cap \partial D$.

Lemma 5 [3]. *At least one straight line supporting for E passes through every point of the boundary of a convex set $E \subset \mathbb{R}^2$.*

Lemma 6 [3]. *If a convex set $E \subset \mathbb{R}^n$ has only smooth points of the boundary, then the unique straight line supporting for E and passing through every point $y \in \partial E$ is a continuous (in the natural topology) function of y.*

Lemma 7. *Suppose that $E \subset \mathbb{R}^2$ is an arbitrary convex set and that points $y_1, y_2 \in \partial E$ are such that all points of the closed part of the boundary $\partial E_{y_1y_2} \subset \partial E$ between y_1 and y_2 are smooth. Then there exists a convex set $E' \subset \mathbb{R}^2$ such that all points of its boundary are smooth and, in addition, $\partial E_{y_1y_2} \subset \partial E'$.*

Proof. We now draw unique supporting straight lines $\gamma(y_1)$ and $\gamma(y_2)$ at the points $y_1, y_2 \in \partial E$.

If the straight lines $\gamma(y_1)$ and $\gamma(y_2)$ intersect at a certain point y or are parallel, then we construct circles S^1 and S^2 such that $\gamma(y_l),\ l = 1, 2,$ touch $S^k,\ k = 1, 2,$ at certain points y_l^k. In addition, $y_1 \in y_1^1 y_1^2,\ y_2 \in y_2^1 y_2^2,$ $y \notin y_1^1 y_1^2$, and $y \notin y_2^1 y_2^2$ (see Fig. 2a). Assume that the arcs $S^1_{y_1^1 y_2^1}$ and $S^2_{y_1^2 y_2^2}$ of the circles S^1 and S^2 between the points $y_1^1,\ y_2^1$ and $y_1^2,\ y_2^2$, respectively, are such that the set bounded by the curve $S^1_{y_1^1 y_2^1} \cup y_1^1 y_1^2 \cup S^2_{y_1^2 y_2^2} \cup y_2^1 y_2^2$ is convex. Moreover, all its points are smooth. Further, we consider a set E_1 bounded by the curve $\partial E_{y_1y_2} \cup y_1^1 y_1^2 \cup S^2_{y_1^2 y_2^2} \cup y_2^1 y_2^2$ and a set E_2 bounded by the curve $S^1_{y_1^1 y_2^1} \cup y_1^1 y_1^2 \cup \partial E_{y_1y_2} \cup y_2^1 y_2^2$. One of the sets $(E_1$ or $E_2)$ is just the required set E'.

If the straight lines $\gamma(y_1)$ and $\gamma(y_2)$ coincide, then the set bounded by the curve $\partial E_{y_1 y_2} \cup y_1 y_2$ is just the required set E' (see Fig. 2b).

Corollary 2. *Let $E \subset \mathbb{R}^2$ be an arbitrary convex set and let points $y_1, y_2 \in \partial E$ be such that all points of the closed part of the boundary $\partial E_{y_1 y_2} \subset \partial E$ lying between y_1 and y_2 are smooth. Then the unique supporting straight line for E that passes through every point $y \in \partial E_{y_1 y_2}$ is a continuous function of y.*

Lemma 8 [3]. *If the set $E \subset \mathbb{R}^2$ is convex, $y \in \overline{E}$, and $x \in \operatorname{Int} E$, then*

$$\overline{xy} \setminus \{y\} \subset \operatorname{Int} E.$$

Corollary 3. *Let $E \subset \mathbb{R}^2$ be an open convex set and let points $y_1, y_2 \in \partial E$ be such that the straight line $\gamma(y_1, y_2)$ passing through these points crosses the set E. Then the open segment $y_1 y_2 \subset E$.*

Proof. Let $x \in \gamma(y_1, y_2) \cap E$. Thus, by Lemma 8,

$$\overline{xy_1} \setminus \{y_1\} \subset E \quad \text{and} \quad \overline{xy_2} \setminus \{y_2\} \subset E.$$

Hence,

$$(\overline{xy_1} \setminus \{y_1\}) \cup (\overline{xy_2} \setminus \{y_2\}) = y_1 y_2 \subset E.$$

In the plane $\mathbb{R}^2$, we introduce a polar coordinate system (r, φ) and compactify $\mathbb{R}^2$ by the infinitely remote points (∞, φ), $\varphi \in [0; 2\pi]$. The compactified plane is denoted by $\overline{\mathbb{R}^2}$. A ***broken line*** $a_1, a_2, \ldots, a_k$, where $a_1, a_k \in \overline{\mathbb{R}^2}$, $a_2, \ldots, a_{k-1} \in \mathbb{R}^2$, $k \geq 3$, is defined as a bounded or unbounded curve without self-intersections formed by the segments $a_1 a_2$, $a_2 a_3, \ldots, a_{k-1} a_k$ such that, in addition, for any $l \in \{2, \ldots, k-1\}$, the segments $a_{l-1} a_l$ and $a_l a_{l+1}$ do not lie on the same straight line. If $a_1 \equiv a_k$, $a_1, a_k \in \mathbb{R}^2$, then the broken line is called ***closed***.

Definition 13. *A closed convex set in the plane $\mathbb{R}^2$ with finitely many vertices and closed or unbounded broken boundary is called a **convex polygon**.*

3. Properties of the Sets of 1-Nonconvexity Points from the Class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$

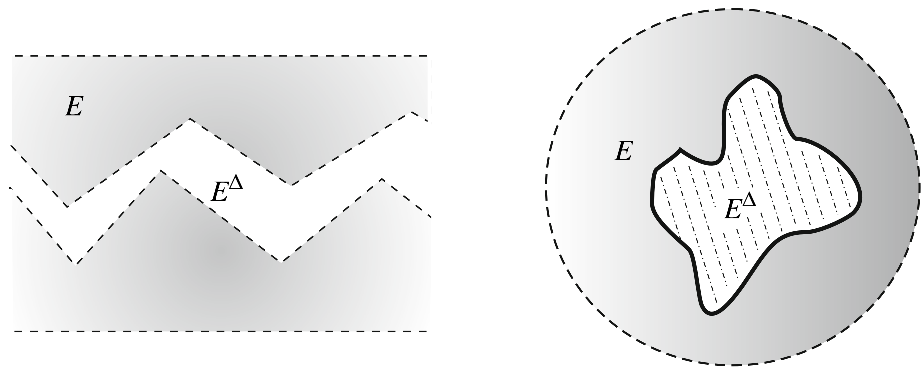
3.1. Some Properties of the Sets from the Class $\mathbf{WC_m^n} \setminus \mathbf{C_m^n}$, $n \geq 2$, $1 \leq m < n$. We first prove the following key fact:

Lemma 9. *For any set $E \subset \mathbb{R}^n$,*

$$\operatorname{conv}_m E = E \cup E_m^{\triangle}, \quad m = 1, 2, \ldots, n-1. \tag{1}$$

Proof. Let $x \in \operatorname{conv}_m E$. Thus, by Definition 3, the point x belongs to all m-convex sets containing E. Since, for any point $y \in \mathbb{R}^2 \setminus (E \cup E_m^{\triangle})$, there exists an m-plane that passes through y and does not cross the set E, this m-plane does not cross the set $E \cup E_m^{\triangle}$ and, hence, $E \cup E_m^{\triangle}$ is m-convex. Thus, we get $x \in E \cup E_m^{\triangle}$.

Let $x \in E \cup E_m^{\triangle}$. An arbitrary m-convex set K that contains E also contains the set $E_m^{\triangle}$ because, otherwise, an arbitrary m-plane passing through $y \in E_m^{\triangle} \setminus K$ intersects the set E and, hence, also the set $K \supset E$. This contradicts the fact that the set K is m-convex. Therefore, $x \in K \supset E \cup E_m^{\triangle}$ and, thus, x belongs to the intersection of all m-convex sets K that contain E, i.e., $x \in \operatorname{conv}_m E$.



(a) **(b)**

Fig. 3

Theorem 3. *Suppose that the set $E \subset \mathbb{R}^n$ is bounded and not m-convex, $1 \le m < n$. Then the set $E^\triangle_m$ is bounded.*

Proof. Since the set E is bounded, it is contained in a certain ball $B \subset \mathbb{R}^n$, which is m-convex, $1 \le m < n$. Thus, according to Definition 3 and relation (1), we have

$$B \supset \mathrm{conv}_m E \supset E^\triangle_m.$$

In Fig. 3a, we present an unbounded non-1-convex set $E \subset \mathbb{R}^2$ such that the set $E^\triangle$ is also unbounded.

The sets from the class $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$ are of especial interest. In particular, the m-convex hull of the open set E from the class $\mathbf{WC^n_m} \setminus \mathbf{C^n_m}$, $1 \le m < n$, is disconnected unlike the convex hull, which is connected for any set $E \subset \mathbb{R}^n$. Indeed, since $E \in \mathbf{WC^n_m}$ and E is open, we get $E^\triangle_m \subset \mathbb{R}^n \setminus \overline{E}$ but this immediately implies that the set $E \cup E^\triangle_m$ is disconnected.

Further, we consider only the set $E^\triangle$ of 1-nonconvexity points of the open set $E \subset \mathbb{R}^2$.

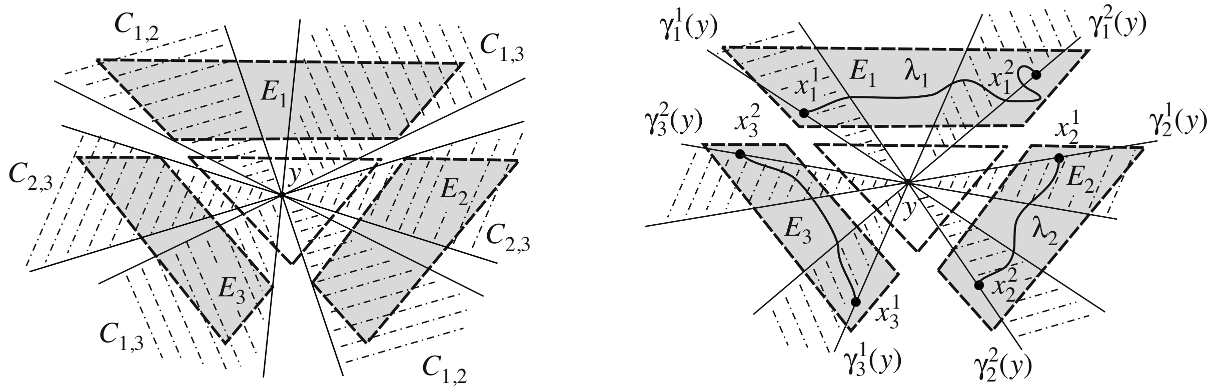
3.2. Open Sets from the Class $\mathbf{WC^2_1} \setminus \mathbf{C^2_1}$. In Fig. 3b, we present a 1-nonconvex open set $E \subset \mathbb{R}^2$ for which the set $E^\triangle$ is closed. A somewhat different result can be formulated for the open sets from the class $\mathbf{WC^2_1} \setminus \mathbf{C^2_1}$ with finitely many connected components:

Theorem 4. *Suppose that the open set $E \subset \mathbb{R}^2$ belongs to the class $\mathbf{WC^2_1} \setminus \mathbf{C^2_1}$ and consists of finitely many components. Then $E^\triangle$ is open.*

Proof. Let y be an arbitrary point of the set $E^\triangle$. We show that this is an interior point.

Since E is weakly 1-convex, we have $y \notin \partial E$. Thus, there exists a number $\varepsilon_1 > 0$ such that $U(y, \varepsilon_1) \subset (\mathbb{R}^n \setminus \overline{E})$.

Let E_i, $i = 1, \ldots, k$, be components of the set E and let $C_{i,j} = C_y E_i \cap C_y E_j$, $i, j = 1, \ldots, k$ (see Fig. 4a). Since y is a 1-nonconvexity point of the set E, for any fixed index $i \in \{1, \ldots, k\}$, there exist indices $j(i) \in \{1, \ldots, k\}$ such that $C_{i,j(i)} \ne \varnothing$. Thus, $C_{i,j(i)}$ is also a union of vertical angles. Since the supporting cones $C_y E_i$, $i = 1, \ldots, k$, are open, we decrease the values of their angles so that $C_{i,j(i)}$ remain nonempty. By $\widetilde{C}_y E_i$, $i = 1, \ldots, k$, we denote the decreased cones. Then $\widetilde{C}_y E_i \subset C_y E_i$. By virtue of Lemma 1, the components E_i, $i = 1, \ldots, k$, are convex. Hence, according to Corollary 1, the boundary of each cone $C_y E_i$ consists of



(a) **(b)**

Fig. 4

one or two straight lines crossing at the point y. Then the boundary $\widetilde{C}_y E_i$ is formed by two straight lines crossing at the point y. We denote these lines by $\gamma_i^1(y)$ and $\gamma_i^2(y)$ (see Fig. 4b). Moreover, $\gamma_i^1(y), \gamma_i^2(y) \subset C_y E_i$. Hence, by Definition 9,

$$\gamma_i^1(y) \cap E_i \neq \varnothing \quad \text{and} \quad \gamma_i^2(y) \cap E_i \neq \varnothing.$$

Let

$$x_i^1 \in \gamma_i^1(y) \cap E_i, \quad x_i^2 \in \gamma_i^2(y) \cap E_i, \quad i = 1, \dots, k.$$

We construct the curves $\lambda_i \subset E_i,\ i = \overline{1,k}$, connecting the points x_i^1 and x_i^2. Then, for any straight line $\gamma(y)$ passing through the point y, there exists an index $i \in \{1, \dots, k\}$ such that $\gamma(y) \cap \lambda_i \neq \varnothing$.

Consider a function

$$d_j(x) = \inf_{x^0 \in \partial E_j} |x - x^0|, \quad x \in E_j, \quad j = \overline{1,k}.$$

It is continuous in the domain $E_j,\ j = \overline{1,k}$. Thus, its restriction to the compact set $\lambda_j,\ j = \overline{1,k}$, attains its minimum $d_j > 0$ in this compact set, i.e.,

$$d_j = \min_{x \in \lambda_j} d_j(x), \quad j = \overline{1,k}.$$

Since E has finitely many components, there exists

$$d = \min_{j=\overline{1,k}} d_j > 0.$$

Hence, for any point $x \in \lambda_j,\ j = \overline{1,k}$, its neighborhood $U(x,d) \subset E$. Let $\varepsilon = \min\{\varepsilon_1, d\}$. We consider a neighborhood $U(y,\varepsilon)$. Let $z \in U(y,\varepsilon)$ and let $\gamma(z)$ be an arbitrary straight line passing though the point z (see Fig. 5). We draw a straight line $\gamma(y)$ parallel to the straight line $\gamma(z)$. This line crosses a curve $\lambda_q,\ q \in \{1, \dots, k\}$ at a certain point $x' \in \gamma(y) \cap \lambda_q$. Since

$$U(x',\varepsilon) \subseteq U(x',d) \subset E \qquad \text{and} \qquad \gamma(z) \cap U(x',\varepsilon) \neq \varnothing,$$

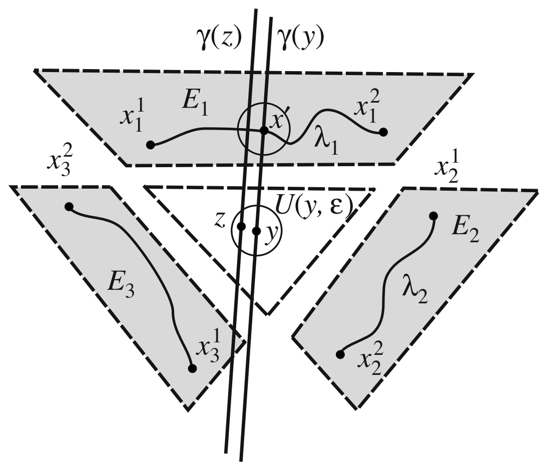


Fig. 5

we have $\gamma(z) \cap E \neq \varnothing$. Since the point z is chosen arbitrarily, this means that all points of the neighborhood $U(y, \varepsilon)$ are 1-nonconvexity points of the set E. Thus, y is an internal point of the set $E^{\triangle}$.

Corollary 4. *Assume that an open set $E \subset \mathbb{R}^2$ belongs to the class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ and consists of finitely many components. Then any connected component of the set $E^{\triangle}$ is convex.*

Proof. It is clear that any connected component of the set $E^{\triangle}$ is a connected component of the set $\mathrm{conv}_1 E$. Since $\mathrm{conv}_1 E$ is open and, by Definition 3, 1-convex, it is also weakly 1-convex. Hence, by Lemma 1, all components of $\mathrm{conv}_1 E$ and, therefore, the components of $E^{\triangle}$, are convex.

The corollary is proved.

Since, by Theorem 4, the points of $\partial E^{\triangle}$ do not belong to the set $E^{\triangle}$, for any point $y \in \partial E^{\triangle}$, there exists a straight line $\gamma(y)$ that does not intersect the set E. In addition, by virtue of Lemma 3, in view of the convexity of components of the set $E^{\triangle}$, for each component of this set, there exist points of the boundary of this component such that the analyzed component possesses a unique supporting straight line at each point of this kind. Hence, the following lemma is true:

Lemma 10. *Suppose that an open set $E \subset \mathbb{R}^2$ with finitely many components belongs to the class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ and that $\widetilde{E^{\triangle}}$ is an arbitrary connected component of the set $E^{\triangle}$. The following assertions are true:*

(i) if a straight line $\gamma(y)$, $y \in \partial \widetilde{E^{\triangle}}$, does not cross the set E, then this straight line is supporting for $\widetilde{E^{\triangle}}$ at the point y;

(ii) if $\gamma(y)$ is a unique supporting straight line for the component $\widetilde{E^{\triangle}}$ at the point $y \in \partial \widetilde{E^{\triangle}}$, then

$$\gamma(y) \cap E = \varnothing.$$

Proof. (i) Since the straight line $\gamma(y)$ does not cross the set E, it does not cross the set $E^{\triangle}$ and, hence, does not intersect $\widetilde{E^{\triangle}}$, which is open by Theorem 4. Thus, by Lemma 4, $\gamma(y)$ is supporting for the set $\widetilde{E^{\triangle}}$ at the point y.

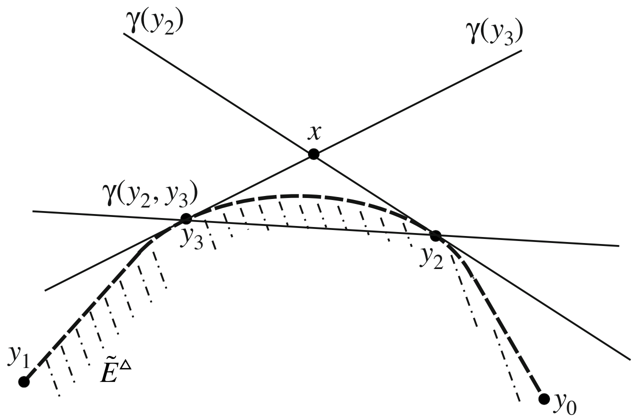


Fig. 6

(ii) By Theorem 4, the component $\widetilde{E^{\triangle}}$ is open. Hence, $\gamma(y)\cap\widetilde{E^{\triangle}}=\varnothing$. Moreover, by the condition, $\gamma(y)$ is the unique straight line that passes through the point y and does not cross $\widetilde{E^{\triangle}}$. Therefore, this is the unique straight line whose intersection with E can be empty. Since $\widetilde{E^{\triangle}}$ is open, we also conclude that, at the point y, there is a straight line that does not cross the set E. Therefore, $\gamma(y)\cap E=\varnothing$.

Theorem 5. *Suppose that the open set $E\subset\mathbb{R}^2$ with finitely many components belongs to the class $\mathbf{WC_1^2}\setminus\mathbf{C_1^2}$ and that $\widetilde{E^{\triangle}}$ is an arbitrary connected component of the set $E^{\triangle}$. Assume that the connected subset*

$$\partial\widetilde{E^{\triangle}}_*\subseteq\partial\widetilde{E^{\triangle}}$$

is such that all points of $\partial\widetilde{E^{\triangle}}_*$ are smooth. Then $\partial\widetilde{E^{\triangle}}_*\neq\partial\widetilde{E^{\triangle}}$ and $\partial\widetilde{E^{\triangle}}_*$ is either a segment or a ray.

Proof. First, we note that the subset $\partial\widetilde{E^{\triangle}}_*$ cannot be a straight line because, otherwise, the straight lines passing through the points of $\widetilde{E^{\triangle}}$ and parallel to the straight line $\partial\widetilde{E^{\triangle}}_*$ are contained in $\widetilde{E^{\triangle}}$ and do not cross the set E, which contradicts Definition 7.

We prove the theorem by contradiction. Assume that either the component $\widetilde{E^{\triangle}}$ is such that all points $\partial\widetilde{E^{\triangle}}$ are smooth or there exist points $y_0,\ y_1\in\partial\widetilde{E^{\triangle}}$ such that a part of the boundary located between these points $\partial\widetilde{E^{\triangle}}_*\subset\partial\widetilde{E^{\triangle}}$ is neither a segment, nor a ray. Then there exist points $y_2,y_3\in\partial\widetilde{E^{\triangle}}_*,\ y_2\neq y_3$, such that a part of $\partial\widetilde{E^{\triangle}}_*$ located between the points y_2 and y_3 and denoted by $\partial\widetilde{E^{\triangle}}_{y_2y_3}$ does not contain segments because, otherwise, $\partial\widetilde{E^{\triangle}}_*$ is either a straight line, or a segment, or a ray, which contradicts the assertion proved above and our assumption, or a broken line, which contradicts the statement that all points of $\partial\widetilde{E^{\triangle}}_*$ are smooth.

Let $\gamma(y)$ be the unique supporting straight line for the set $\widetilde{E^{\triangle}}$ at every point $y\in\partial\widetilde{E^{\triangle}}_{y_2y_3}$. Then, by Lemma 10, $\gamma(y),\ y\in\partial\widetilde{E^{\triangle}}_{y_2y_3}$, does not cross the set E. Since, by Corollary 2, the straight line $\gamma(y)$ continuously depends on the point $y\in\partial\widetilde{E^{\triangle}}_{y_2y_3}$, without loss of generality, we can assume that the points y_2 and y_3 are sufficiently close in order to guarantee the intersection of the straight lines $\gamma(y_2)$ and $\gamma(y_3)$ at a certain point x. Then the straight lines $\gamma(y_2)$ and $\gamma(y_3)$ form the boundary of the cone $C_x\widetilde{E^{\triangle}}$. Since the straight line $\gamma(y)$ continuously depends on the point $y\in\partial\widetilde{E^{\triangle}}_{y_2y_3}$ and $\gamma(y)$ does not cross the set E, we conclude that

$$E\cap(\mathbb{R}^2\setminus C_x\widetilde{E^{\triangle}})=\varnothing.$$

We draw a straight line $\gamma(y_2, y_3)$ through the points y_2 and y_3 (Fig. 6). Then

$$\gamma(y_2, y_3) = \gamma_{y_2} \cup y_2y_3 \cup \gamma_{y_3},$$

where $\gamma_{y_2}, \gamma_{y_3} \subset \gamma(y_2, y_3)$ are closed rays with origins at the points y_2 and y_3, respectively, such that

$$\gamma_{y_2} \cap \gamma_{y_3} = \varnothing.$$

Moreover, $\gamma_{y_2}, \gamma_{y_3} \subset (\mathbb{R}^2 \setminus C_x\widetilde{E^{\triangle}})$. In addition, since, by Corollary 4, the set $\widetilde{E^{\triangle}}$ is convex, by virtue of Corollary 3, the open segment

$$y_2y_3 \subset \widetilde{E^{\triangle}} \subset \mathbb{R}^2 \setminus \overline{E}.$$

Hence, $\gamma(y_2, y_3) \cap E = \varnothing$. On the other hand, $\gamma(y_2, y_3) \cap E \neq \varnothing$ because $\gamma(y_2, y_3)$ passes through points of the set $\widetilde{E^{\triangle}}$.

Thus, we arrive at a contradiction and, hence, our assumption is not true. Hence, the bounded part of $\partial\widetilde{E^{\triangle}}$ all point of which are smooth is a segment and the unbounded part is a ray.

Corollary 5. *Assume that an open set $E \subset \mathbb{R}^2$ with finitely many components belongs to the class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ and that $\widetilde{E^{\triangle}}$ is an arbitrary connected component of the set $E^{\triangle}$. Then:*

(i) $\partial\widetilde{E^{\triangle}}$ contains at least one vertex $\widetilde{E^{\triangle}}$;

(ii) a part of the boundary of the set $\widetilde{E^{\triangle}}$ lying between any two its neighboring vertices is a segment;

(iii) the unbounded part of the boundary of the set $\widetilde{E^{\triangle}}$ with one vertex is a ray.

Lemma 11. *Suppose that an open set $E \subset \mathbb{R}^2$ with finitely many connected components belongs to the class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ and that $a \in \partial\widetilde{E^{\triangle}}$ is a vertex of the connected component $\widetilde{E^{\triangle}}$ of the set $E^{\triangle}$. Then the set $\mathbb{R}^2 \setminus C_a\widetilde{E^{\triangle}}$ contains at least one connected component of the set E.*

Proof. By Corollary 5, $\partial\widetilde{E^{\triangle}}$ is a broken line. Consider straight lines $\gamma'(a)$ and $\gamma''(a)$ that contain neighboring sides of the broken line $\partial\widetilde{E^{\triangle}}$ with a common vertex a. Let $b' \in \gamma'(a)$ and $b'' \in \gamma''(a)$ be points of these two sides different from the vertices. Then the straight lines $\gamma'(a)$ and $\gamma''(a)$ are the unique supporting straight lines of the set $\widetilde{E^{\triangle}}$ at the points b' and b'', respectively, that do not cross the set E (by Lemma 10). The straight lines $\gamma'(a)$ and $\gamma''(a)$ also form the boundary of the set $C_a\widetilde{E^{\triangle}}$ (Fig. 7a).

We now draw a straight line $\gamma(b', b'')$ through the points b' and b''. Then

$$\gamma(b', b'') = \gamma_{b'} \cup b'b'' \cup \gamma_{b''},$$

where $b'b''$ is an open segment contained, by virtue of Corollary 3, in $\widetilde{E^{\triangle}} \subset \mathbb{R}^2 \setminus \overline{E}$ and $\gamma_{b'}$ and $\gamma_{b''}$ are closed rays on the straight line $\gamma(b', b'')$ with origins at the points b' and b'', respectively, such that $\gamma_{b'} \cap \gamma_{b''} = \varnothing$ and, hence, such that $\gamma_{b'}, \gamma_{b''} \subset \left(\mathbb{R}^2 \setminus C_a\widetilde{E^{\triangle}}\right)$. Since $\gamma(b', b'')$ passes through points of the set $\widetilde{E^{\triangle}}$, by the definition of the set $E^{\triangle}$, we get

$$\gamma(b', b'') \cap E \neq \varnothing.$$

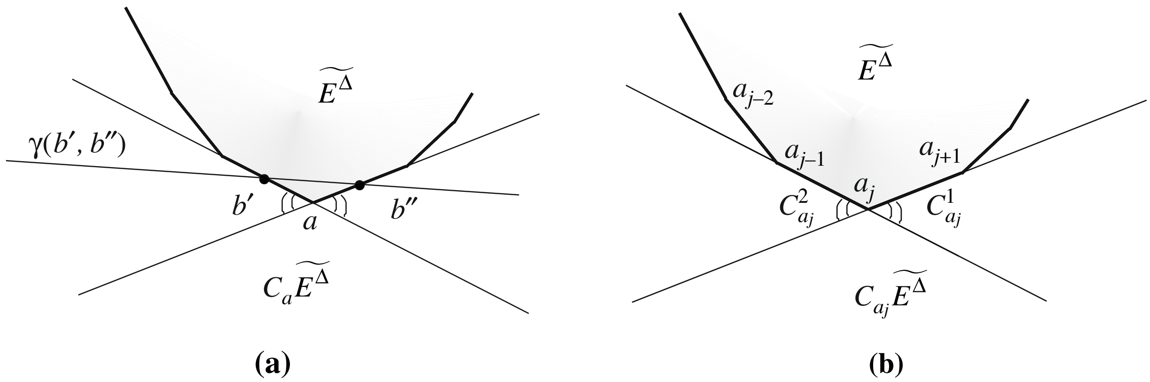


Fig. 7

Thus, $\left(\mathbb{R}^2 \setminus C_a\widetilde{E^\triangle}\right) \cap E \neq \varnothing$. Since

$$\partial\left(\mathbb{R}^2 \setminus C_a\widetilde{E^\triangle}\right) = \gamma'(a) \cup \gamma''(a),$$

we get

$$\partial\left(\mathbb{R}^2 \setminus C_a\widetilde{E^\triangle}\right) \cap E = \varnothing.$$

Therefore, the set $\mathbb{R}^2 \setminus C_a\widetilde{E^\triangle}$ contains at least one connected component of the set E.

Remark 1. In particular, Lemma 11 shows that the condition of uniqueness of supporting line in Assertion (ii) of Lemma 10 is essential. Indeed, all straight lines $\gamma(a) \subset \mathbb{R}^2 \setminus C_a\widetilde{E^\triangle}$ are supporting for $\widetilde{E^\triangle}$ at the point a. Therefore, in view of the fact that E is open, infinitely many straight lines $\gamma(a) \subset \mathbb{R}^2 \setminus C_a\widetilde{E^\triangle}$ cross the set E.

Theorem 6. *Suppose that an open set $E \subset \mathbb{R}^2$ with finitely many components s belongs to the class $\mathbf{WC^2_1} \setminus \mathbf{C^2_1}$. Then any component of the set $E^\triangle$ has finitely many vertices p and, moreover, $p \leq 2s$.*

Proof. Let $\widetilde{E^\triangle}$ be an arbitrary connected component of the set $E^\triangle$. By Lemma 3 and condition (i) of Corollary 5, p is either finite or countable. We enumerate the vertices of $\widetilde{E^\triangle}$ by traversing the boundary of the set $\widetilde{E^\triangle}$ anticlockwise and denote them by $a_1, a_2, \ldots, a_p$. Consider the set $\mathbb{R}^2 \setminus C_{a_j}\widetilde{E^\triangle}$, $j = 1, 2, \ldots, p$, obtained as the union of two closed vertical angles. The angles located to the right and to the left of the point a_j relative to the chosen direction of traversing are denoted by $C^1_{a_j}$ and $C^2_{a_j}$, respectively. Then

$$\mathbb{R}^2 \setminus C_{a_j}\widetilde{E^\triangle} = C^1_{a_j} \cup C^2_{a_j}, \qquad j = 1, 2, \ldots, p$$

(see Fig. 7b).

We now show that any two open angles from the family of curves $\operatorname{Int} C^1_{a_j}$, $j = \overline{1, p}$, do not intersect. Consider an angle $C^1_{a_1}$ (Fig. 8a). Its boundary splits the plane into two connected components: $\operatorname{Int} C^1_{a_1}$ and $\mathbb{R}^2 \setminus C^1_{a_1}$. Moreover, $\widetilde{E^\triangle} \subset \mathbb{R}^2 \setminus C^1_{a_1}$. Further, we consider the angle $C^1_{a_2}$. One its side belongs to the side of the angle $C^1_{a_1}$ and the second side contains the segment $a_2a_3 \subset \partial\widetilde{E^\triangle}$. Therefore, $\operatorname{Int} C^1_{a_2} \subset \mathbb{R}^2 \setminus C^1_{a_1}$ and, hence,

$$\operatorname{Int} C^1_{a_1} \cap \operatorname{Int} C^1_{a_2} = \varnothing.$$

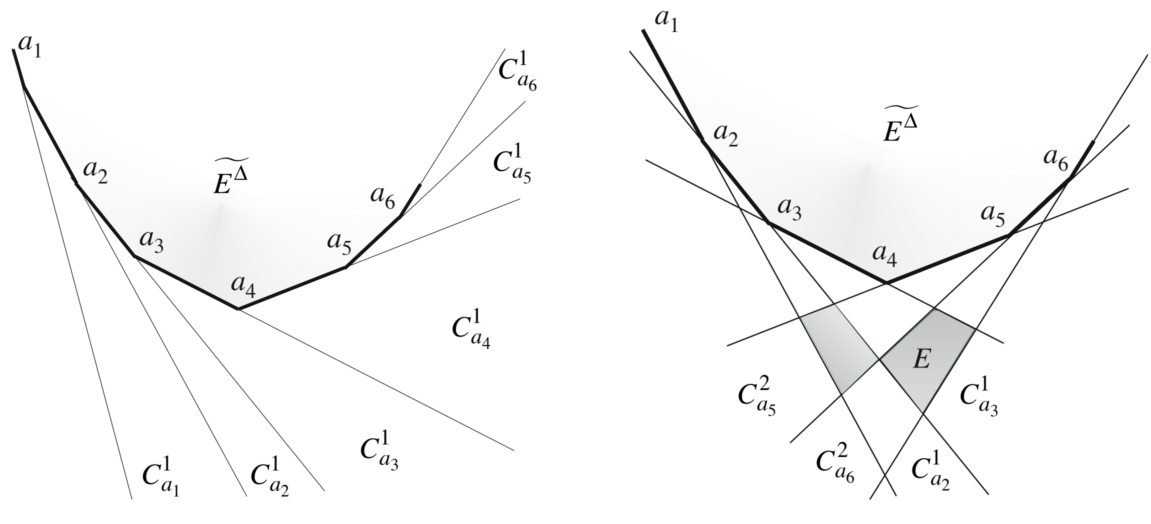


Fig. 8

Further, we consider the set $C^1_{a_1a_2} := C^1_{a_1} \cup C^1_{a_2}$. Its boundary $\partial C^1_{a_1a_2}$ splits the plane into two components: $\operatorname{Int} C^1_{a_1a_2}$ and $\mathbb{R}^2 \setminus C^1_{a_1a_2}$. Moreover, $\widetilde{E^\triangle} \subset \mathbb{R}^2 \setminus C^1_{a_1a_2}$. Finally, consider the angle $C^1_{a_3}$. One of its sides belongs to $\partial C^2_{a_1a_2}$ and the second side contains the segment $a_3a_4 \subset \partial\widetilde{E^\triangle}$. Therefore, $\operatorname{Int} C^1_{a_3} \subset \mathbb{R}^2 \setminus C^1_{a_1a_2}$ and, hence,

$$\operatorname{Int} C^1_{a_1a_2} \cap \operatorname{Int} C^1_{a_3} = \varnothing.$$

Reasoning similarly for the remaining angles $C^1_{a_j},\ j = \overline{3,p}$, we obtain

$$\operatorname{Int} C^1_{a_1a_2\ldots a_{k-1}} \cap \operatorname{Int} C^1_{a_k} = \varnothing, \quad k = 2,\ldots,p, \tag{2}$$

where

$$C^1_{a_1a_2\ldots a_{k-1}} := \bigcup_{j=1}^{k-1} C^1_{a_j}.$$

This implies that

$$\operatorname{Int} C^1_{a_i} \cap \operatorname{Int} C^1_{a_j} = \varnothing, \quad i,j = \overline{1,p}, \quad i \neq j.$$

Indeed, if we assume that there exist indices $l,q \in \{1,\ldots,p\}$ (here, without loss of generality, we set $l < q$) such that $\operatorname{Int} C^1_{a_l} \cap \operatorname{Int} C^1_{a_q} \neq \varnothing$, then we get

$$\operatorname{Int} C^1_{a_l} \subset \operatorname{Int} C^1_{a_1a_2\ldots a_{q-1}} \quad \text{and} \quad \operatorname{Int} C^1_{a_1a_2\ldots a_{q-1}} \cap \operatorname{Int} C^1_{a_q} \neq \varnothing,$$

which contradicts condition (2).

Thus, any two open angles from the family of angles $\operatorname{Int} C^2_{a_j},\ j = \overline{1,p}$, do not intersect as vertical for the corresponding angles $\operatorname{Int} C^1_{a_j},\ j = \overline{1,p}$. However, these families may intersect between themselves. Moreover, it is clear that any three (or more) angles $\operatorname{Int} C^k_{a_j},\ k = 1, 2,\ j = 1, 2, \ldots, p,$ do not simultaneously intersect. Hence, each component $E_i,\ k = \overline{1, s},$ can be simultaneously contained in at most two sets $\mathbb{R}^2 \setminus C_{a_j}\widetilde{E^\triangle},\ j = \overline{1, p}$ (see Fig. 8b).

By virtue of Lemma 11, every set $\mathbb{R}^2 \setminus C_{a_j}\widetilde{E^\triangle},\ j = \overline{1, p},$ contains a component of the set E. Hence, there exist a mapping (not necessarily unambiguous) of the entire set of vertices $A := \{a_1, a_2, \ldots, a_p\}$ into the set of components $\{E_1, E_2, \ldots, E_s\}$. Moreover, according to the results established above, every component E_i, $k = \overline{1, s}$, is simultaneously associated with at most two vertices from A. Therefore, $p \le 2s$. Since, by the condition of the theorem, the number of components s of the set E is finite, p is also finite.

Theorem 7. *Suppose that a bounded open set $E \subset \mathbb{R}^2$ with finitely many components belongs to the class $\mathbf{WC^2_1} \setminus \mathbf{C^2_1}$. Then every connected component $\widetilde{E^\triangle}$ of the set $E^\triangle$ is the interior of a convex polygon.*

Proof. By Theorem 4, Corollary 4, and Theorem 6, $\widetilde{E^\triangle}$ is an open convex set with finitely many vertices. By Corollary 5, $\partial\widetilde{E^\triangle}$ is a broken line. Thus, by Definition 13, the set $\widetilde{E^\triangle} \cup \partial\widetilde{E^\triangle}$ is a convex polygon. It is clear that

$$\operatorname{Int}(\widetilde{E^\triangle} \cup \partial\widetilde{E^\triangle}) = \widetilde{E^\triangle}.$$

We now show that the converse statement is also true.

Proposition 1. *For any convex polygon P, there is an open set E from the class $\mathbf{WC^2_1} \setminus \mathbf{C^2_1}$ such that $E^\triangle = \operatorname{Int} P$.*

We first prove several auxiliary assertions:

Lemma 12. *Suppose that an open set $E \in \mathbf{WC^2_1}$ and that $\{\gamma_1, \gamma_2, \ldots, \gamma_s\}$ is a finite set of arbitrary straight lines. Then*

$$E \setminus \bigcup_j \gamma_j \in \mathbf{WC^2_1}.$$

Proof. Without loss of generality, we assume that $E \cap \gamma_j \ne \varnothing,\ j = \overline{1, s}$. Let

$$x \in \partial\Big(E \setminus \bigcup_j \gamma_j\Big).$$

Then $x \in \partial E$ or $x \in \bigcup_j \gamma_j$. If $x \in \partial E$, then there exists a straight line $\gamma(x)$ that does not cross the set E and, hence, does not cross the set $\Big(E \setminus \bigcup_j \gamma_j\Big) \subset E$. If $x \in \bigcup_j \gamma_j$, then $x \in \gamma_q,\ q \in \{1, \ldots, s\}$. Thus, the straight line that does not cross $\Big(E \setminus \bigcup_j \gamma_j\Big)$ is γ_q. Hence, the set $E \setminus \bigcup_j \gamma_j \in \mathbf{WC^2_1}$.

Lemma 13. *Suppose that an open set $E \notin \mathbf{C^2_1}$ and $\{\gamma_1, \gamma_2, \ldots, \gamma_s\}$ is a finite set of arbitrary straight lines. Then*

$$\Big(E \setminus \bigcup_j \gamma_j\Big)^\triangle = E^\triangle \setminus \bigcup_j \gamma_j.$$

Proof. If $E^{\triangle} \subset \bigcup\limits_j \gamma_j$, then, on the one hand, $E^{\triangle} \setminus \bigcup\limits_j \gamma_j = \varnothing$ but, on the other hand,

$$\Big(E \setminus \bigcup_j \gamma_j\Big)^{\triangle} \subset \mathbb{R}^2 \setminus \Big(E \setminus \bigcup_j \gamma_j\Big) = (\mathbb{R}^2 \setminus E) \cup \Big(\bigcup_j \gamma_j\Big).$$

A straight line that does not cross the set E (and, hence, does not cross the set $E \setminus \bigcup\limits_j \gamma_j \subset E$) passes through every point $x \in (\mathbb{R}^2 \setminus E) \setminus \bigcup\limits_j \gamma_j$. If $x \in \bigcup\limits_j \gamma_j$, then $x \in \gamma_q$ for some $q \in \{1, 2, \ldots, s\}$ and $\gamma_q \cap \Big(E \setminus \bigcup\limits_j \gamma_j\Big) = \varnothing$, i.e.,

$$E \setminus \bigcup_j \gamma_j \in \mathbf{C_1^2} \qquad \text{and} \qquad \Big(E \setminus \bigcup_j \gamma_j\Big)^{\triangle} = E^{\triangle} \setminus \bigcup_j \gamma_j = \varnothing.$$

Let $x \in E^{\triangle} \setminus \bigcup\limits_j \gamma_j \neq \varnothing$. Since $x \in E^{\triangle}$, the straight line $\gamma(x)$ crosses E. Further, since $x \notin \bigcup\limits_j \gamma_j$, the straight line $\gamma(x)$ does not coincide with any straight line γ_j, $j = \overline{1, s}$. Thus, for any $j = \overline{1, s}$, either $\gamma(x) \cap \gamma_j = \varnothing$ or the straight lines $\gamma(x)$ and γ_j intersect at one point. Since E is open, the set $\gamma(x) \cap E$ is also open with respect to its affine hull. Hence, the set

$$(\gamma(x) \cap E) \setminus \bigcup_j (\gamma(x) \cap \gamma_j) = (\gamma(x) \cap E) \setminus \Big(\gamma(x) \cap \bigcup_j (\gamma_j)\Big) = \gamma(x) \cap \Big(E \setminus \bigcup_j \gamma_j\Big)$$

is also open with respect to its affine hull and, therefore, nonempty, i.e., $x \in \Big(E \setminus \bigcup\limits_j \gamma_j\Big)^{\triangle}$.

Let $x \in \Big(E \setminus \bigcup\limits_j \gamma_j\Big)^{\triangle}$. Then any straight line $\gamma(x)$ crosses the set $E \setminus \bigcup\limits_j \gamma_j$. Since $E \setminus \bigcup\limits_j \gamma_j \subset E$, we get $\gamma(x) \cap E \neq \varnothing$. Therefore, $x \in E^{\triangle}$. Moreover, $x \notin \bigcup\limits_j \gamma_j$ because, otherwise, there exists $q \in \{1, \ldots, s\}$ such that $x \in \gamma_q$ and $\gamma_q \cap \Big(E \setminus \bigcup\limits_j \gamma_j\Big) = \varnothing$, which contradicts the choice of the point x. Thus, $x \in E^{\triangle} \setminus \bigcup\limits_j \gamma_j$.

Corollary 6. *Assume that an open set $E \subset \mathbb{R}^2$ belongs to the class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ and consists of finitely many components and that $\{\gamma_1, \gamma_2, \ldots, \gamma_s\}$ is a finite set of arbitrary straight lines. Then*

$$E \setminus \bigcup_j \gamma_j \in \mathbf{WC_1^2} \setminus \mathbf{C_1^2}$$

and, in addition,

$$\Big(E \setminus \bigcup_j \gamma_j\Big)^{\triangle} = E^{\triangle} \setminus \bigcup_j \gamma_j. \tag{3}$$

Proof. By Lemma 12, $E \setminus \bigcup\limits_j \gamma_j \in \mathbf{WC_1^2}$. Since the set E is open, by virtue of Theorem 4, $E^{\triangle}$ is also open. Then the set $E^{\triangle} \setminus \bigcup\limits_j \gamma_j$ is also open and, hence, nonempty and, according to Lemma 13, condition (3) is satisfied, i.e., $E \setminus \bigcup\limits_j \gamma_j \notin \mathbf{C_1^2}$.

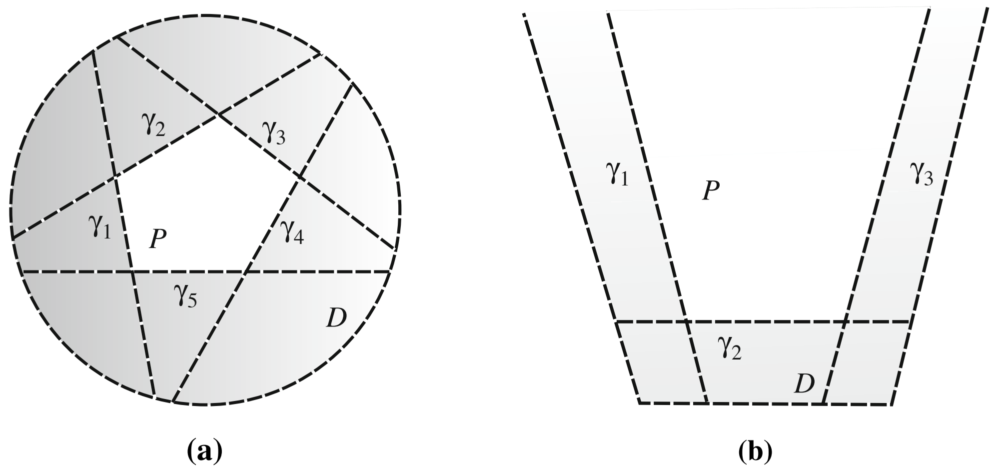


Fig. 9

Theorem 8. *Suppose that the set* $E := (D \setminus \operatorname{Int} P) \setminus \bigcup_{k=1}^{s} \gamma_k \subset \mathbb{R}^2$ *is given. Here,* P *is a convex polygon;* $\gamma_k,\ k = \overline{1, s}$, *are straight lines containing the sides of* P, *and* D *is an arbitrary open convex set that contains* P. *Then* $E \in \mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ *and* $E^{\triangle} = \operatorname{Int} P$.

Proof. The boundary of the set E is formed by the points of ∂D (such that, by virtue of Lemma 5, the straight lines that do not cross D and, hence, do not cross the set E pass through these points) and the segments lying on the straight lines which, by construction, do not cross E. Hence, the set E is weakly 1-convex [see Fig. 9a (with a bounded polygon P) and Fig. 9b (with an unbounded polygon)].

Consider the set $D \setminus P$. It is open and not 1-convex. Moreover,

$$(D \setminus P)^{\triangle} = P. \tag{4}$$

Then

$$(D \setminus P) \setminus \bigcup_{k=1}^{s} \gamma_k = E$$

and, by Lemma 13, in view of (4), we get

$$E^{\triangle} = (D \setminus P)^{\triangle} \setminus \bigcup_{k=1}^{s} \gamma_k = P \setminus \bigcup_{k=1}^{s} \gamma_k = \operatorname{Int} P.$$

Thus, Proposition 1 now directly follows from Theorem 8.

Lemma 14. *Suppose that the set* $E \notin \mathbf{C_1^2}$ *and* $P^0 \subseteq E^{\triangle}$. *Then*

$$\left(E \cup P^0\right)^{\triangle} = E^{\triangle} \setminus P^0.$$

Proof. If $P^0 = E^{\triangle}$, which is equivalent to $E^{\triangle} \setminus P^0 = \varnothing$, then $E \cup P^0 = \operatorname{conv}_1 E$ and, therefore,

$$\left(E \cup P^0\right)^{\triangle} = \varnothing.$$

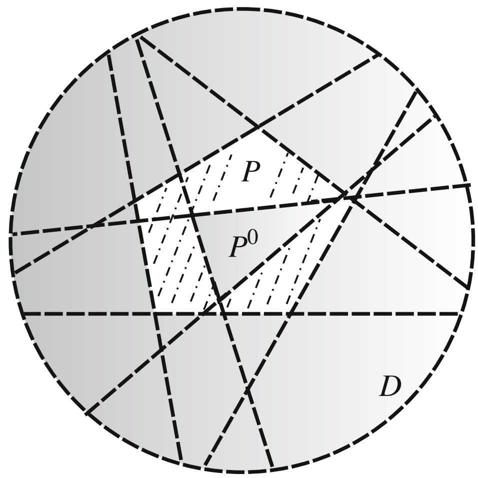


Fig. 10

Now let $P^0 \subsetneq E^{\triangle}$ and $x \in E^{\triangle} \setminus P^0$. Since $x \in E^{\triangle}$, any straight line $\gamma(x)$ crosses the set E and, hence, the set $E \cup P^0$, i.e., $x \in \left(E \cup P^0\right)^{\triangle}$. Let $x \in \left(E \cup P^0\right)^{\triangle}$. Then any straight line $\gamma(x)$ crosses either E or P^0. However, if $\gamma(x) \cap P^0 \neq \varnothing$, then $\gamma(x) \cap E \neq \varnothing$ because $P^0 \subsetneq E^{\triangle}$. Hence, $x \in E^{\triangle}$ and, by Definition 7, $x \notin P^0$. Therefore, $x \in E^{\triangle} \setminus P^0$.

Theorem 9. *For any convex polygon P^0, there exists an open set E_* from the class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ such that $E_*^{\triangle}$ also belongs to the class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ and $(E_*^{\triangle})^{\triangle} = \operatorname{Int} P^0$.*

Proof. Let P be an arbitrary convex polygon such that $P^0 \subset \operatorname{Int} P$ and let D be an arbitrary open convex set containing P (Fig. 10). Further, we construct straight lines $\gamma_k^0,\ k = \overline{1,p}$, containing the sides of P^0 and the straight lines $\gamma_k,\ k = \overline{1,s}$, containing the sides of P. Thus, by Theorem 8, the sets

$$E^0 := (\operatorname{Int} P \setminus \operatorname{Int} P^0) \setminus \bigcup_{k=1}^{p} \gamma_k^0,$$
$$E := (D \setminus \operatorname{Int} P) \setminus \bigcup_{k=1}^{s} \gamma_k \tag{5}$$

belong to the class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ and

$$(E^0)^{\triangle} = \operatorname{Int} P^0, \qquad E^{\triangle} = \operatorname{Int} P. \tag{6}$$

By virtue of Lemma 14, in view of relation (6), we get

$$(E \cup \operatorname{Int} P^0)^{\triangle} = E^{\triangle} \setminus \operatorname{Int} P^0 = \operatorname{Int} P \setminus \operatorname{Int} P^0. \tag{7}$$

We now consider the set

$$E_* := (E \cup \operatorname{Int} P^0) \setminus \bigcup_{k=1}^{p} \gamma_k^0.$$

By virtue of Lemma 13 and relations (7) and (5), we obtain

$$E_*^{\triangle} = (E \cup \operatorname{Int} P^0)^{\triangle} \setminus \bigcup_{k=1}^{p} \gamma_k^0 = \big(\operatorname{Int} P \setminus \operatorname{Int} P^0\big) \setminus \bigcup_{k=1}^{p} \gamma_k^0 = E^0 \neq \varnothing.$$

Thus, $E_* \notin \mathbf{C_1^2}$ and, by using (6), we get $(E_*^{\triangle})^{\triangle} = \operatorname{Int} P^0$.

We now show that $E_* \in \mathbf{WC_1^2}$. First, we note that

$$E_* = (E \cup \operatorname{Int} P^0) \setminus \bigcup_{k=1}^{p} \gamma_k^0 = \left(E \setminus \bigcup_{k=1}^{p} \gamma_k^0\right) \cup \operatorname{Int} P^0$$

and

$$\partial E_* = \partial\left(E \setminus \bigcup_{k=1}^{p} \gamma_k^0\right) \cup \partial(\operatorname{Int} P^0).$$

By Corollary 6, the open set $E \setminus \bigcup\limits_{k=1}^{p} \gamma_k^0$ belongs to the class $\mathbf{WC_1^2} \setminus \mathbf{C_1^2}$ and

$$\left(E \setminus \bigcup_{k=1}^{p} \gamma_k^0\right)^{\triangle} = E^{\triangle} \setminus \bigcup_{k=1}^{p} \gamma_k^0.$$

Since $\operatorname{Int} P^0 \subset E^{\triangle} \setminus \bigcup\limits_{k=1}^{p} \gamma_k^0$, the straight line $\gamma(x)$ that passes through the point

$$x \in \partial\left(E \setminus \bigcup_{k=1}^{p} \gamma_k^0\right)$$

and does not intersect the set $E \setminus \bigcup\limits_{k=1}^{p} \gamma_k^0$ also does not intersect $\operatorname{Int} P^0$ and, hence, the set E_*. If $x \in \partial(\operatorname{Int} P^0)$, then there exists $q \in \{1, \ldots, p\}$ such that $x \in \gamma_q^0$ and $\gamma_q^0 \cap E_* = \varnothing$. Thus, for any point $x \in \partial E_*$, there exists a straight line $\gamma(x)$ that does not cross the set E_*.

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